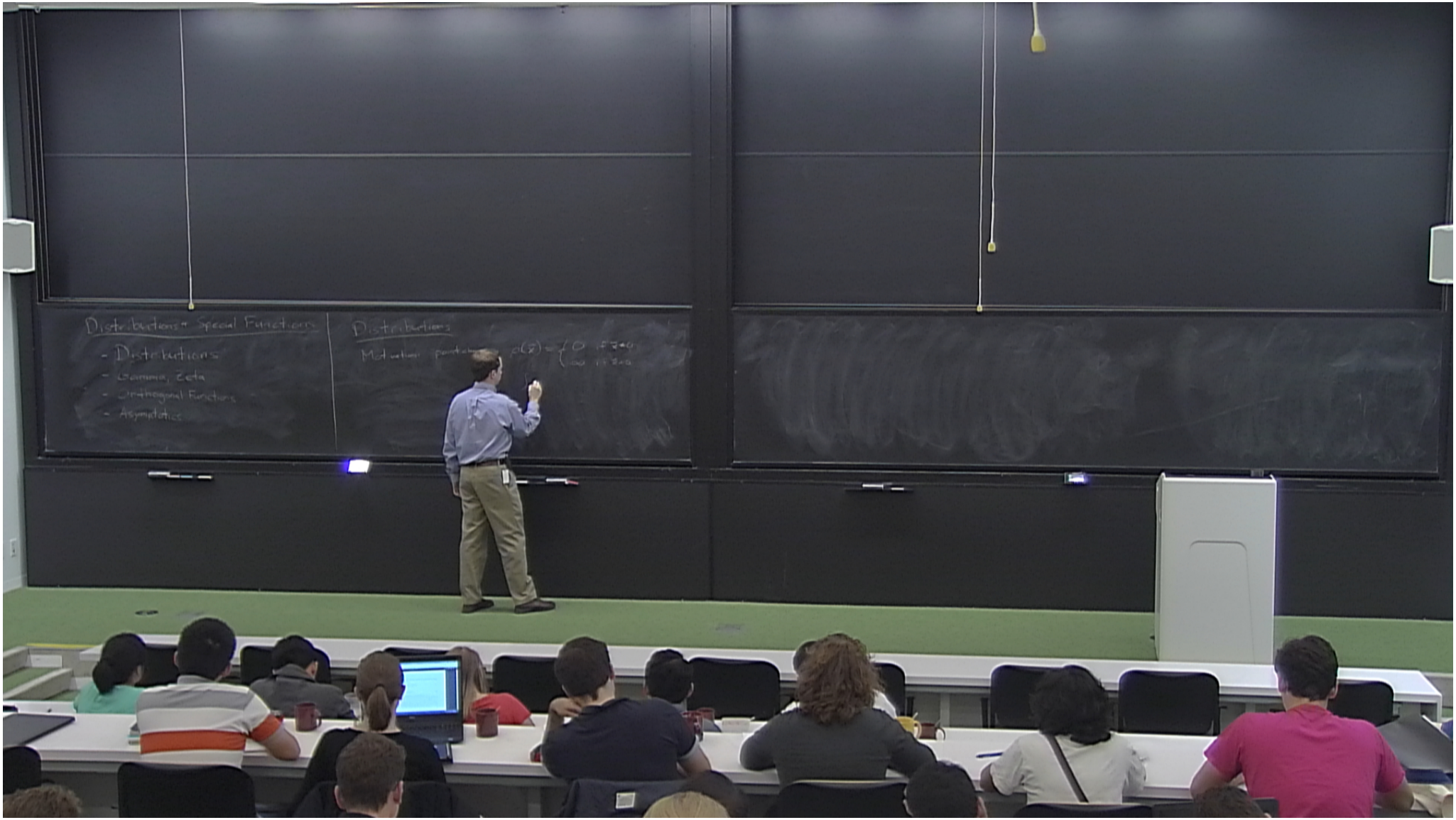
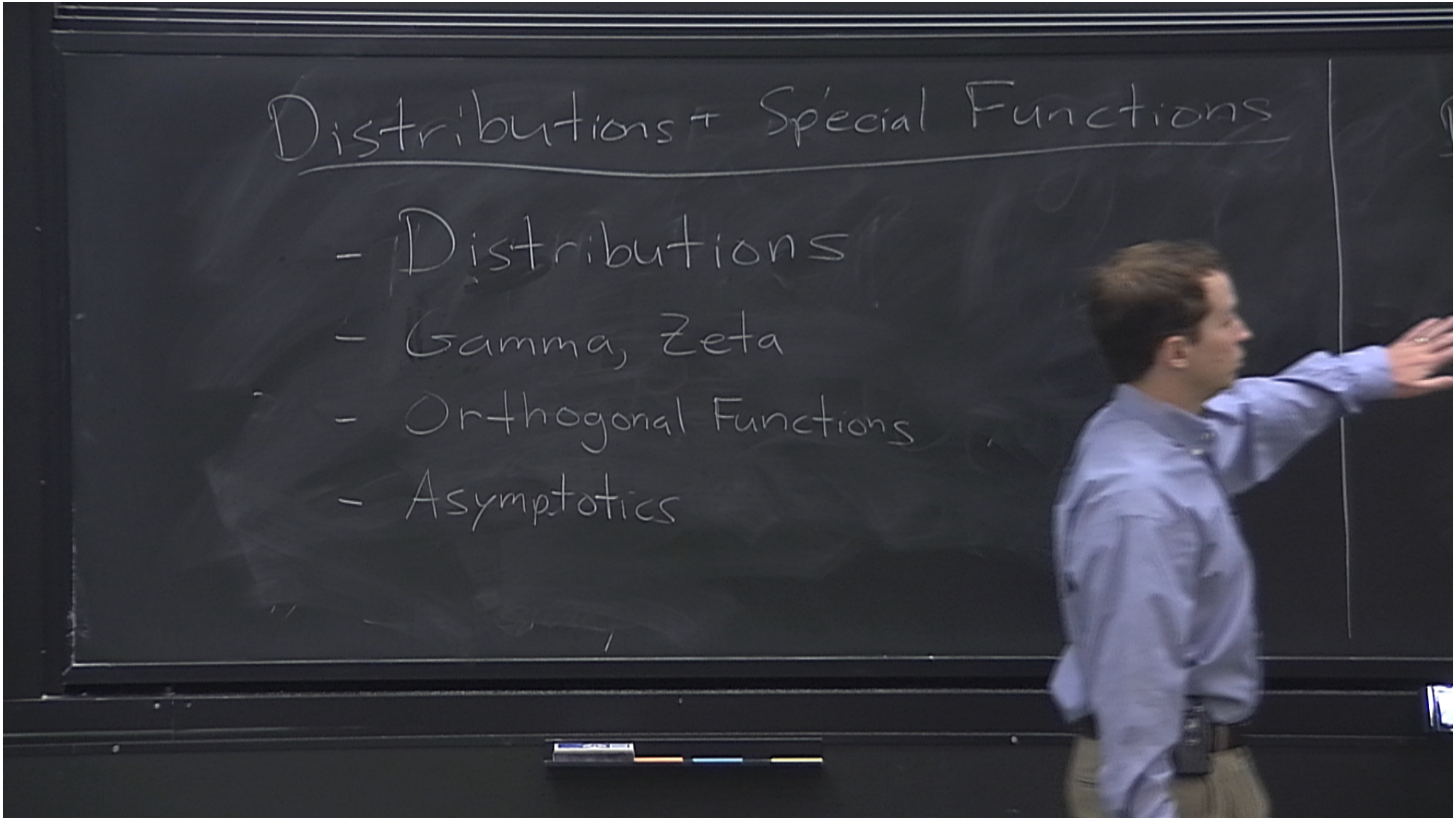
Title: PSI 2015/2016 Special Functions - Dan Wohns -1

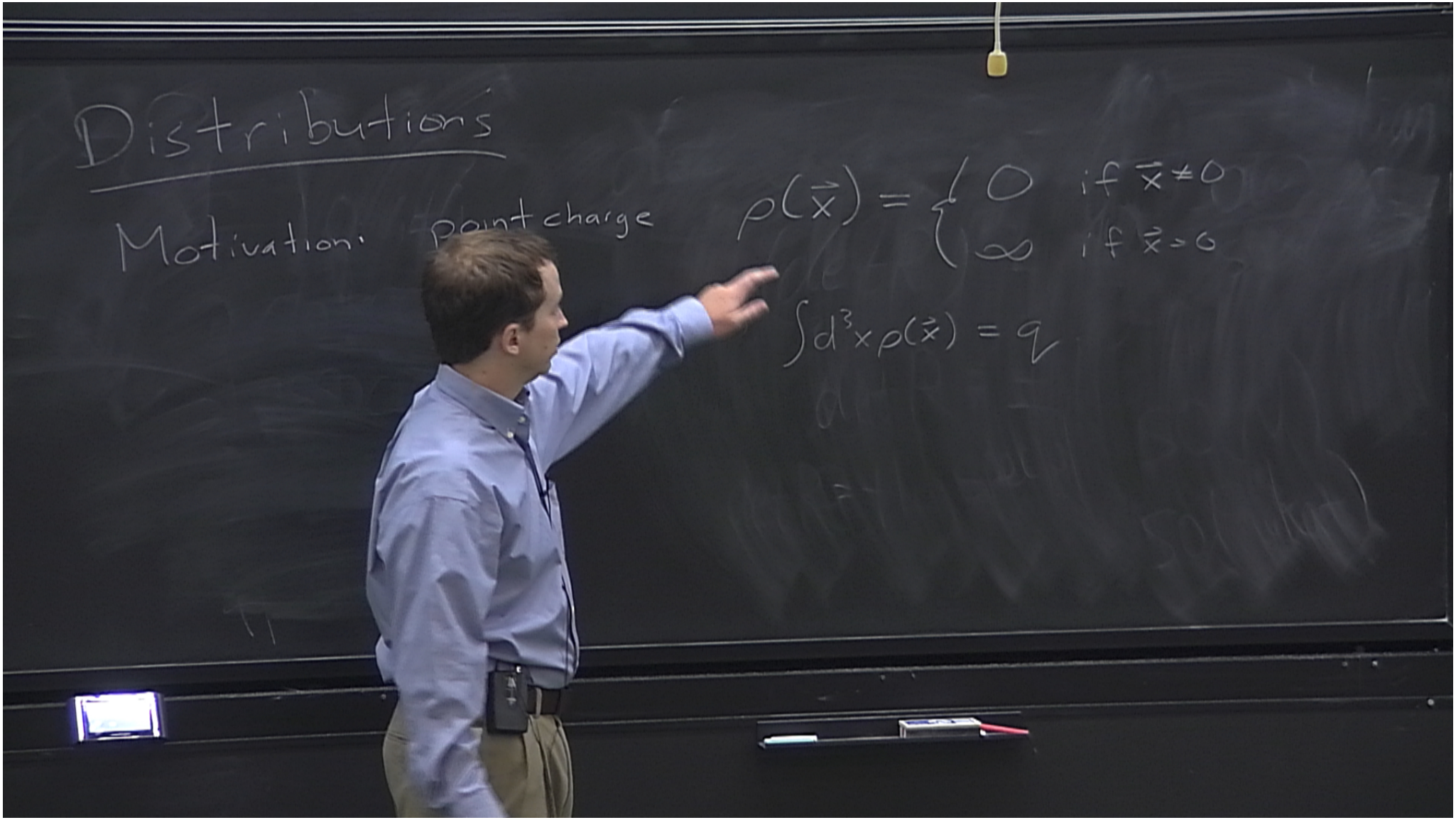
Date: Aug 31, 2015 10:30 AM

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Abstract:


Distributions + Special Functions
- Distributions
- Gamma, Zeta
- Orthogonal Functions
- Asymptotics
Distributions


Distributions + Special Functions
- Distributions
- Gamma, Zeta
- Orthogonal Functions
- Asymptotics


Distributions
Motivation: point charge
$\rho(\vec{x}) = \begin{cases} 0 & \text{if } \vec{x} \neq 0 \\ \infty & \text{if } \vec{x} = 0 \end{cases}$
$\int d^3x \, \rho(\vec{x}) = q$

Distributions

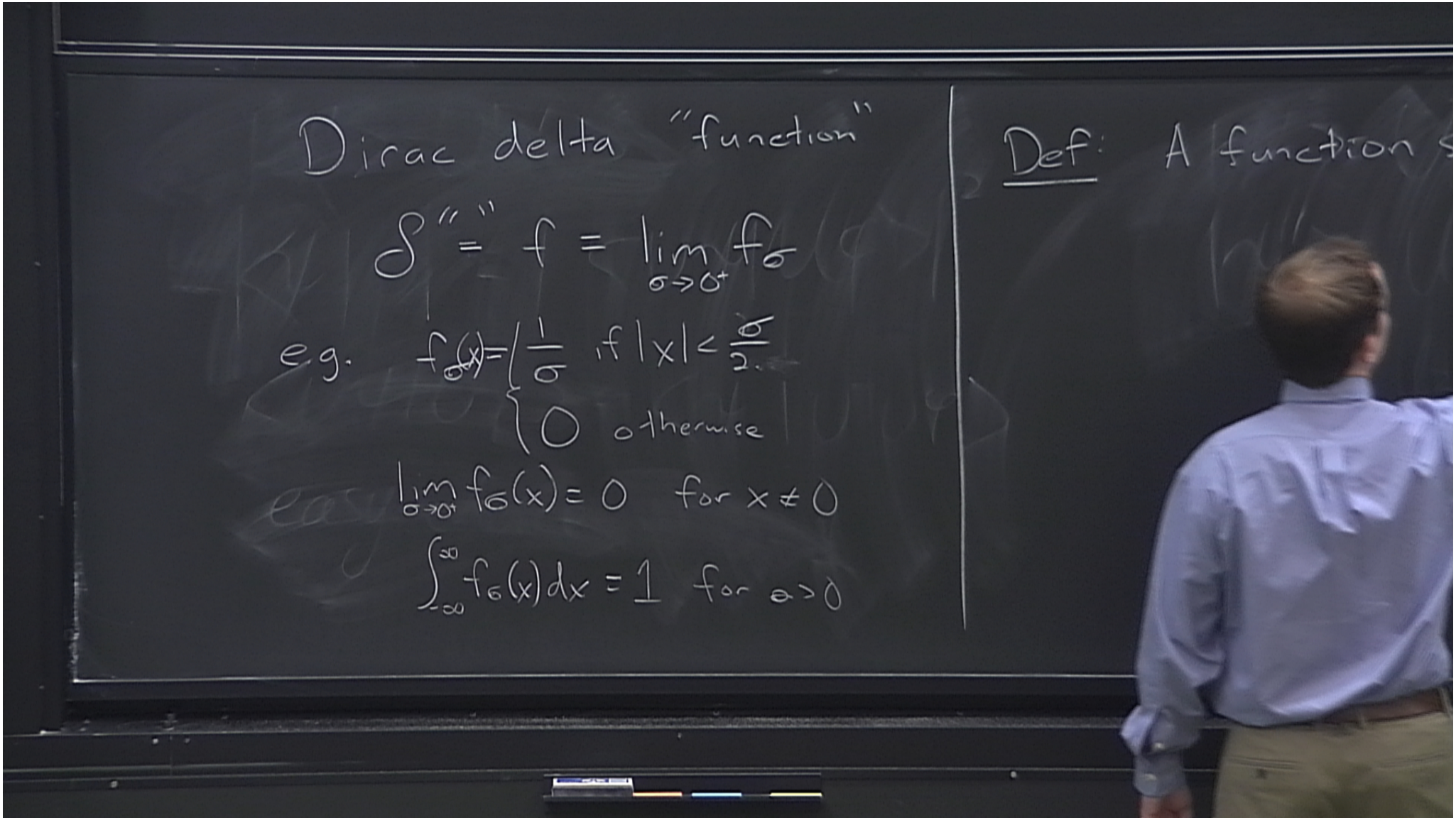
Motivation: • point charge $\rho(\vec{x}) = \begin{cases} 0 & \text{if } \vec{x} \neq 0 \\ \infty & \text{if } \vec{x} = 0 \end{cases}$

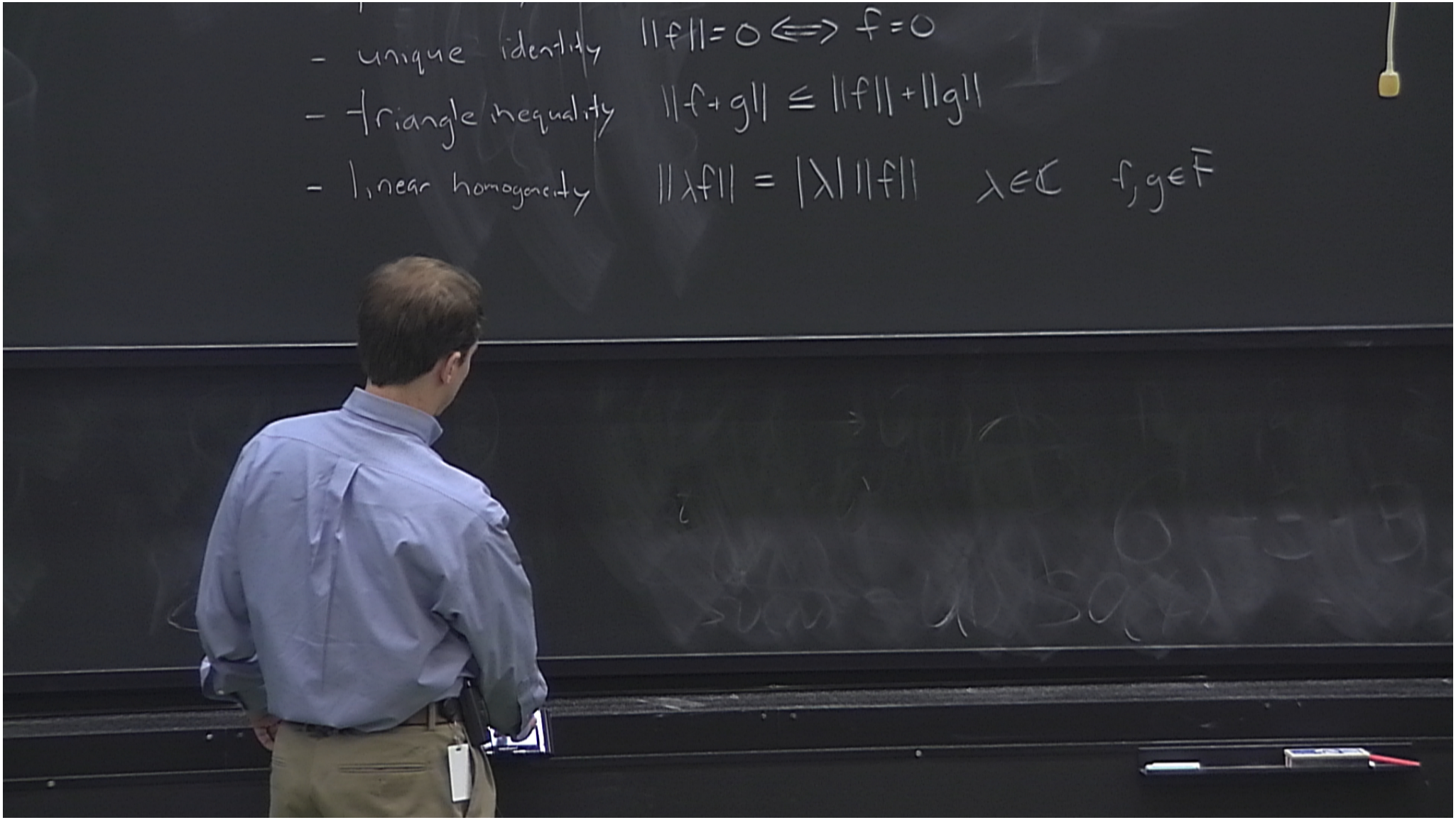
$\int d^3x \, \rho(\vec{x}) = q$

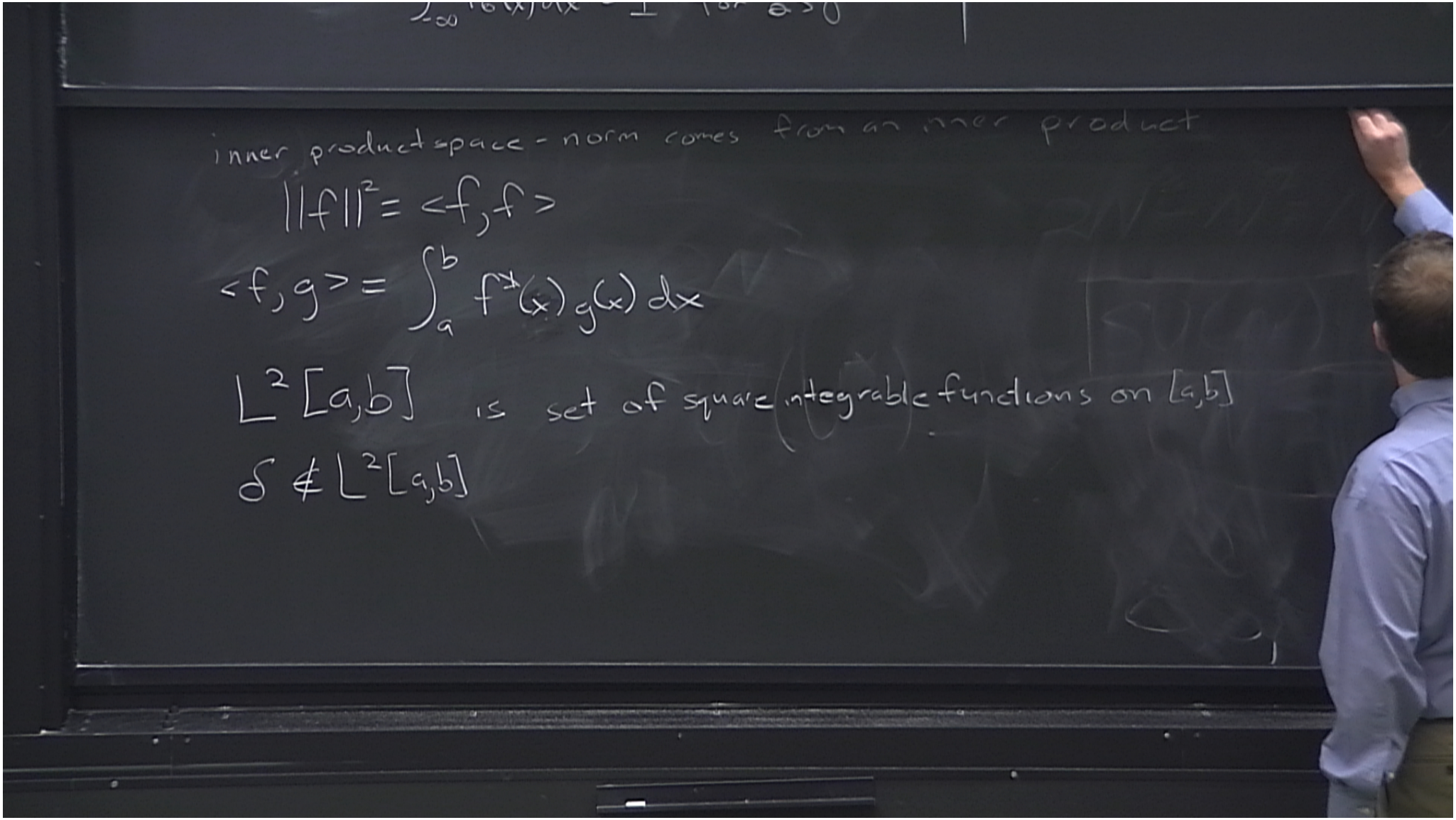
- $\frac{d}{dx}|x|$?
- $\int_{\mathbb{R}} \frac{1}{x} dx$?

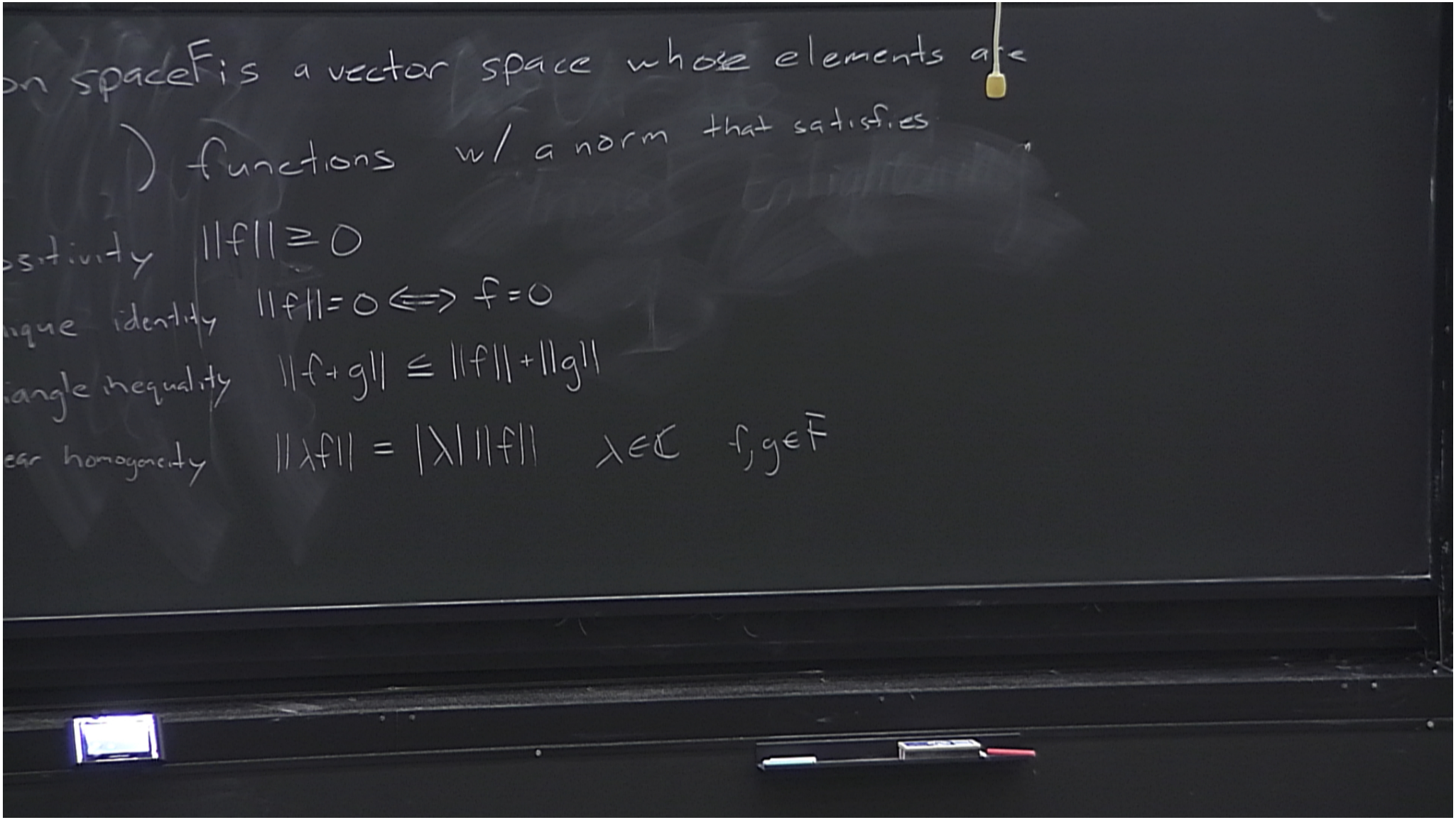
• limits of sequences of functions

• variation problems


Dirac delta "function"
δ "=" $f = \lim_{\sigma \to 0^+} f_\sigma$
e.g. $f_\sigma(x) = \begin{cases} \frac{1}{\sigma} & \text{if } |x| < \frac{\sigma}{2} \\ 0 & \text{otherwise} \end{cases}$
$\lim_{\sigma \to 0^+} f_\sigma(x) = 0$ for $x \neq 0$
$\int_{-\infty}^{\infty} f_\sigma(x)\,dx = 1$ for $\sigma > 0$
Def: A function


- unique identity $\|f\| = 0 \iff f = 0$
- triangle inequality $\|f+g\| \leq \|f\| + \|g\|$
- linear homogeneity $\|\lambda f\| = |\lambda| \|f\|$ $\lambda \in \mathbb{C}$ $f, g \in F$


inner product space - norm comes from an inner product
$\|f\|^2 = \langle f, f \rangle$
$\langle f, g \rangle = \int_a^b f^*(x)\, g(x)\, dx$
$L^2[a,b]$ is set of square integrable functions on $[a,b]$
$\delta \notin L^2[a,b]$


on space F is a vector space whose elements are
) functions w/ a norm that satisfies
positivity $\|f\| \geq 0$
nique identity $\|f\| = 0 \iff f = 0$
iangle inequality $\|f+g\| \leq \|f\| + \|g\|$
ear homogeneity $\|\lambda f\| = |\lambda| \|f\|$ $\lambda \in \mathbb{C}$ $f, g \in F$

$f_1(x) = 0 \iff \|f_1\| = 0$

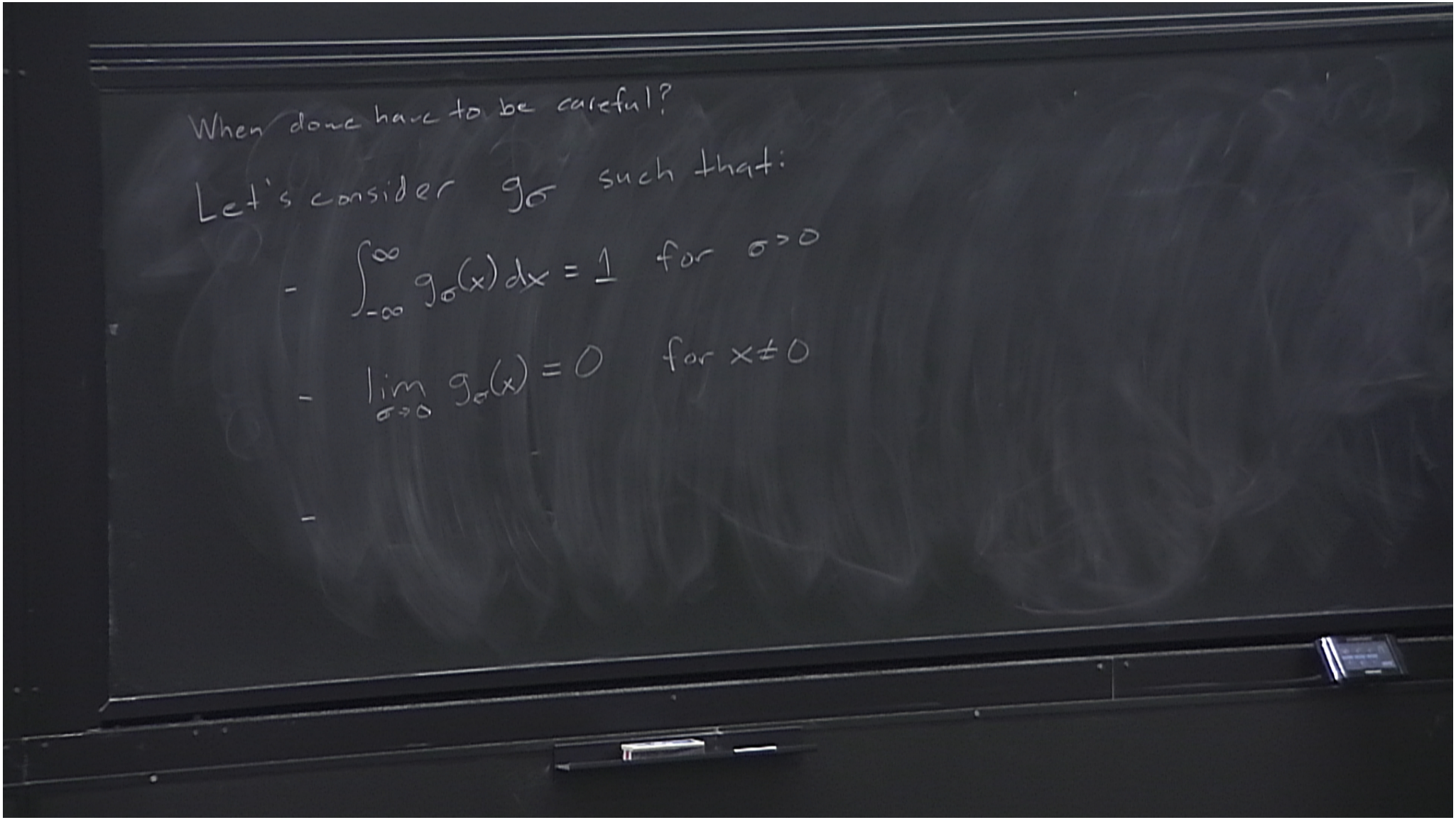
$$f_2(x) = \begin{cases} 0 & x \neq 1 \\ \pi & x = 1 \end{cases} \qquad \|f_2\| = 0$$

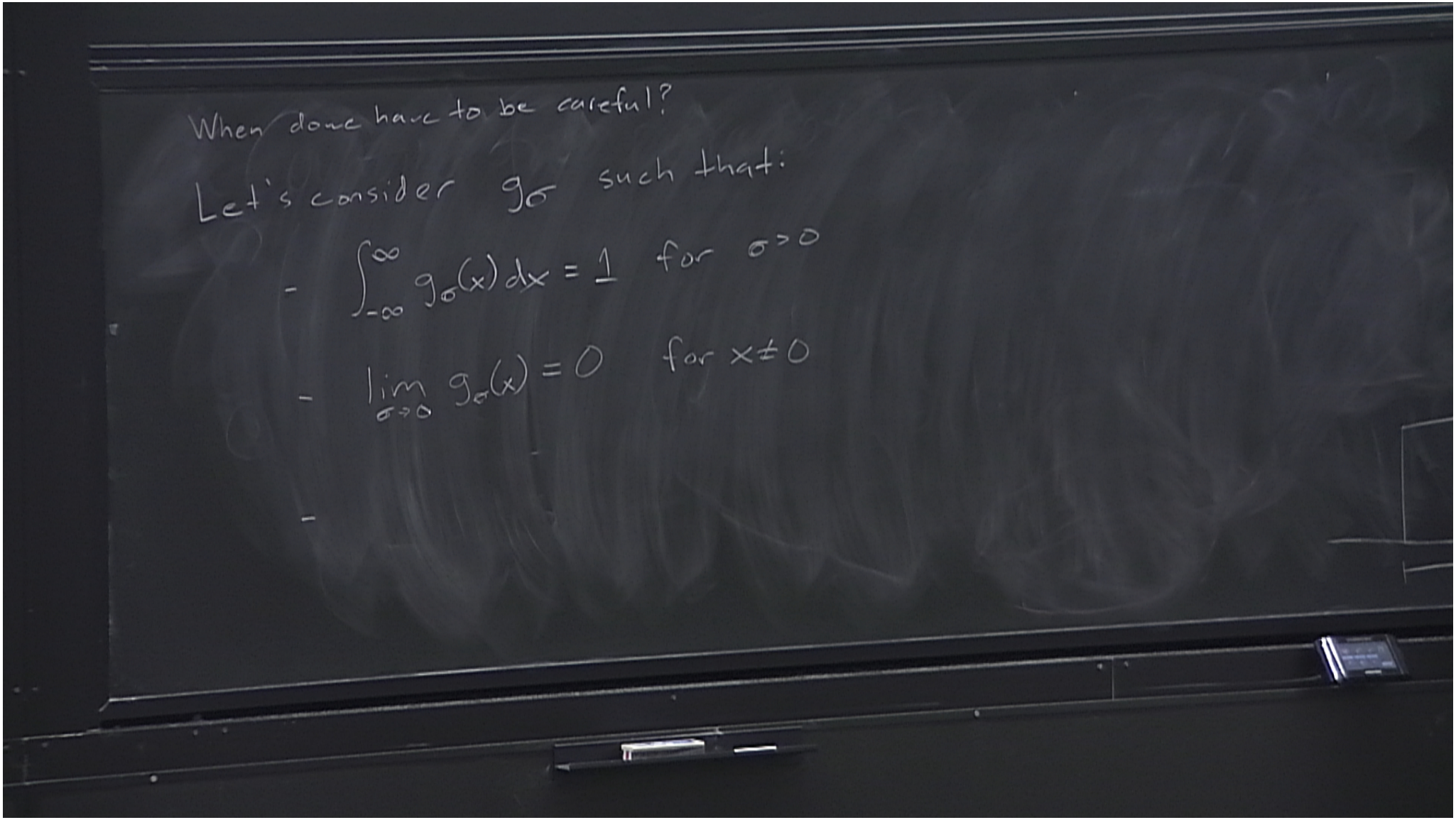
unique identity means we need to identify these functions

$L^2[-\infty, \infty]$ $\qquad \|f_\sigma\|^2 = \int_{-\infty}^{\infty} f_\sigma^2(x)\,dx$ $\qquad \|f\|$ is undefined

$$= \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{\sigma^2}\,dx$$

$$= \frac{1}{\sigma}$$

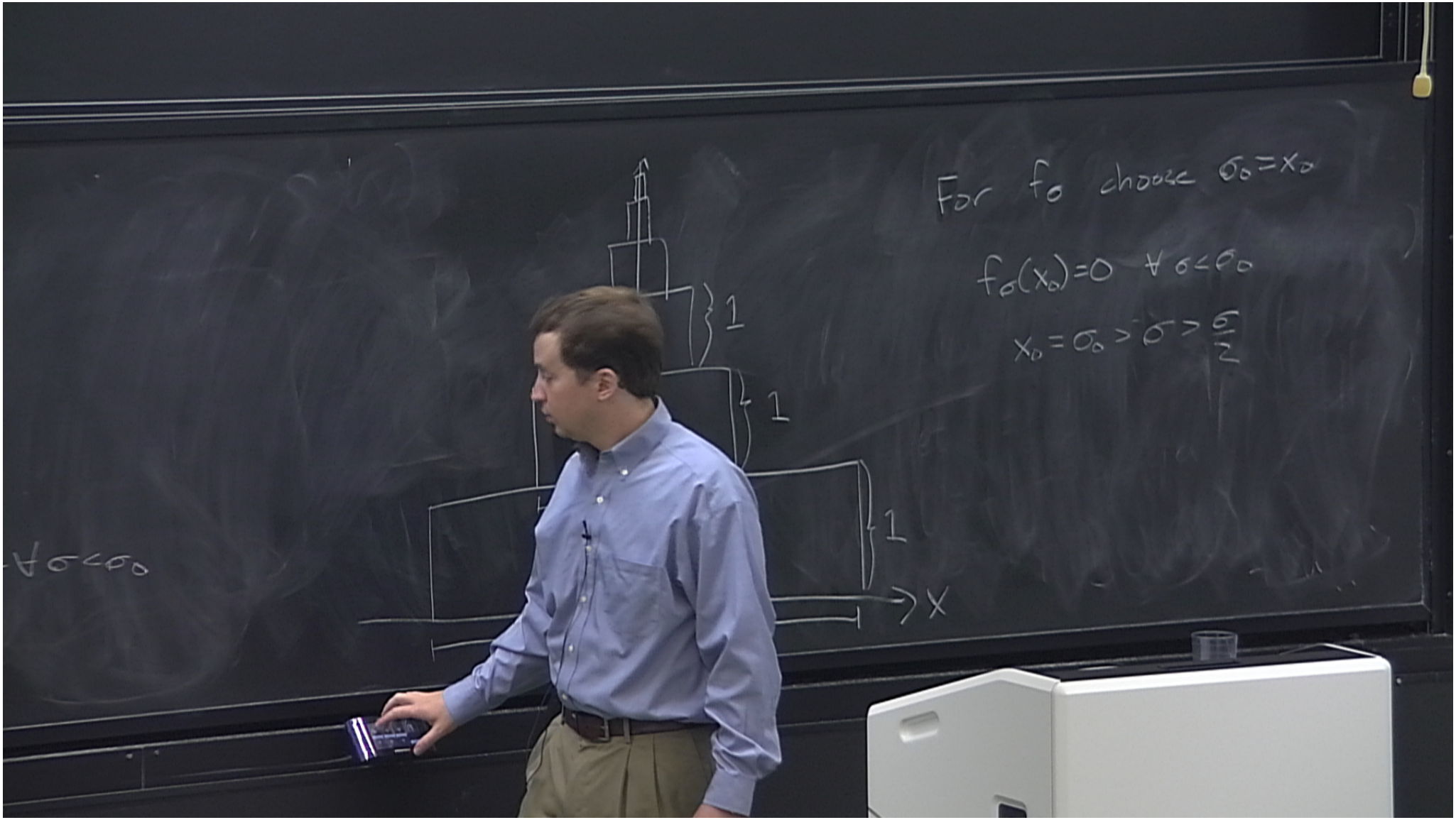

When done have to be careful?
Let's consider g_σ such that:
- $\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1$ for $\sigma > 0$
- $\lim_{\sigma \to 0} g_\sigma(x) = 0$ for $x \neq 0$
-


When does have to be careful?
Let's consider g_σ such that:
- $\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1$ for $\sigma > 0$
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When do we have to be careful?

Let's consider g_σ such that:

- $\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1$ for $\sigma > 0$
- $\lim_{\sigma \to 0} g_\sigma(x) = 0$ for $x \neq 0$
- For each x_0 + each ε, $\exists \sigma_0$ such that $\forall \sigma < \sigma_0$ $g_\sigma(x_0)$ fits in the boxes


For f_σ choose $\sigma_0 = x_0$
$f_\sigma(x_0) = 0 \quad \forall \sigma < \sigma_0$
$x_0 = \sigma_0 > \sigma > \frac{\sigma}{2}$
1
1
1
X

$g(x) = \lim_{\sigma \to 0} g_\sigma(x)$ fits in the boxes

$$\int_{-\infty}^{\infty} g(x)\,dx < A = \epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{4} + \cdots = 2\epsilon$$

$$\int_{-\infty}^{\infty} \lim_{\sigma \to 0} g_\sigma(x)\,dx = 0 \neq \underline{1} = \lim_{\sigma \to 0} \int_{-\infty}^{\infty} g_\sigma(x)\,dx$$

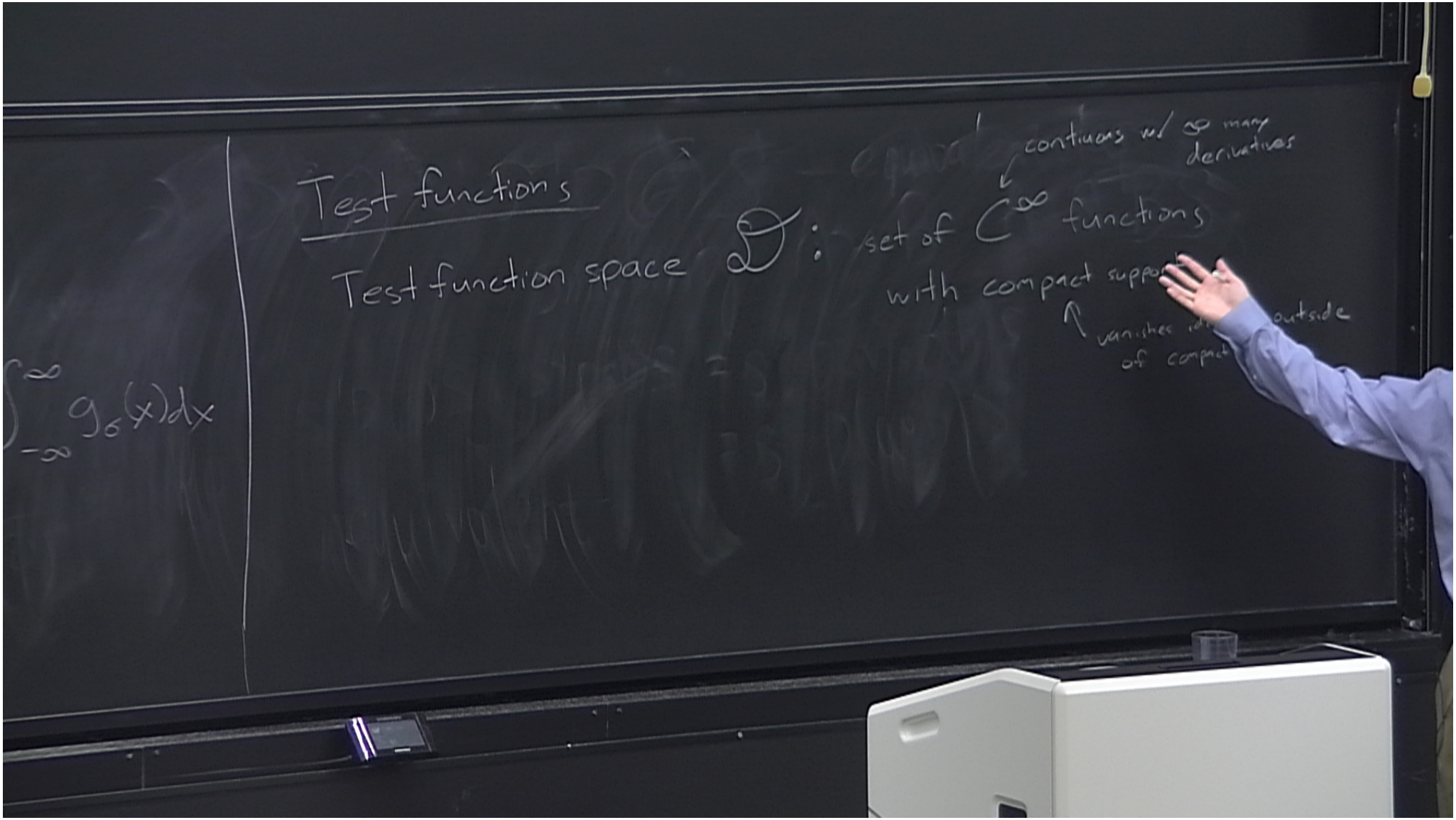
order matters!

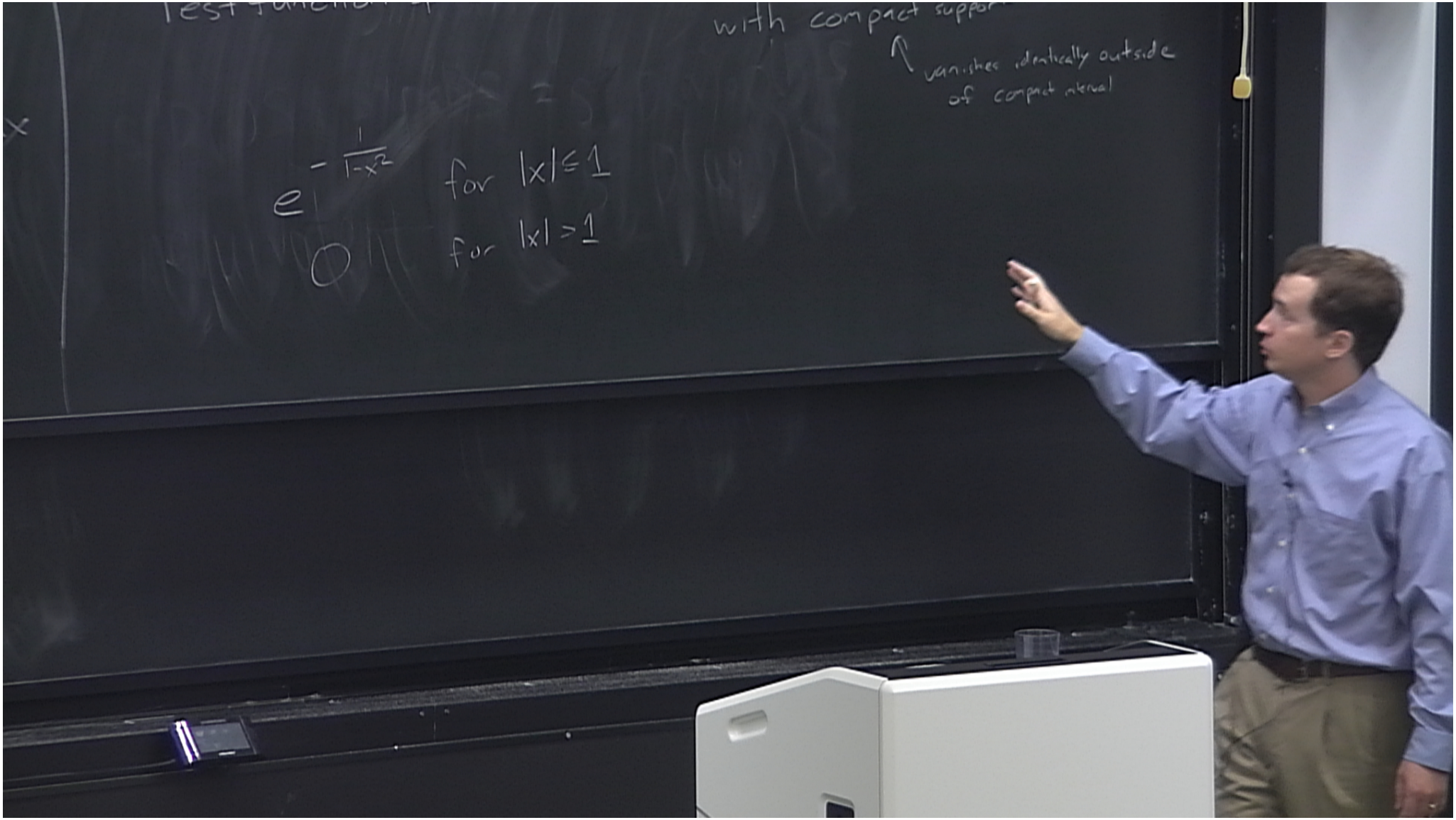
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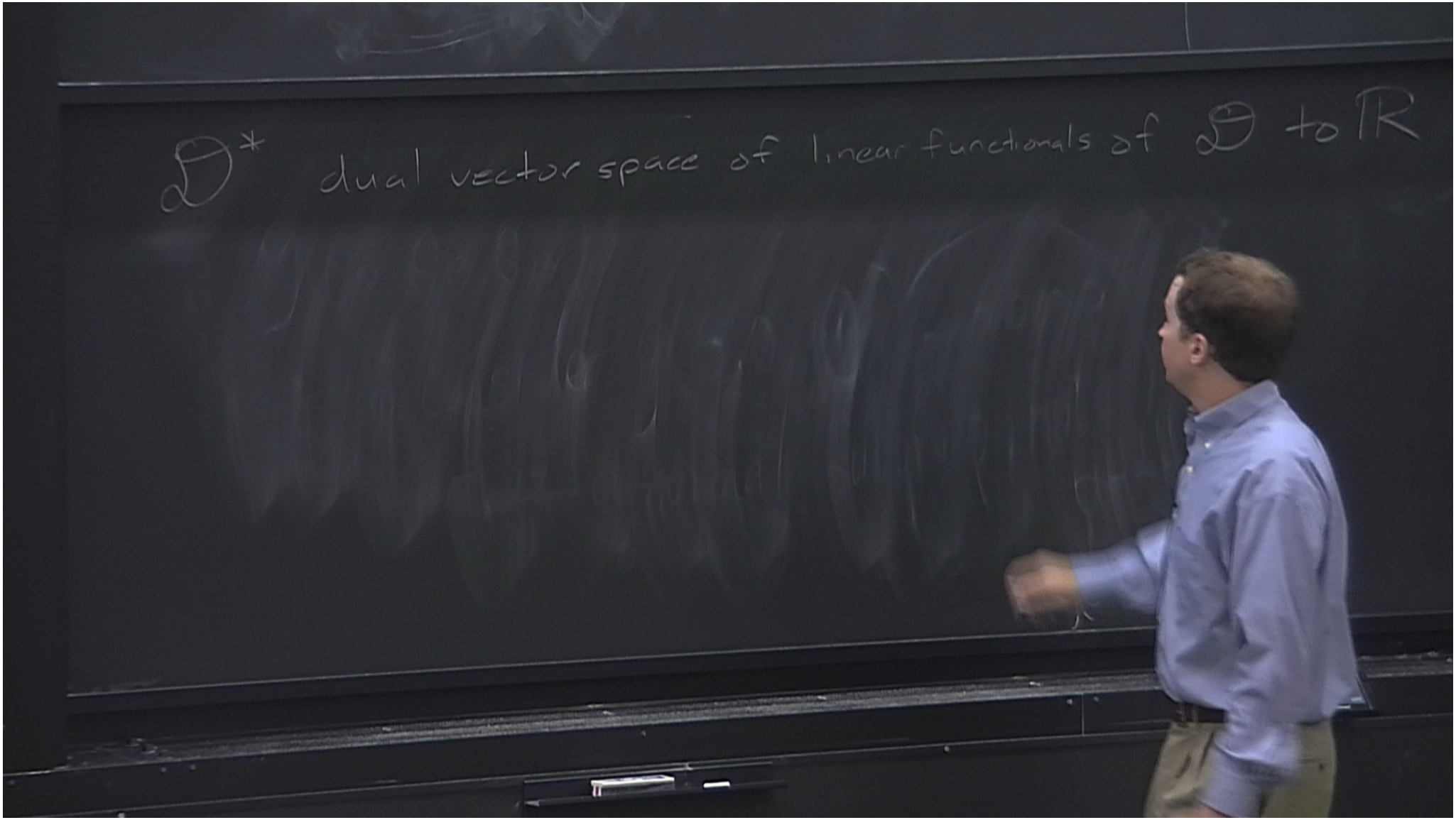
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$$\int_{-\infty}^{\infty} \lim_{\sigma \to 0} g_\sigma(x)\,dx = 0 \neq 1 = \lim_{\sigma \to 0} \int_{-\infty}^{\infty} g_\sigma(x)\,dx$$

order matters!


Test functions
Test function space $\mathcal{D}$: set of C^∞ functions
continuous w/ ∞ many derivatives
with compact support
$\int_{-\infty}^{\infty} g_\sigma(x)dx$


with compact
vanishes identically outside
of compact interval
$e^{-\frac{1}{1-x^2}}$ for $|x| \leq 1$
0 for $|x| > 1$


$\mathcal{D}^*$ dual vector space of linear functionals of $\mathcal{D}$ to $\mathbb{R}$

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$u \in \mathcal{D}^*$ $\quad u(\alpha\varphi + \beta\psi) = \alpha u(\varphi) + \beta u(\psi)$ $\quad \alpha, \beta \in \mathbb{R}$ $\quad \varphi, \psi \in \mathcal{D}$

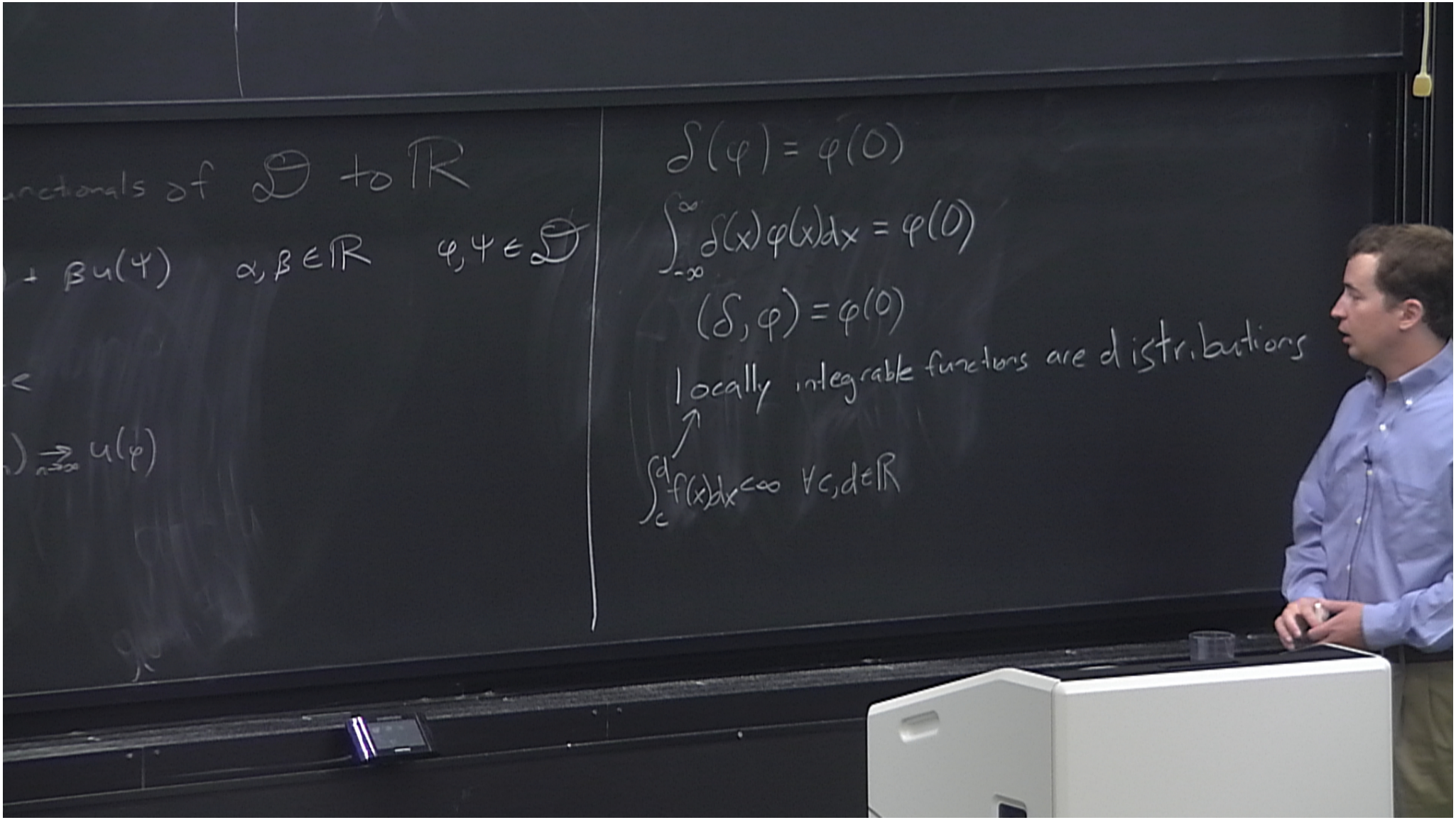
$\mathcal{D}^*$ dual vector space of linear functionals of $\mathcal{D}$ to $\mathbb{R}$

$u \in \mathcal{D}^*$ $\quad u(\alpha\varphi + \beta\psi) = \alpha u(\varphi) + \beta u(\psi) \quad \alpha, \beta \in \mathbb{R} \quad \varphi, \psi \in \mathcal{D}$

$\mathcal{D}'$ is largest continuous dual space

If $\|\varphi_n - \varphi\| \underset{n\to\infty}{\to} 0$ then $u(\varphi_n) \underset{n\to\infty}{\to} u(\varphi)$

distribution is a functional that is
- linear
- continuous
- finite


functionals of $\mathcal{D}$ to $\mathbb{R}$
$+ \beta u(\psi)$ $\alpha, \beta \in \mathbb{R}$ $\varphi, \psi \in \mathcal{D}$
$\delta(\varphi) = \varphi(0)$
$\int_{-\infty}^{\infty} \delta(x)\varphi(x)dx = \varphi(0)$
$(\delta, \varphi) = \varphi(0)$
locally integrable functions are distributions
$\int_c^d f(x)dx < \infty \quad \forall c, d \in \mathbb{R}$

$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$ for $a > 0$

$u(\varphi) = \varphi^2(0)$ X not linear $\quad u(\varphi+\psi) \neq u(\varphi)+u(\psi)$

$v(\varphi) = \varphi(0) + \varphi'(0)$ ✓ finite

✓ continuous

✓ linear

Der



(4)

Derivatives

$$u'(x) = \lim_{\epsilon \to 0} \frac{u(x+\epsilon) - u(x)}{\epsilon}$$

u is not defined at a point

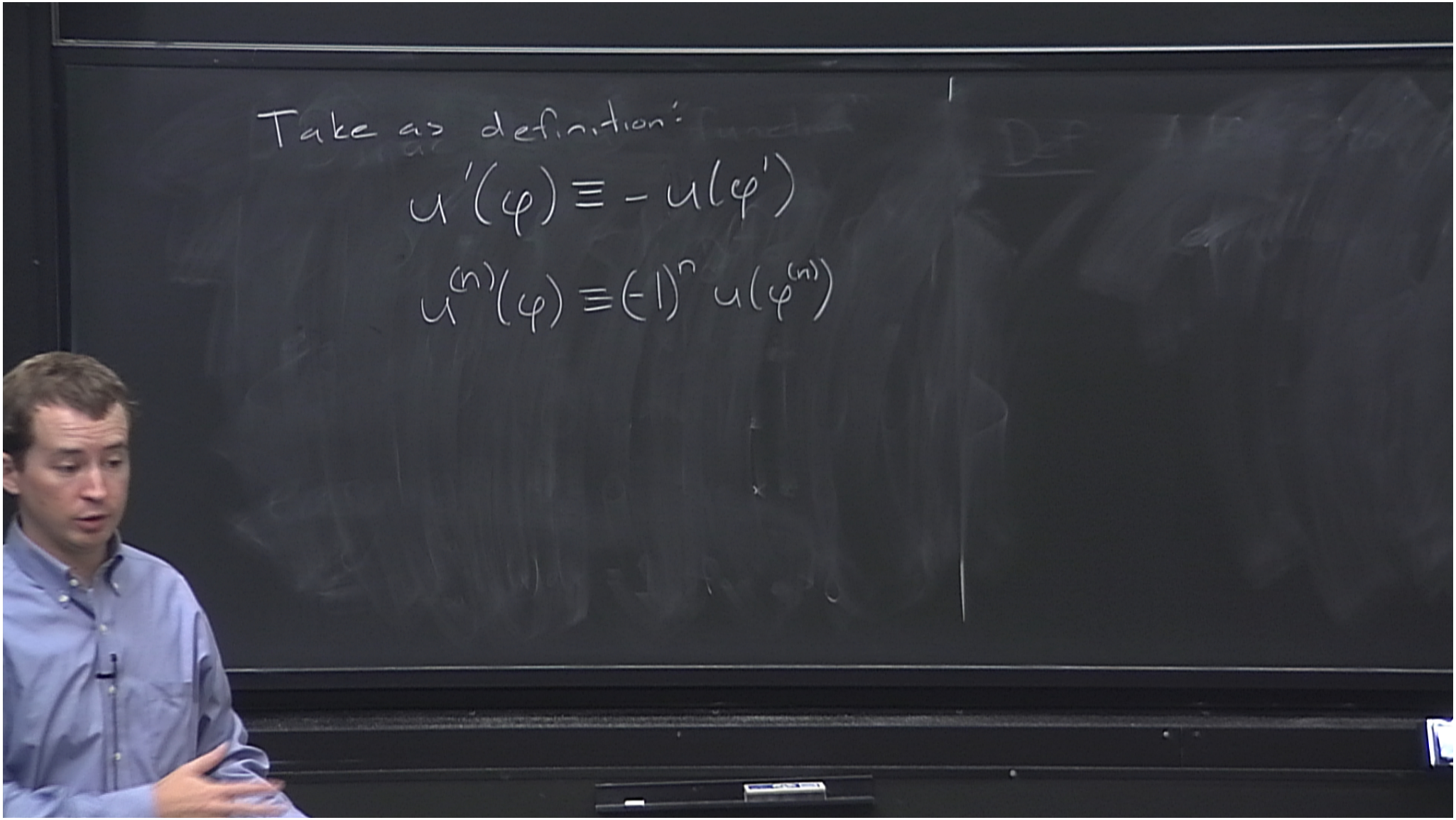
$$u'(\varphi) = \int_{-\infty}^{\infty} u'(x)\,\varphi(x)\,dx$$

$$= u(x)\varphi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)\varphi'(x)\,dx$$

ok if u is a function

$$= -\int_{-\infty}^{\infty} u(x)\varphi'(x)\,dx$$

since $\varphi(x)$ is a test function


Take as definition:
$u'(\varphi) \equiv -u(\varphi')$
$u^{(n)}(\varphi) \equiv (-1)^n u(\varphi^{(n)})$

Example: Heaviside function $H(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$

$$H(\varphi) = \int_{-\infty}^{\infty} H(x)\,\varphi(x)\,dx$$

$$= \int_{0}^{\infty} \varphi(x)\,dx$$ finite since φ is bounded + has finite support

(also locally integrable)

...ner product

$f_1(x) = 0 \iff \|f_1\| = 0$

$f_2(x) = \begin{cases} 0 & x \neq 1 \\ \pi & x = 1 \end{cases}$ $\|f_2\| = 0$

$$\Theta'(\varphi) = -\Theta(\varphi')$$

$$\Theta' = \delta$$

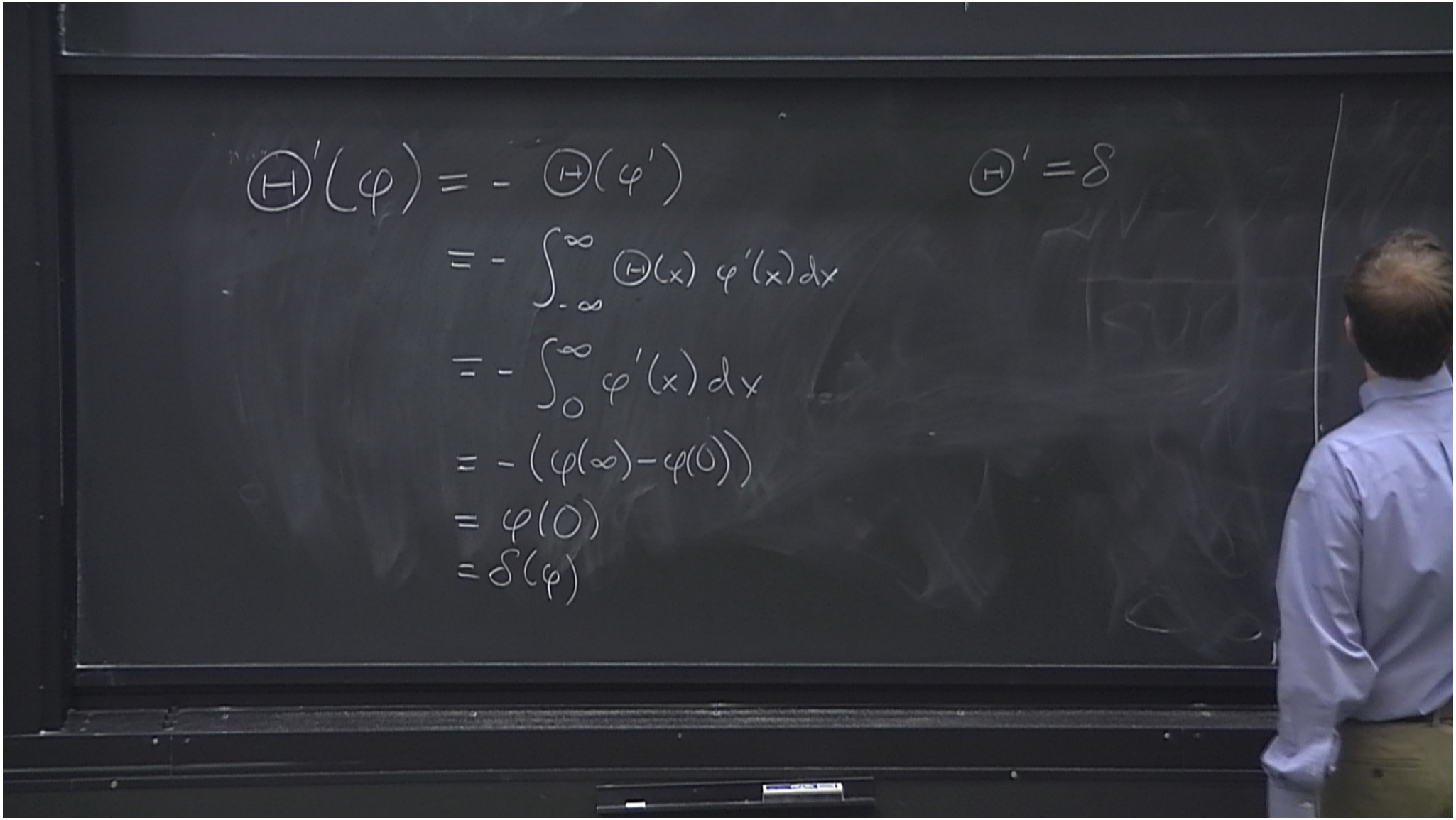
$$= -\int_{-\infty}^{\infty} \Theta(x)\,\varphi'(x)\,dx$$

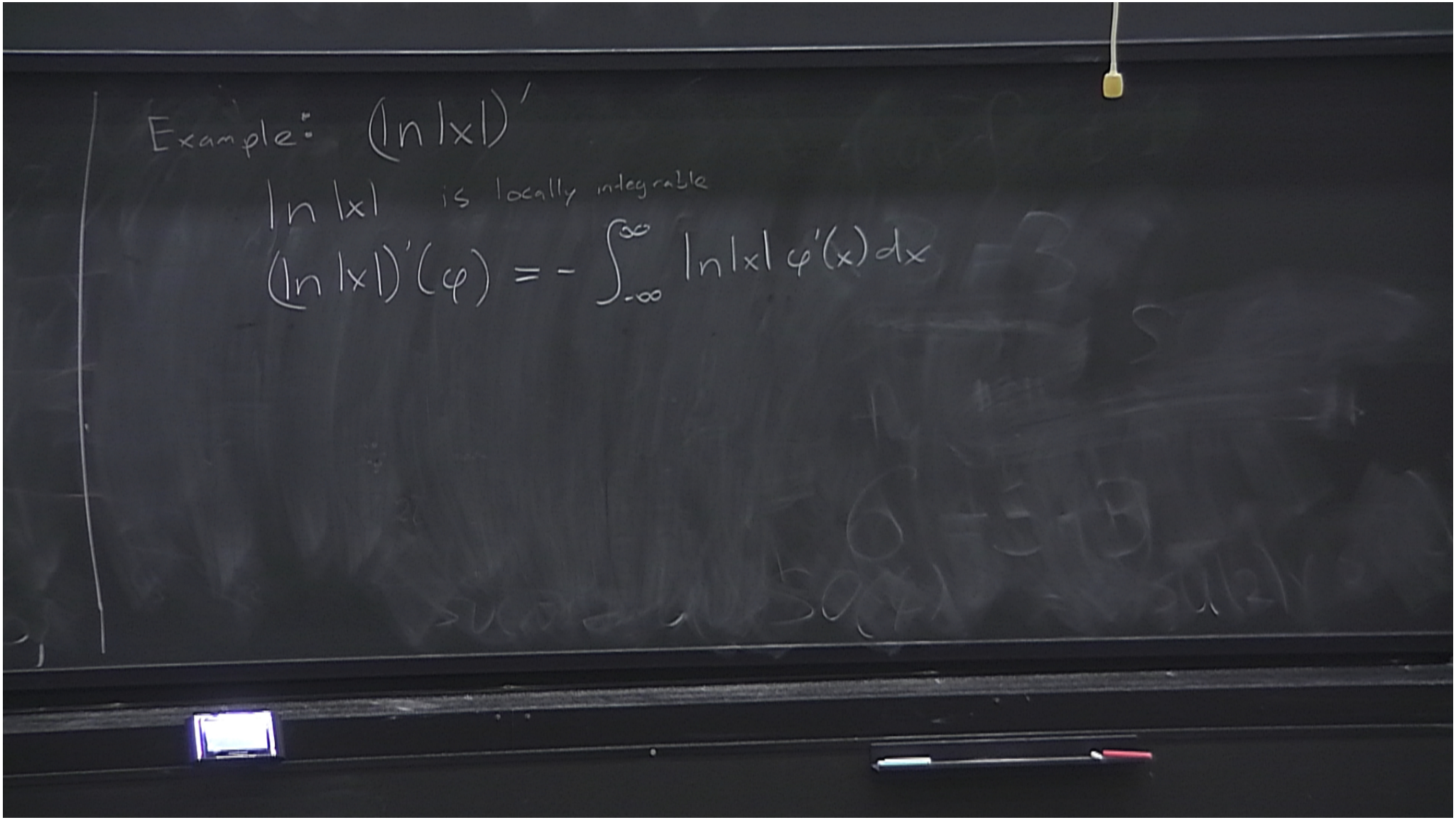
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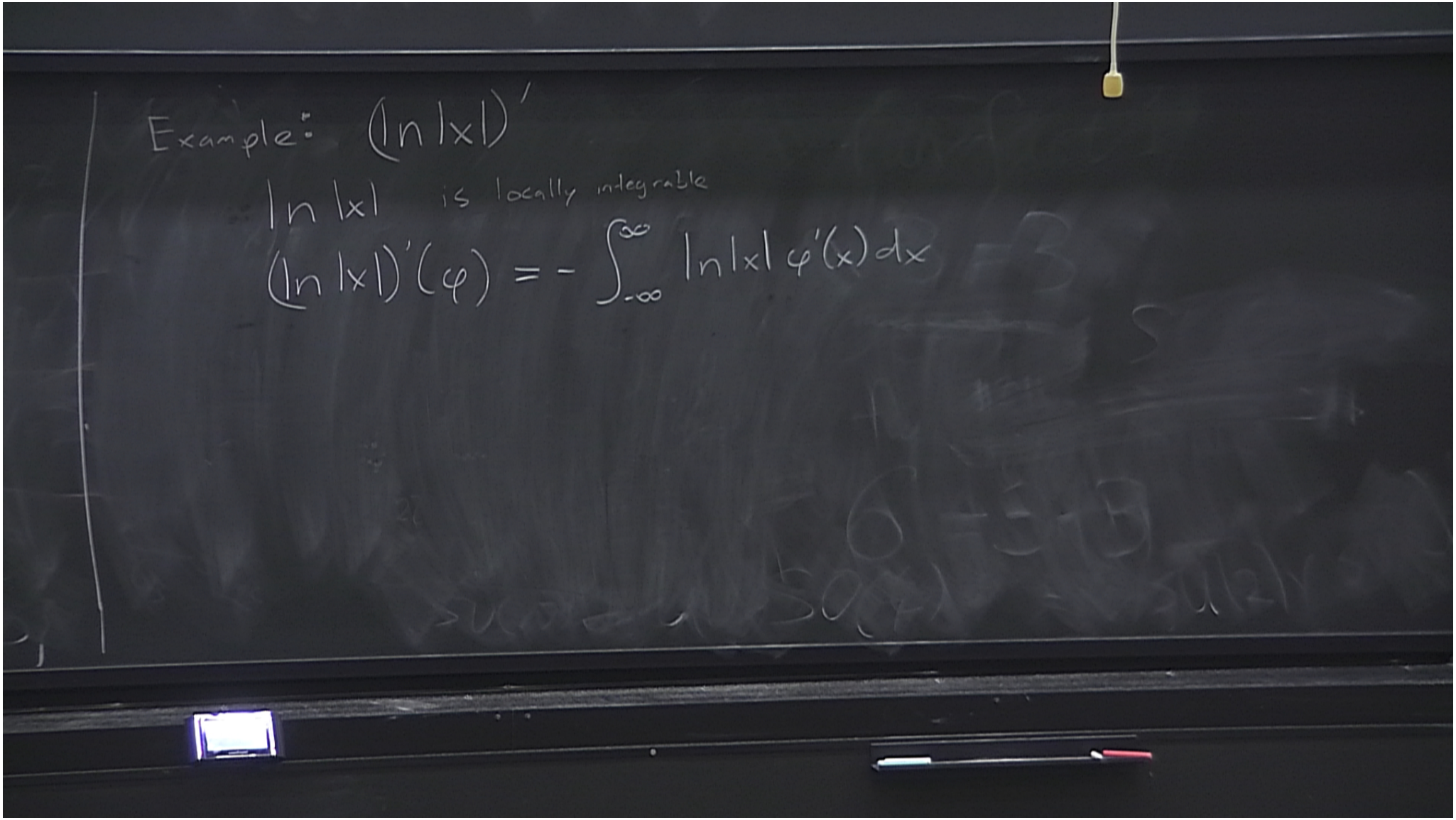
$$= -(\varphi(\infty) - \varphi(0))$$

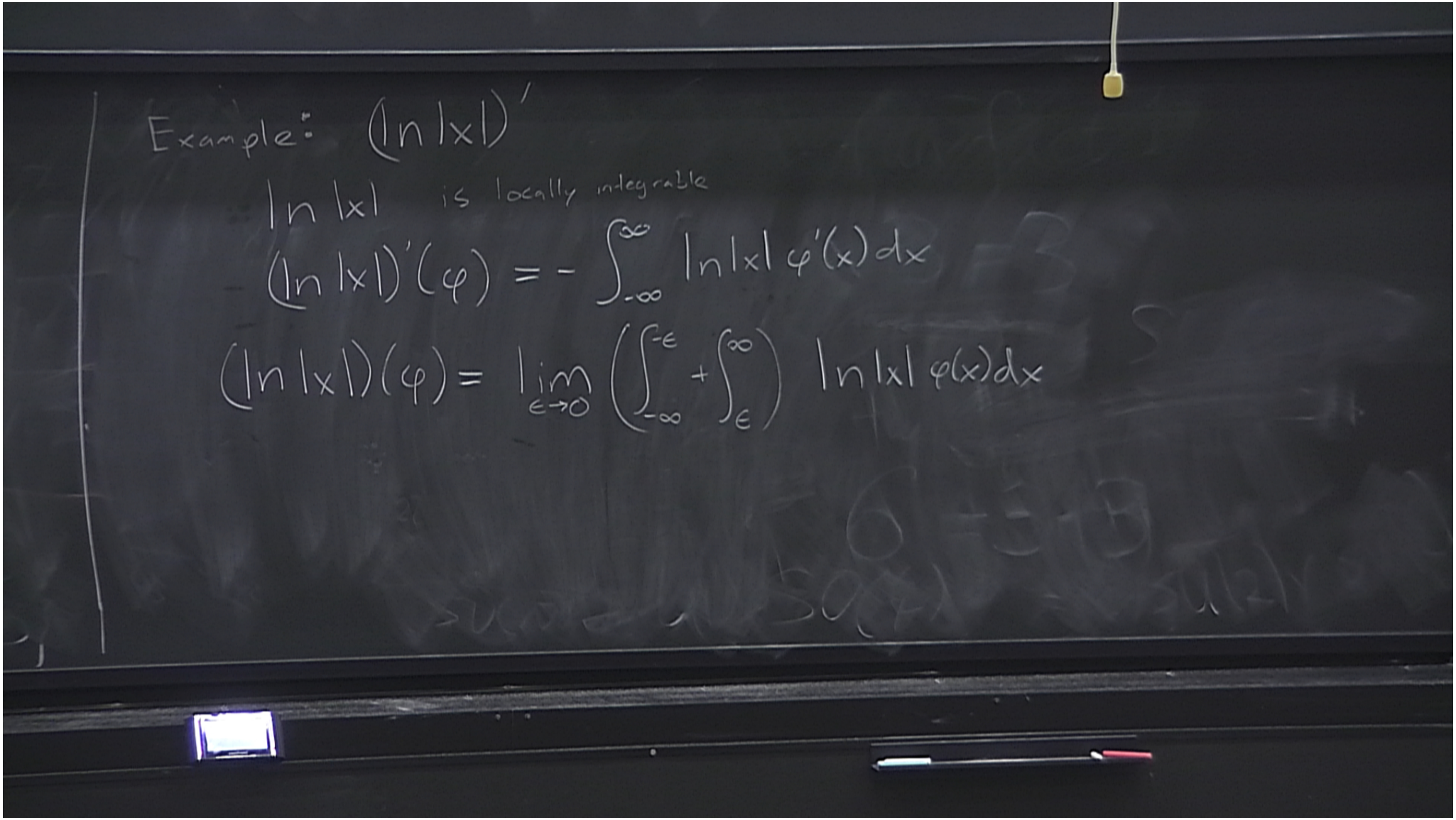
$$= \varphi(0)$$

$$= \delta(\varphi)$$


$\Theta'(\varphi) = -\Theta(\varphi')$
$\Theta' = \delta$
$= -\int_{-\infty}^{\infty} \Theta(x)\,\varphi'(x)\,dx$
$= -\int_{0}^{\infty} \varphi'(x)\,dx$
$= -(\varphi(\infty) - \varphi(0))$
$= \varphi(0)$
$= \delta(\varphi)$


Example: $(\ln|x|)'$
$\ln|x|$ is locally integrable
$(\ln|x|)'(\varphi) = -\int_{-\infty}^{\infty} \ln|x|\,\varphi'(x)\,dx$


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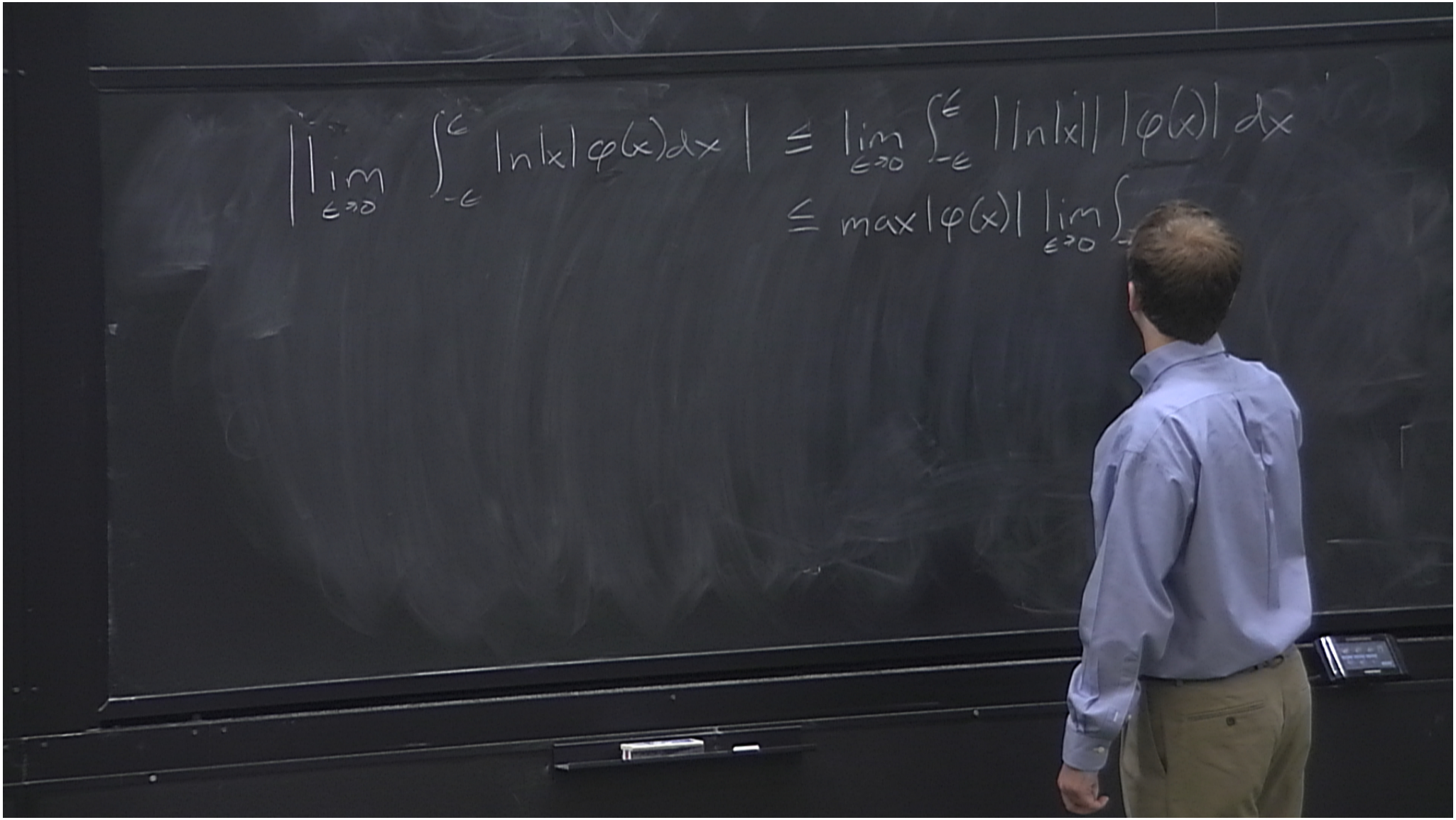

Example: $(\ln|x|)'$
$\ln|x|$ is locally integrable
$(\ln|x|)'(\varphi) = -\int_{-\infty}^{\infty} \ln|x|\,\varphi'(x)\,dx$
$(\ln|x|)(\varphi) = \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) \ln|x|\,\varphi(x)\,dx$

Example: $(\ln|x|)'$

$\ln|x|$ is locally integrable

$$(\ln|x|)'(\varphi) = -\int_{-\infty}^{\infty} \ln|x|\,\varphi'(x)\,dx$$

$$(\ln|x|)(\varphi) = \lim_{\epsilon \to 0} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \ln|x|\,\varphi(x)\,dx$$


$\left| \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \ln|x| \varphi(x) dx \right| \leq \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} |\ln|x|| \, |\varphi(x)| \, dx$
$\leq \max|\varphi(x)| \lim_{\epsilon \to 0}$

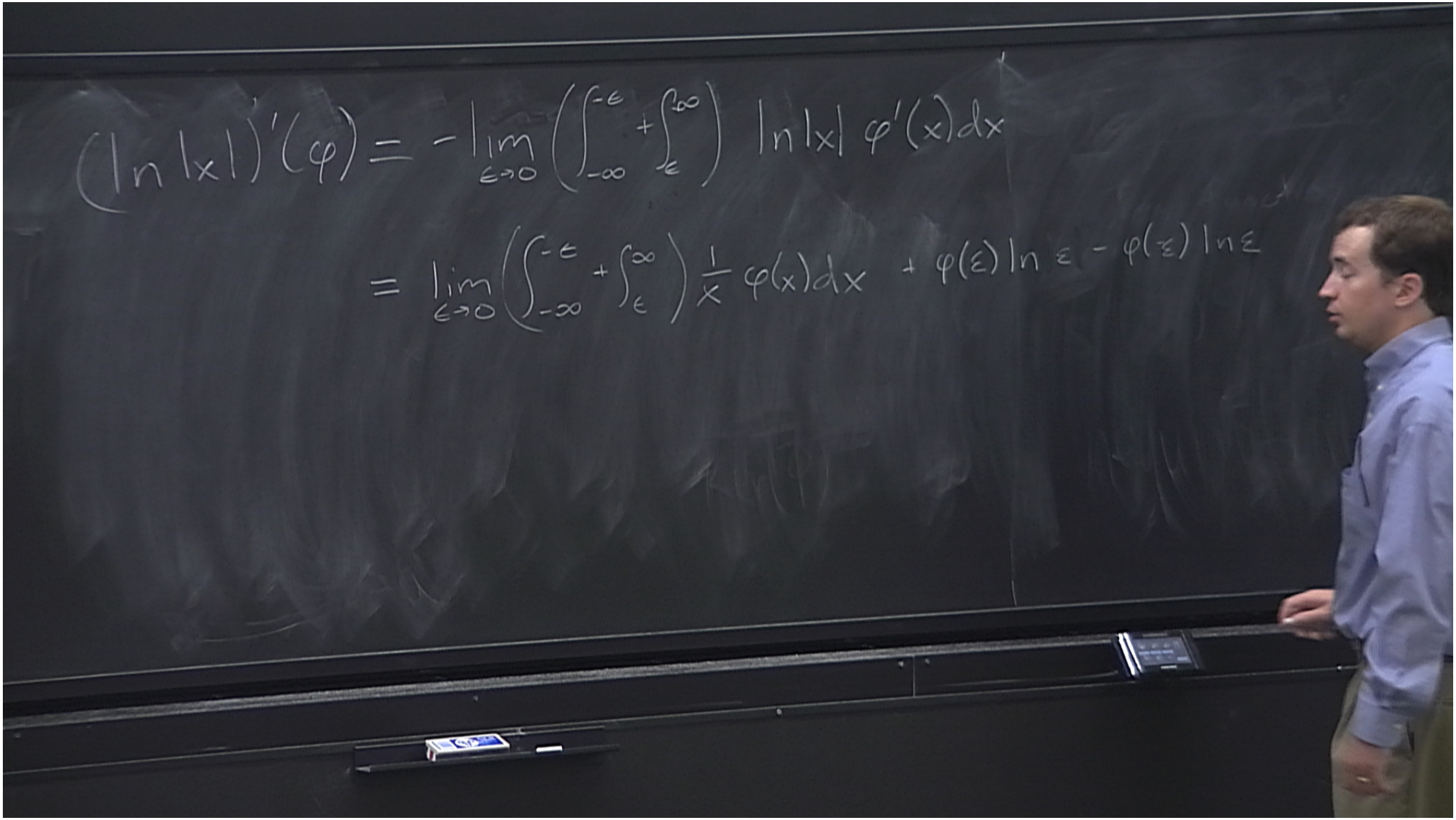
$$\left|\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\ln|x|\,\varphi(x)\,dx\right| \le \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}|\ln|x||\,|\varphi(x)|\,dx$$

$$\le \max|\varphi(x)|\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}|\ln|x||\,dx$$

$$= \max|\varphi(x)|\lim_{\epsilon\to 0}\int_{0}^{\epsilon}(-2)\ln x\,dx$$

$$= \max|\varphi(x)|\lim_{\epsilon\to 0}(-2)\,x(\ln x-1)\Big|_0^{\epsilon}$$

$$= 0$$


$(\ln|x|)'(\varphi) = -\lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) \ln|x|\,\varphi'(x)\,dx$
$= \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) \frac{1}{x}\varphi(x)\,dx + \varphi(\epsilon)\ln\epsilon - \varphi(-\epsilon)\ln\epsilon$