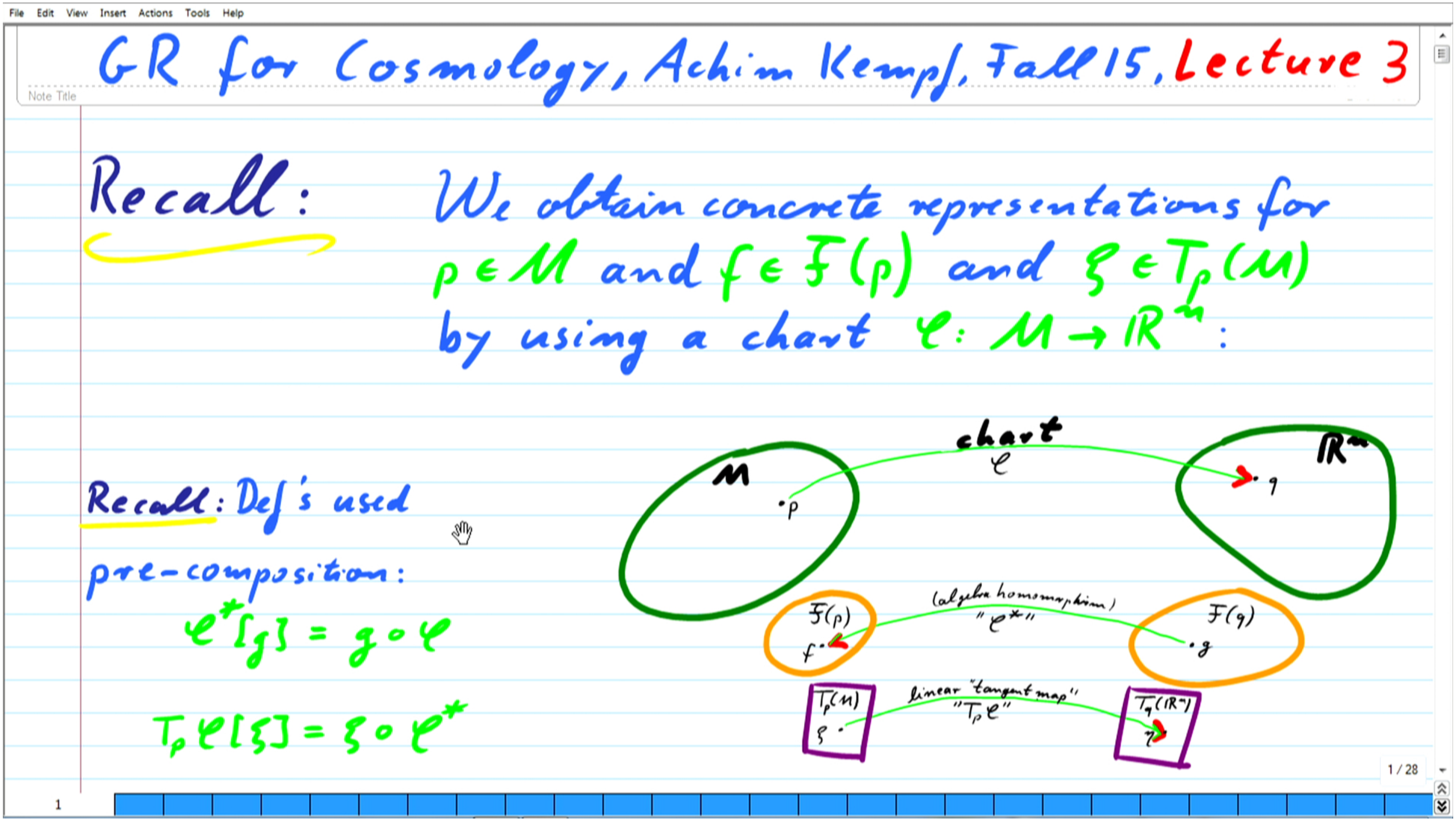
Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 3

Date: Sep 21, 2015 01:30 PM

URL: http://pirsa.org/15090004

Abstract: <p>Course Description coming soon.</p>



GR for Cosmology, Achim Kempf, Fall 15, Lecture 3

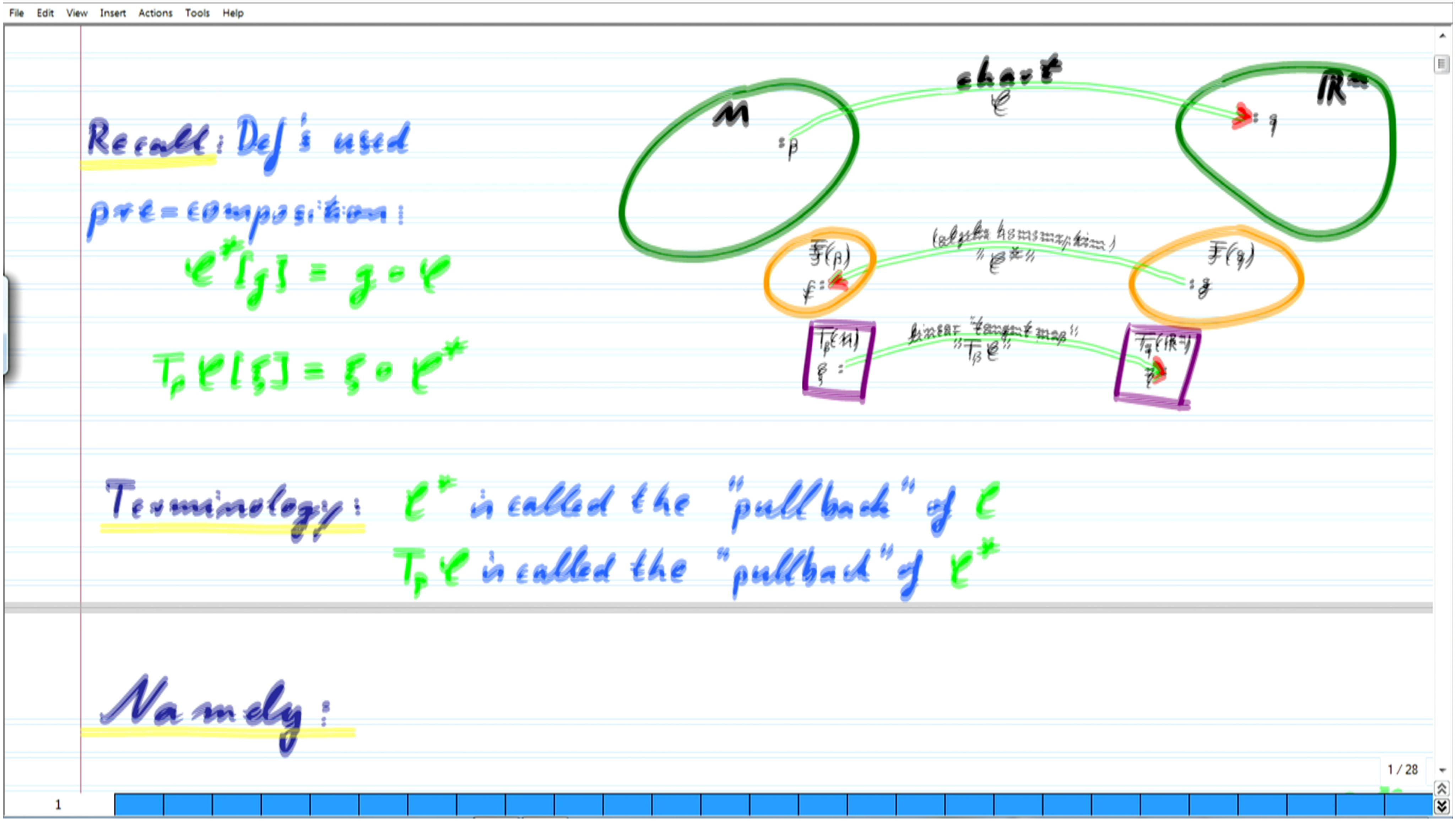
Recall: We obtain concrete representations for $p \in M$ and $f \in \mathcal{F}(p)$ and $\xi \in T_p(M)$ by using a chart $\varphi : M \to \mathbb{R}^n$:

Recall: Def's used

pre-composition:

$$\varphi^*[g] = g \circ \varphi$$

$$T_p\varphi[\xi] = \xi \circ \varphi^*$$



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$$\varphi^*[g] = g \circ \varphi$$

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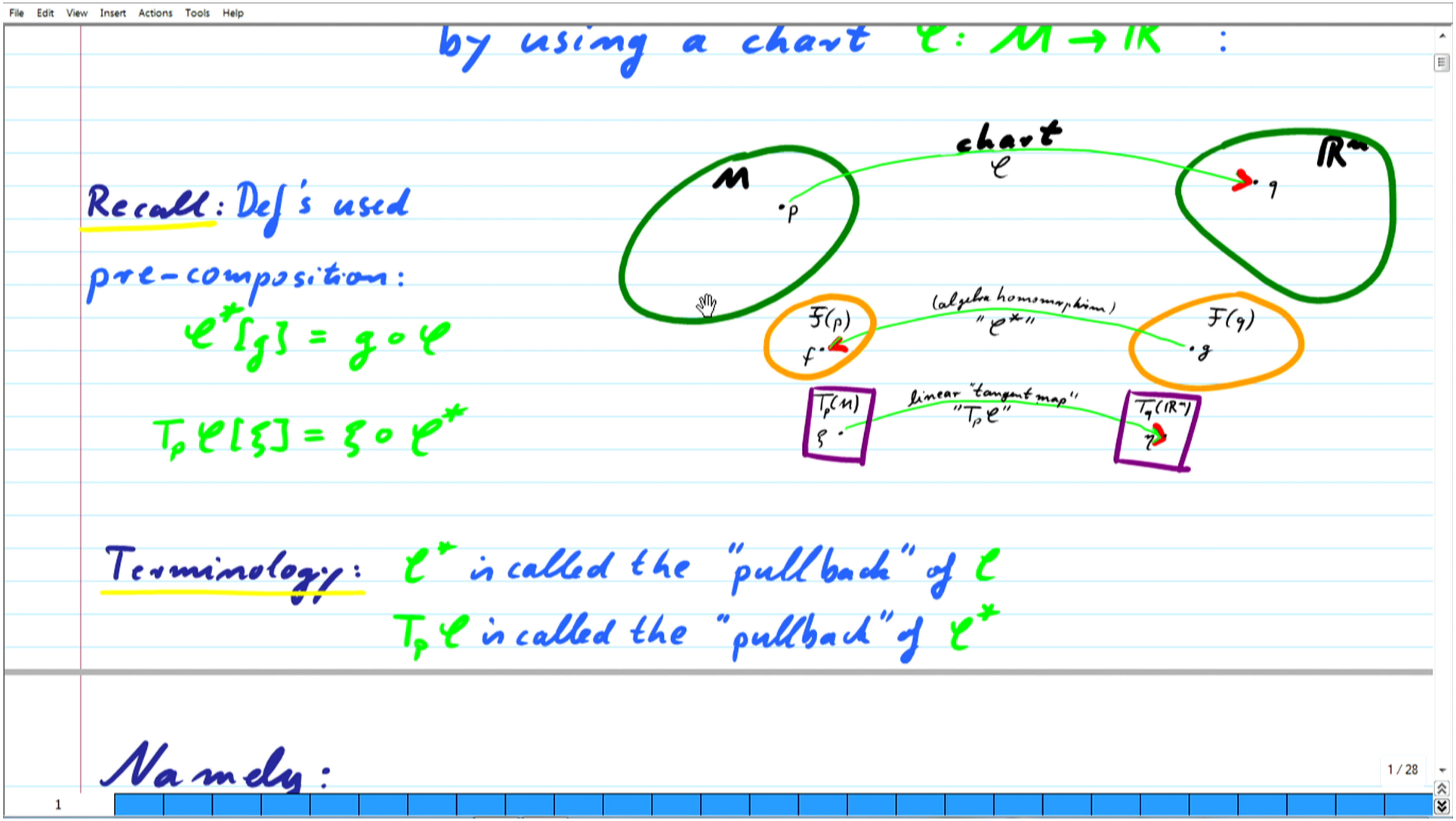
Terminology: φ^* is called the "pull back" of φ

$T_p\varphi$ is called the "pullback" of φ^*

Namely:



by using a chart $\varphi: M \to \mathbb{R}^n$:

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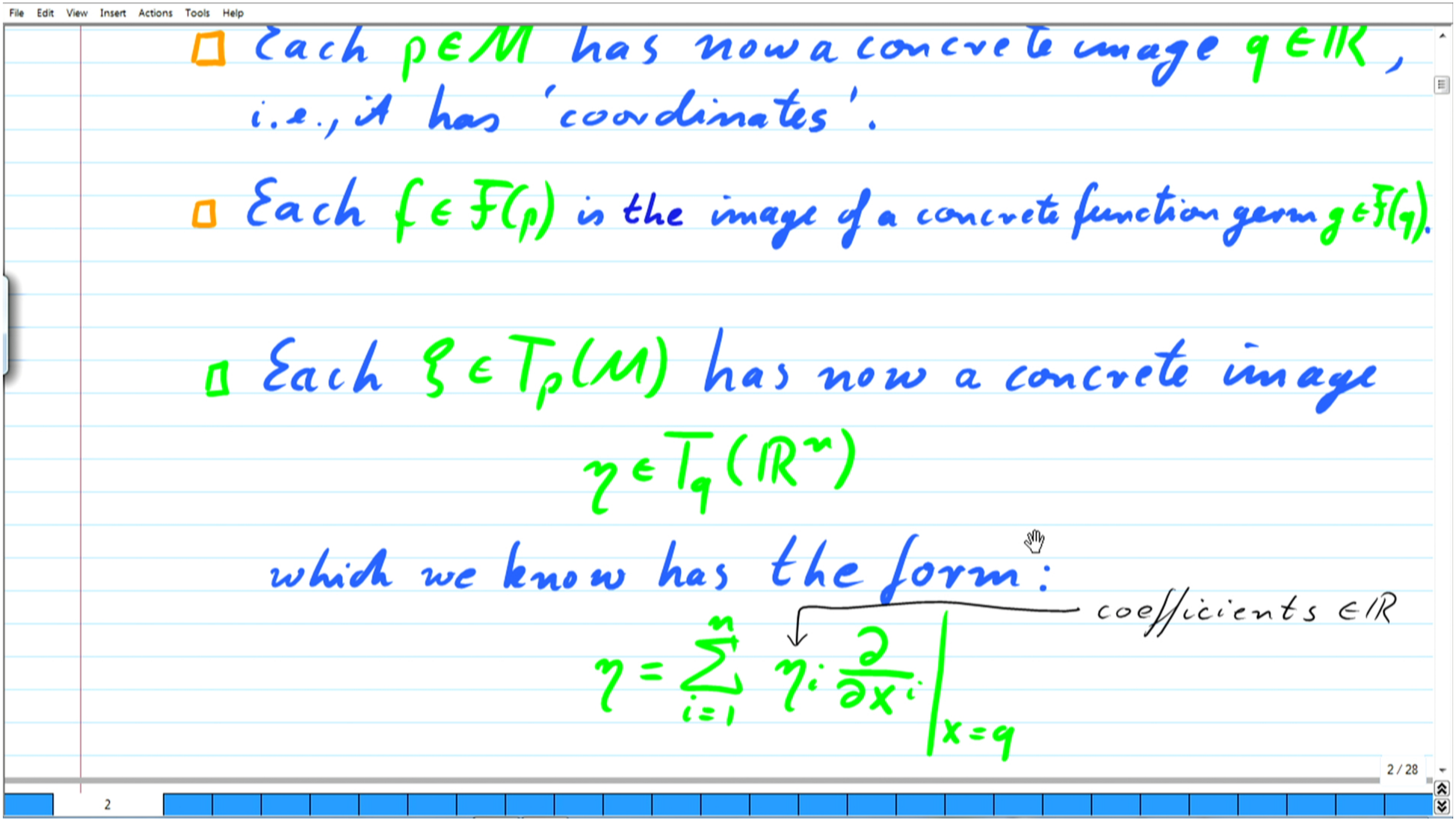


☐ Each $p \in M$ has now a concrete image $q \in \mathbb{R}$, i.e., it has 'coordinates'.

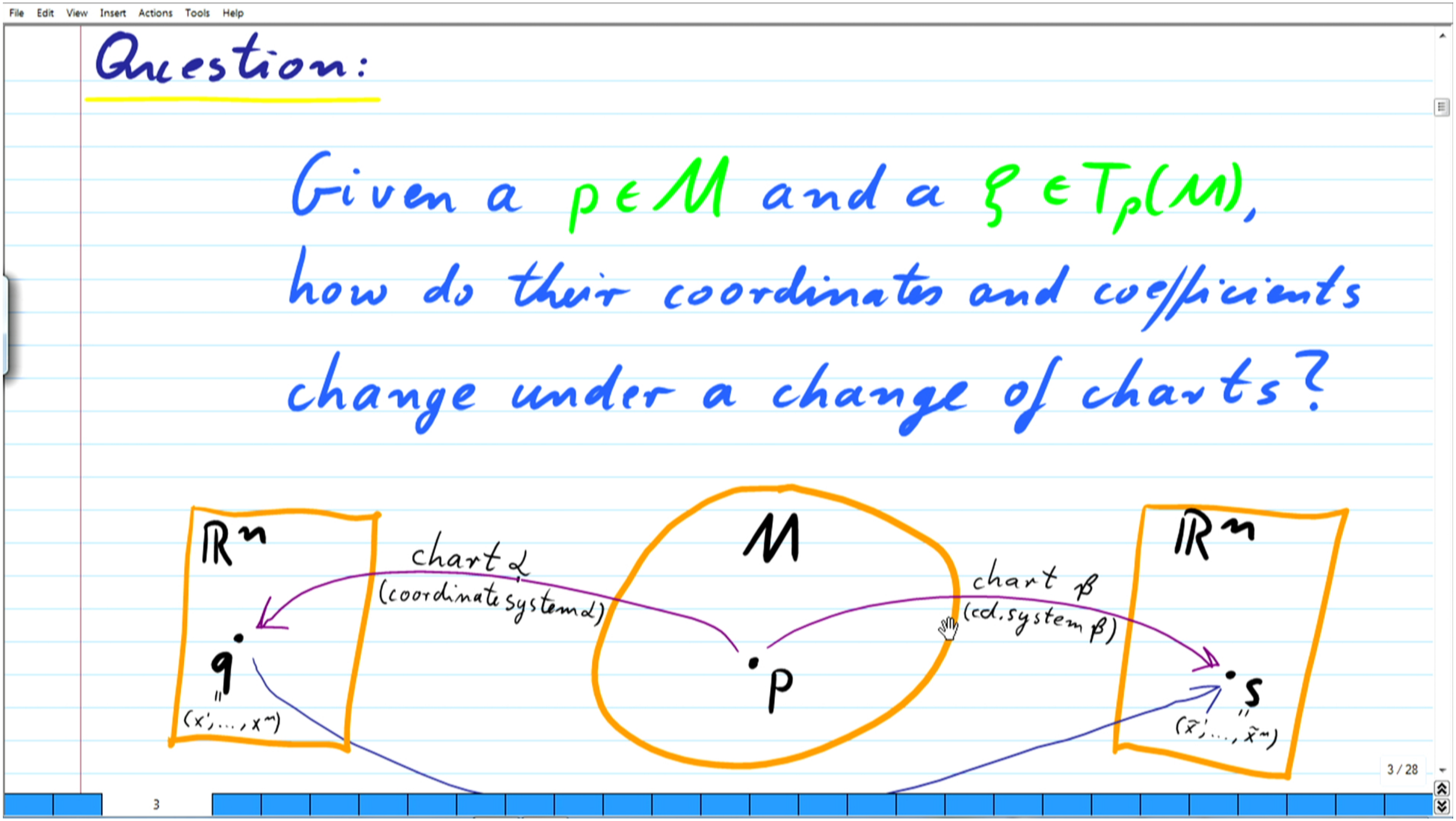
☐ Each $f \in \mathcal{F}(p)$ is the image of a concrete function germ $g \in \mathcal{F}(q)$.

☐ Each $\xi \in T_p(M)$ has now a concrete image

$$\eta \in T_q(\mathbb{R}^n)$$

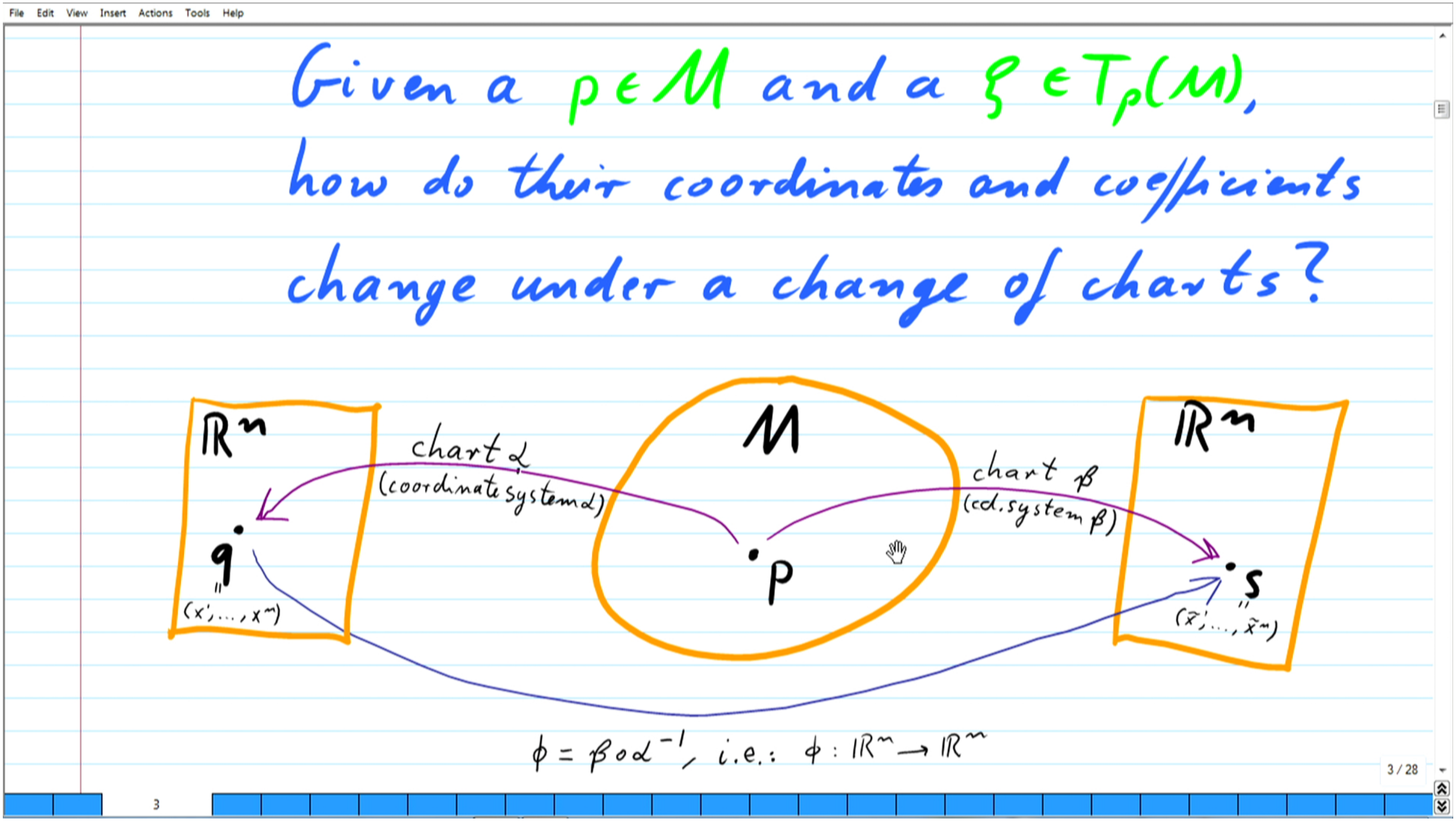
which we know has the form:

$$\eta = \sum_{i=1}^{n} \overset{\text{coefficients } \in \mathbb{R}}{\eta^i} \frac{\partial}{\partial x^i}\Big|_{x=q}$$



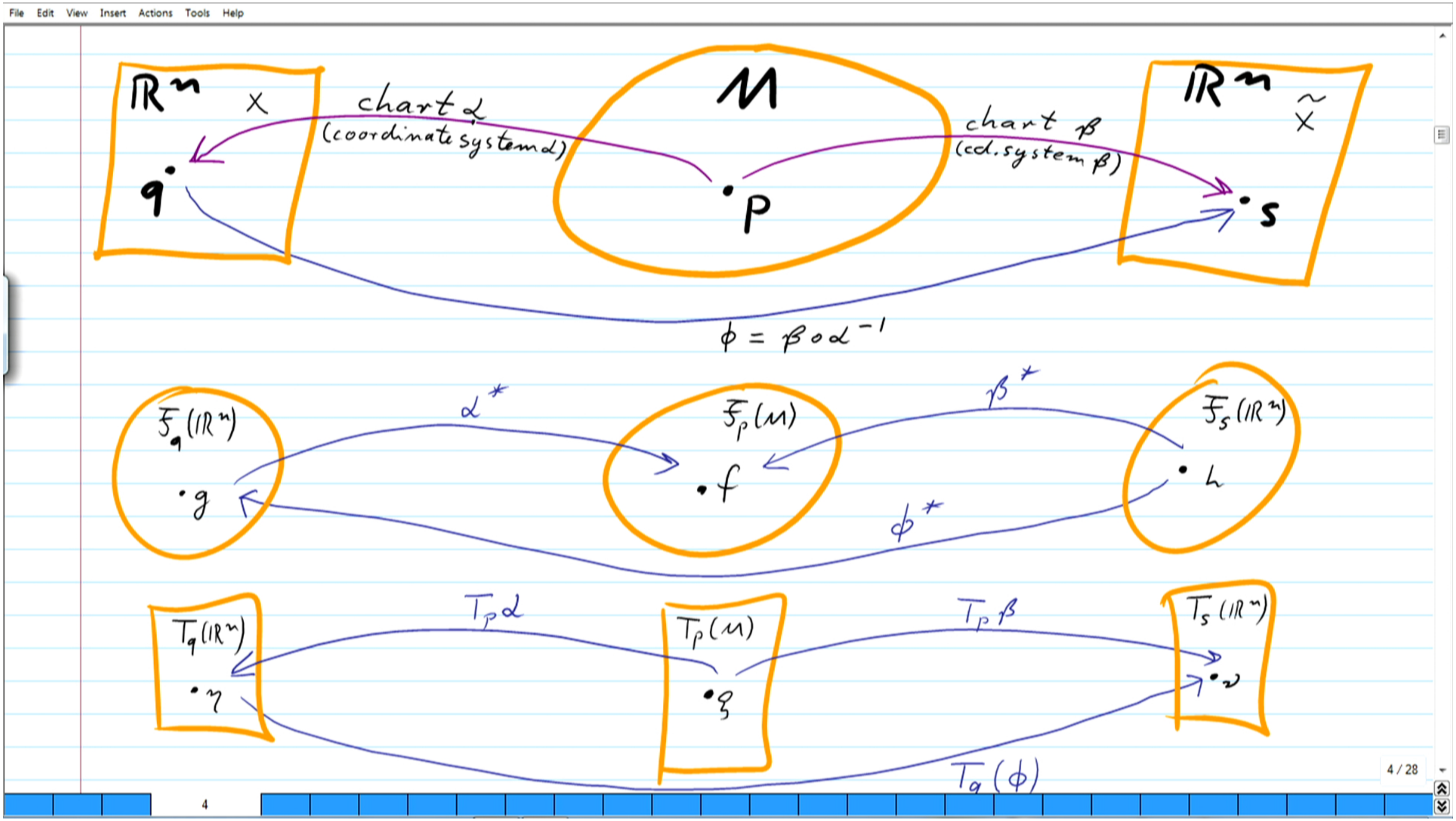
Question:

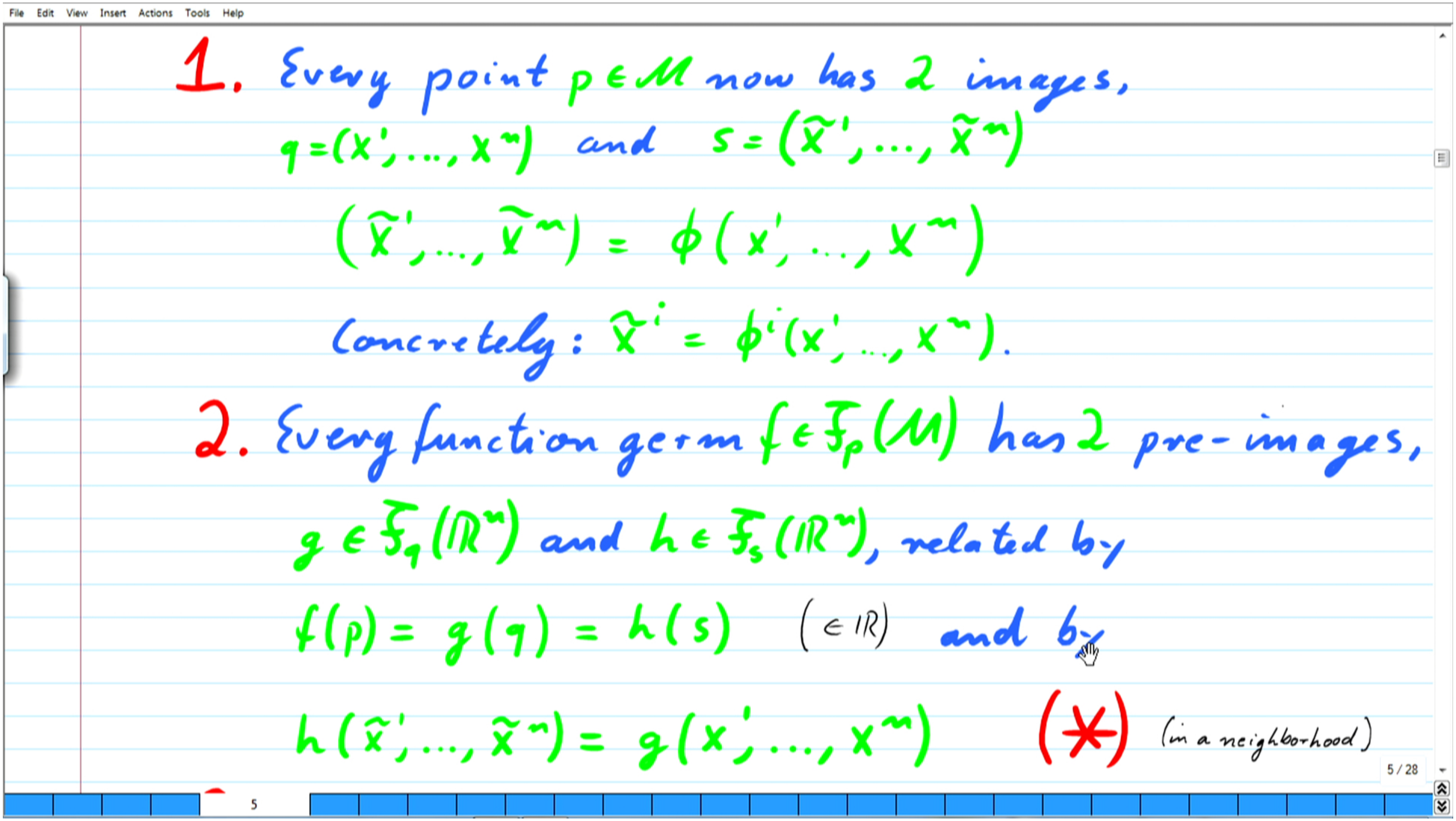
Given a $p \in M$ and a $\xi \in T_p(M)$, how do their coordinates and coefficients change under a change of charts?



Given a $p \in M$ and a $\xi \in T_p(M)$, how do their coordinates and coefficients change under a change of charts?

$\phi = \beta \circ \alpha^{-1}$, i.e.: $\phi : \mathbb{R}^n \to \mathbb{R}^n$


File Edit View Insert Actions Tools Help
$\mathbb{R}^n$
X
q
chart α
(coordinate system α)
M
p
chart β
(cd. system β)
$\mathbb{R}^n$
$\tilde{X}$
s
$\phi = \beta \circ \alpha^{-1}$
$\mathcal{F}_q(\mathbb{R}^n)$
g
α^*
$\mathcal{F}_p(M)$
f
β^*
$\mathcal{F}_s(\mathbb{R}^n)$
h
ϕ^*
$T_q(\mathbb{R}^n)$
η
$T_p\alpha$
$T_p(M)$
ξ
$T_p\beta$
$T_s(\mathbb{R}^n)$
ν
$T_q(\phi)$
4 / 28
4



1. Every point $p \in \mathcal{M}$ now has 2 images,

$q = (x^1, \dots, x^n)$ and $s = (\tilde{x}^1, \dots, \tilde{x}^n)$

$$(\tilde{x}^1, \dots, \tilde{x}^n) = \phi(x^1, \dots, x^n)$$

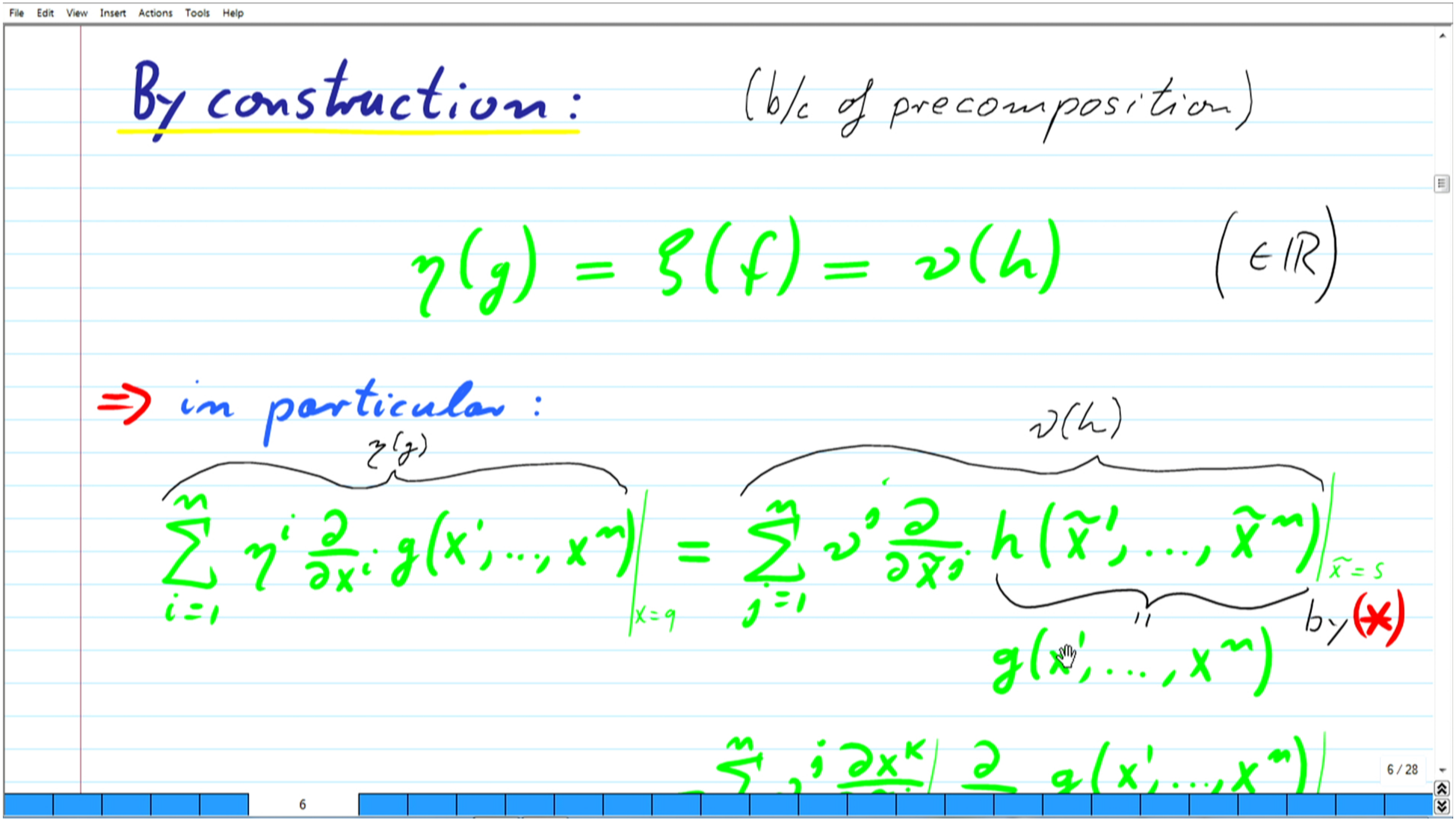
Concretely: $\tilde{x}^i = \phi^i(x^1, \dots, x^n)$.

2. Every function germ $f \in \mathcal{F}_p(\mathcal{M})$ has 2 pre-images,

$g \in \mathcal{F}_q(\mathbb{R}^n)$ and $h \in \mathcal{F}_s(\mathbb{R}^n)$, related by

$f(p) = g(q) = h(s)$ $(\in \mathbb{R})$ and by

$h(\tilde{x}^1, \dots, \tilde{x}^n) = g(x^1, \dots, x^n)$ (*) (in a neighborhood)



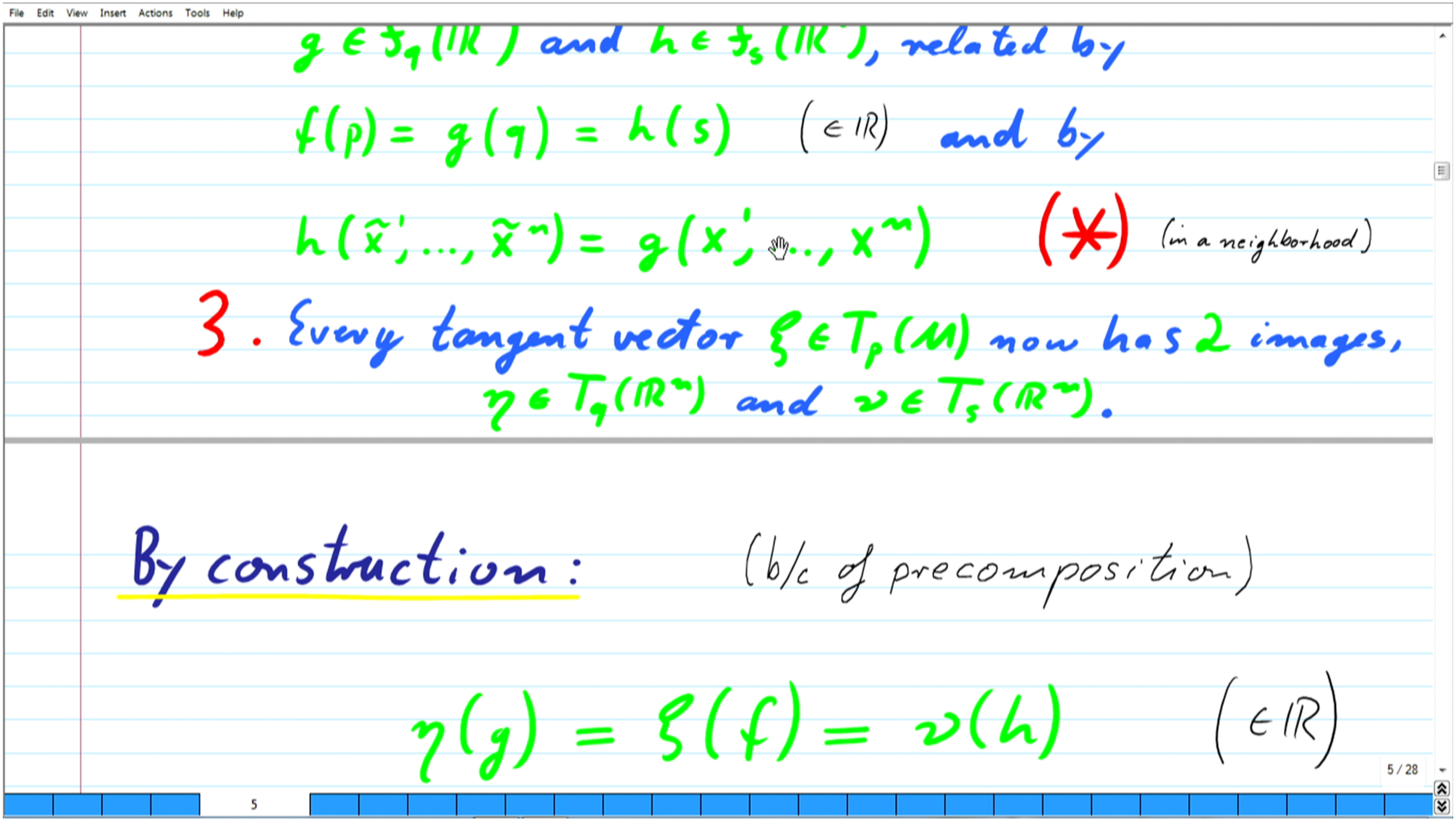
By construction: (b/c of precomposition)

$$\eta(g) = \xi(f) = \nu(h) \qquad (\in \mathbb{R})$$

⇒ in particular:

$$\overbrace{\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} g(x^1,\dots,x^n)}^{\eta(g)}\Big|_{x=q} = \overbrace{\sum_{j=1}^{n} \nu^j \frac{\partial}{\partial \tilde{x}^j} \underbrace{h(\tilde{x}^1,\dots,\tilde{x}^n)}_{\overset{''}{g(x^1,\dots,x^n)}}}^{\nu(h)}\Big|_{\tilde{x}=s} \text{ by } (*)$$

$$= \sum^{n} \nu^j \frac{\partial x^k}{}\Big| \frac{\partial}{} g(x^1,\dots,x^n)\Big|$$



$g \in \mathcal{F}_q(\mathbb{R}\ )$ and $h \in \mathcal{F}_s(\mathbb{R}\ )$, related by

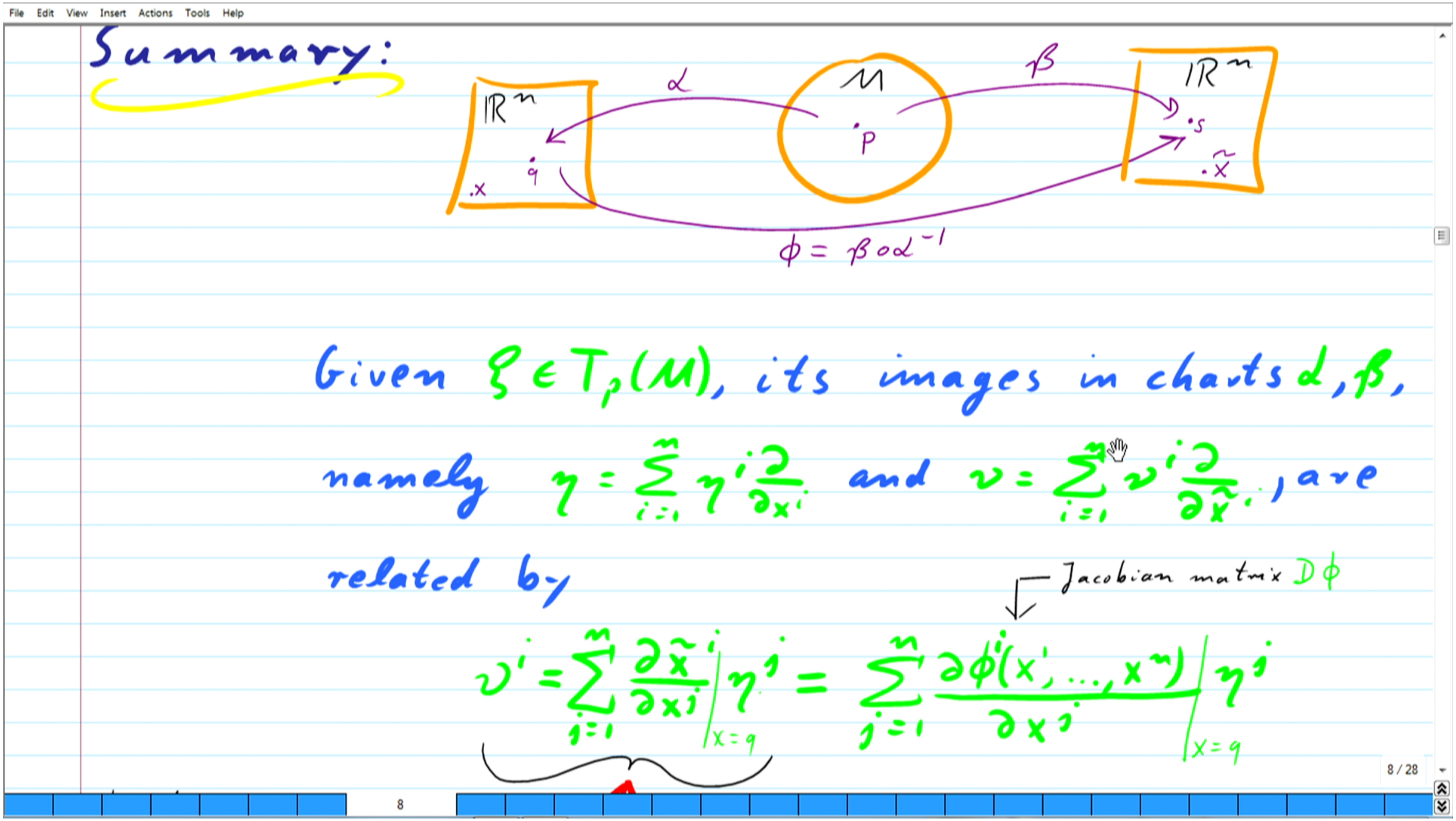
$f(p) = g(q) = h(s) \quad (\in \mathbb{R})$ and by

$$h(\tilde{x}^1, \ldots, \tilde{x}^n) = g(x^1, \ldots, x^n) \qquad (*)$$ (in a neighborhood)

3. Every tangent vector $\xi \in T_p(M)$ now has 2 images, $\eta \in T_q(\mathbb{R}^n)$ and $\nu \in T_s(\mathbb{R}^n)$.

By construction: (b/c of precomposition)

$$\eta(g) = \xi(f) = \nu(h) \qquad (\in \mathbb{R})$$

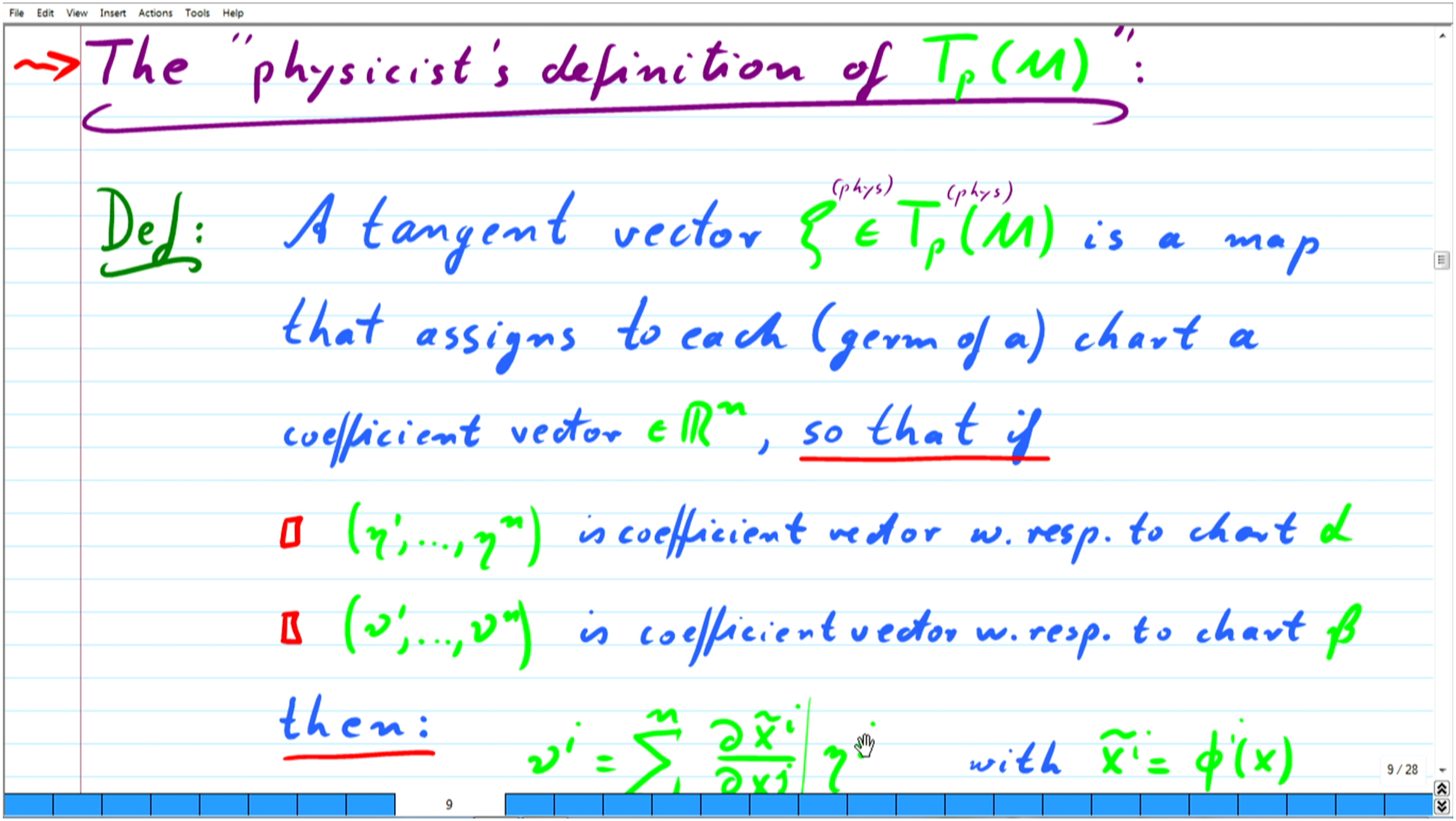


Summary:

Given $\xi \in T_p(M)$, its images in charts α, β, namely $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}$ and $\nu = \sum_{i=1}^{n} \nu^i \frac{\partial}{\partial \tilde{x}^i}$, are related by

Jacobian matrix $D\phi$

$$\nu^i = \underbrace{\sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}\Big|_{x=q} \eta^j}_{} = \sum_{j=1}^{n} \frac{\partial \phi^i(x^1, \dots, x^n)}{\partial x^j}\Big|_{x=q} \eta^j$$

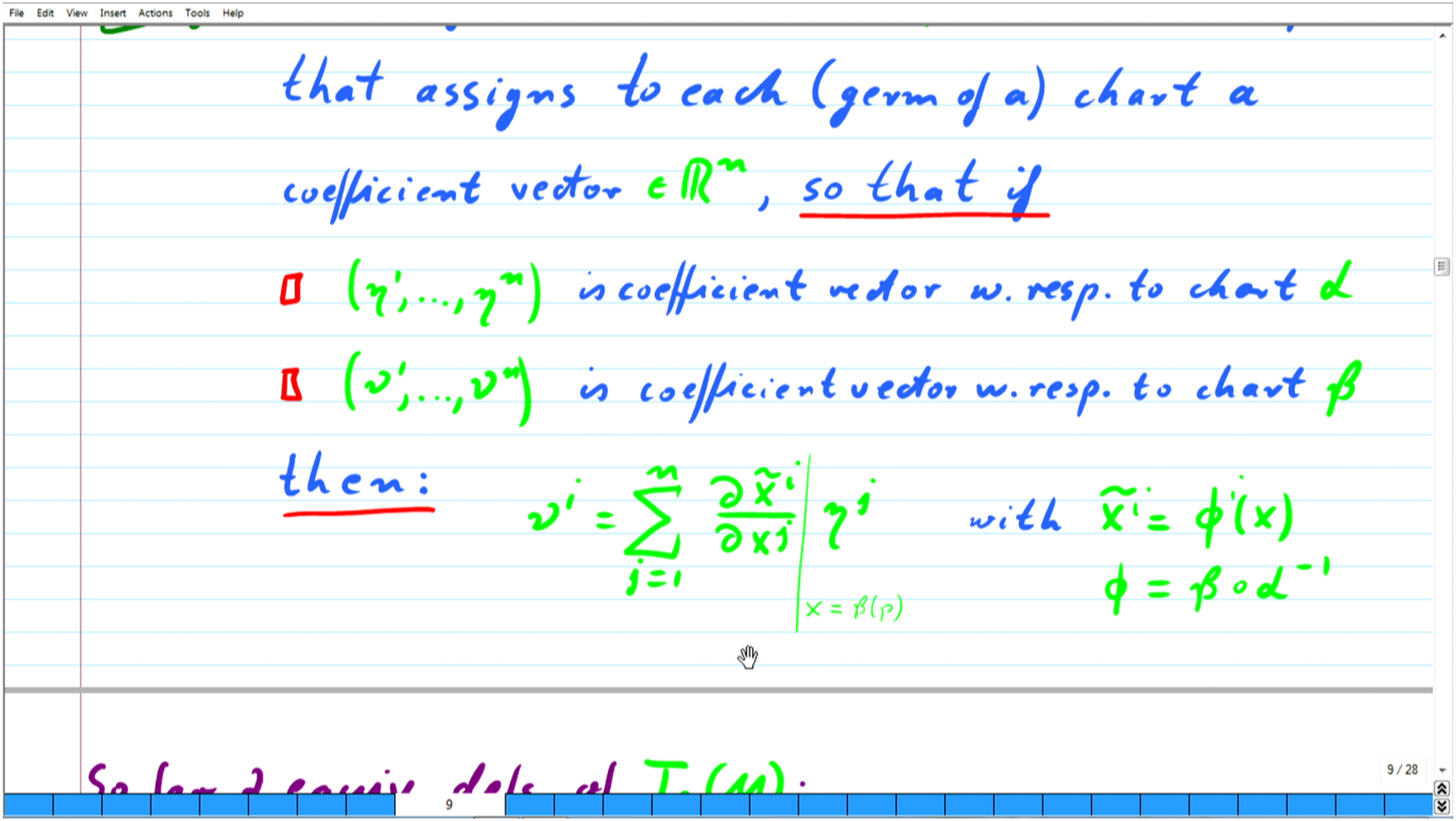


→ The "physicist's definition of $T_p(M)$":

Def: A tangent vector $\xi^{(phys)} \in T_p^{(phys)}(M)$ is a map that assigns to each (germ of a) chart a coefficient vector $\in \mathbb{R}^n$, so that if

- ☐ $(\eta^1, \dots, \eta^n)$ is coefficient vector w. resp. to chart α
- ☐ $(\nu^1, \dots, \nu^n)$ is coefficient vector w. resp. to chart β

then: $\nu^i = \sum_{j}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}\Big| \eta^j$ with $\tilde{x}^i = \phi^i(x)$



that assigns to each (germ of a) chart a coefficient vector $\in \mathbb{R}^n$, so that if

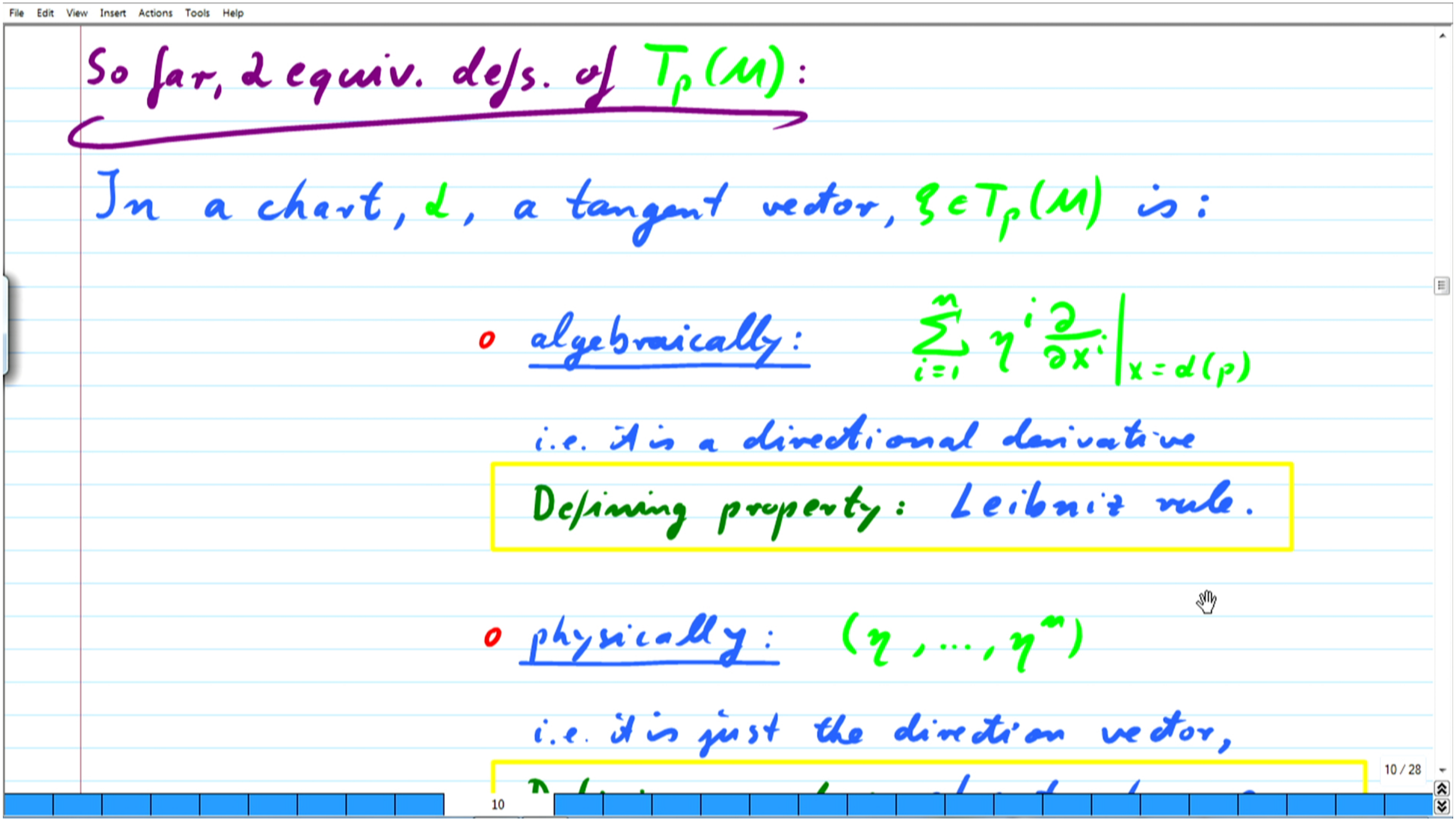
☐ $(\eta^1,\dots,\eta^n)$ is coefficient vector w. resp. to chart α

☐ $(v^1,\dots,v^n)$ is coefficient vector w. resp. to chart β

then:

$$v^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}\Bigg|_{x=\beta(p)} \eta^j \qquad \text{with } \tilde{x}^i = \phi^i(x), \quad \phi = \beta \circ \alpha^{-1}$$

So far 2 equiv. defs of $T_p(M)$:



So far, 2 equiv. defs. of $T_p(M)$:

In a chart, α, a tangent vector, $\xi \in T_p(M)$ is:

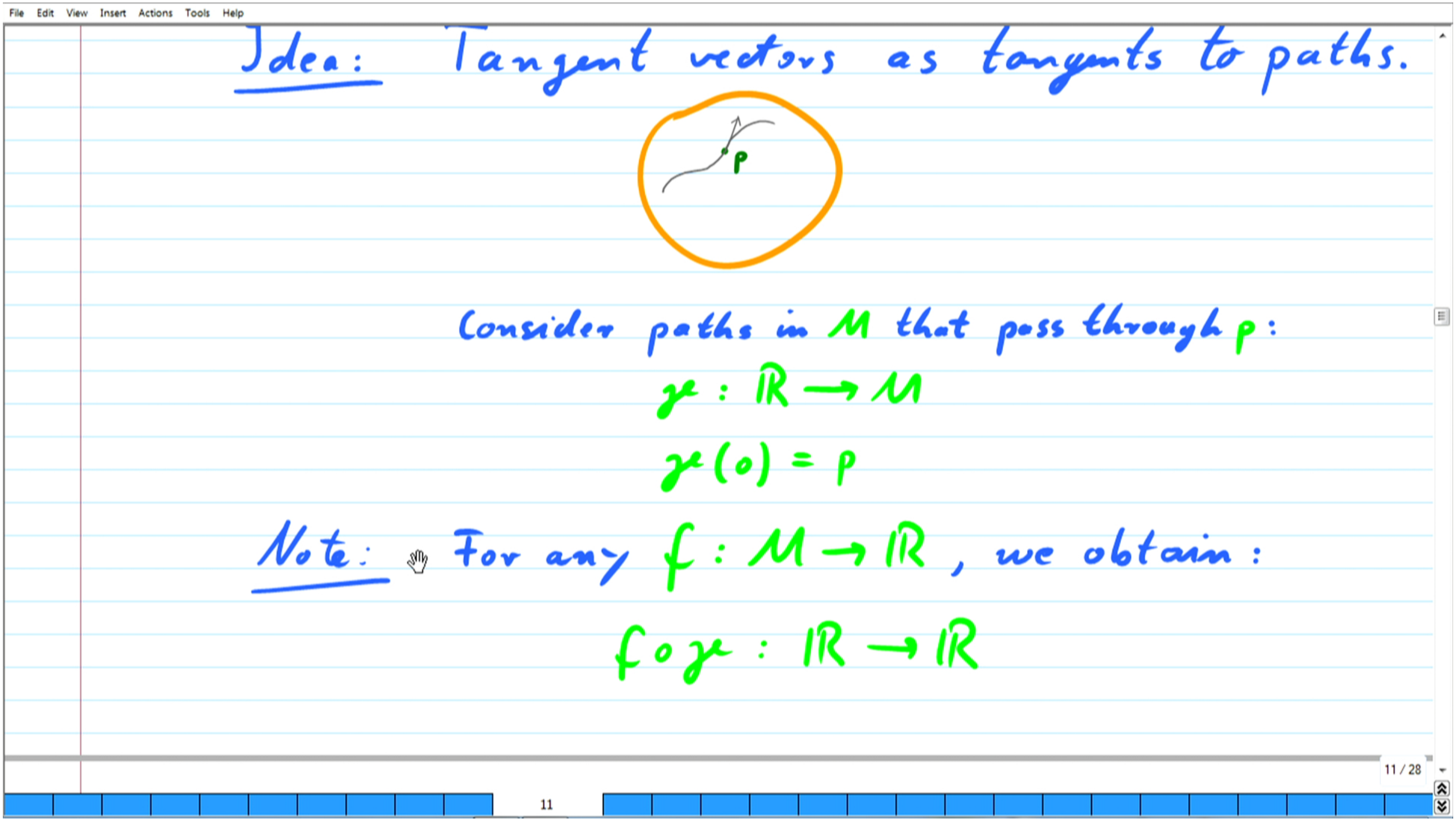
- algebraically: $\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=\alpha(p)}$

  i.e. it is a directional derivative

  Defining property: Leibniz rule.

- physically: $(\eta, \dots, \eta^n)$

  i.e. it is just the direction vector,



Idea: Tangent vectors as tangents to paths.

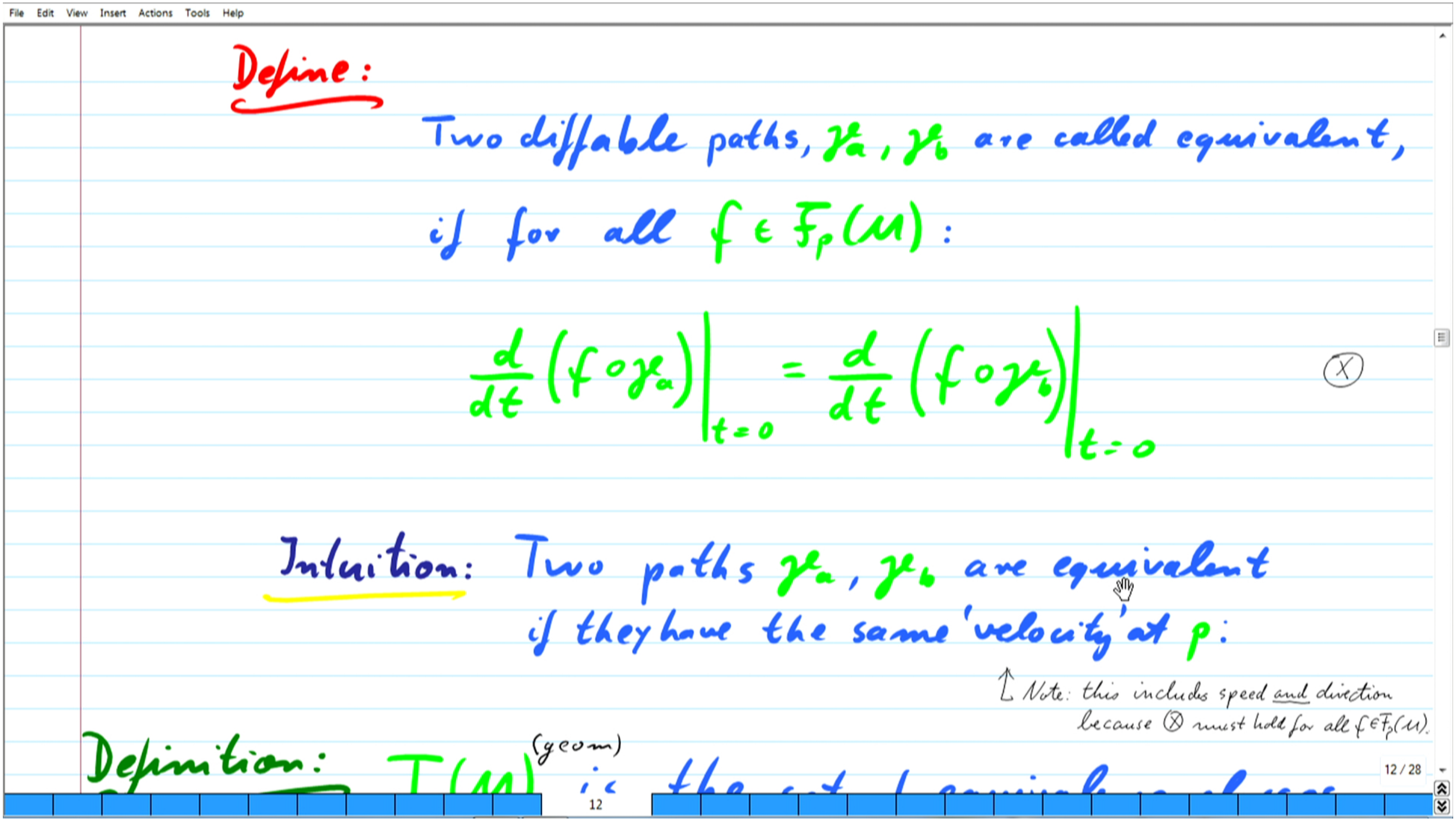
Consider paths in M that pass through p:

$$\gamma : \mathbb{R} \to M$$

$$\gamma(0) = p$$

Note: For any $f : M \to \mathbb{R}$, we obtain:

$$f \circ \gamma : \mathbb{R} \to \mathbb{R}$$



Define:

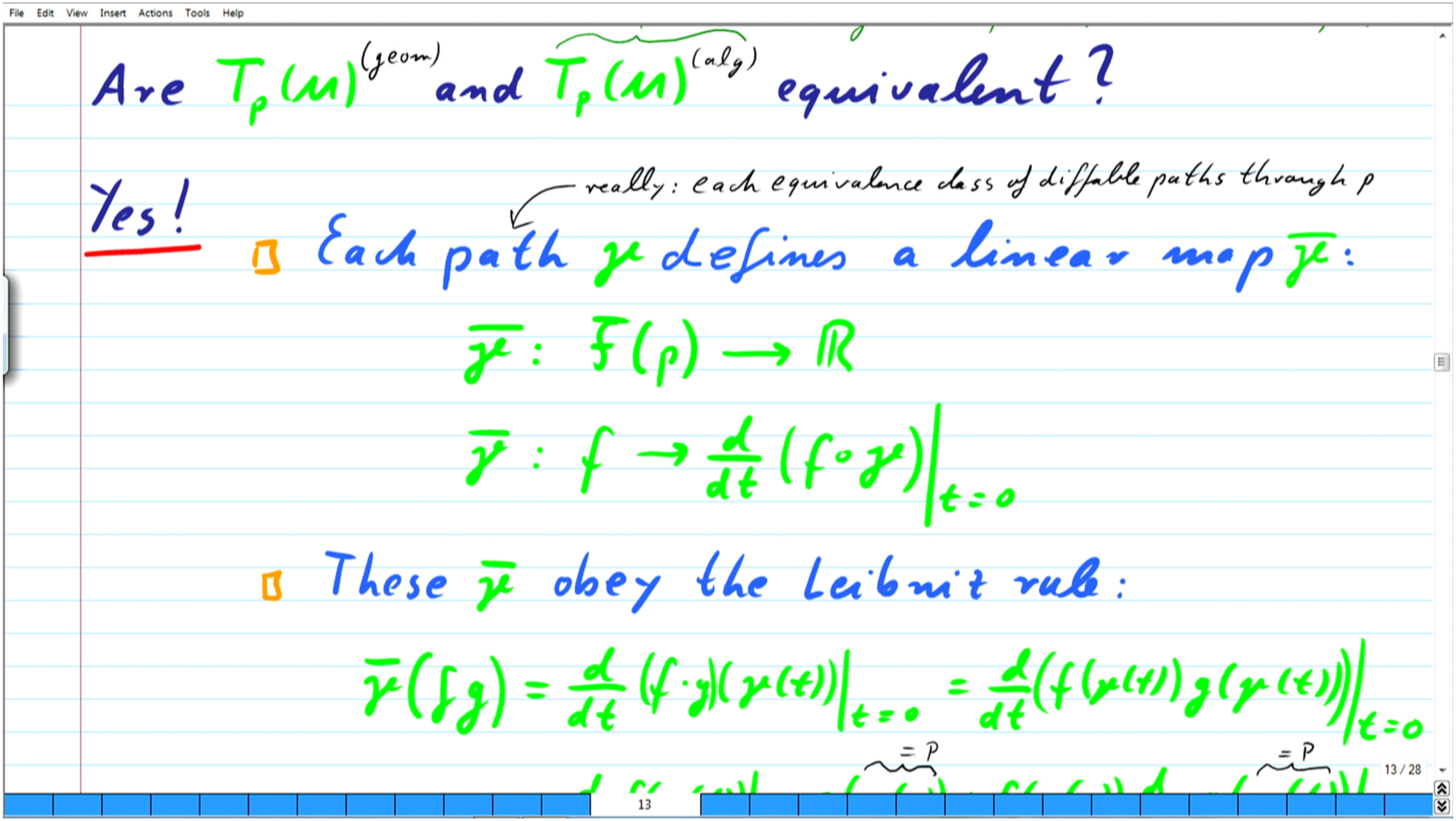
Two diffable paths, γ_a, γ_b are called equivalent, if for all $f \in \mathcal{F}_p(\mathcal{M})$:

$$\frac{d}{dt}\left(f \circ \gamma_a\right)\Big|_{t=0} = \frac{d}{dt}\left(f \circ \gamma_b\right)\Big|_{t=0} \qquad \textcircled{X}$$

Intuition: Two paths γ_a, γ_b are equivalent if they have the same 'velocity' at p:

↑ Note: this includes speed and direction because $\textcircled{X}$ must hold for all $f \in \mathcal{F}_p(\mathcal{M})$.

Definition: $T(\mathcal{M})^{(geom)}$ is the set of equivalence classes



Are $T_p(M)^{(geom)}$ and $T_p(M)^{(alg)}$ equivalent?

Yes!

really: each equivalence class of diff'able paths through p

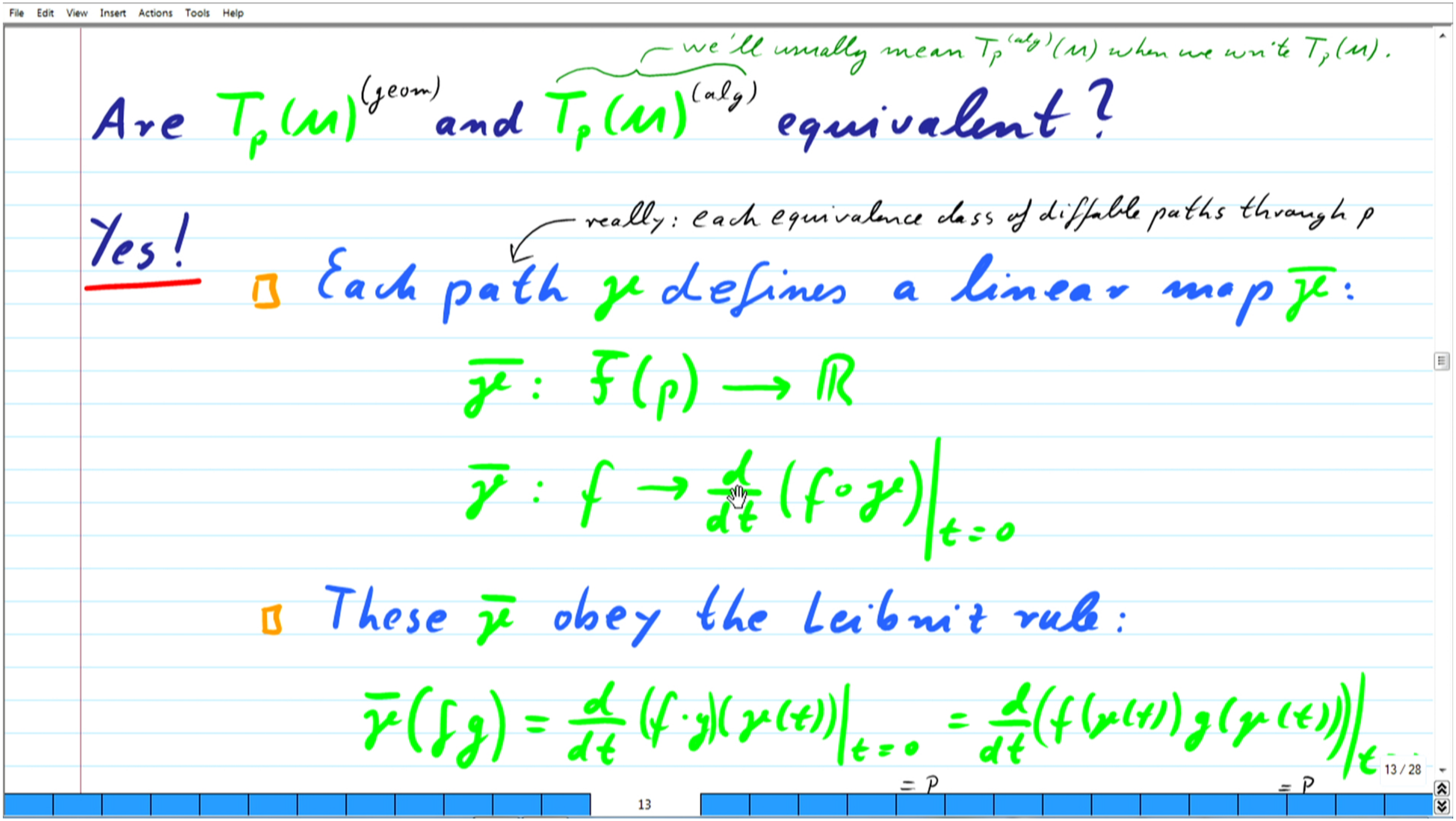
▫ Each path γ defines a linear map $\bar{\gamma}$:

$$\bar{\gamma} : \mathcal{F}(p) \longrightarrow \mathbb{R}$$

$$\bar{\gamma} : f \to \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}$$

▫ These $\bar{\gamma}$ obey the Leibniz rule:

$$\bar{\gamma}(fg) = \frac{d}{dt}(f \cdot g)(\gamma(t))\Big|_{t=0} = \frac{d}{dt}\big(f(\gamma(t))\,g(\gamma(t))\big)\Big|_{t=0}$$



we'll usually mean $T_p^{(alg)}(M)$ when we write $T_p(M)$.

Are $T_p(M)^{(geom)}$ and $T_p(M)^{(alg)}$ equivalent?

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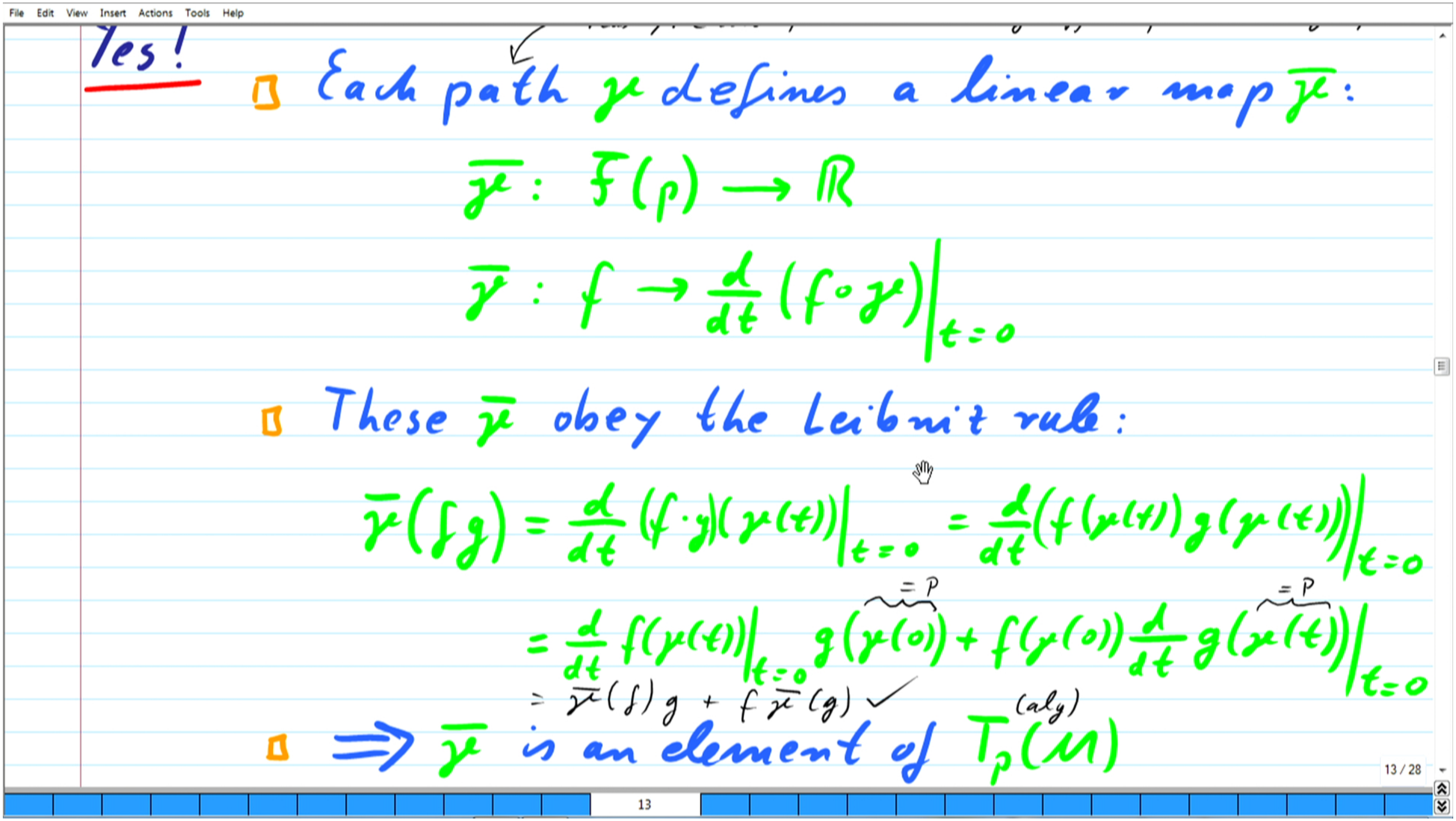
$$\bar{\gamma}(fg) = \frac{d}{dt}(f \cdot g)(\gamma(t))\Big|_{t=0} = \frac{d}{dt}\big(f(\gamma(t))\, g(\gamma(t))\big)\Big|_{t}$$

$= p$ $\quad$ $= p$



Yes!

- Each path γ defines a linear map $\bar{\gamma}$:

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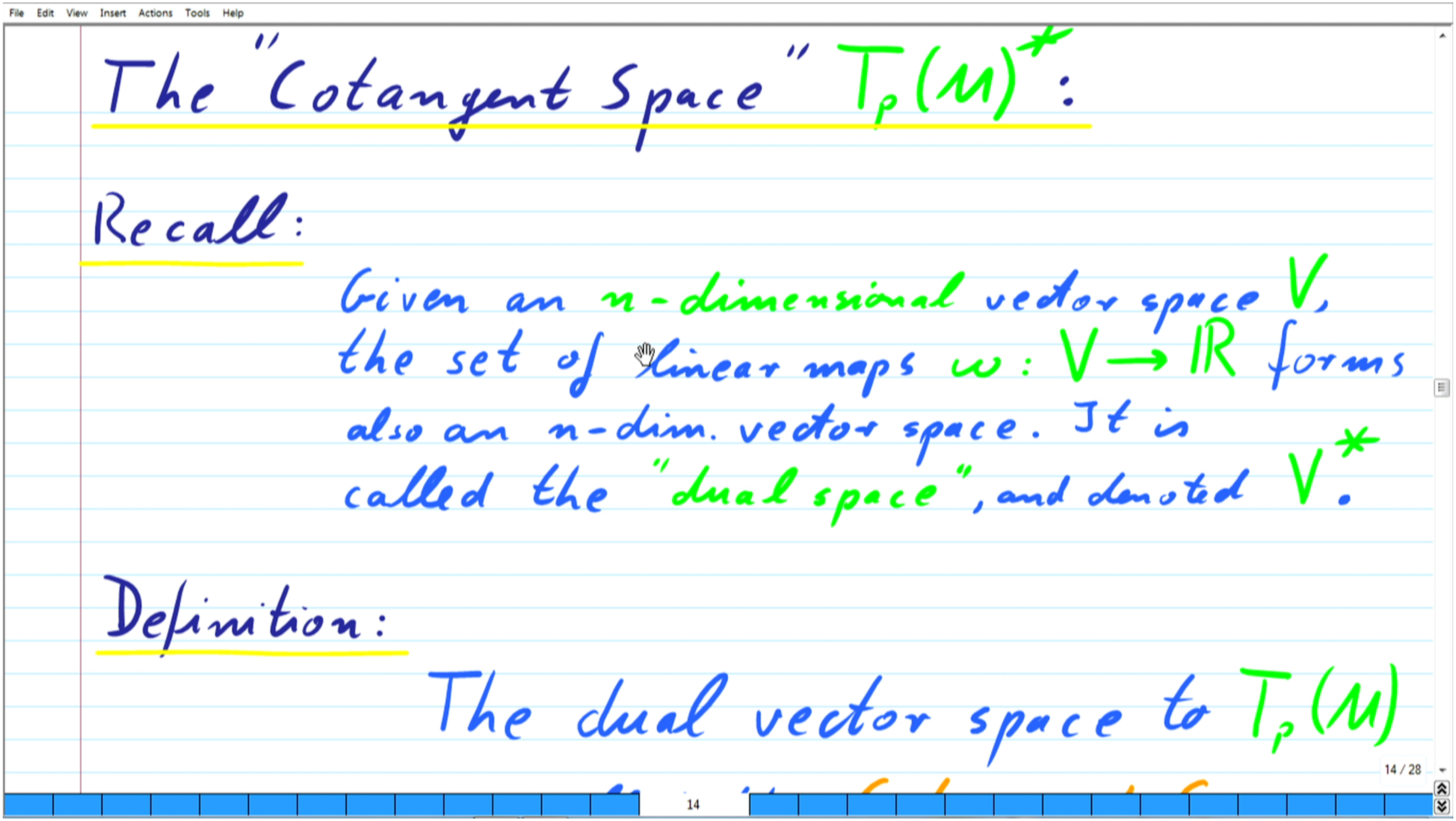
- These $\bar{\gamma}$ obey the Leibnitz rule:

$$\bar{\gamma}(fg) = \frac{d}{dt}(f \cdot g)(\gamma(t))\Big|_{t=0} = \frac{d}{dt}(f(\gamma(t))\, g(\gamma(t)))\Big|_{t=0}$$

$$= \frac{d}{dt} f(\gamma(t))\Big|_{t=0} g(\overbrace{\gamma(0)}^{=p}) + f(\gamma(0)) \frac{d}{dt} g(\overbrace{\gamma(t)}^{=p})\Big|_{t=0}$$

$$= \bar{\gamma}(f)\, g + f\, \bar{\gamma}(g) \checkmark$$

- $\Rightarrow \bar{\gamma}$ is an element of $T_p^{(alg)}(\mathcal{M})$



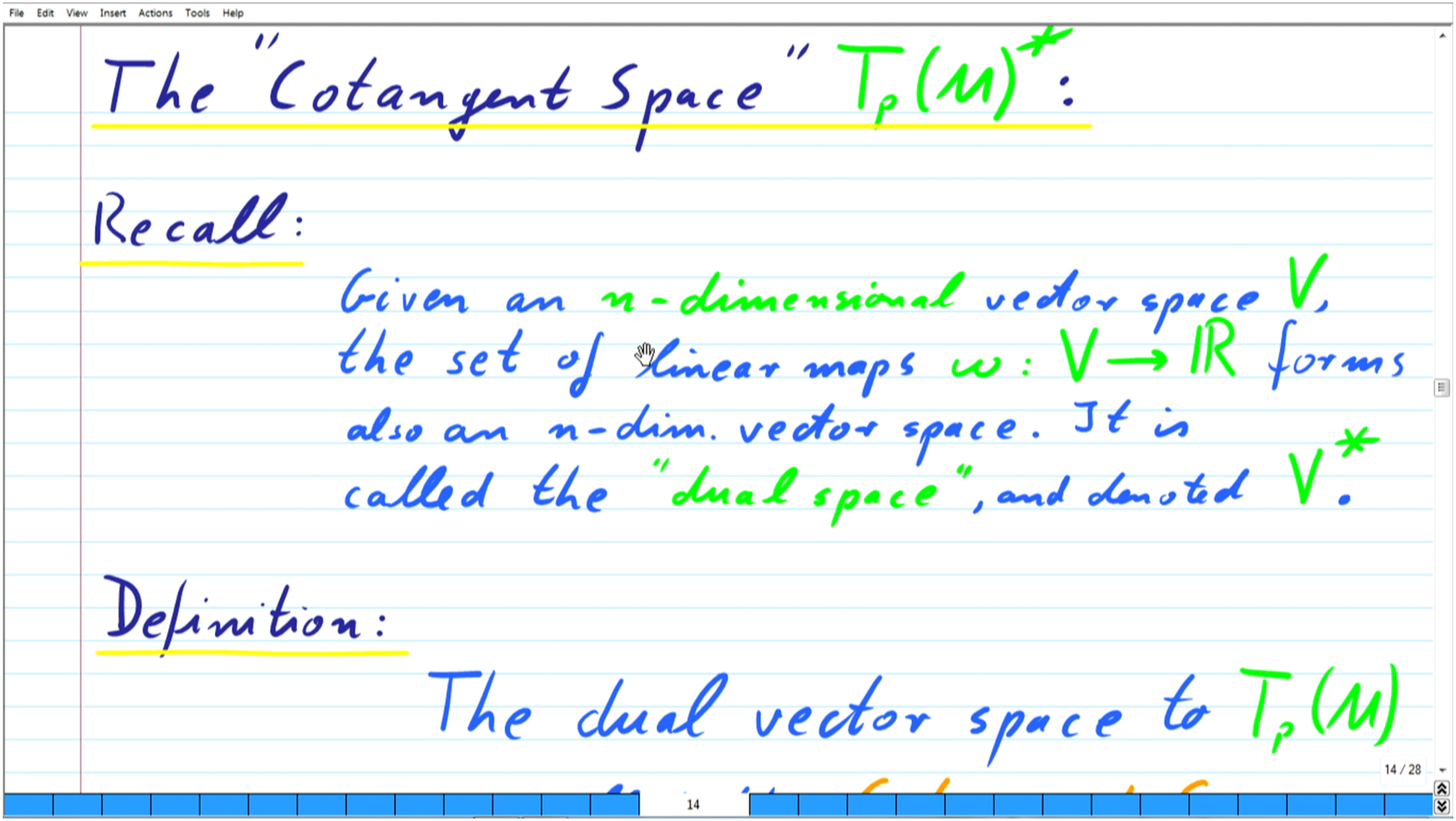
The "Cotangent Space" $T_p(M)^*$:

Recall:

Given an n-dimensional vector space V, the set of linear maps $\omega : V \to \mathbb{R}$ forms also an n-dim. vector space. It is called the "dual space", and denoted V^*.

Definition:

The dual vector space to $T_p(M)$



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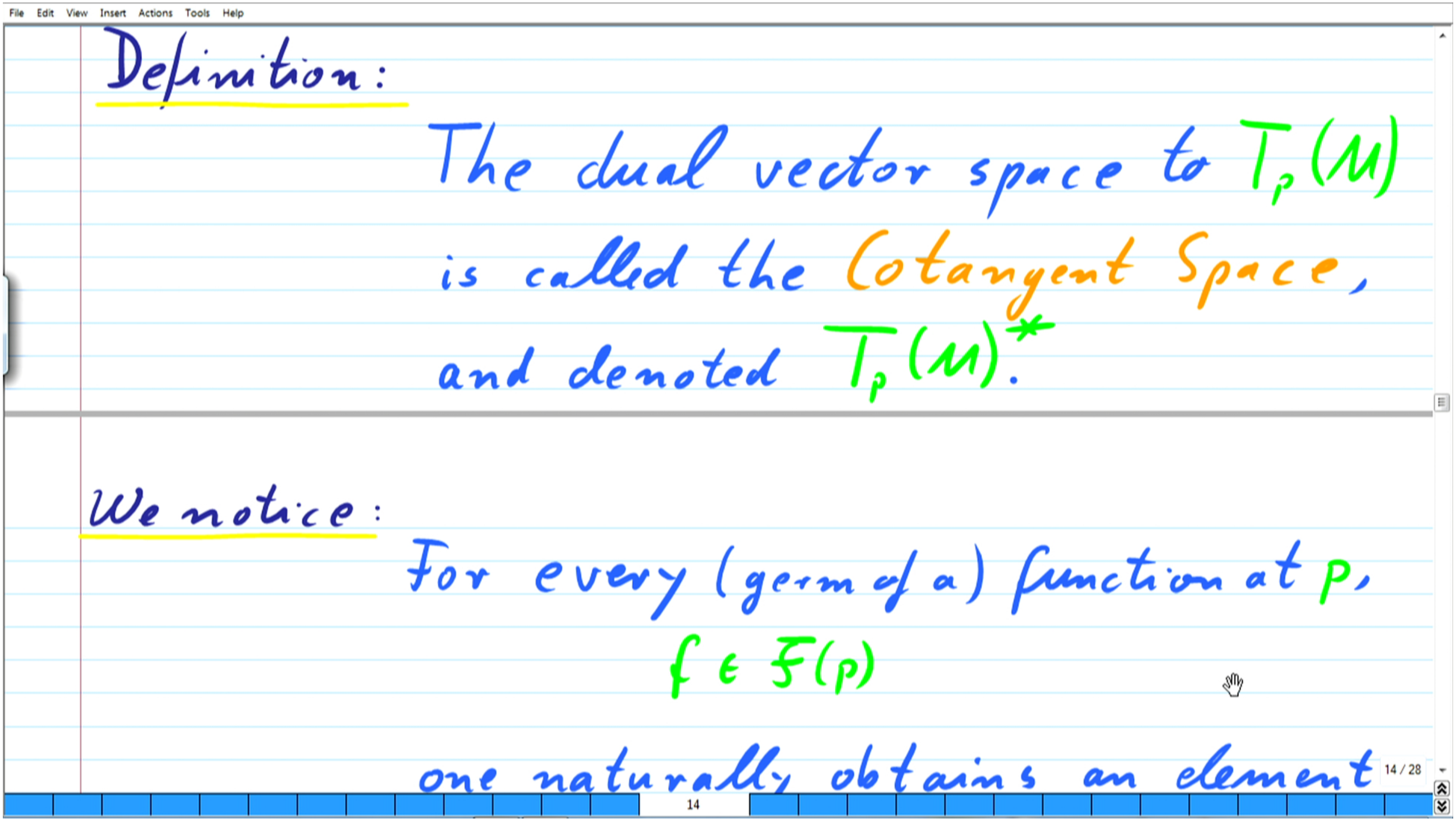


Definition:

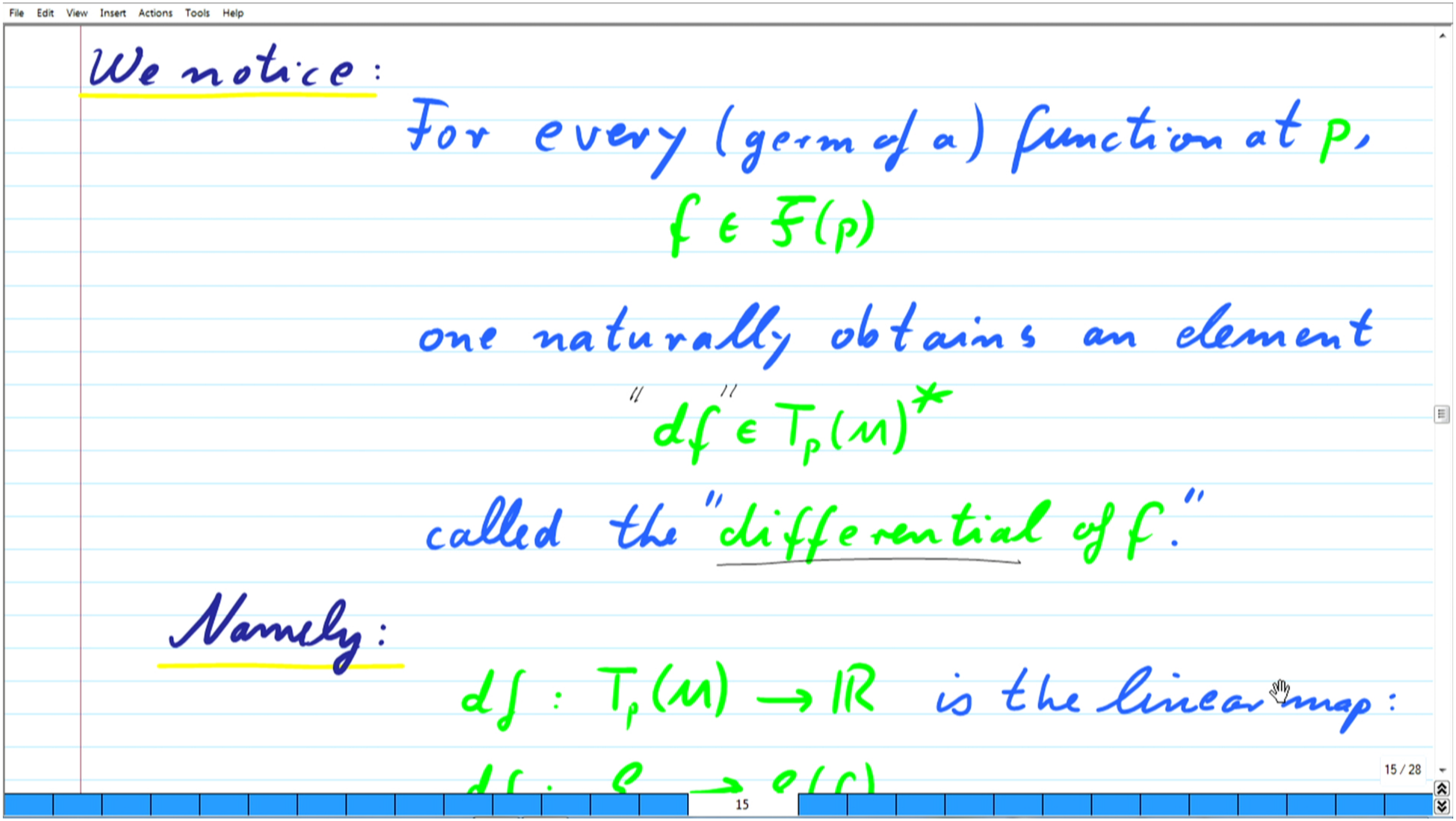
The dual vector space to $T_p(M)$ is called the Cotangent Space, and denoted $T_p(M)^*$.

We notice:

For every (germ of a) function at p,

$$f \in \mathcal{F}(p)$$

one naturally obtains an element



We notice:

For every (germ of a) function at p,

$$f \in \mathcal{F}(p)$$

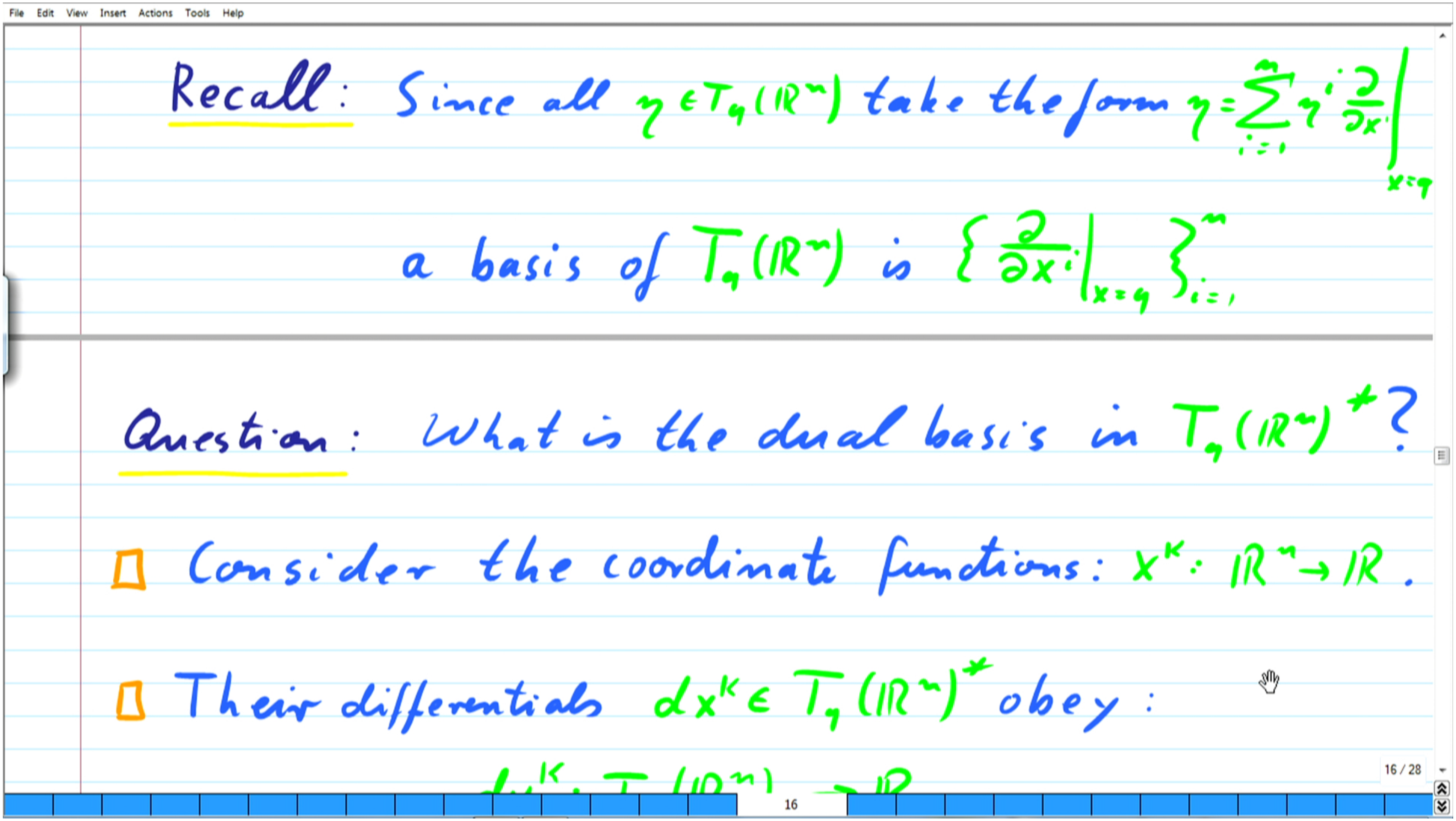
one naturally obtains an element

$$\text{"}df\text{"} \in T_p(M)^*$$

called the "differential of f."

Namely:

$df : T_p(M) \to \mathbb{R}$ is the linear map:



Recall: Since all $\eta \in T_q(\mathbb{R}^n)$ take the form $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=q}$

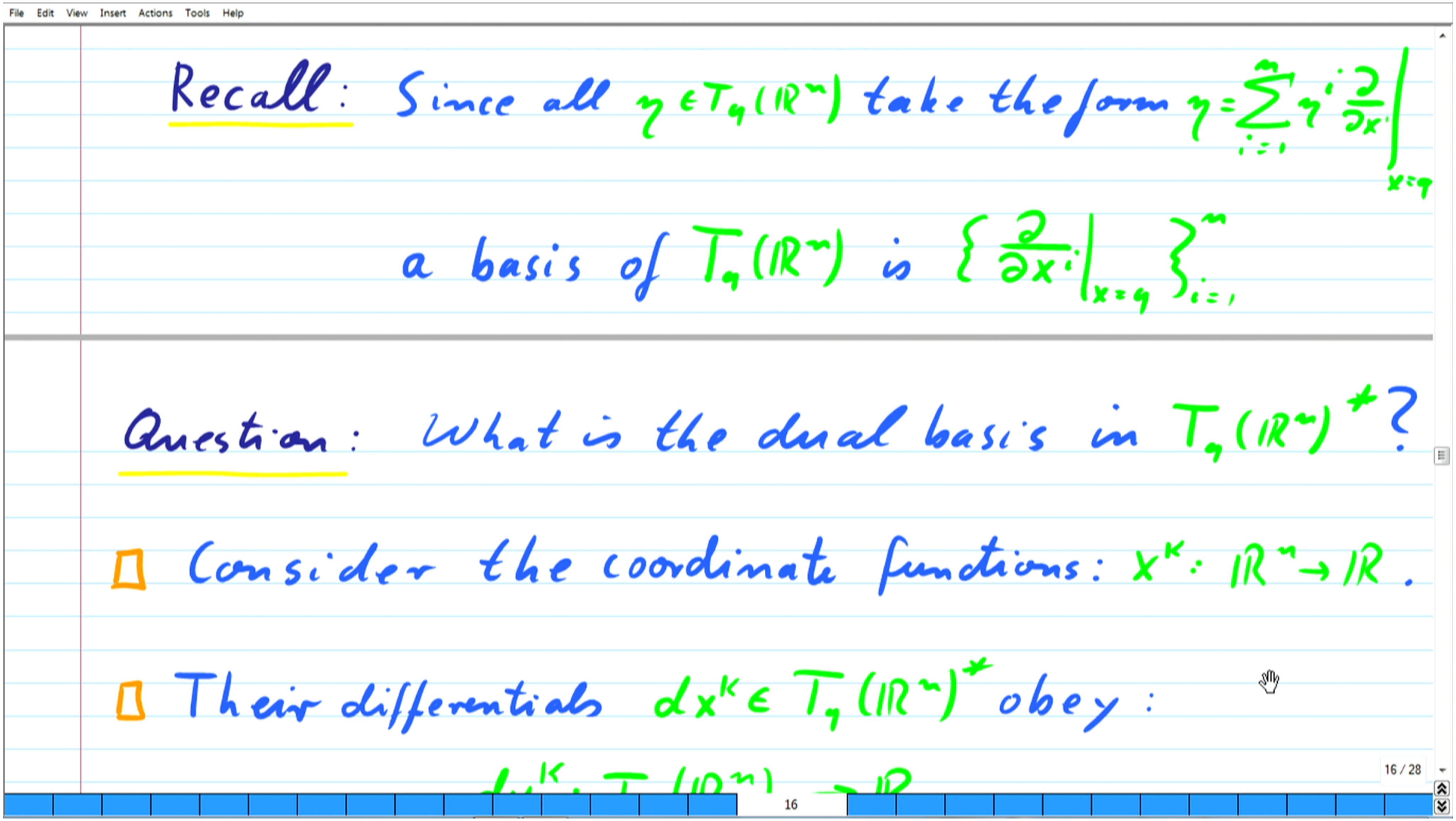
a basis of $T_q(\mathbb{R}^n)$ is $\left\{ \frac{\partial}{\partial x^i}\Big|_{x=q} \right\}_{i=1}^{n}$

Question: What is the dual basis in $T_q(\mathbb{R}^n)^*$?

☐ Consider the coordinate functions: $x^k: \mathbb{R}^n \to \mathbb{R}$.

☐ Their differentials $dx^k \in T_q(\mathbb{R}^n)^*$ obey:

$dx^k: T_q(\mathbb{R}^n) \to \mathbb{R}$



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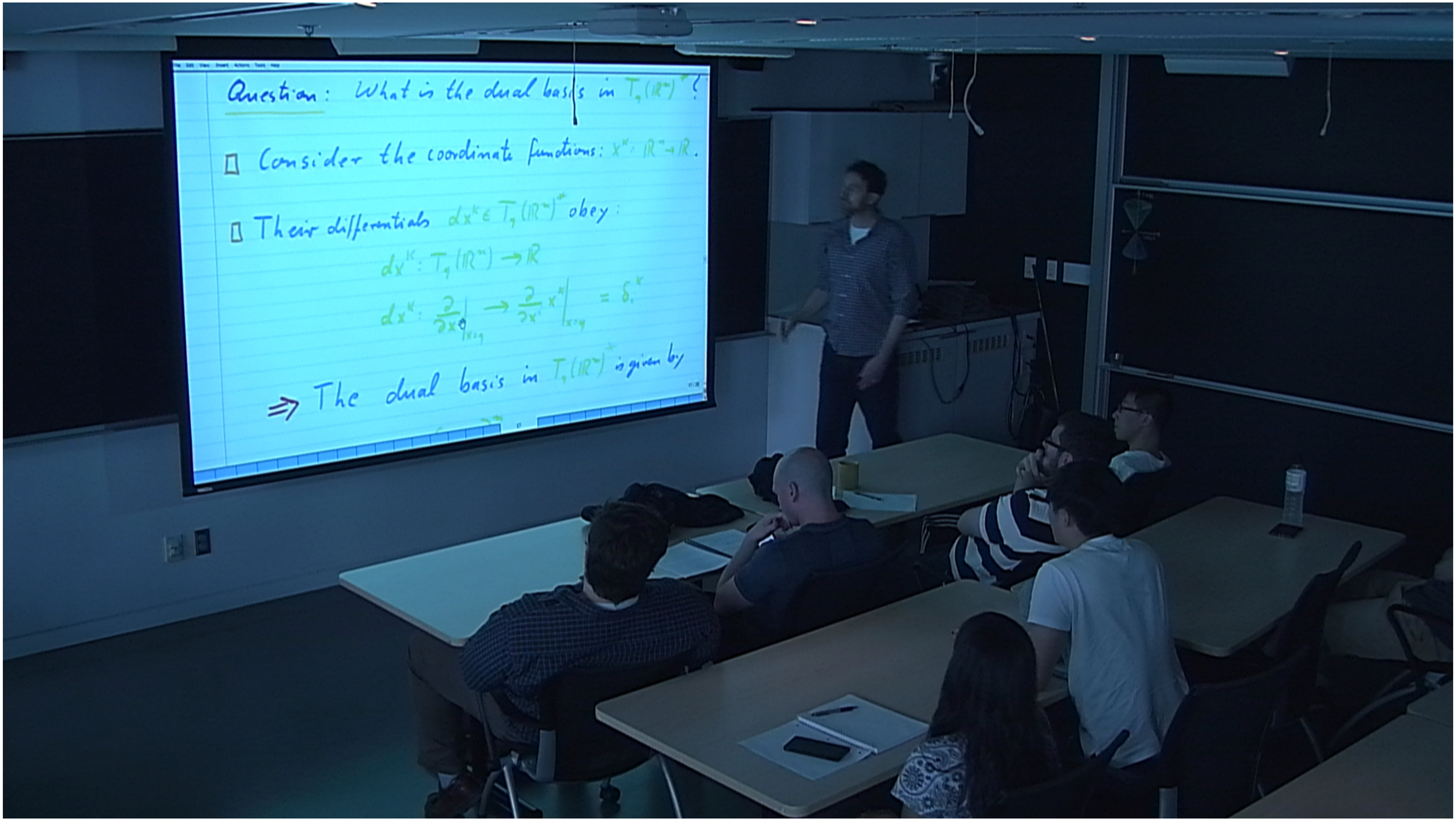
☐ Consider the coordinate functions: $x^K: \mathbb{R}^n \to \mathbb{R}$.

☐ Their differentials $dx^K \in T_q(\mathbb{R}^n)^*$ obey:

$dx^K: T_q(\mathbb{R}^n) \to \mathbb{R}$



Question: What is the dual basis in $T_q(\mathbb{R}^m)^*$?

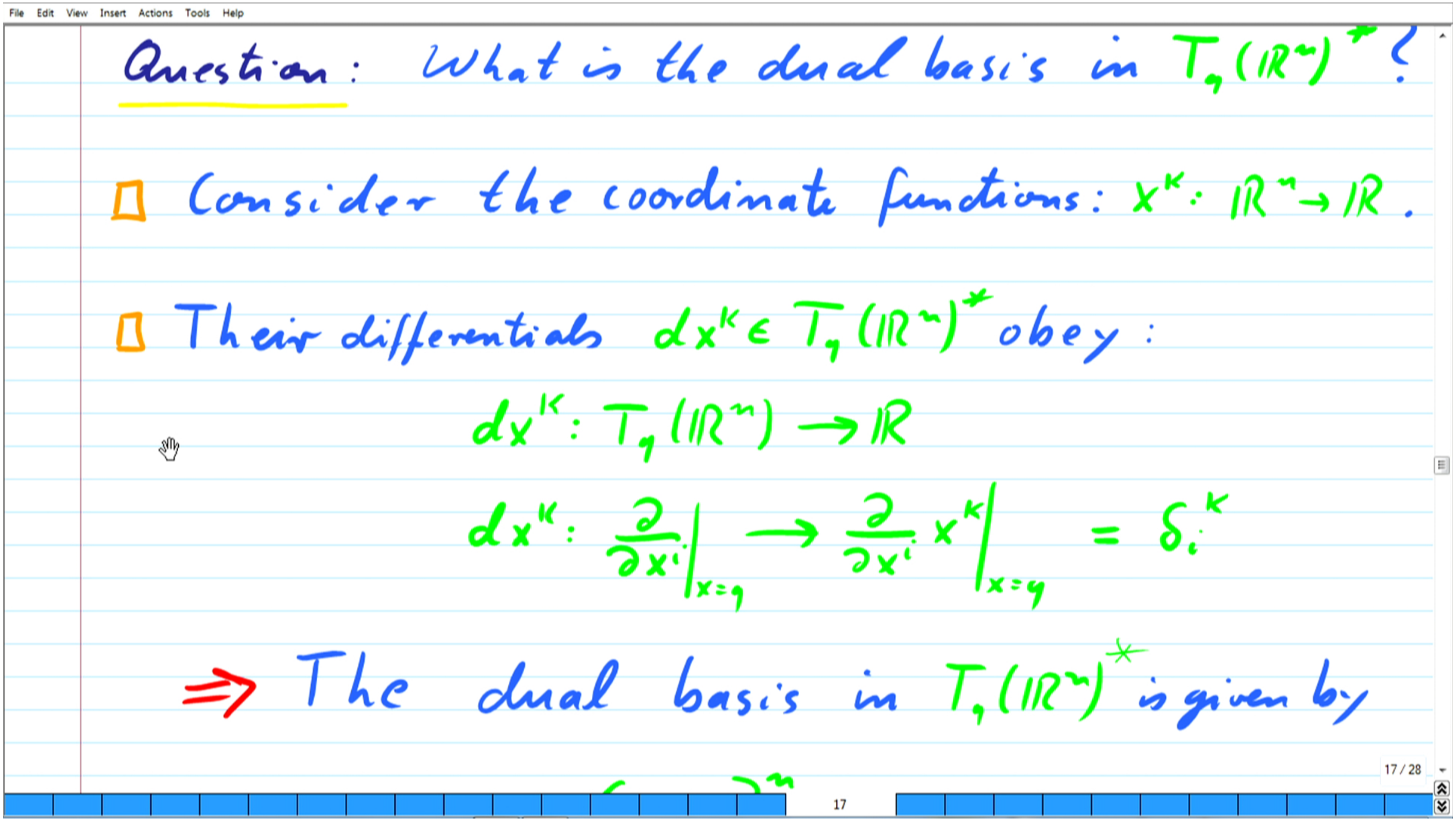
☐ Consider the coordinate functions: $x^k : \mathbb{R}^m \to \mathbb{R}$.

☐ Their differentials $dx^k \in T_q(\mathbb{R}^m)^*$ obey:

$$dx^k : T_q(\mathbb{R}^m) \to \mathbb{R}$$

$$dx^k : \frac{\partial}{\partial x^i}\Big|_{x=q} \to \frac{\partial}{\partial x^i} x^k \Big|_{x=q} = \delta_i^k$$

$\Rightarrow$ The dual basis in $T_q(\mathbb{R}^m)^*$ is given by



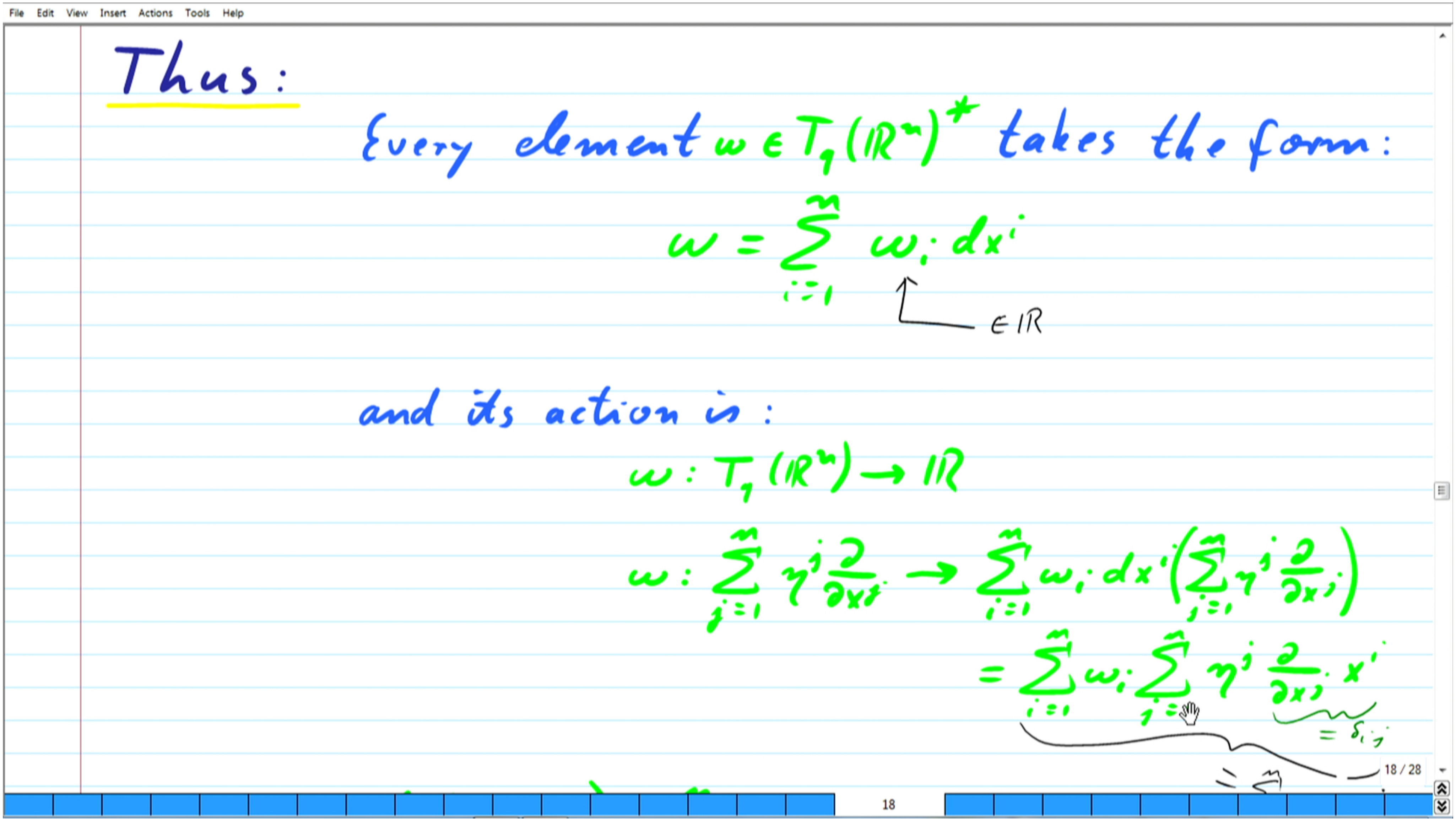
Question: What is the dual basis in $T_q(\mathbb{R}^n)^*$?

- Consider the coordinate functions: $x^k: \mathbb{R}^n \to \mathbb{R}$.
- Their differentials $dx^k \in T_q(\mathbb{R}^n)^*$ obey:

$$dx^k: T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$dx^k: \frac{\partial}{\partial x^i}\Big|_{x=q} \longrightarrow \frac{\partial}{\partial x^i} x^k\Big|_{x=q} = \delta_i^k$$

$\Rightarrow$ The dual basis in $T_q(\mathbb{R}^n)^*$ is given by



Thus:

Every element $\omega \in T_q(\mathbb{R}^n)^*$ takes the form:

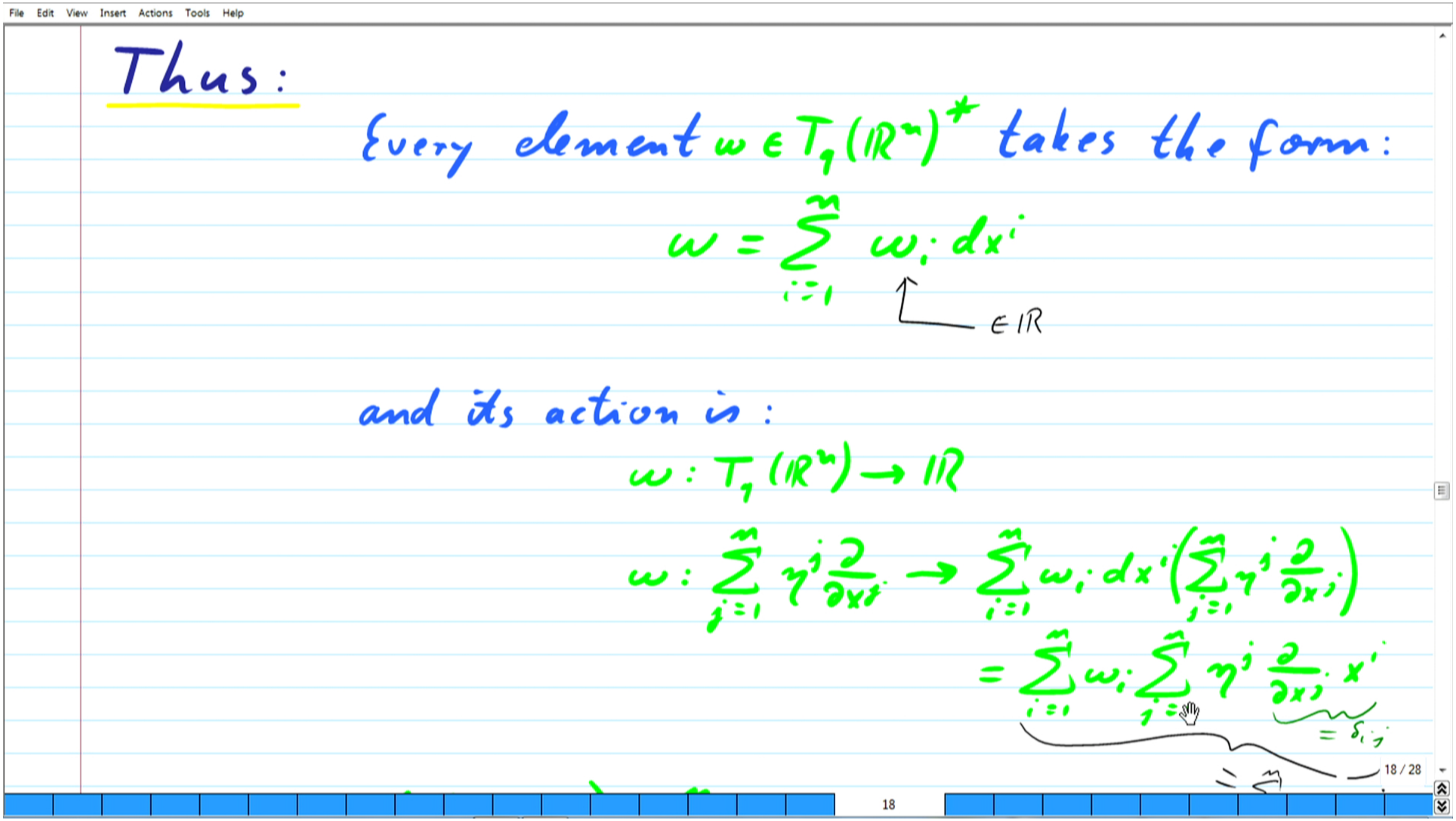
$$\omega = \sum_{i=1}^{n} \underset{\in \mathbb{R}}{\omega_i}\, dx^i$$

and its action is:

$$\omega : T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$\omega : \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j} \to \sum_{i=1}^{n} \omega_i\, dx^i \left(\sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j} \right)$$

$$= \underbrace{\sum_{i=1}^{n} \omega_i \sum_{j=1}^{n} \eta^j \underbrace{\frac{\partial}{\partial x^j} x^i}_{= \delta_{ij}}}_{=}$$



Thus:

Every element $\omega \in T_q(\mathbb{R}^n)^*$ takes the form:

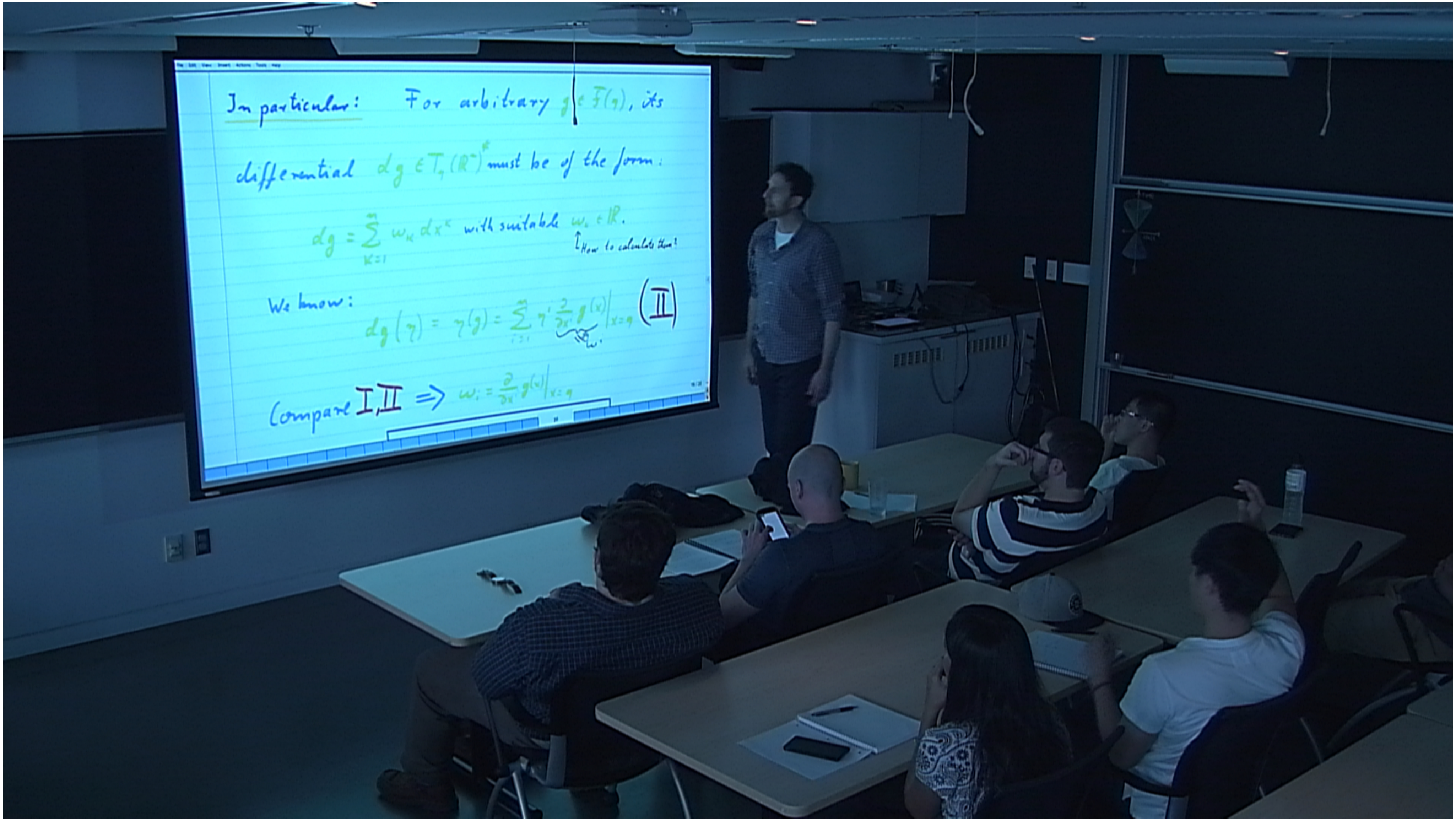
$$\omega = \sum_{i=1}^{m} \underset{\in \mathbb{R}}{\omega_i}\, dx^i$$

and its action is:

$$\omega : T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$\omega : \sum_{j=1}^{m} \eta^j \frac{\partial}{\partial x^j} \to \sum_{i=1}^{m} \omega_i\, dx^i\left(\sum_{j=1}^{m} \eta^j \frac{\partial}{\partial x^j}\right)$$

$$= \underbrace{\sum_{i=1}^{m} \omega_i \sum_{j=1}^{m} \eta^j \underbrace{\frac{\partial}{\partial x^j} x^i}_{= \delta_{ij}}}_{= \sum^{m}}$$



In particular: For arbitrary $g \in \mathcal{F}(\eta)$, its differential $dg \in T_\eta(\mathbb{R}^m)^*$ must be of the form:

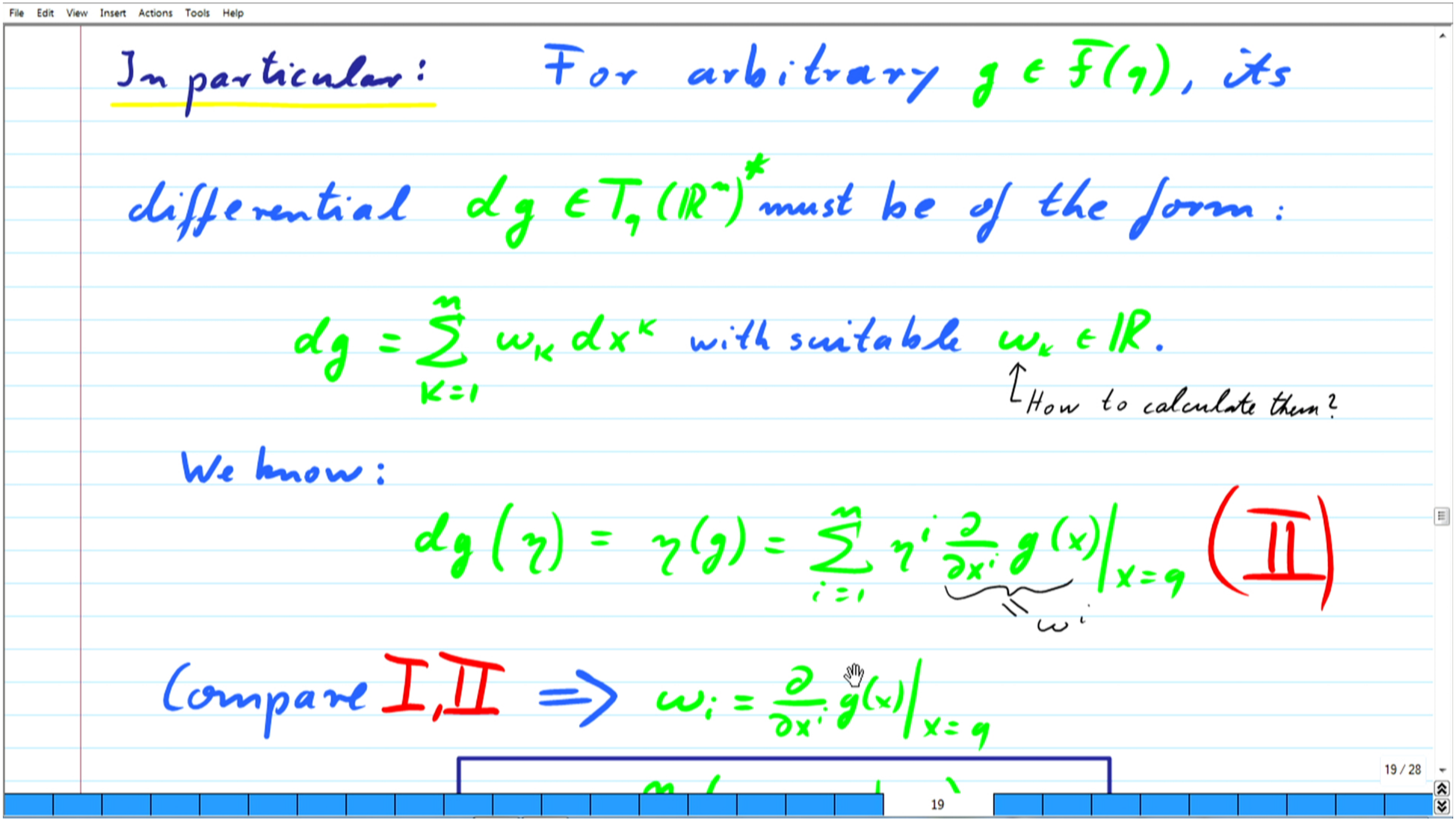
$$dg = \sum_{k=1}^{m} w_k \, dx^k \text{ with suitable } w_k \in \mathbb{R}.$$

↑ How to calculate them?

We know:

$$dg(\eta) = \eta(g) = \sum_{i=1}^{m} \eta^i \underbrace{\frac{\partial}{\partial x^i} g(x)}_{w_i}\Big|_{x=\eta} \quad (\text{II})$$

Compare I, II $\Rightarrow$ $w_i = \frac{\partial}{\partial x^i} g(x)\Big|_{x=\eta}$



In particular: For arbitrary $g \in \mathcal{F}(q)$, its differential $dg \in T_q(\mathbb{R}^n)^*$ must be of the form:

$$dg = \sum_{k=1}^{n} w_k \, dx^k \quad \text{with suitable } w_k \in \mathbb{R}.$$

↑ How to calculate them?

We know:

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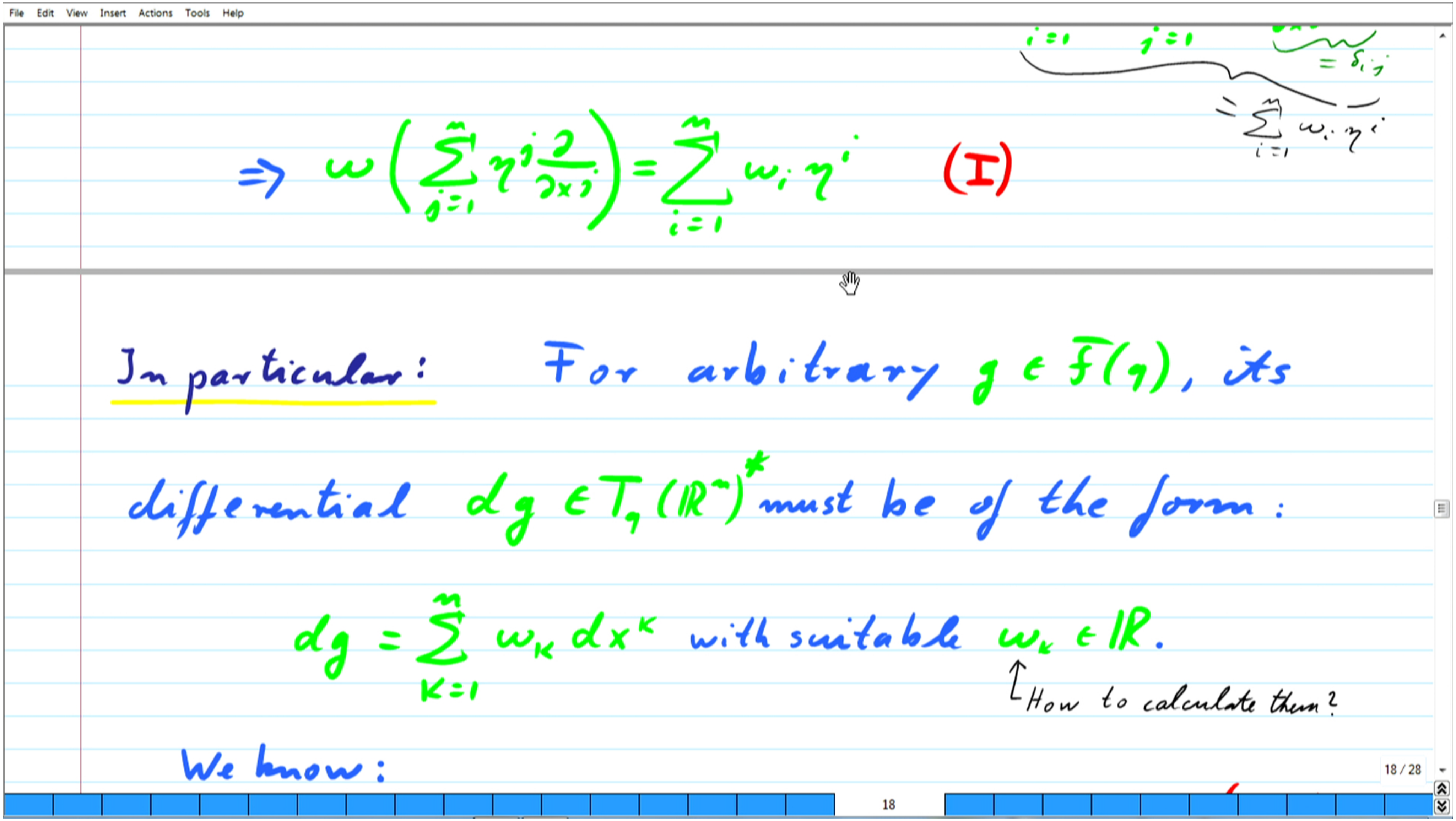

$i=1 \quad j=1 \quad \underbrace{}_{=\delta_{ij}}$

$$= \sum_{i=1}^{n} \omega_i \eta^i$$

$$\Rightarrow \omega\left(\sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} \omega_i \eta^i \qquad \text{(I)}$$

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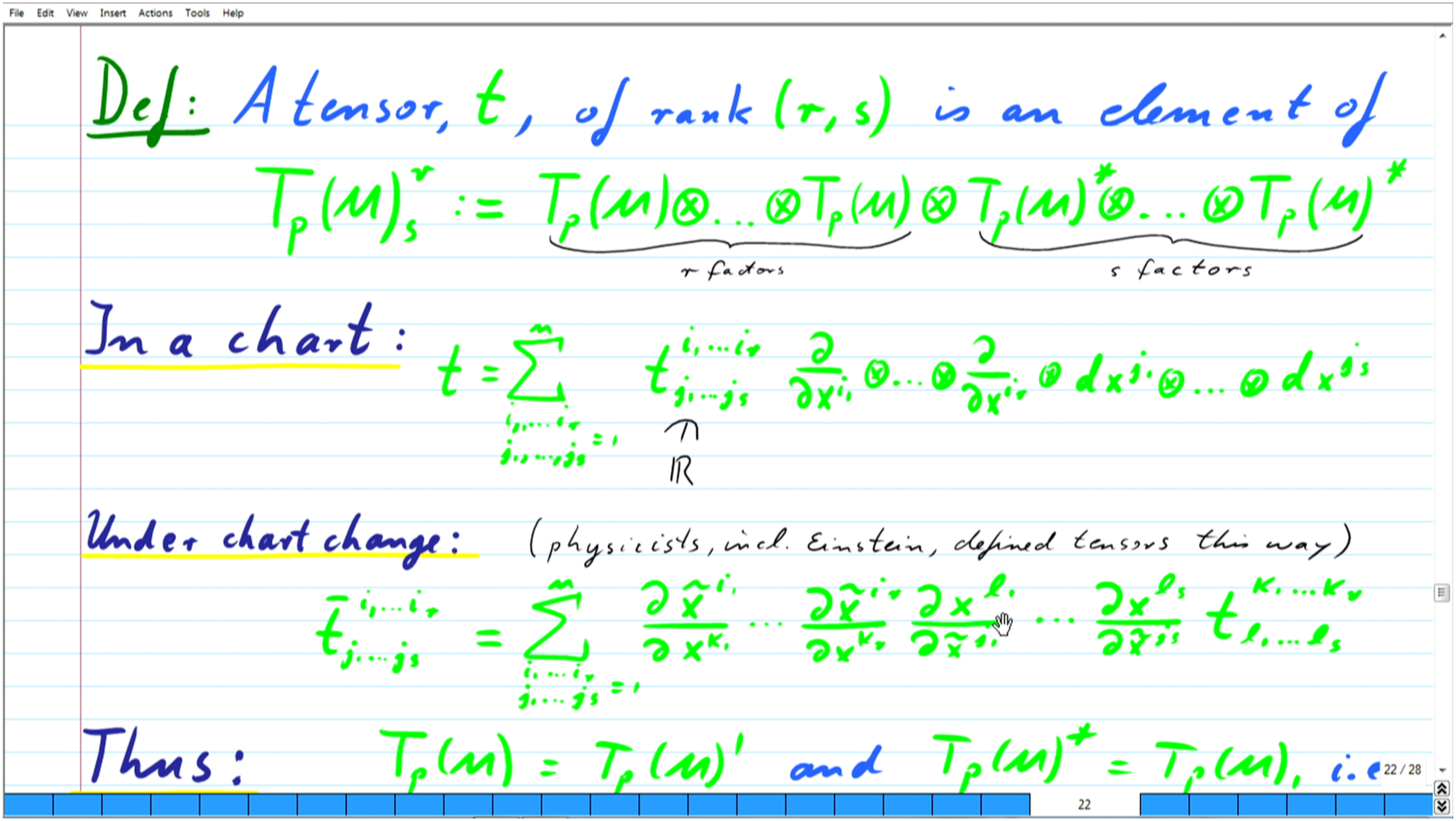


Def: A tensor, t, of rank (r,s) is an element of

$$T_p(M)^r_s := \underbrace{T_p(M)\otimes\ldots\otimes T_p(M)}_{r\text{ factors}}\otimes\underbrace{T_p(M)^*\otimes\ldots\otimes T_p(M)^*}_{s\text{ factors}}$$

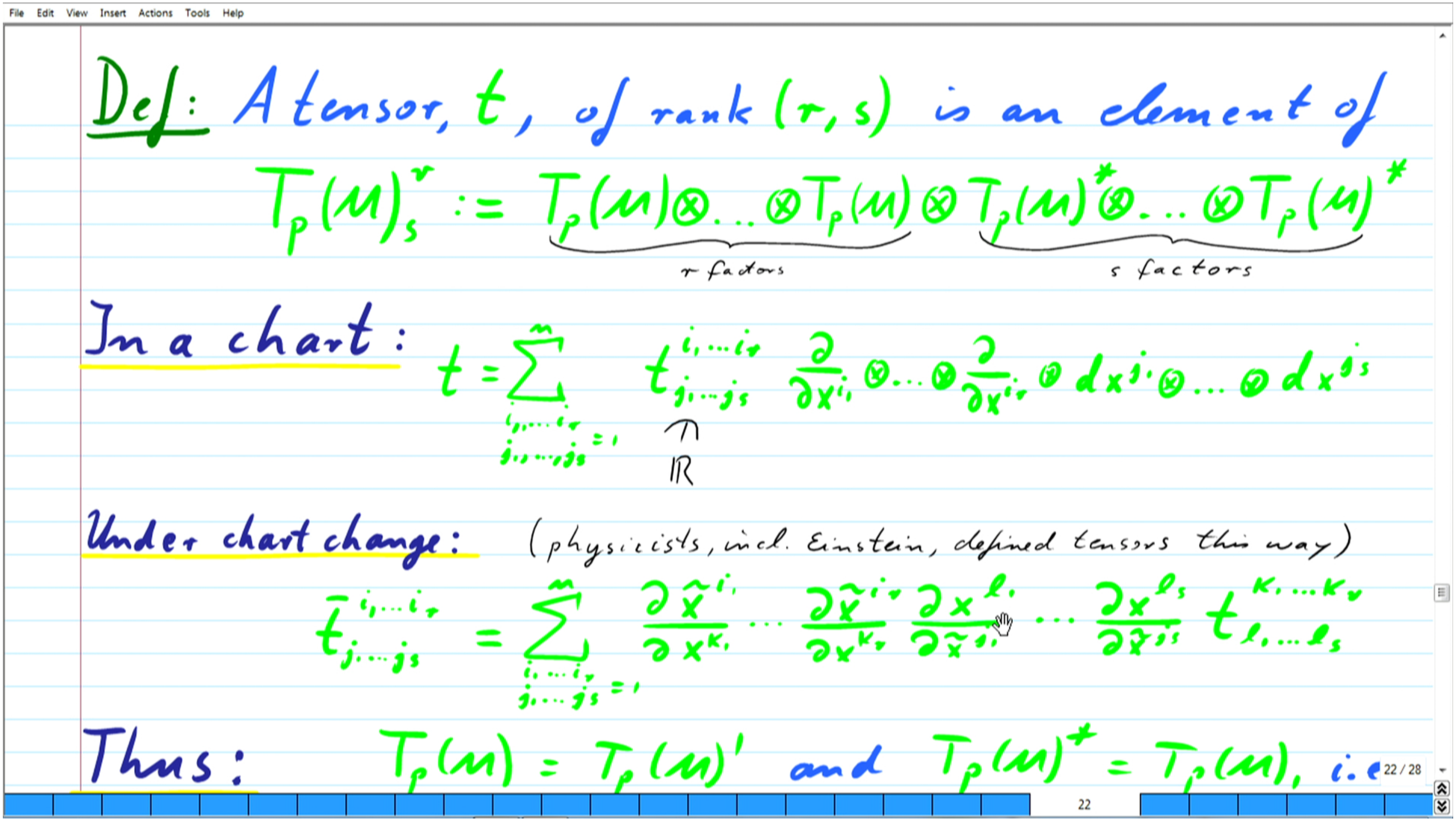
In a chart:

$$t = \sum_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_s=1}}^{n} \underset{\overset{\cap}{\mathbb{R}}}{t^{i_1\ldots i_r}_{j_1\ldots j_s}}\, \frac{\partial}{\partial x^{i_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_s}$$

Under chart change: (physicists, incl. Einstein, defined tensors this way)

$$\tilde t^{i_1\ldots i_r}_{j_1\ldots j_s} = \sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_s=1}}^{n} \frac{\partial\tilde x^{i_1}}{\partial x^{k_1}}\cdots\frac{\partial\tilde x^{i_r}}{\partial x^{k_r}}\frac{\partial x^{l_1}}{\partial\tilde x^{j_1}}\cdots\frac{\partial x^{l_s}}{\partial\tilde x^{j_s}}\, t^{k_1\ldots k_r}_{l_1\ldots l_s}$$

Thus: $T_p(M) = T_p(M)'$ and $T_p(M)^* = T_p(M)$, i.e



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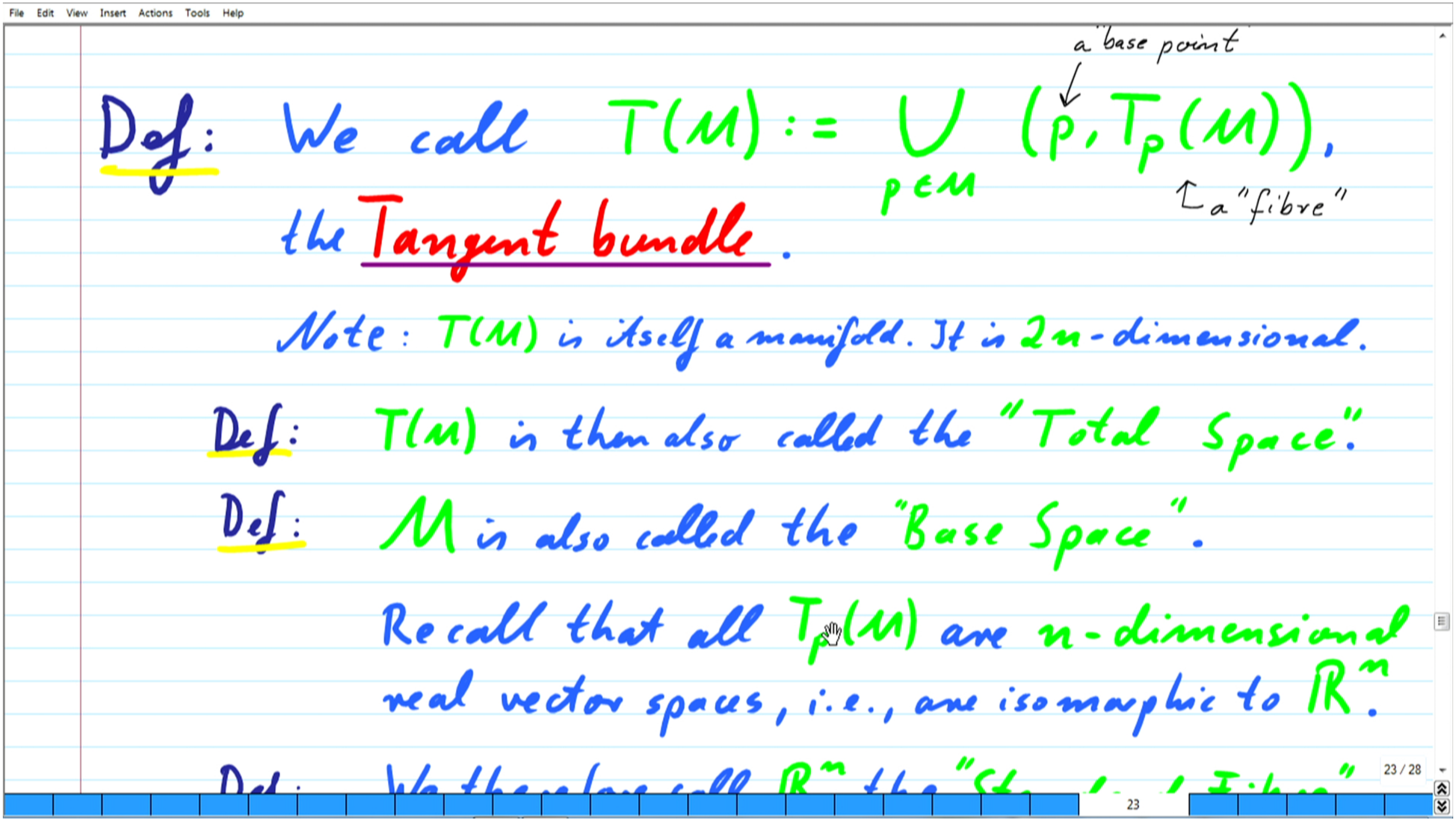


Def: We call $T(M) := \bigcup_{p \in M} (p, T_p(M))$, the **Tangent bundle**.

a "base point" ↓ p; ↖ a "fibre" $T_p(M)$

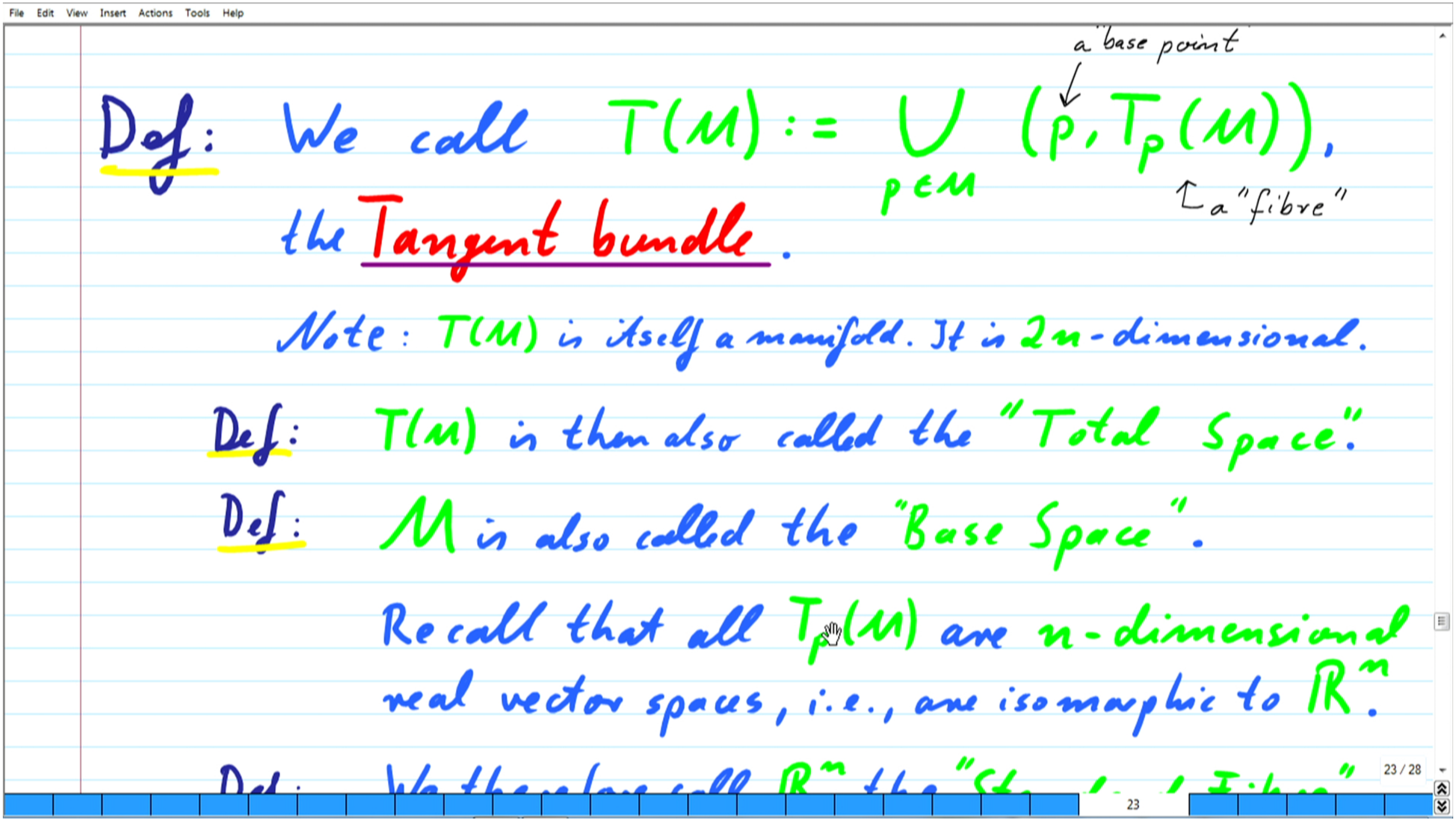
Note: $T(M)$ is itself a manifold. It is $2n$-dimensional.

Def: $T(M)$ is then also called the "Total Space".

Def: M is also called the "Base Space".

Recall that all $T_p(M)$ are n-dimensional real vector spaces, i.e., are isomorphic to $\mathbb{R}^n$.

Def: We therefore call $\mathbb{R}^n$ the "Standard Fibre"



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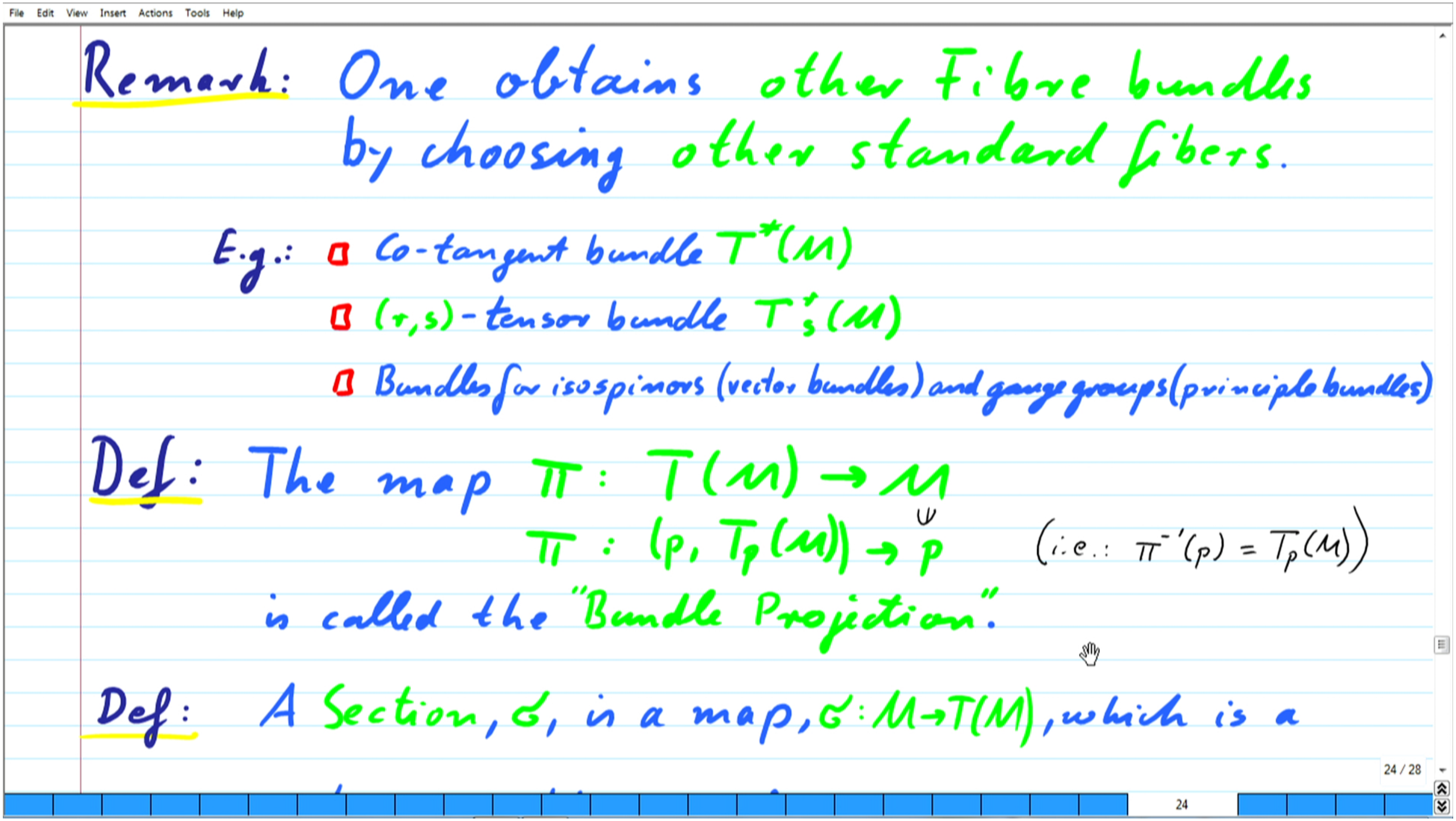


Remark: One obtains other Fibre bundles by choosing other standard fibers.

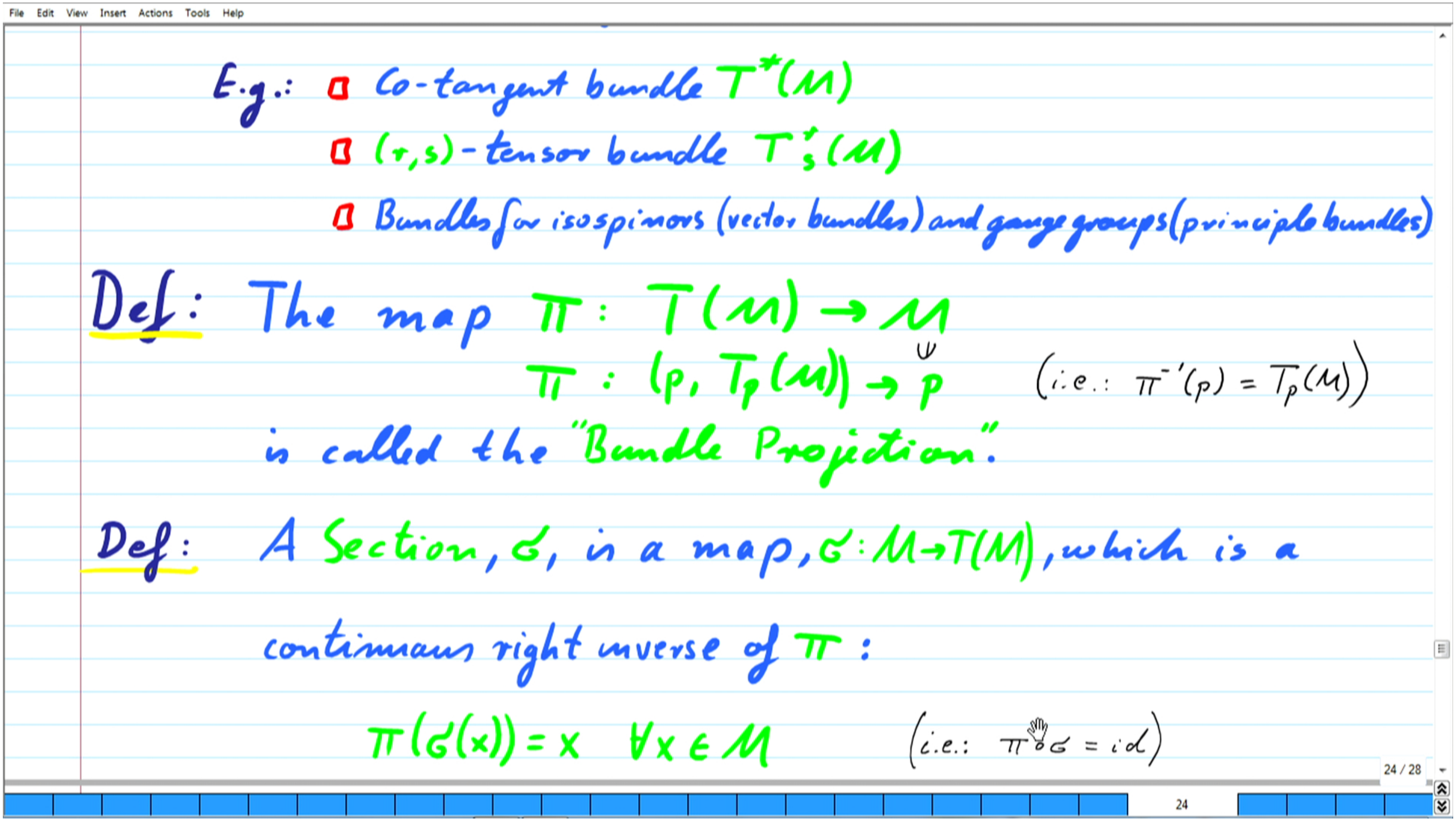
E.g.:
- ☐ Co-tangent bundle $T^*(M)$
- ☐ (r,s)-tensor bundle $T^r_s(M)$
- ☐ Bundles for isospinors (vector bundles) and gauge groups (principle bundles)

Def: The map

$$\pi : T(M) \to M$$
$$\pi : (p, T_p(M)) \to \overset{\cup}{p} \qquad (\text{i.e.: } \pi^{-1}(p) = T_p(M))$$

is called the "Bundle Projection".

Def: A Section, σ, is a map, $\sigma : M \to T(M)$, which is a



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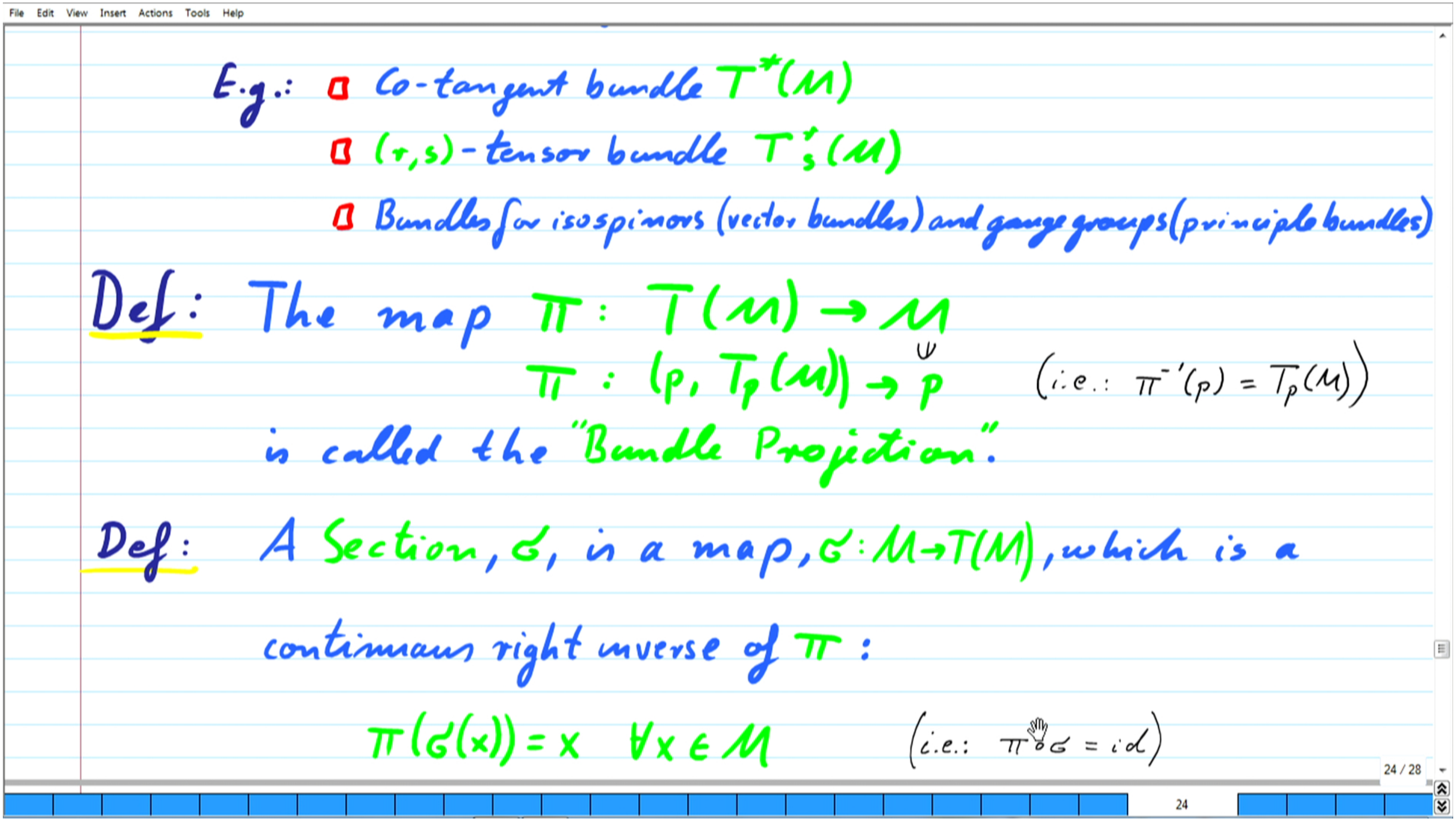
$$\pi(\sigma(x)) = x \quad \forall x \in M \qquad \left(\text{i.e.: } \pi \circ \sigma = id\right)$$



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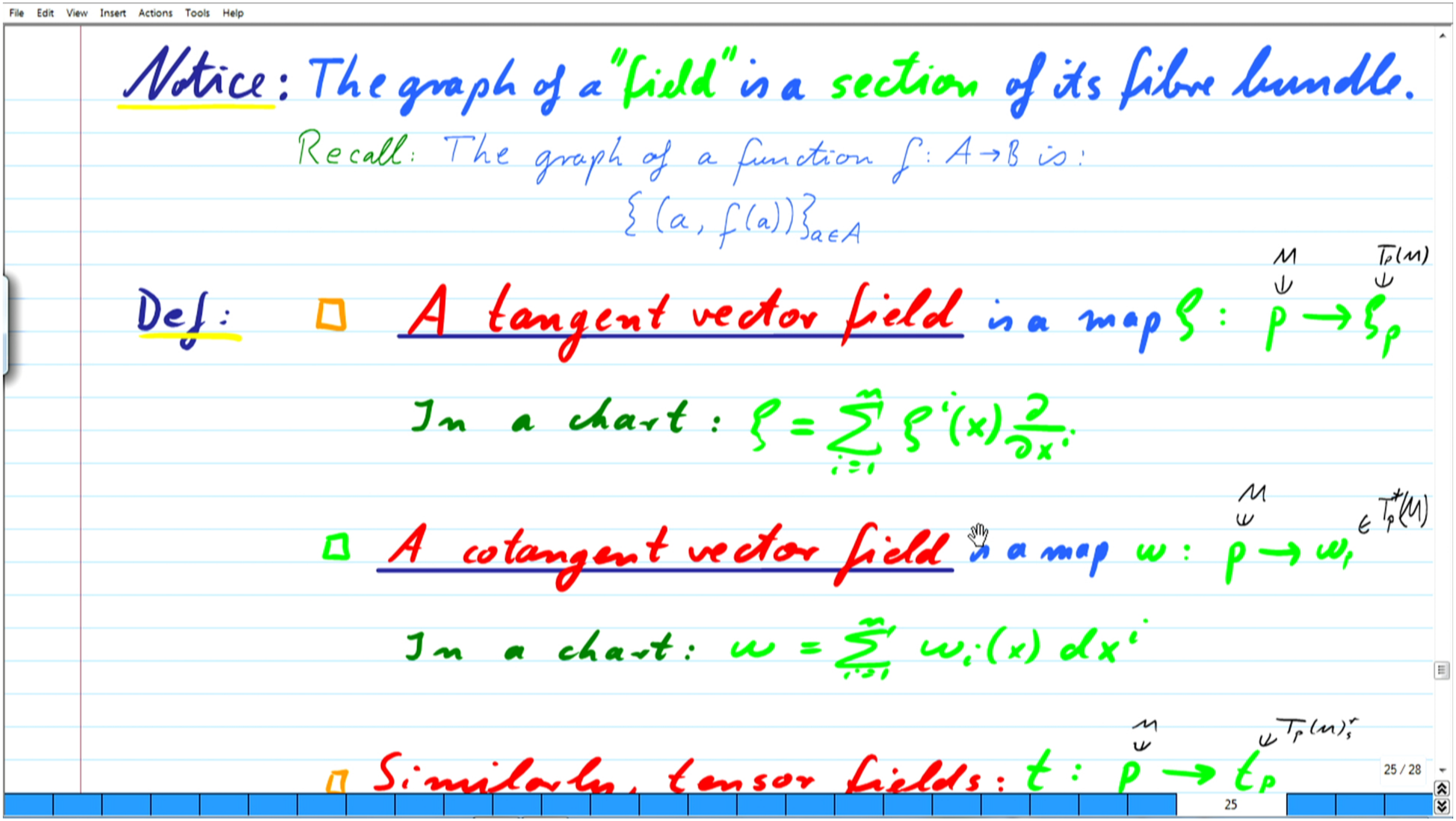

Notice: The graph of a "field" is a section of its fibre bundle.

Recall: The graph of a function $f: A \to B$ is:

$$\{(a, f(a))\}_{a \in A}$$

Def:

☐ A tangent vector field is a map $\xi: \overset{\substack{M \\ \in}}{p} \to \overset{\substack{T_p(M) \\ \in}}{\xi_p}$

In a chart: $\xi = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i}$

☐ A cotangent vector field is a map $\omega: \overset{\substack{M \\ \in}}{p} \to \omega_p \in T_p^*(M)$

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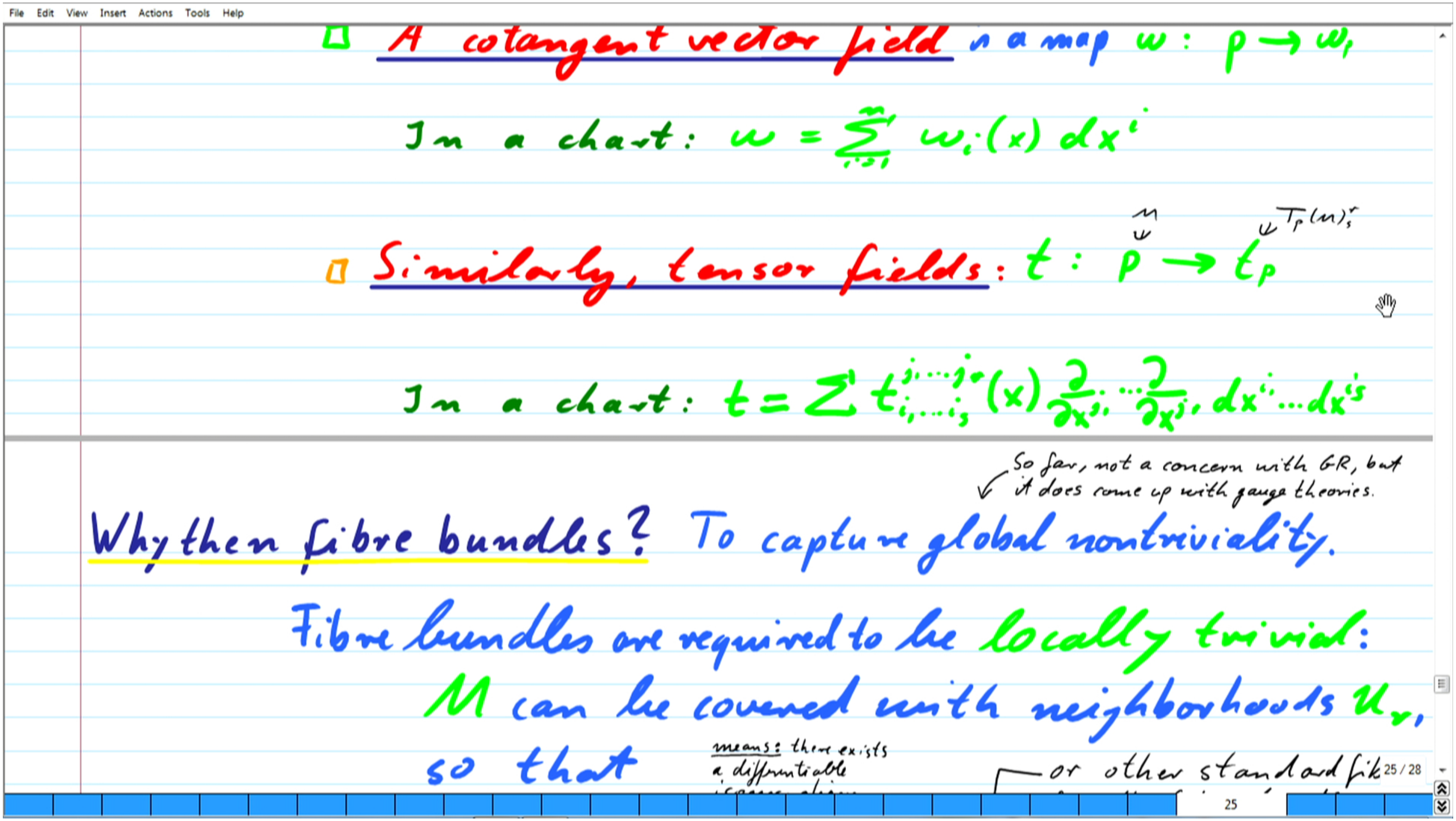
☐ Similarly, tensor fields: $t: \overset{\substack{M \\ \in}}{p} \to \overset{\substack{T_p(M)_s^r \\ \in}}{t_p}$



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So far, not a concern with GR, but it does come up with gauge theories.

Why then fibre bundles? To capture global nontriviality.

Fibre bundles are required to be locally trivial:

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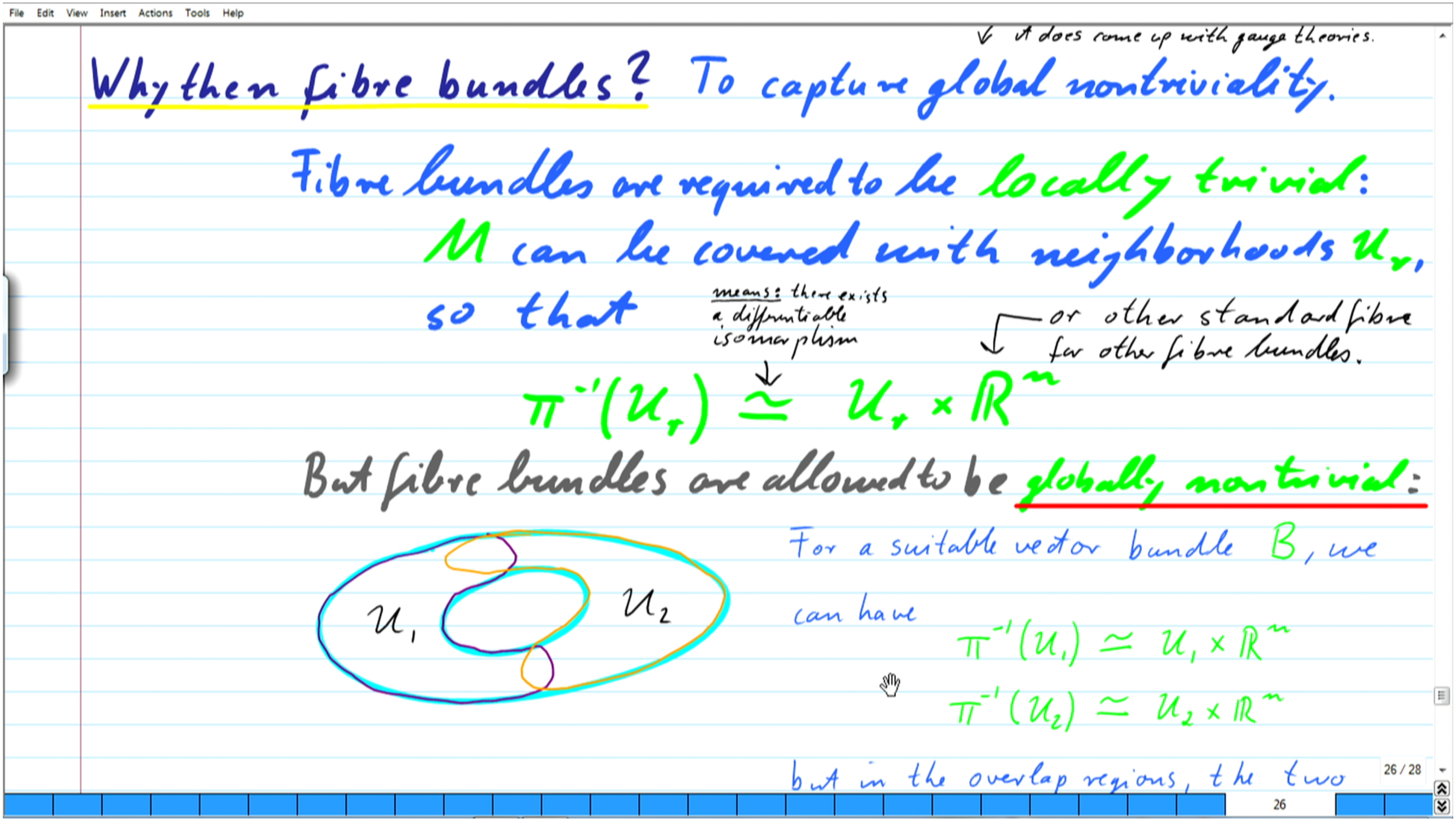

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For a suitable vector bundle B, we can have

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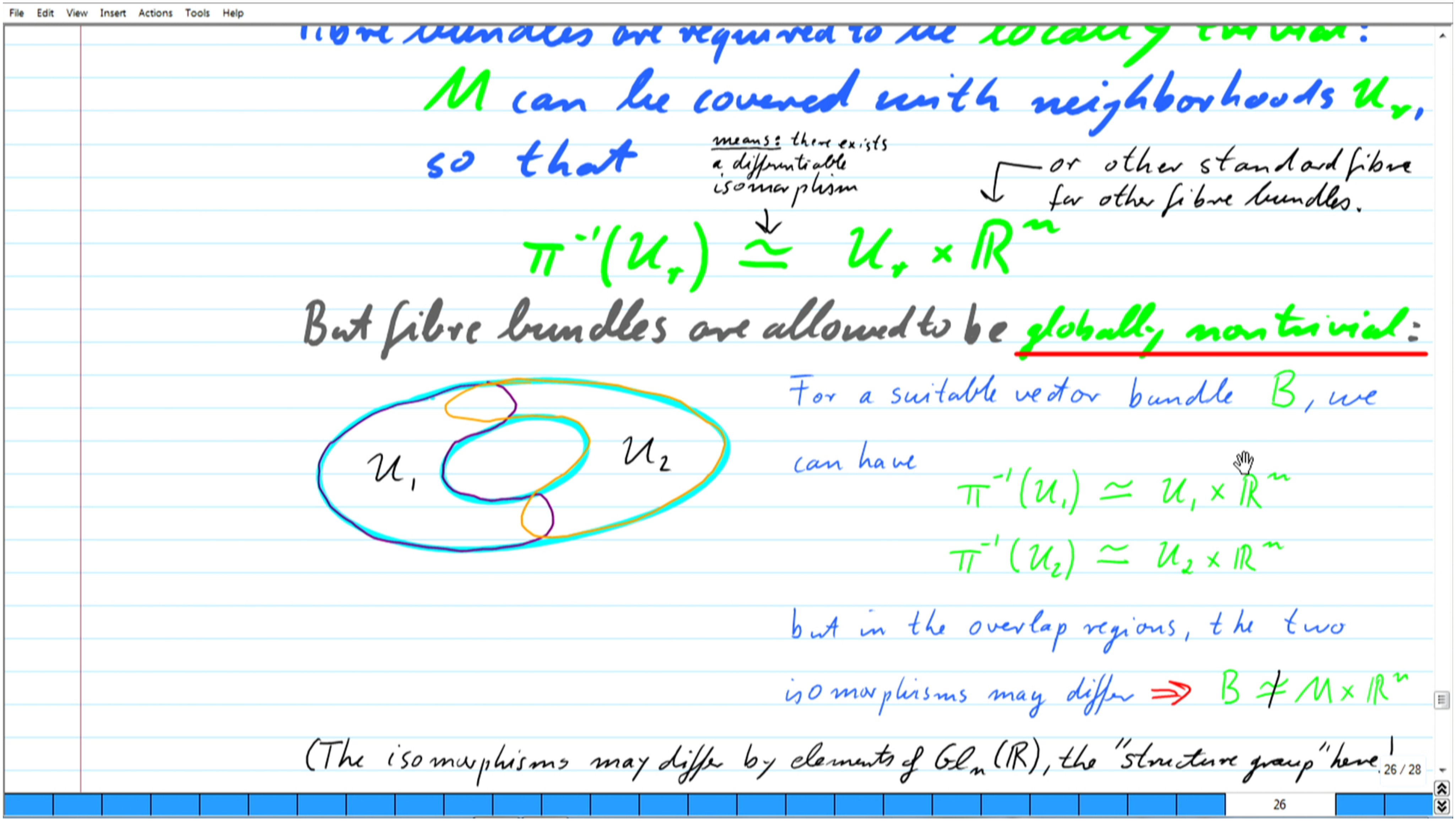

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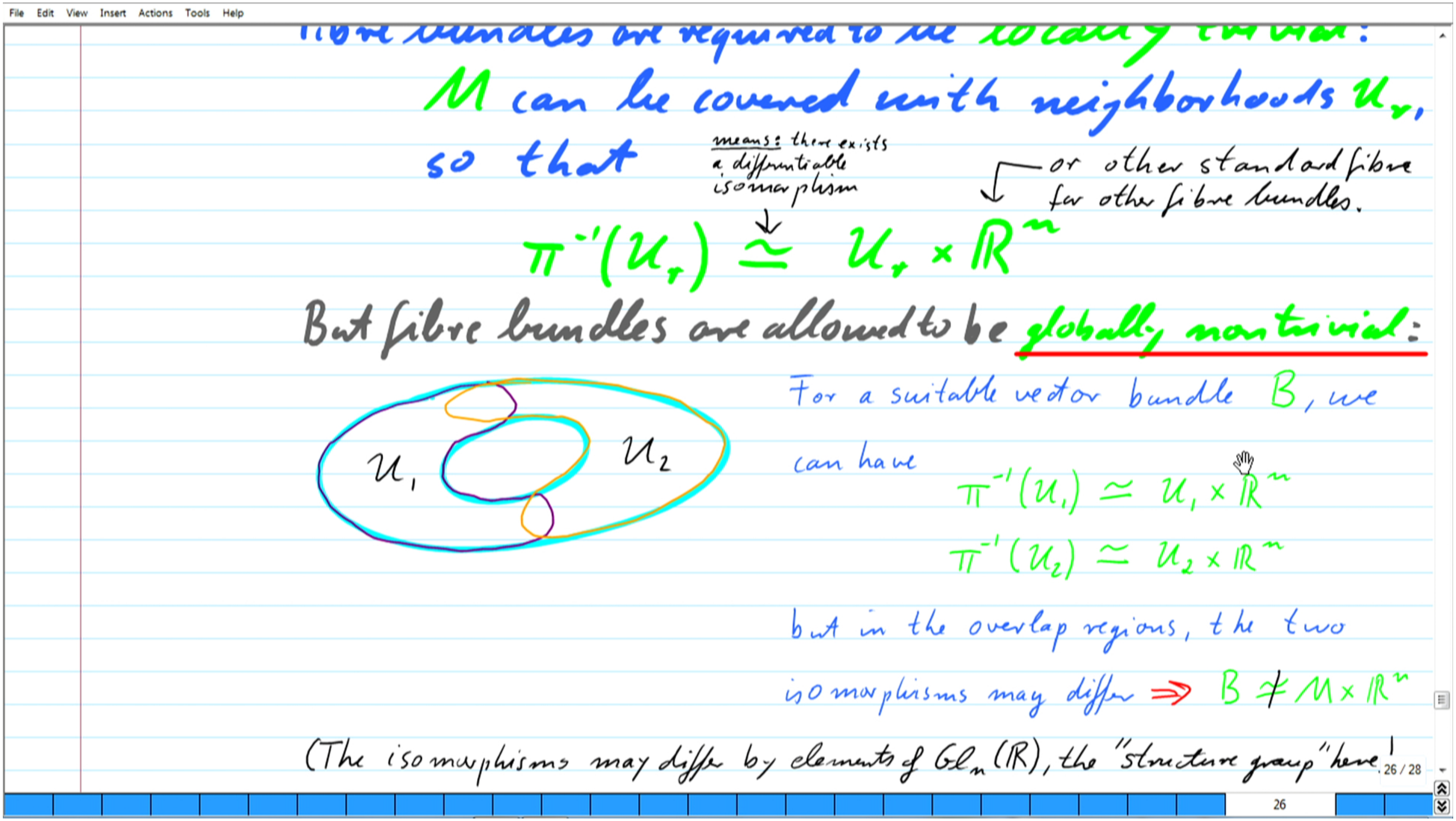

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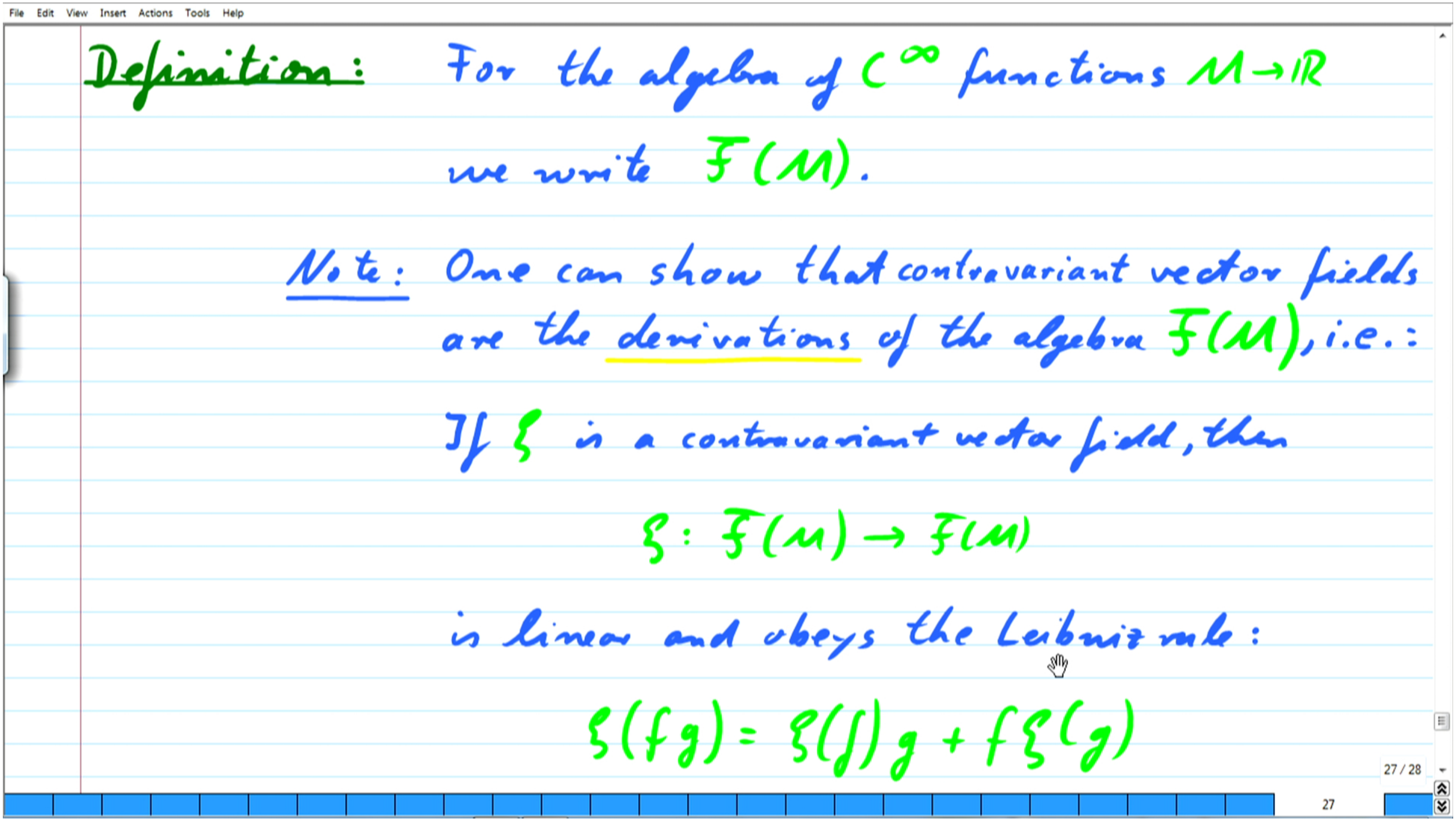


Definition: For the algebra of C^∞ functions $M \to \mathbb{R}$ we write $\mathcal{F}(M)$.

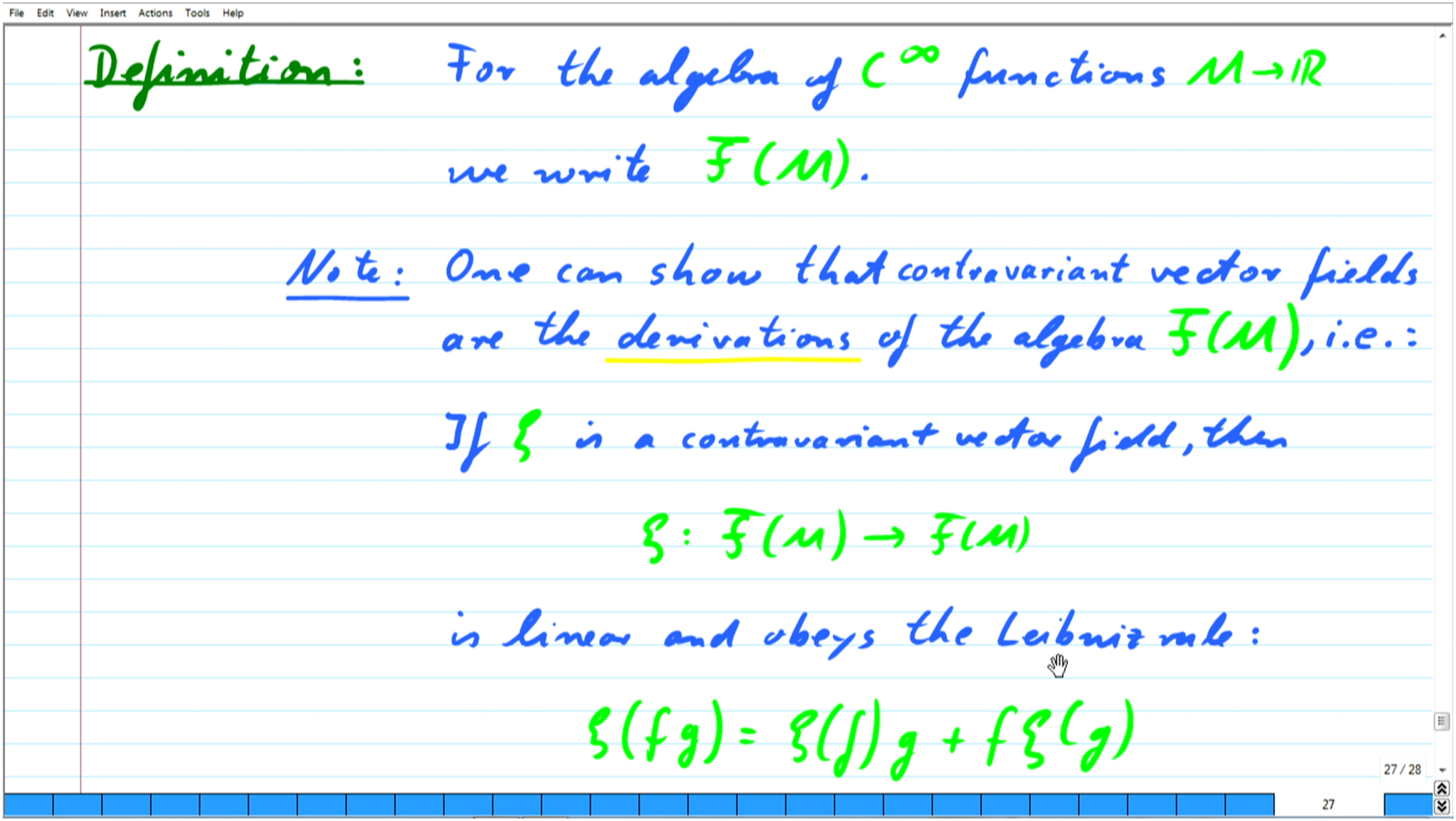
Note: One can show that contravariant vector fields are the derivations of the algebra $\mathcal{F}(M)$, i.e.:

If ξ is a contravariant vector field, then

$$\xi : \mathcal{F}(M) \to \mathcal{F}(M)$$

is linear and obeys the Leibniz rule:

$$\xi(fg) = \xi(f)g + f\xi(g)$$



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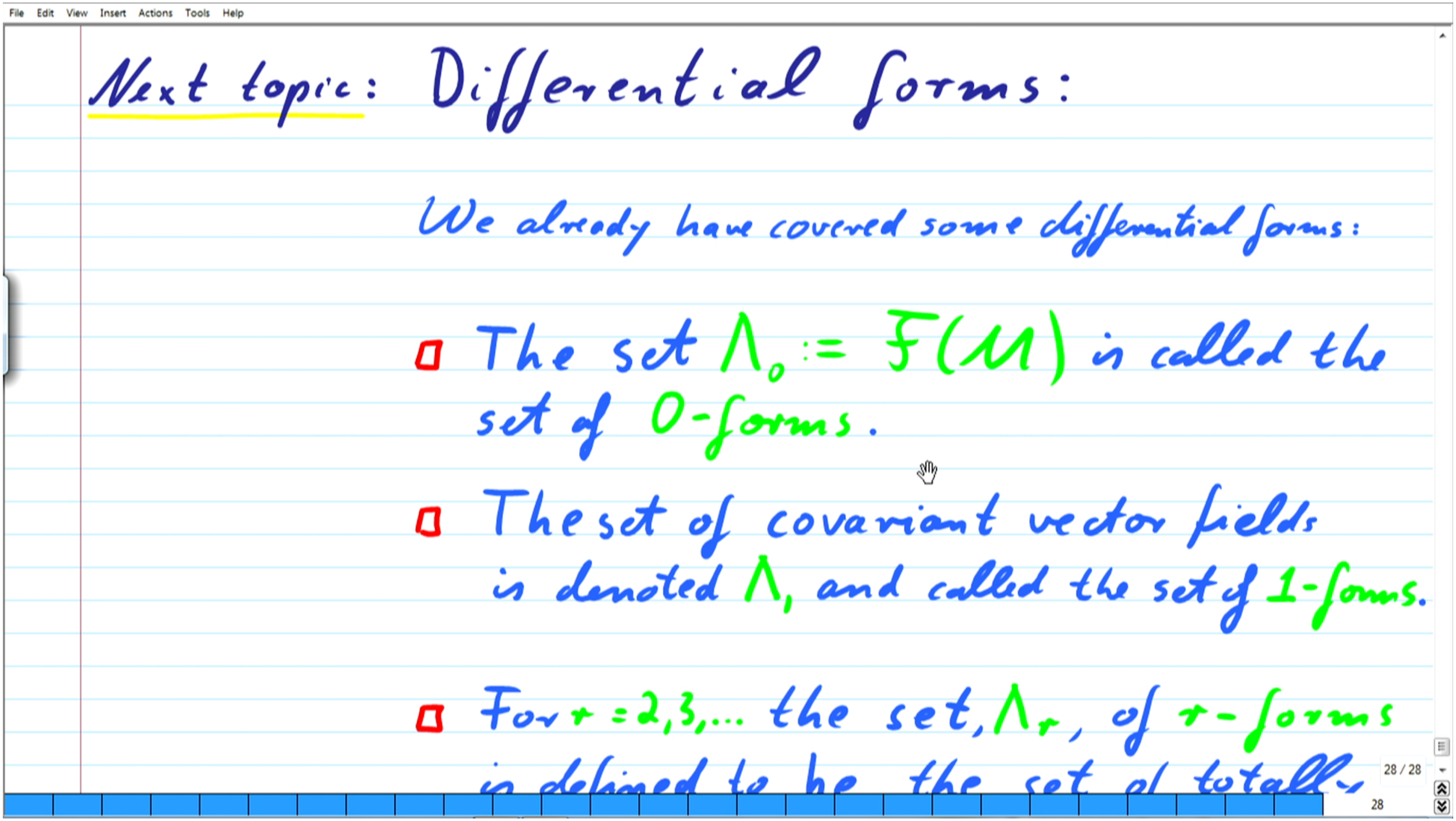


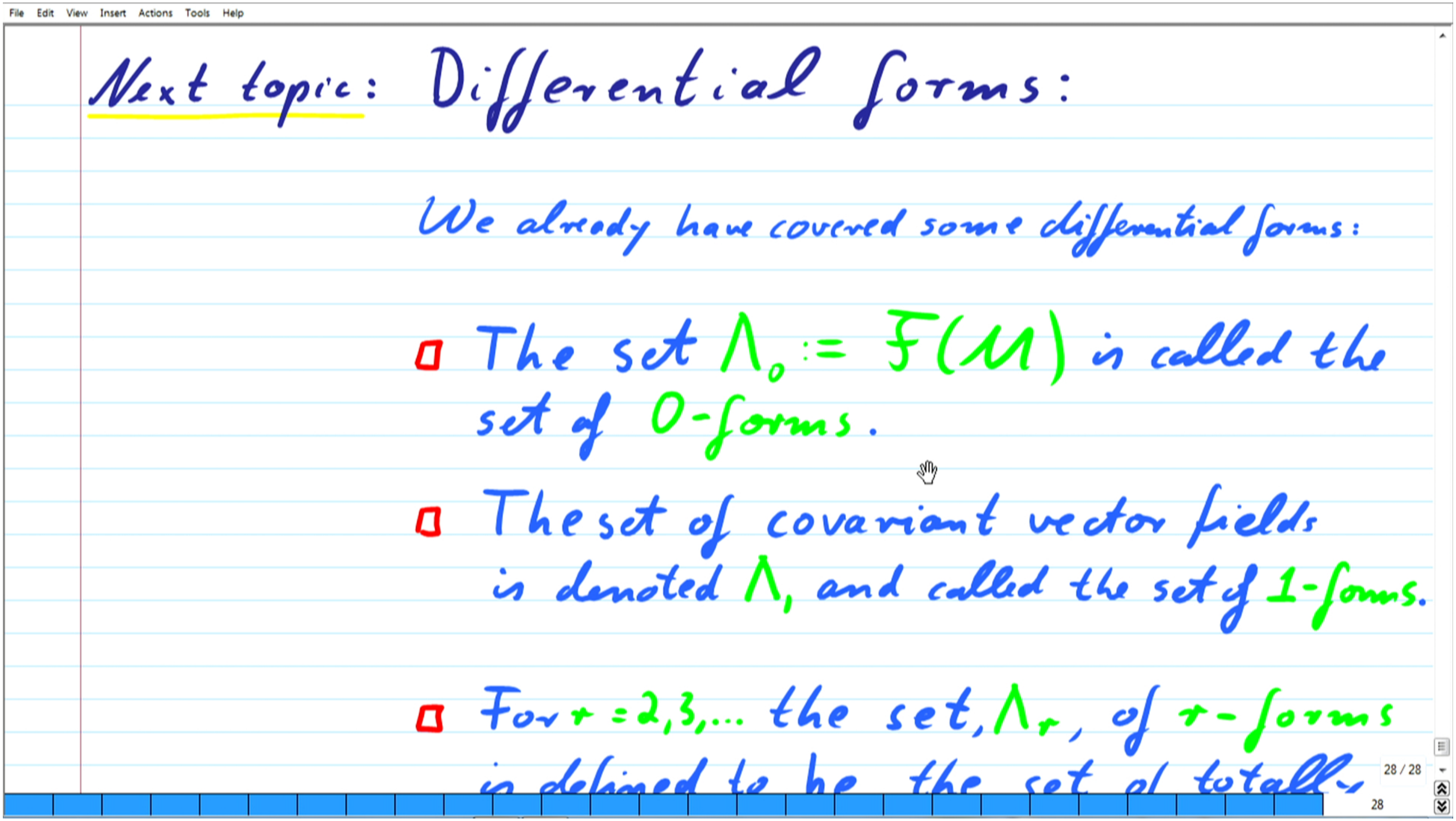
Next topic: Differential forms:

We already have covered some differential forms:

☐ The set $\Lambda_0 := \mathcal{F}(\mathcal{M})$ is called the set of 0-forms.

☐ The set of covariant vector fields is denoted Λ_1 and called the set of 1-forms.

☐ For $r = 2, 3, \ldots$ the set, Λ_r, of r-forms is defined to be the set of totally



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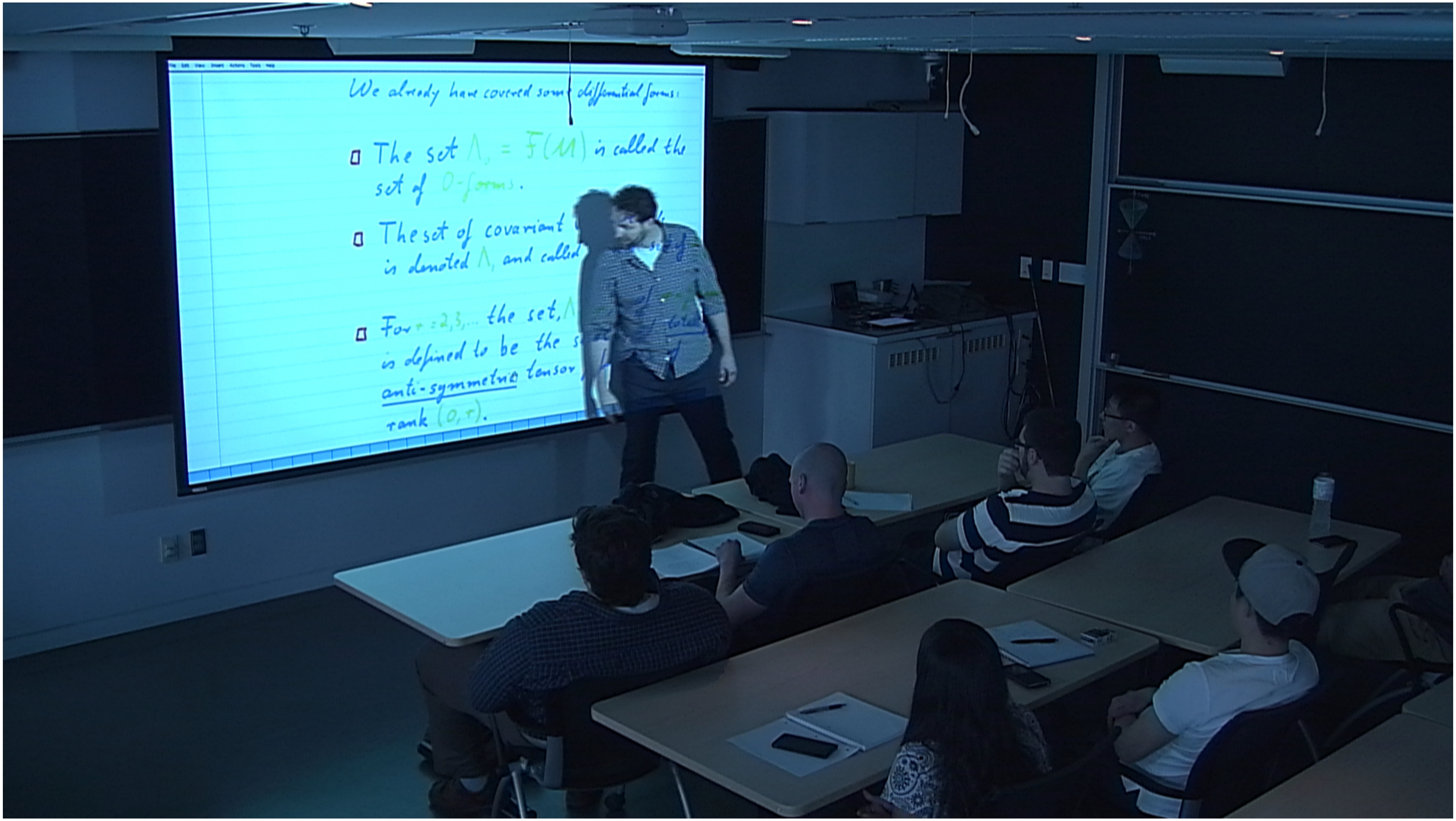

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