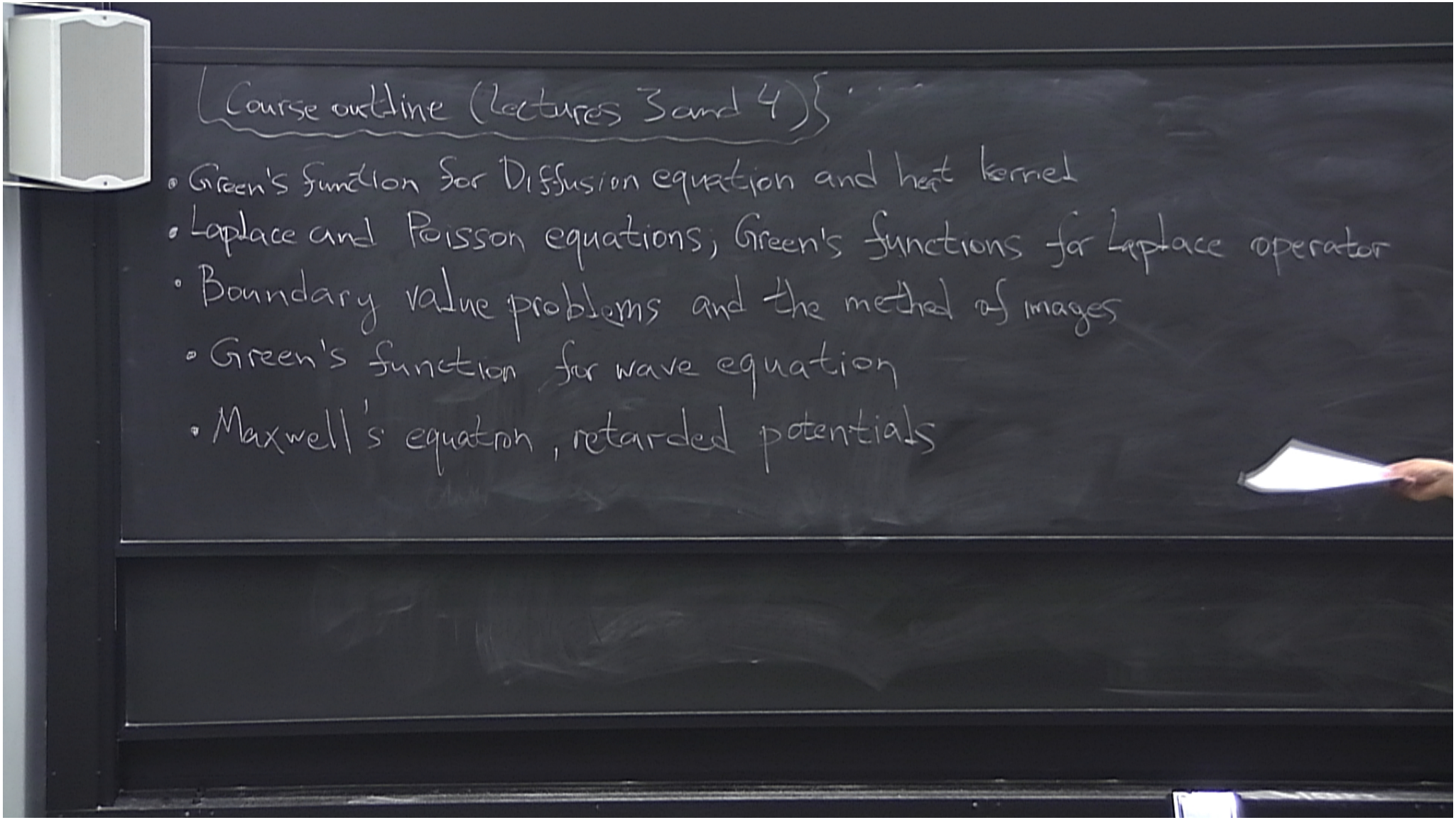
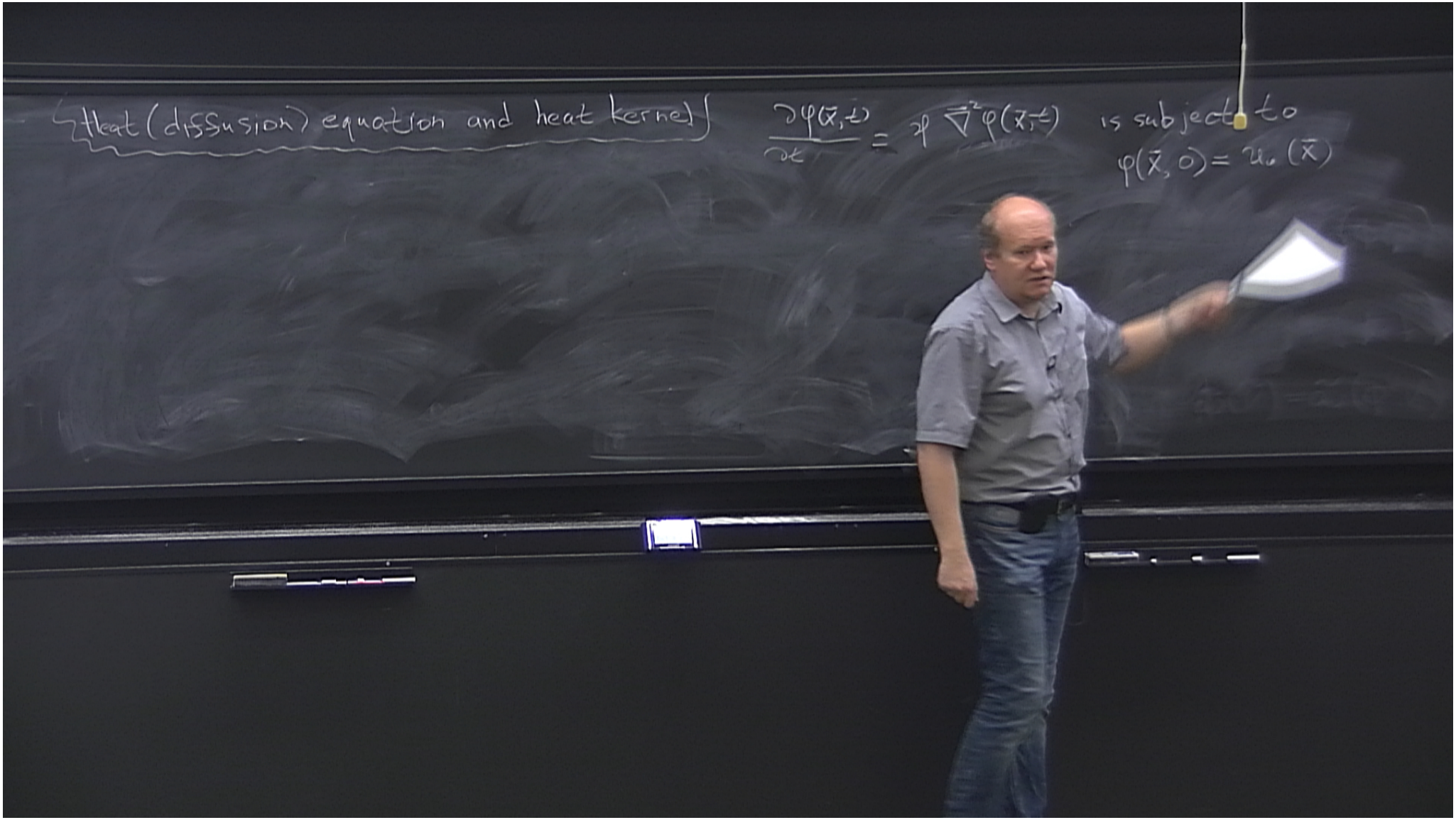
Title: PSI 2015/2016 Greens Functions - Denis Dalidovich - 3

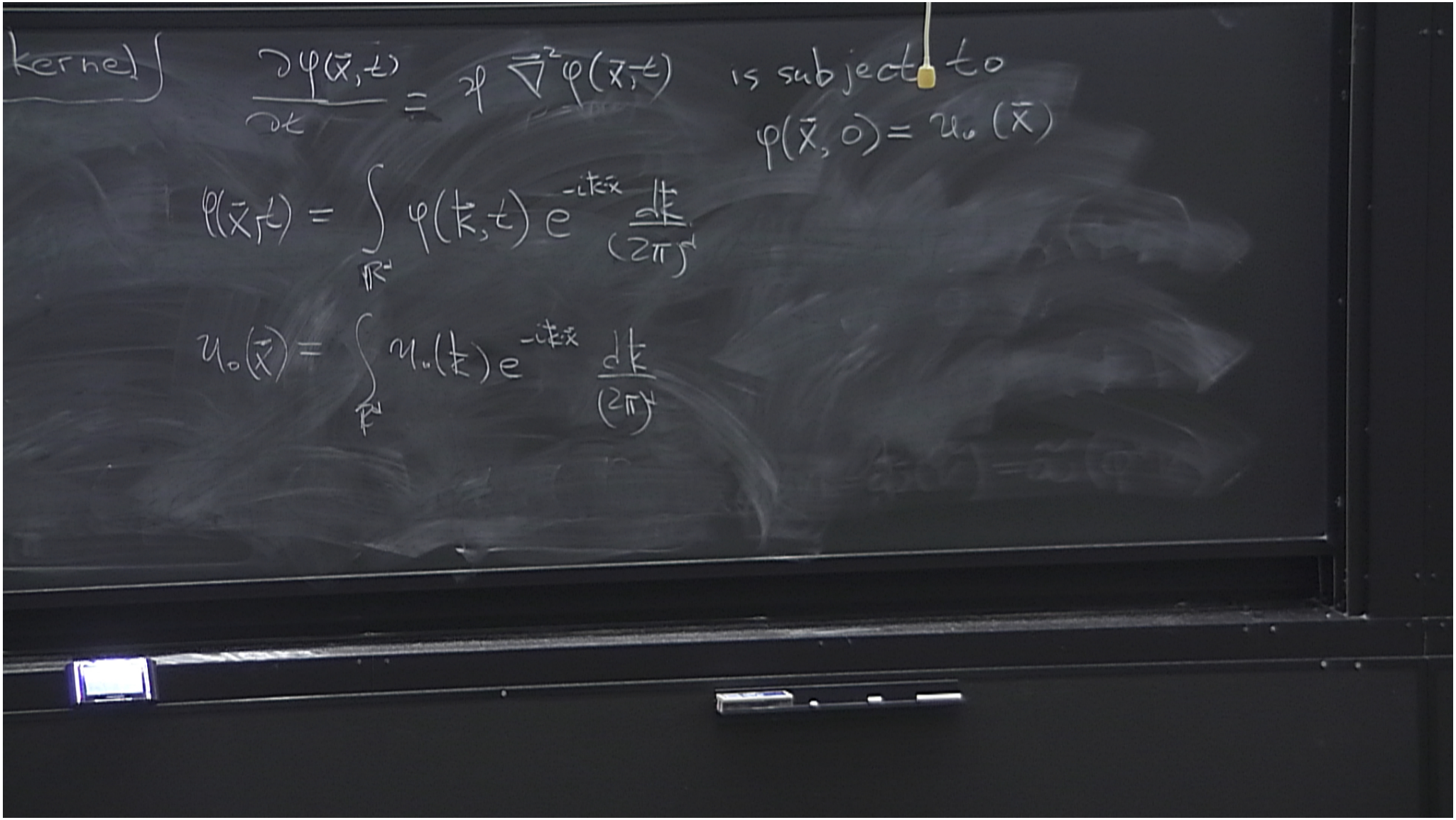
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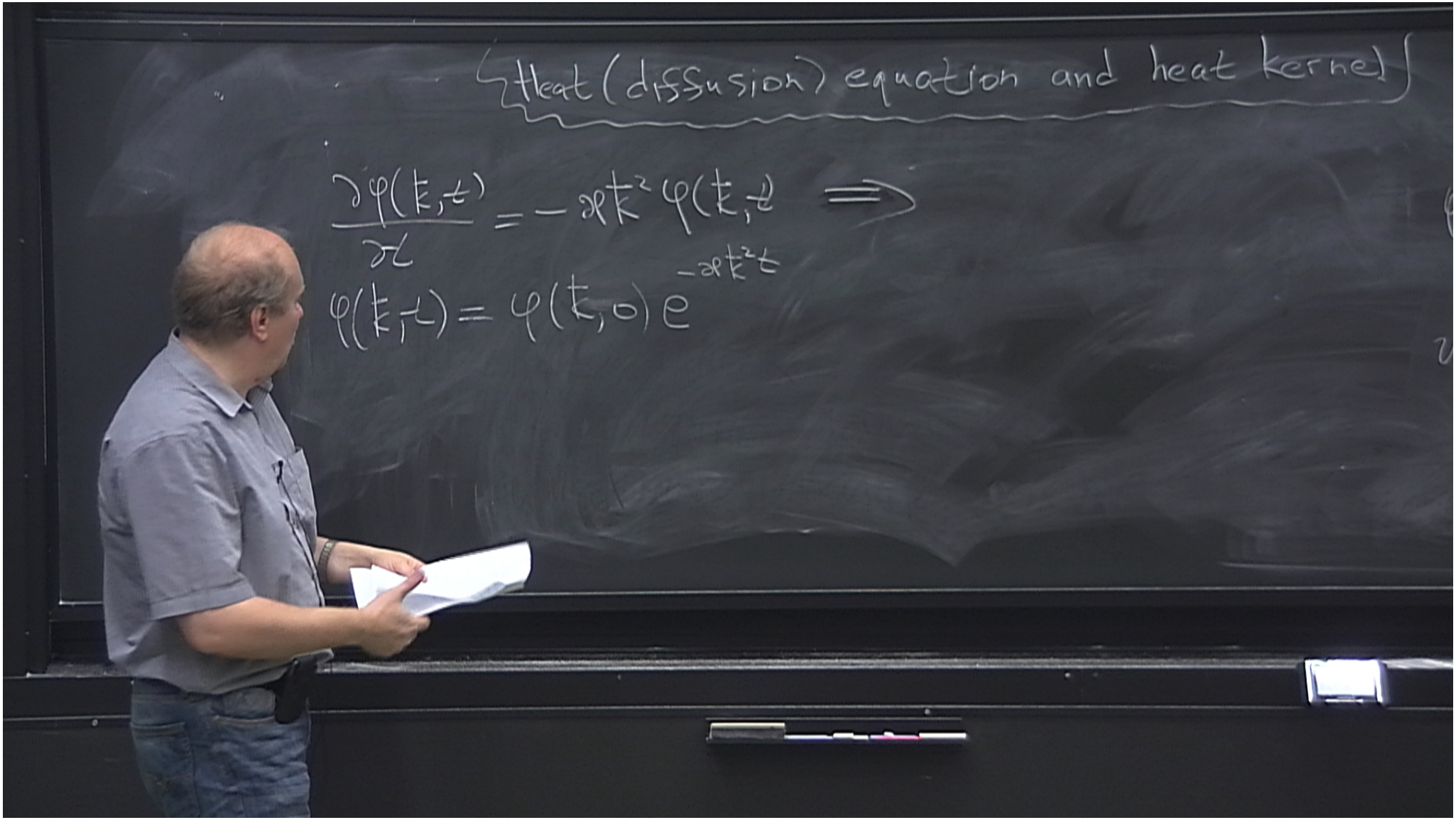
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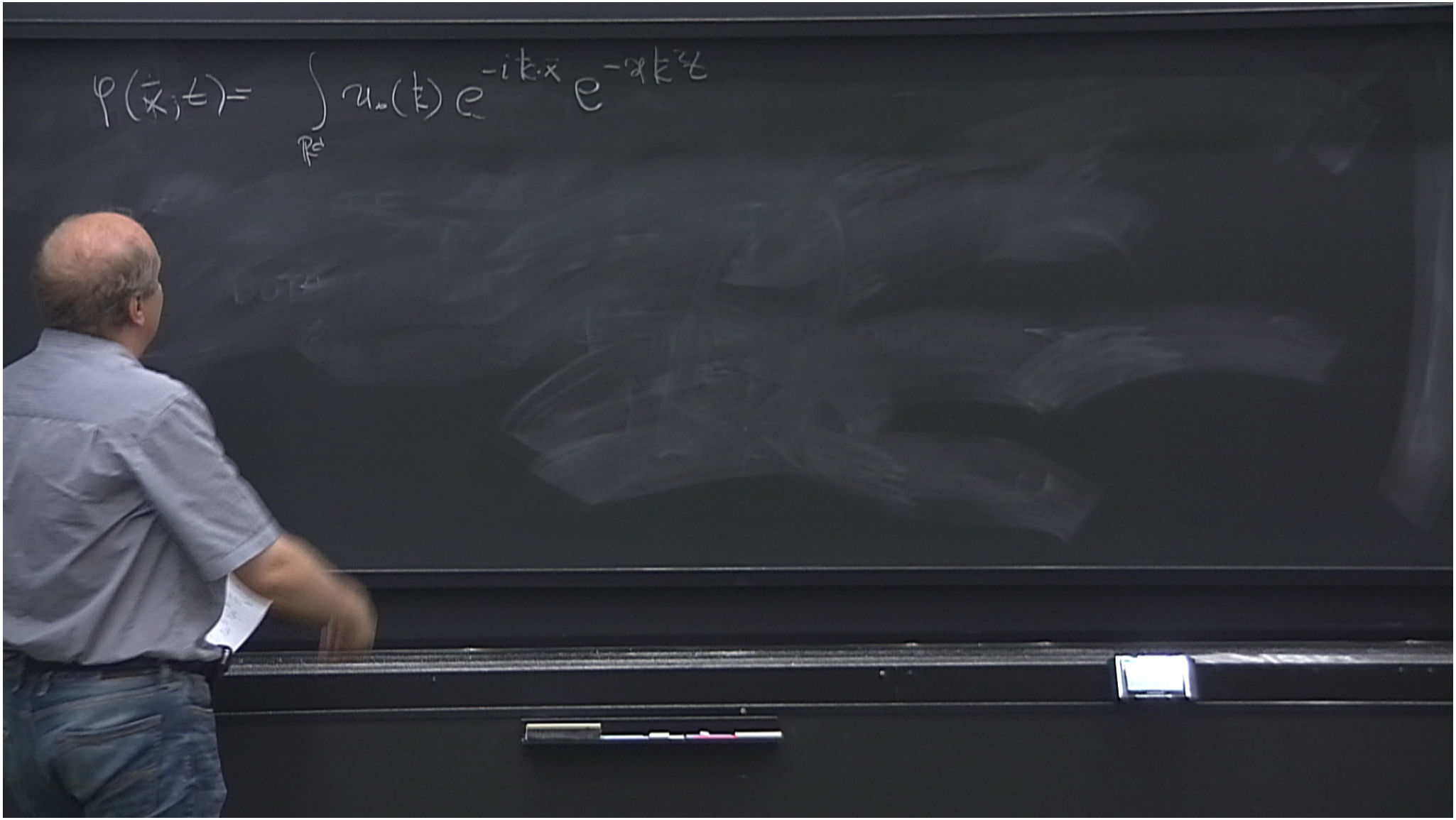
Abstract:

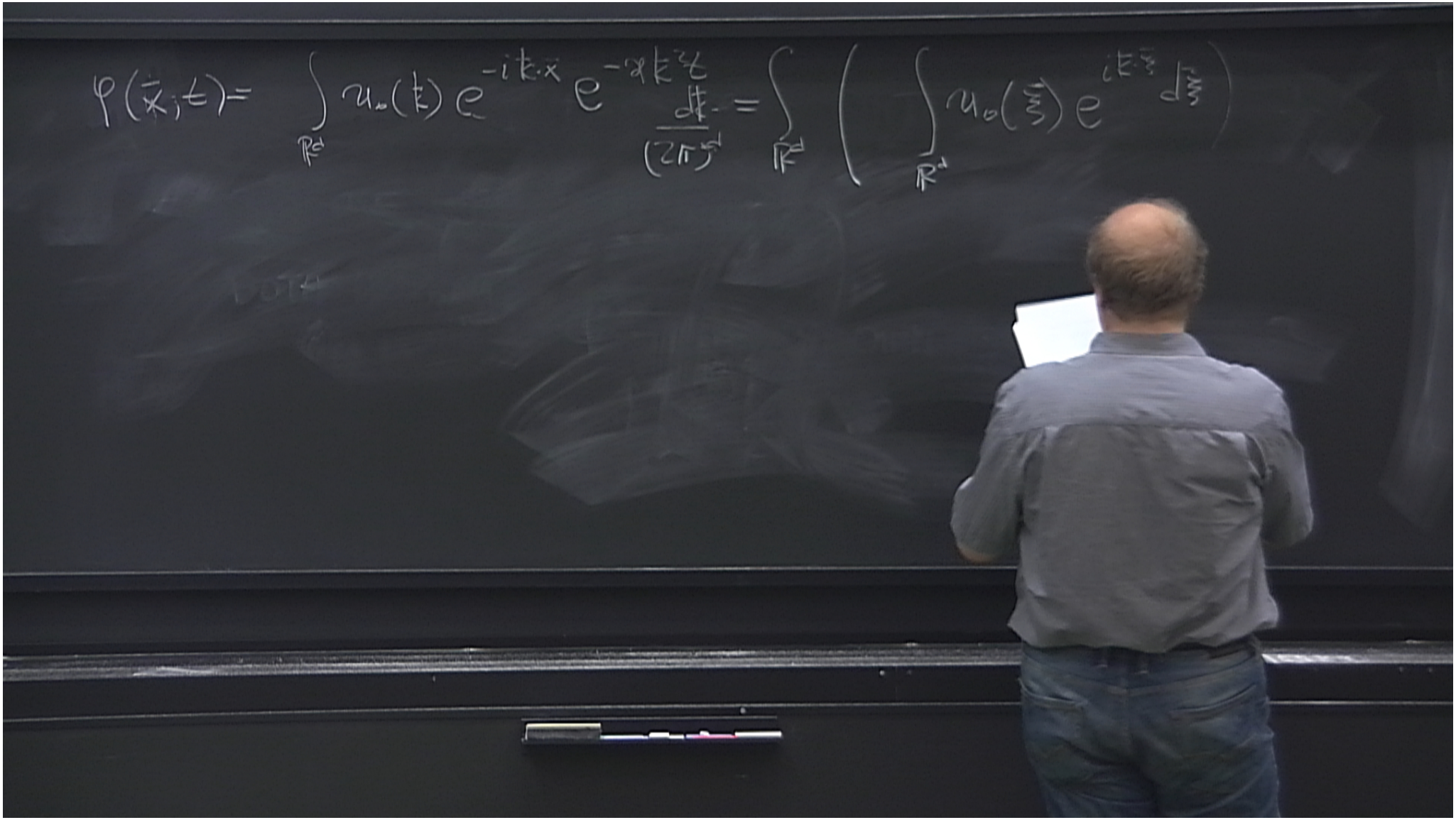

Course outline (lectures 3 and 4)
• Green's function for Diffusion equation and heat kernel
• Laplace and Poisson equations; Green's functions for Laplace operator
• Boundary value problems and the method of images
• Green's function for wave equation
• Maxwell's equation, retarded potentials

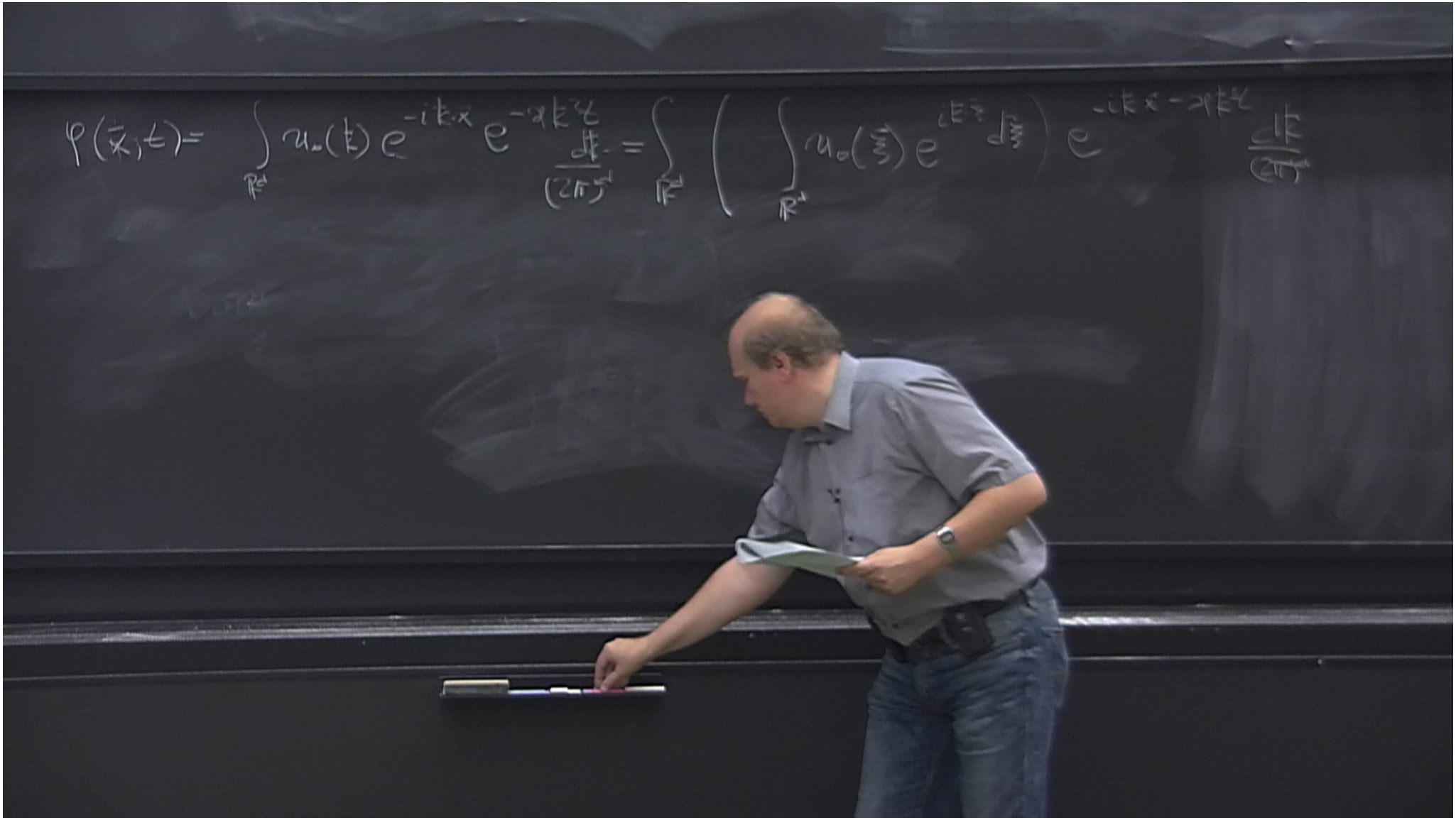

Heat (diffusion) equation and heat kernel
$\frac{\partial \varphi(\vec{x},t)}{\partial t} = \gamma \vec{\nabla}^2 \varphi(\vec{x},t)$ is subject to
$\varphi(\vec{x},0) = u_0(\vec{x})$

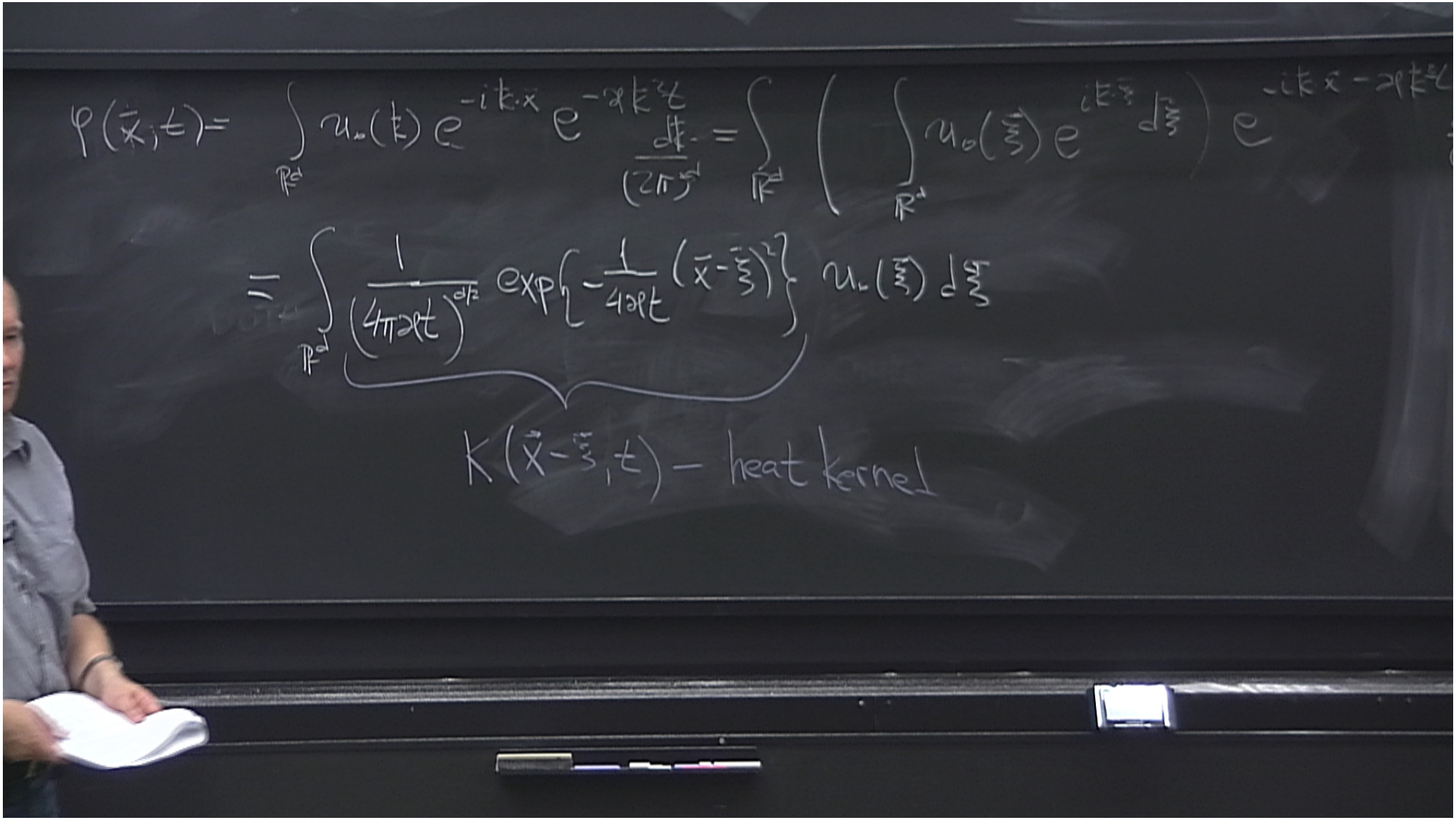

kernel
$\frac{\partial \varphi(\vec{x},t)}{\partial t} = \gamma \vec{\nabla}^2 \varphi(\vec{x},t)$ is subject to
$\varphi(\vec{x},0) = u_0(\vec{x})$
$\varphi(\vec{x},t) = \int_{\mathbb{R}^N} \varphi(\vec{k},t)\, e^{-i\vec{k}\cdot\vec{x}} \frac{d\vec{k}}{(2\pi)^N}$
$u_0(\vec{x}) = \int_{\mathbb{R}^N} u_0(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \frac{d\vec{k}}{(2\pi)^N}$

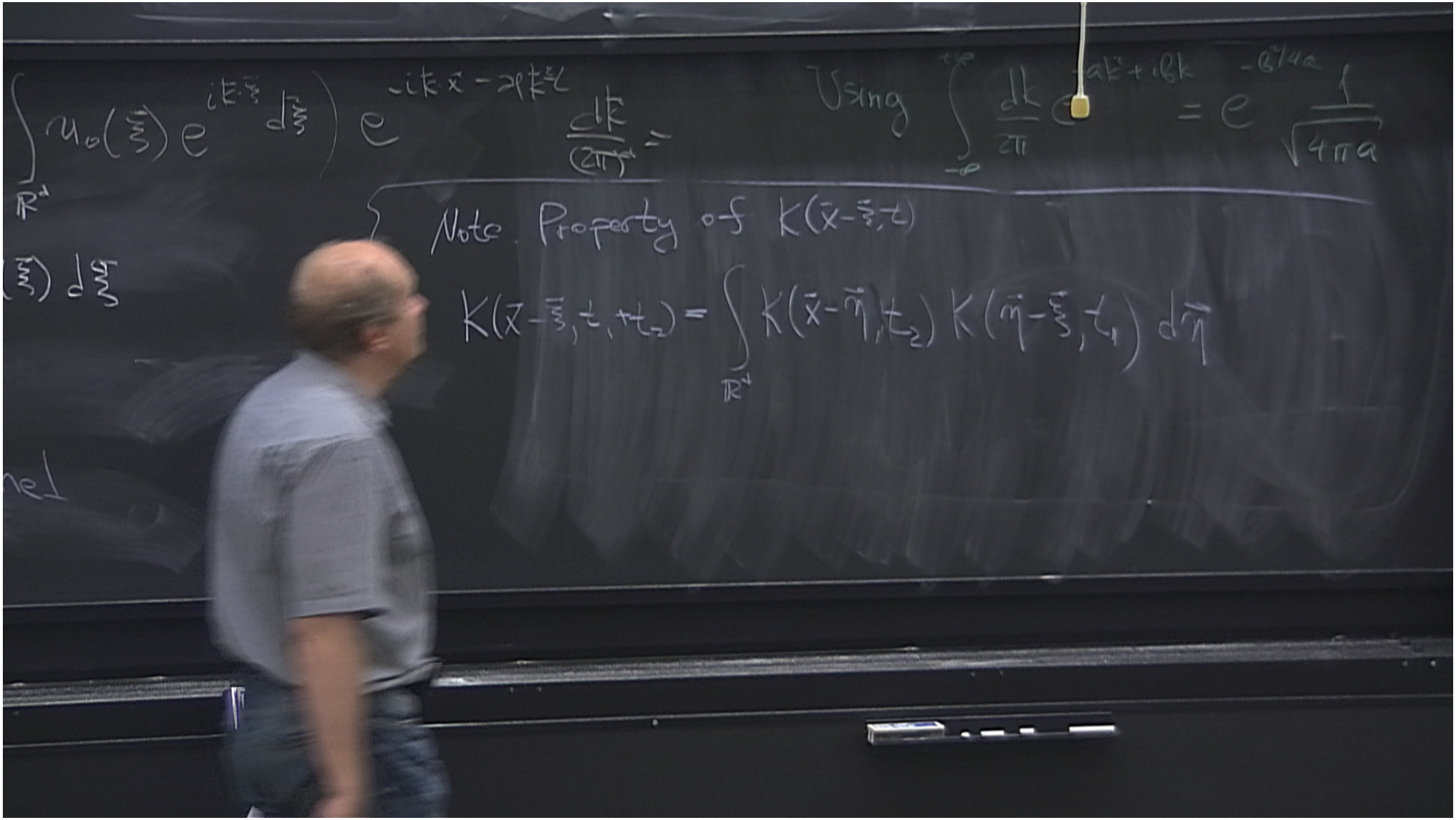

Heat (diffusion) equation and heat kernel
$\frac{\partial \varphi(k,t)}{\partial t} = -\alpha k^2 \varphi(k,t) \Rightarrow$
$\varphi(k,t) = \varphi(k,0) e^{-\alpha k^2 t}$

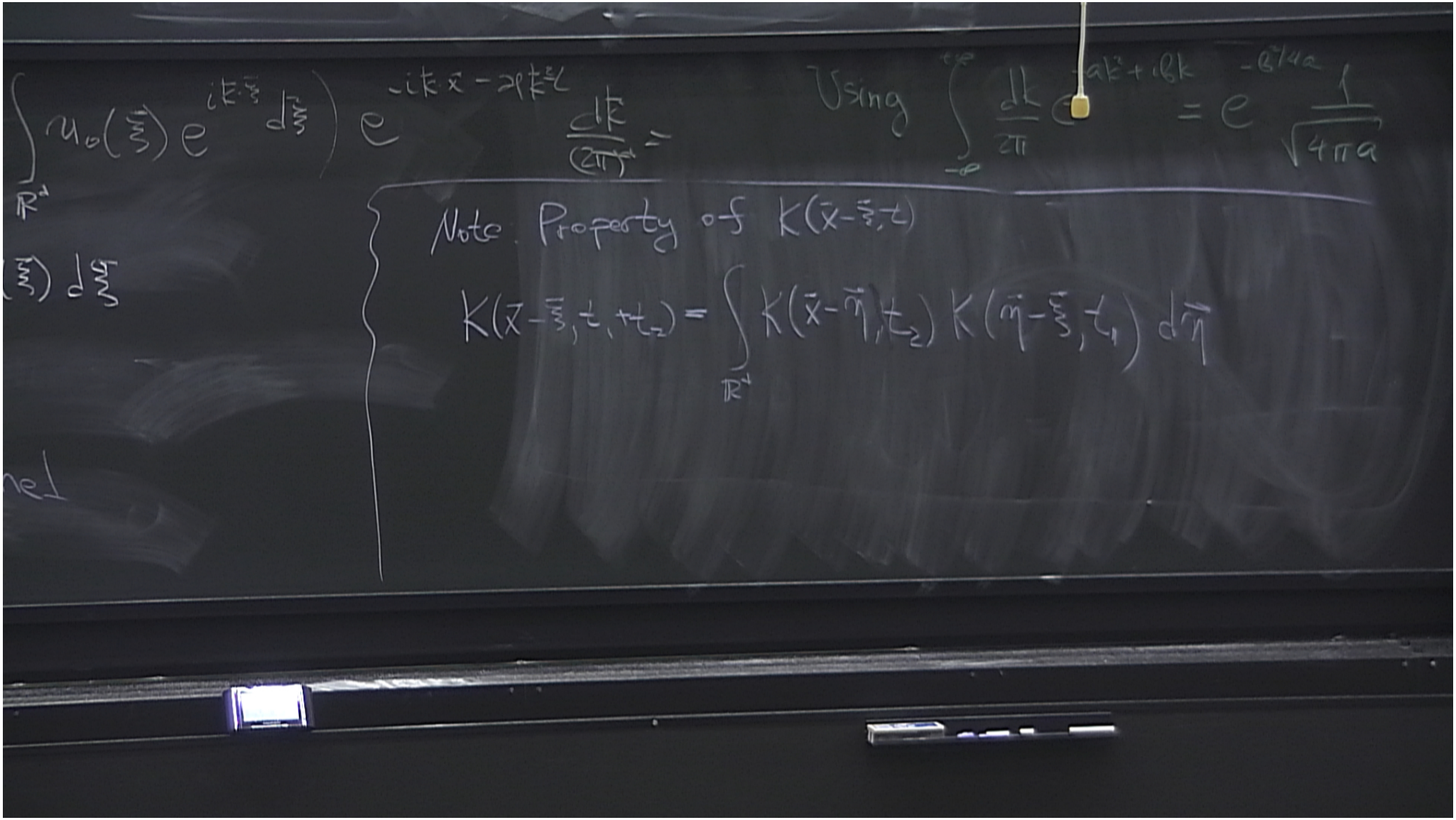

$\varphi(\vec{x},t) = \int_{\mathbb{R}^d} a_-(k)\, e^{-ik\cdot x}\, e^{-\alpha k^2 t}$

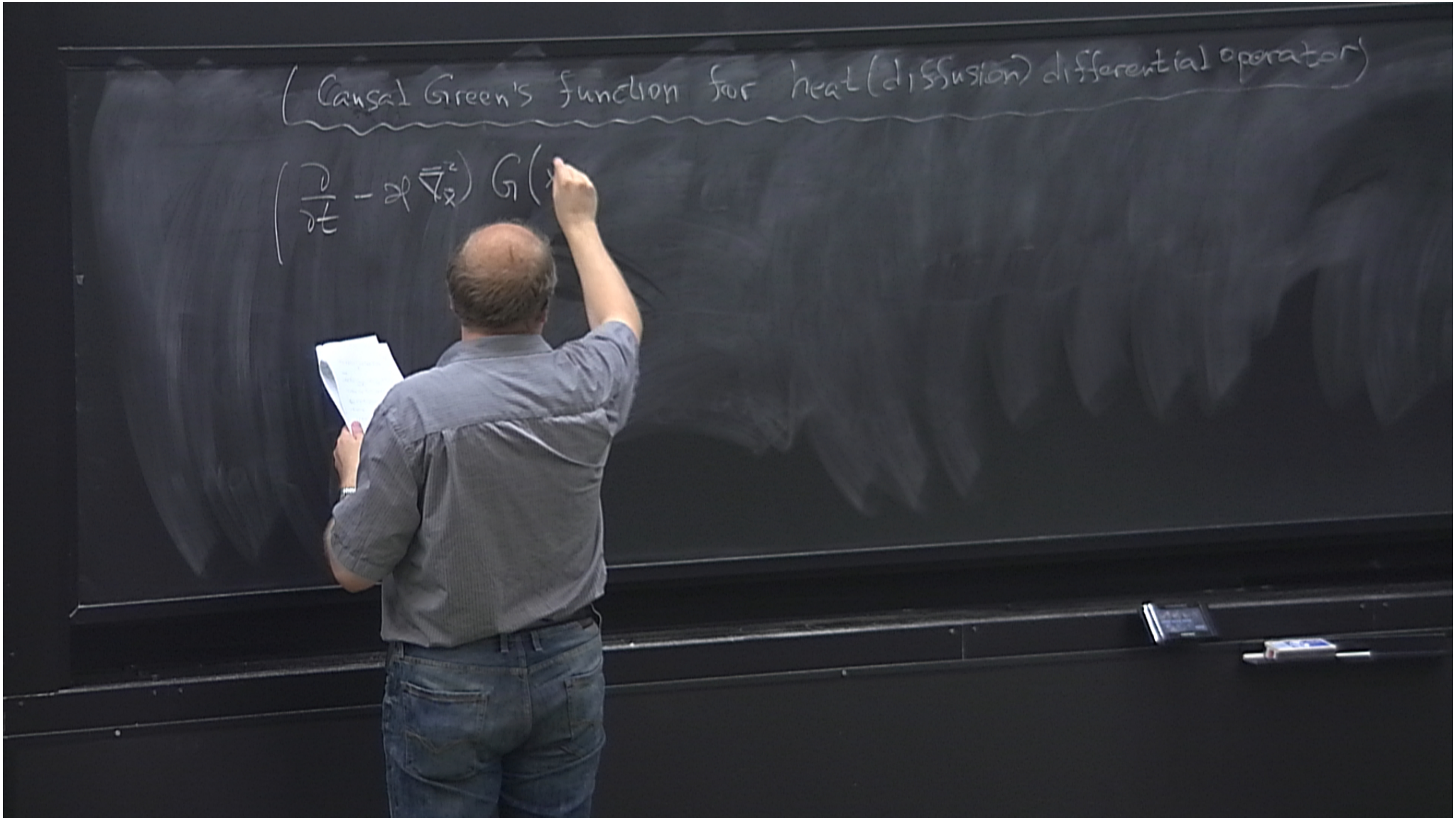

$\varphi(\vec{x},t) = \int_{\mathbb{R}^3} u_0(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}\, e^{-\alpha k^2 t}\, \frac{d\vec{k}}{(2\pi)^3} = \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} u_0(\vec{\xi})\, e^{i\vec{k}\cdot\vec{\xi}}\, d\vec{\xi} \right)$

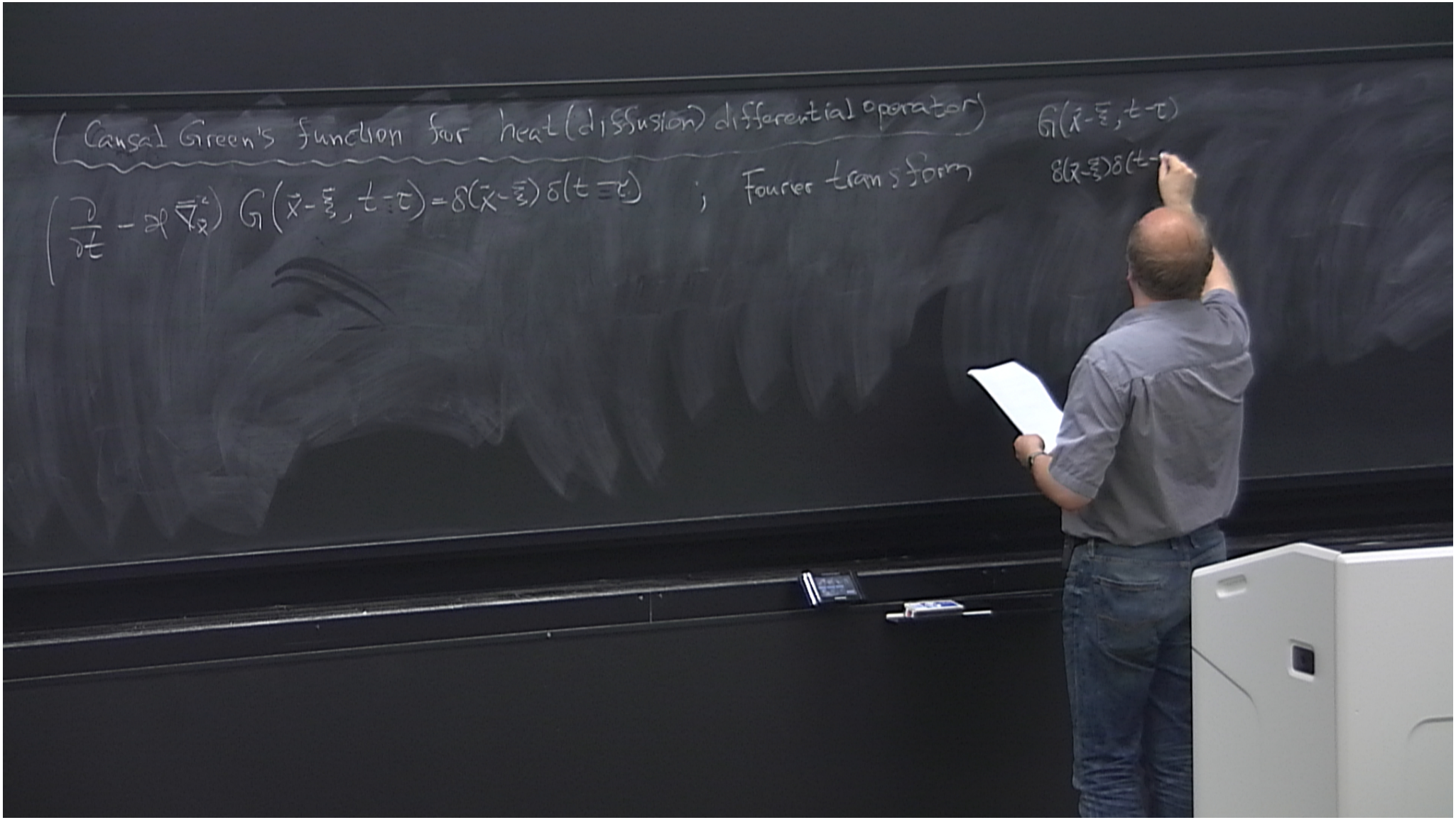

$\varphi(\vec{x},t) = \int_{\mathbb{R}^d} u_0(k)\, e^{-ik\cdot x}\, e^{-\alpha k^2 t}\, \frac{dk}{(2\pi)^d} = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} u_0(\vec{\xi})\, e^{i\xi\cdot} \right) e^{-ik\cdot x - \alpha k^2 t} \frac{dk}{(2\pi)^d}$

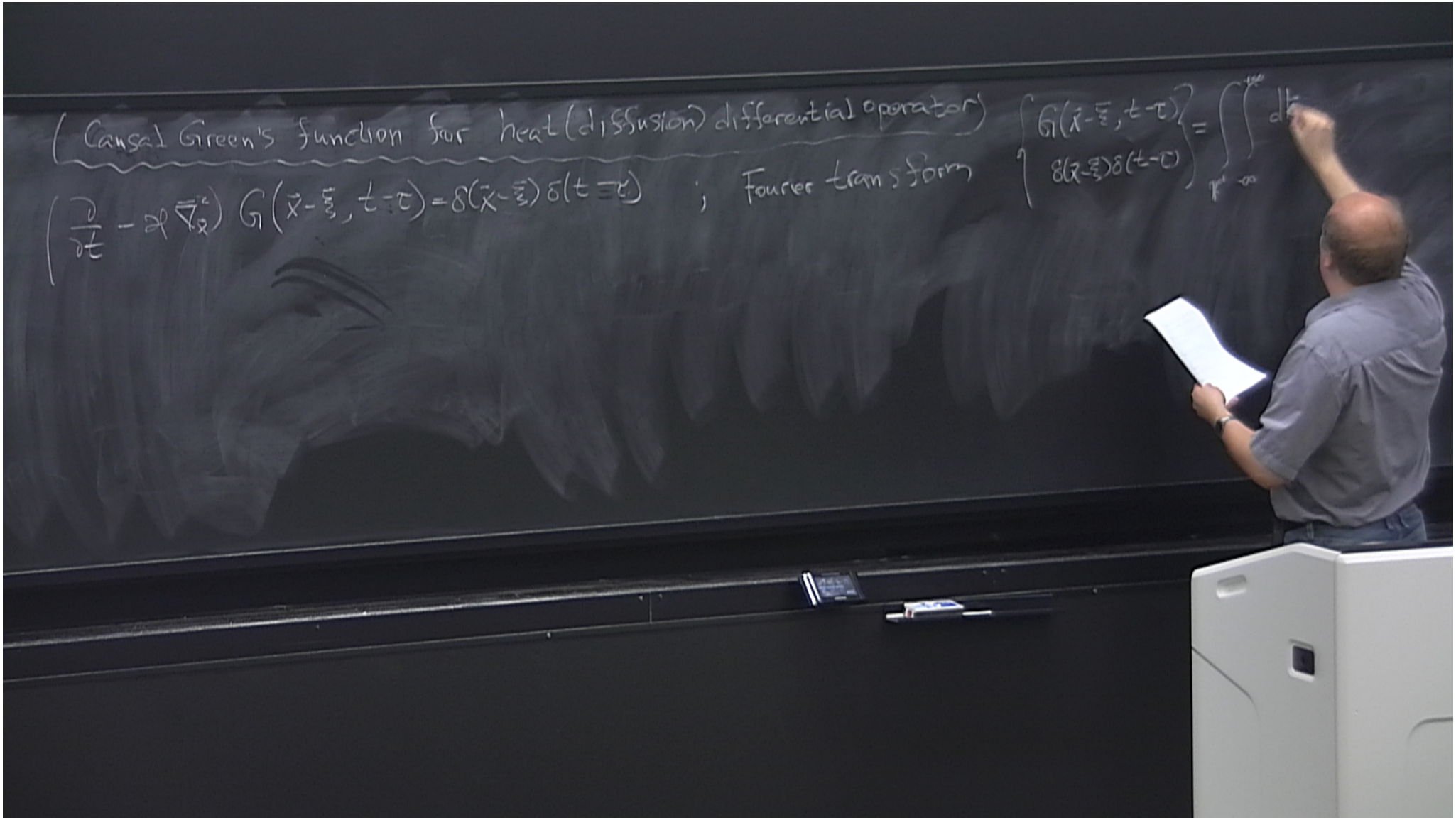

$\varphi(\vec{x},t) = \int_{\mathbb{R}^d} u_0(k)\, e^{-ik\cdot x}\, e^{-\alpha k^2 t}\, \frac{dk}{(2\pi)^d} = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} u_0(\xi)\, e^{ik\cdot\xi}\, d\xi \right) e^{-ikx - \alpha k^2 t}$
$= \int_{\mathbb{R}^d} \underbrace{\frac{1}{(4\pi\alpha t)^{d/2}} \exp\left\{-\frac{1}{4\alpha t}(\vec{x}-\vec{\xi})^2\right\}}\, u_0(\xi)\, d\xi$
$K(\vec{x}-\vec{\xi},t)$ – heat kernel

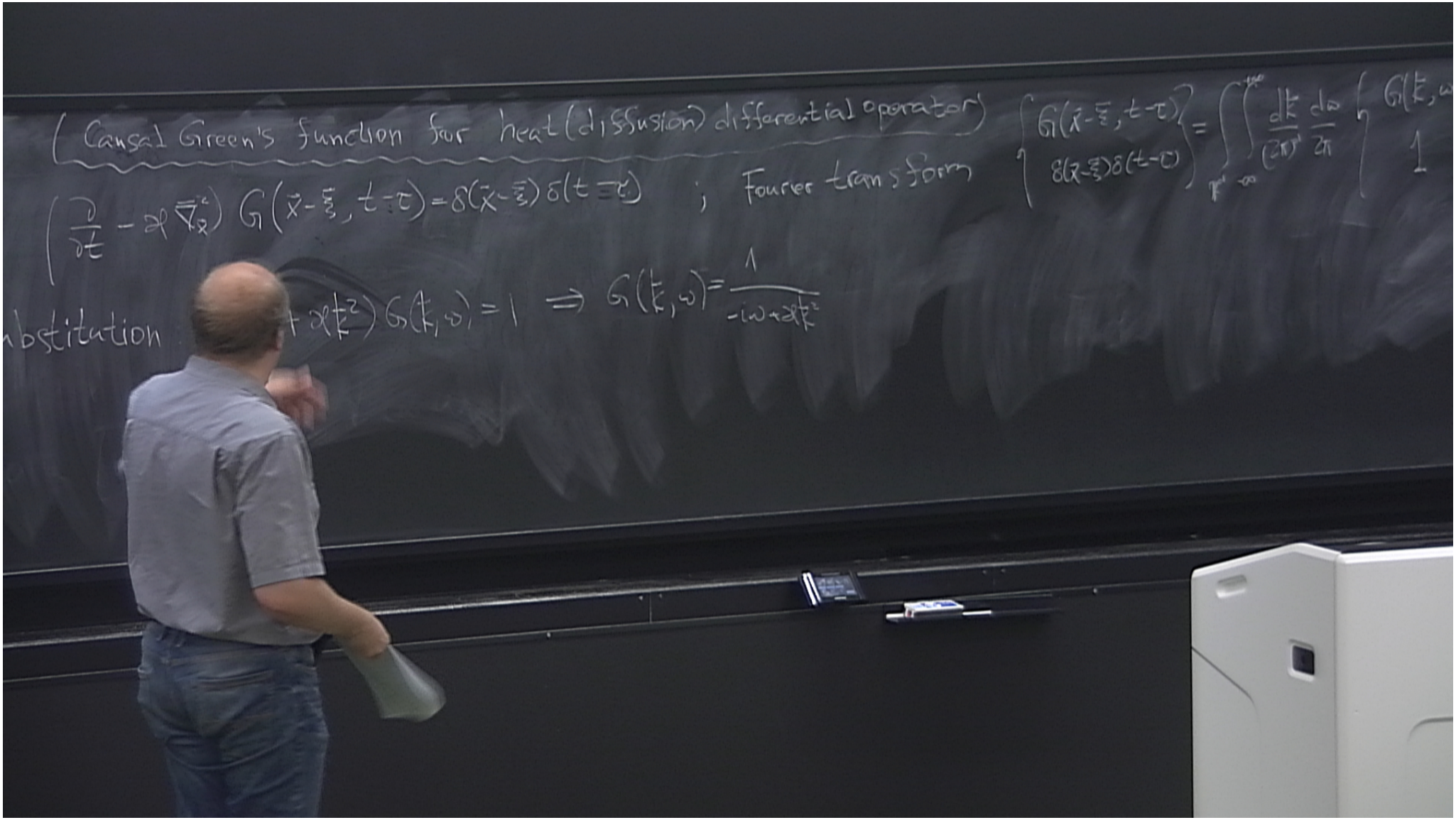

Note Property of $K(\vec{x}-\vec{\xi};t)$
$K(\vec{x}-\vec{\xi};t_1+t_2) = \int_{\mathbb{R}^1} K(\vec{x}-\vec{\eta};t_2)\, K(\vec{\eta}-\vec{\xi};t_1)\, d\vec{\eta}$
Using $\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-ak^2+ibk} = e^{-b^2/4a} \frac{1}{\sqrt{4\pi a}}$

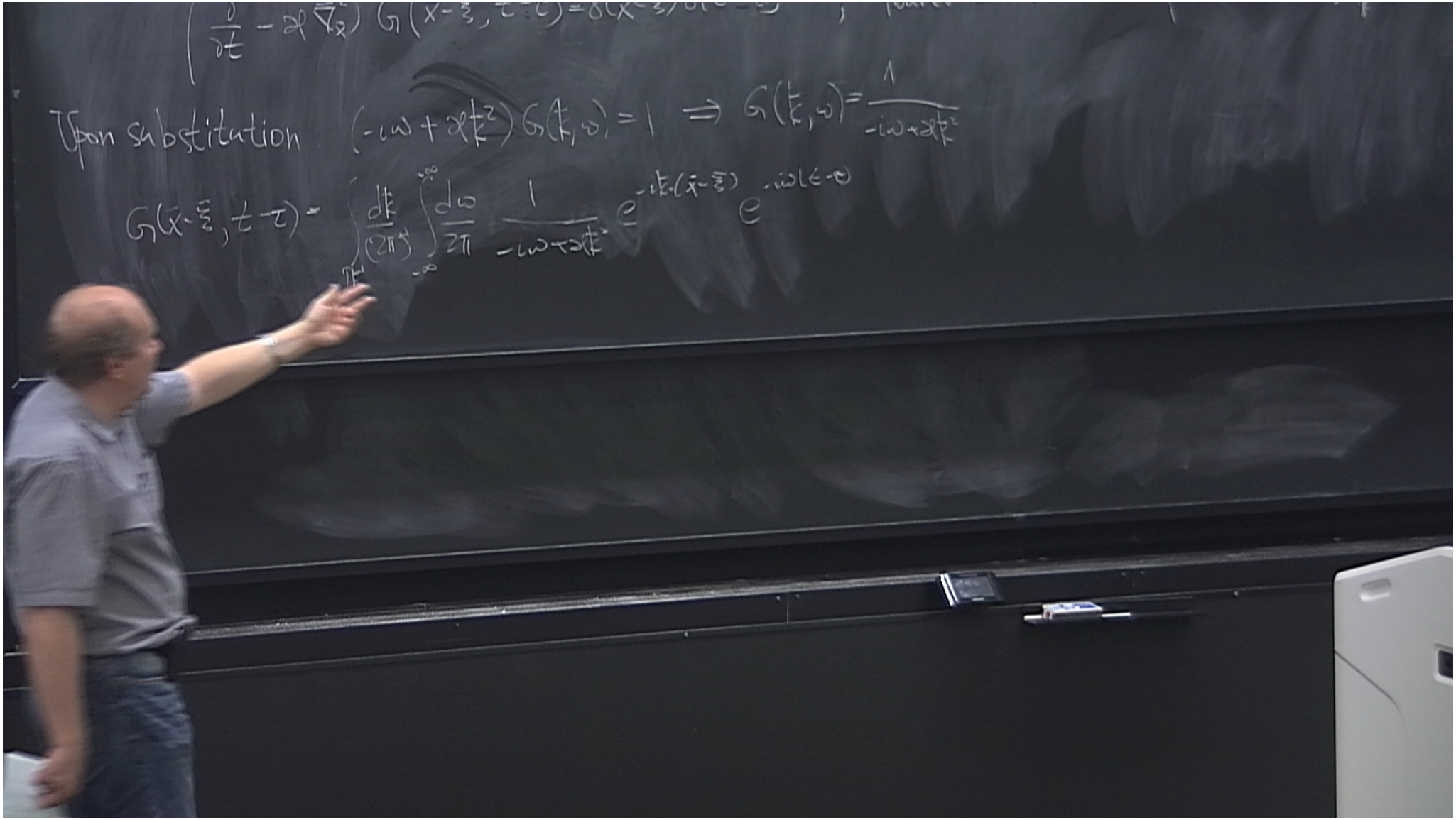

Using
Note Property of $K(\vec{x}-\vec{\xi};t)$
$K(\vec{x}-\vec{\xi};t_1+t_2) = \int_{\mathbb{R}^1} K(\vec{x}-\vec{\eta};t_2)\, K(\vec{\eta}-\vec{\xi};t_1)\, d\vec{\eta}$

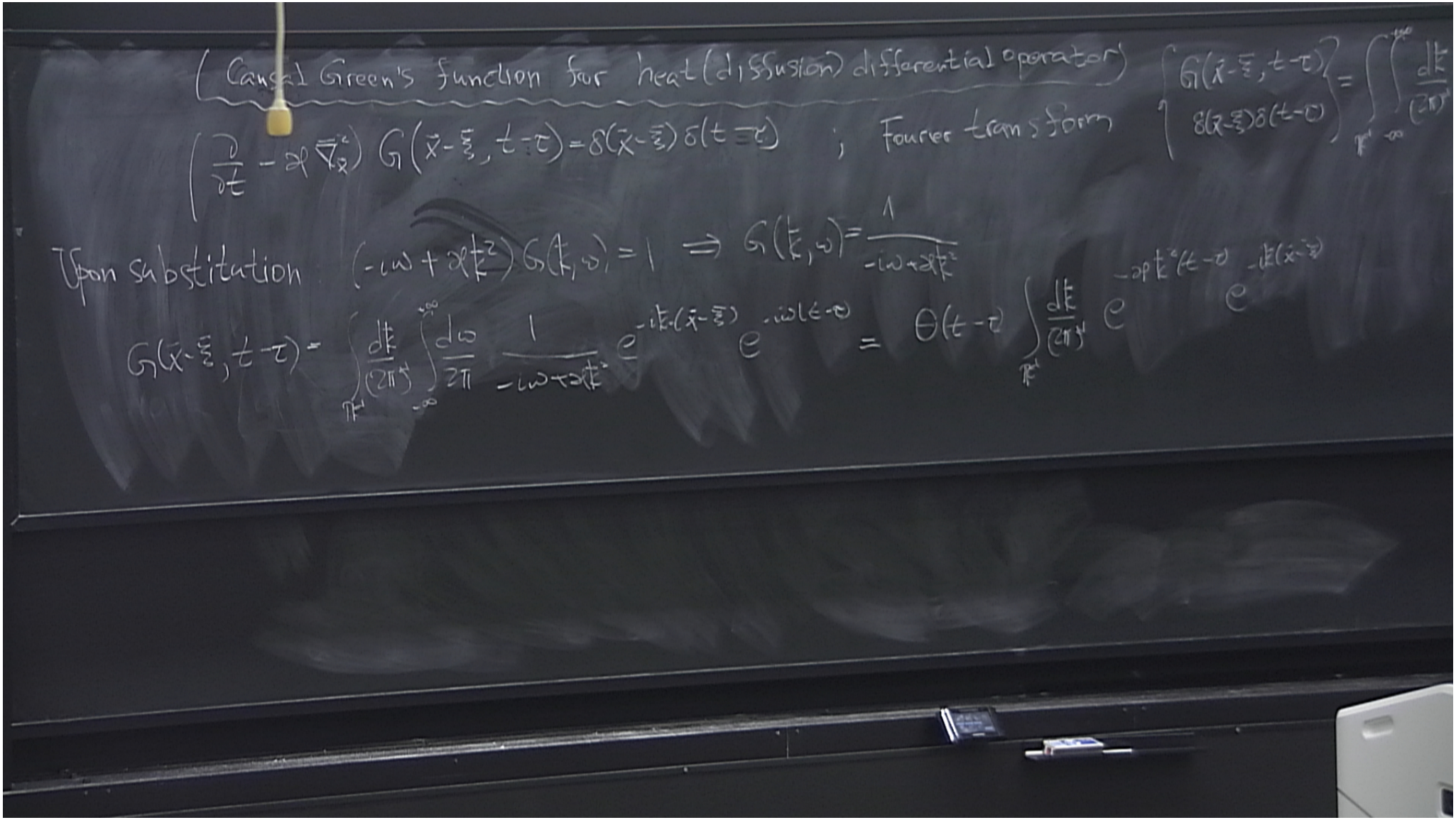

(Causal Green's function for heat (diffusion) differential operator)
$\left(\frac{\partial}{\partial t} - \alpha \nabla_x^2\right) G($

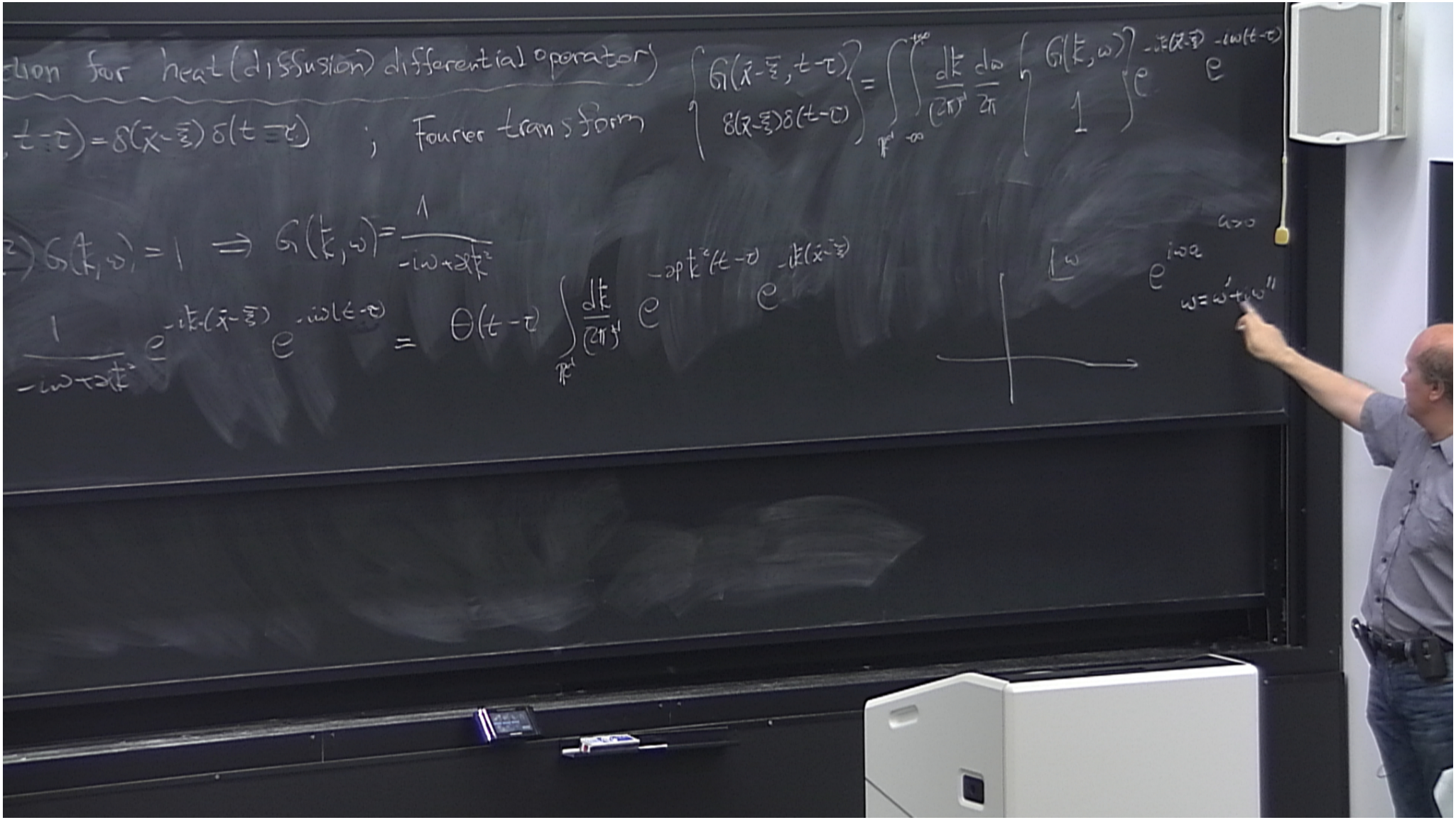

(Causal Green's function for heat (diffusion) differential operator)
$\left(\frac{\partial}{\partial t} - \alpha \nabla_x^2\right) G(\vec{x}-\vec{\xi}, t-\tau) = \delta(\vec{x}-\vec{\xi})\,\delta(t-\tau)$; Fourier transform
$G(\vec{x}-\vec{\xi}, t-\tau)$
$\delta(\vec{x}-\vec{\xi})\,\delta(t-$

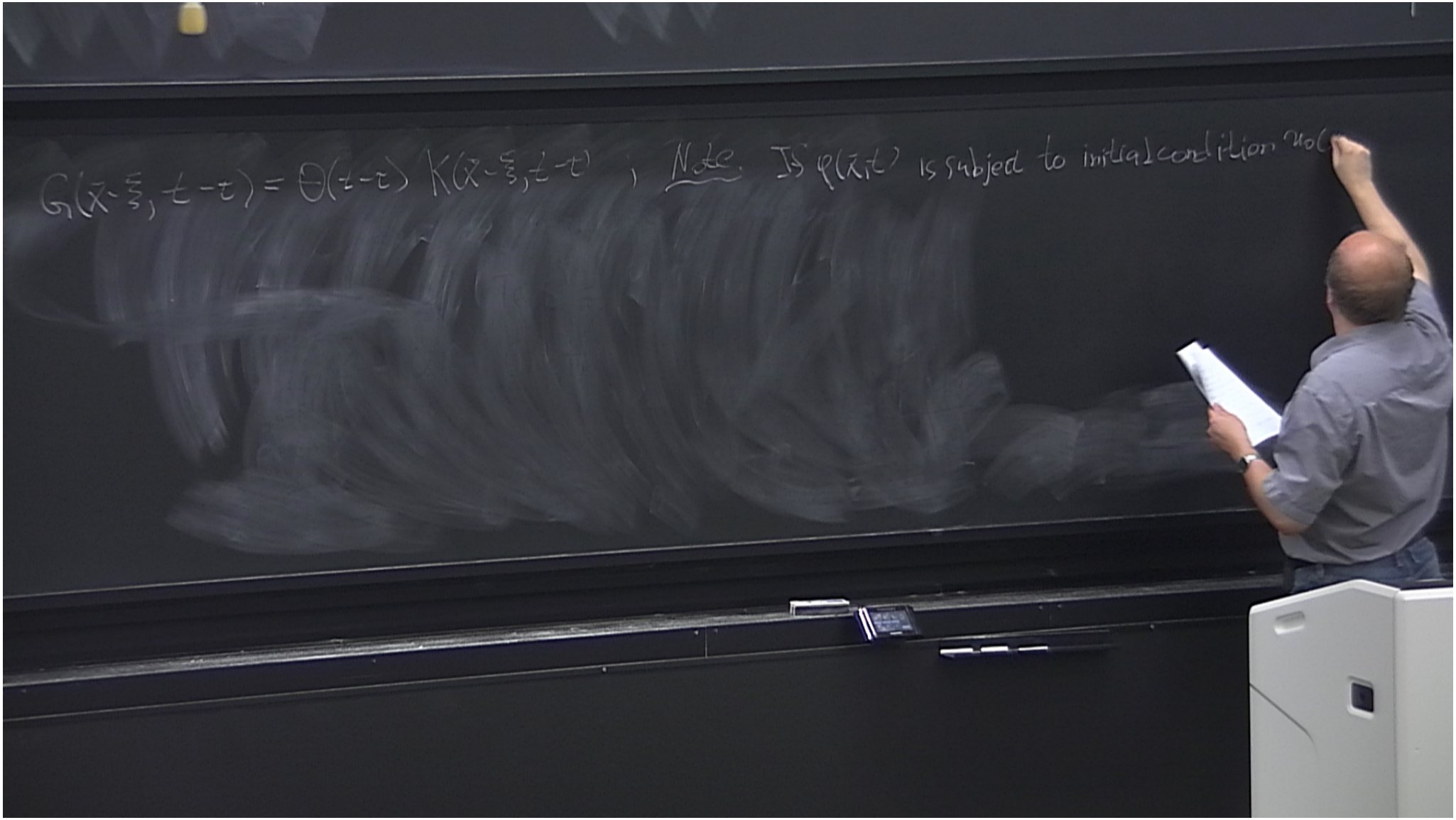

(Causal Green's function for heat (diffusion) differential operator)
$\left(\frac{\partial}{\partial t} - \alpha \nabla_x^2\right) G(\vec{x}-\vec{\xi}, t-\tau) = \delta(\vec{x}-\vec{\xi})\,\delta(t-\tau)$; Fourier transform
$\left\{ \begin{matrix} G(\vec{x}-\vec{\xi}, t-\tau) \\ \delta(\vec{x}-\vec{\xi})\,\delta(t-\tau) \end{matrix} \right\} =$

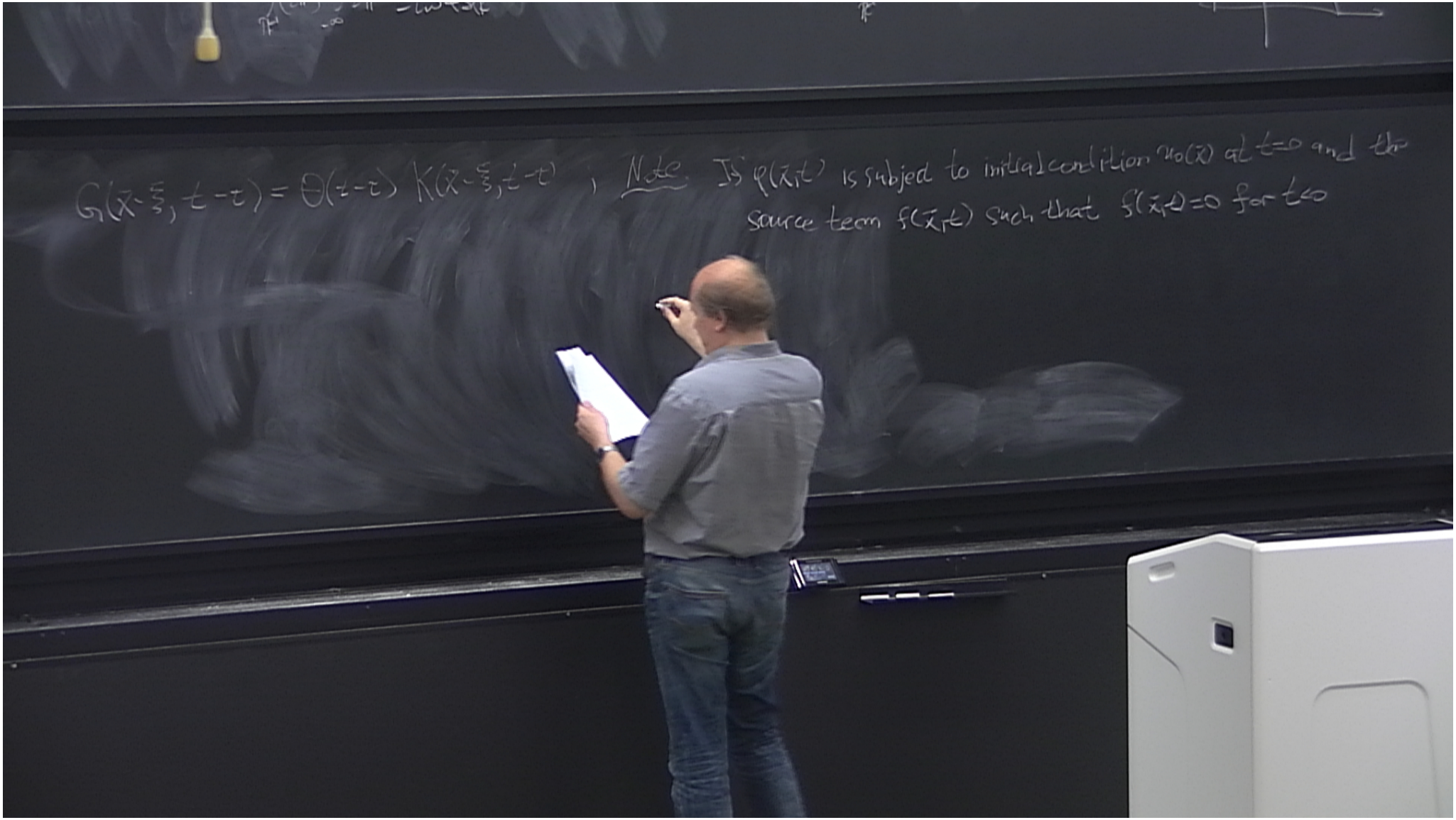

(Causal Green's function for heat (diffusion) differential operator)
$\left(\frac{\partial}{\partial t} - \alpha \vec{\nabla}_x^2\right) G(\vec{x}-\vec{\xi}, t-\tau) = \delta(\vec{x}-\vec{\xi})\,\delta(t-\tau)$; Fourier transform
$\left\{ \begin{matrix} G(\vec{x}-\vec{\xi}, t-\tau) \\ \delta(\vec{x}-\vec{\xi})\,\delta(t-\tau) \end{matrix} \right\} = \int_{\mathbb{R}^3} \int_{-\infty}^{\infty} \frac{d\vec{k}}{(2\pi)^3} \frac{d\omega}{2\pi}$
$\left\{ \begin{matrix} G(\vec{k}, \omega \\ 1 \end{matrix} \right.$
bstitution
$+ \alpha \vec{k}^2)\, G(\vec{k}, \omega) = 1 \Rightarrow G(\vec{k}, \omega) = \frac{1}{-i\omega + \alpha \vec{k}^2}$

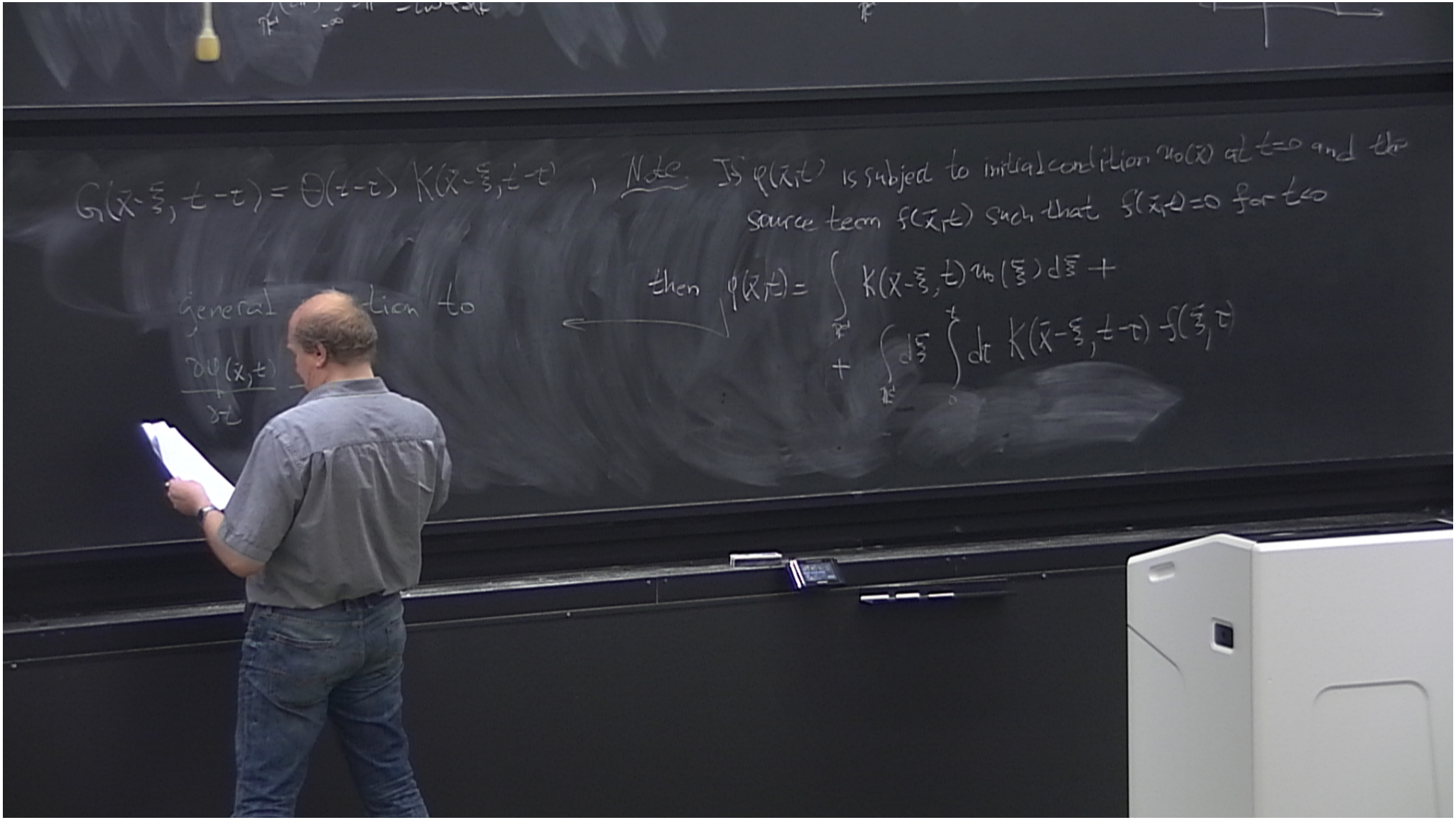

Upon substitution
$\left(\frac{\partial}{\partial t} - \alpha \nabla_x^2\right) G(x-\xi, t-\tau) = \delta(x-\xi)\delta(t-\tau)$
$(-i\omega + \alpha k^2)\, G(k,\omega) = 1 \Rightarrow G(k,\omega) = \frac{1}{-i\omega + \alpha k^2}$
$G(x-\xi, t-\tau) = \int \frac{dk}{(2\pi)^n} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{-i\omega + \alpha k^2} e^{-ik\cdot(x-\xi)} e^{-i\omega(t-\tau)}$

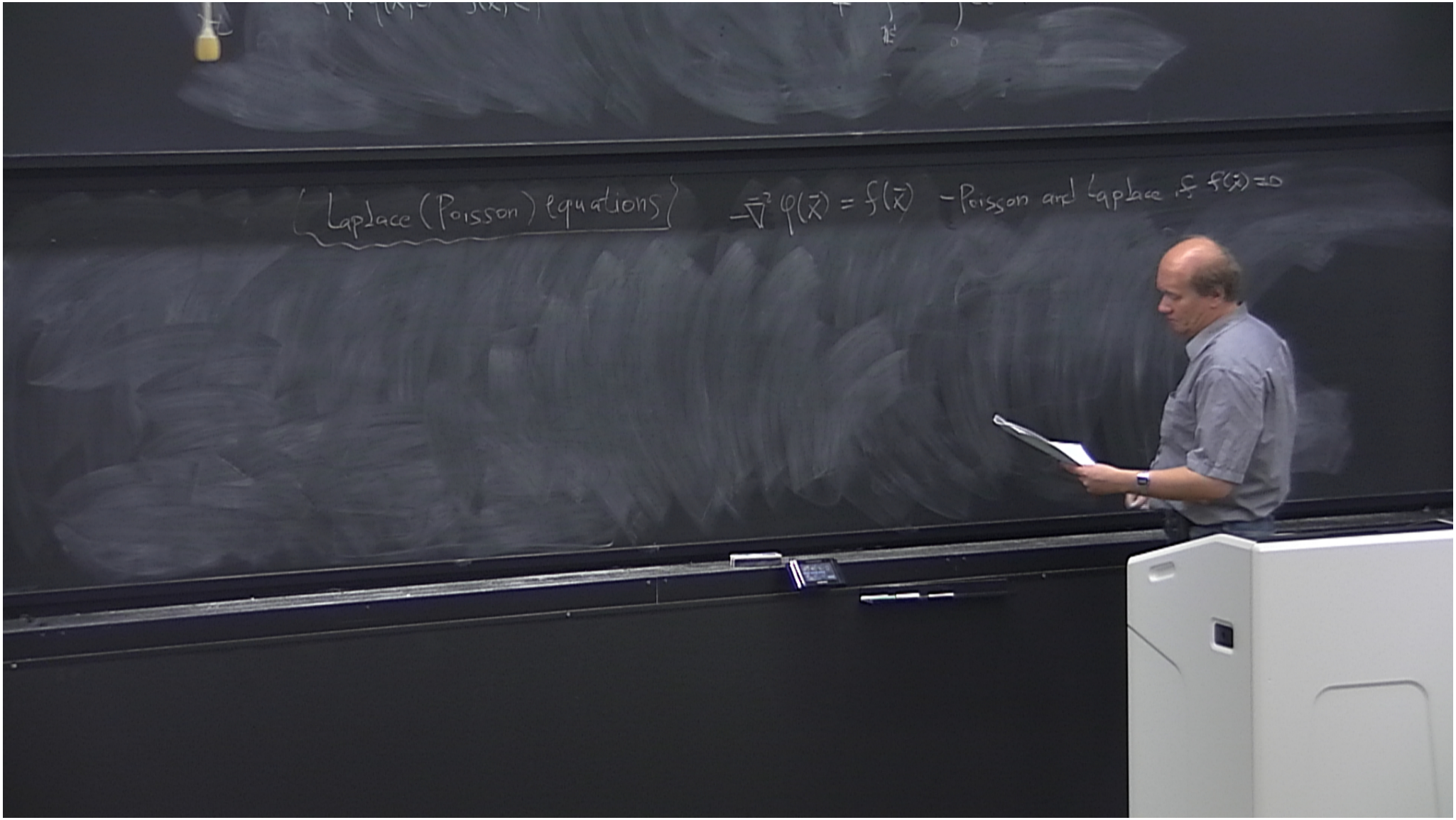

Causal Green's function for heat (diffusion) differential operator
$\left(\frac{\partial}{\partial t} - \alpha \vec{\nabla}_x^2\right) G(\vec{x}-\vec{\xi}, t-\tau) = \delta(\vec{x}-\vec{\xi})\,\delta(t-\tau)$; Fourier transform
$\left\{ \begin{matrix} G(\vec{x}-\vec{\xi}, t-\tau) \\ \delta(\vec{x}-\vec{\xi})\delta(t-\tau) \end{matrix} \right\} = \int \int \frac{d\vec{k}}{(2\pi)^3}$
Upon substitution $(-i\omega + \alpha k^2)\, G(\vec{k}, \omega) = 1 \Rightarrow G(\vec{k}, \omega) = \frac{1}{-i\omega + \alpha k^2}$
$G(\vec{x}-\vec{\xi}, t-\tau) = \int_{\mathbb{R}^3} \frac{d\vec{k}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{-i\omega + \alpha k^2} e^{-i\vec{k}\cdot(\vec{x}-\vec{\xi})} e^{-i\omega(t-\tau)} = \theta(t-\tau) \int_{\mathbb{R}^3} \frac{d\vec{k}}{(2\pi)^3} e^{-\alpha k^2 (t-\tau)} e^{-i\vec{k}(\vec{x}-\vec{\xi})}$

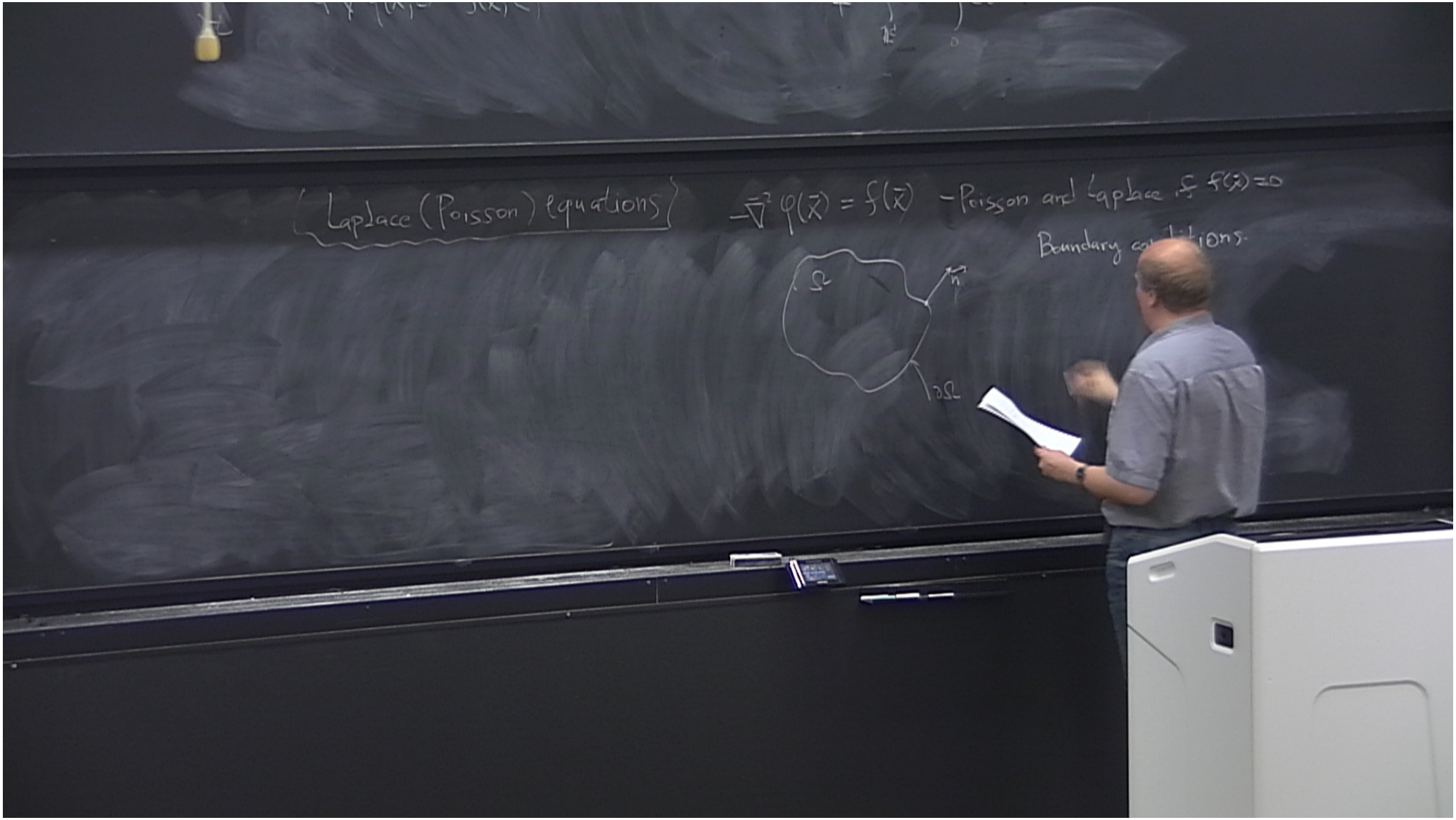

tion for heat (diffusion) differential operator)
Fourier transform
$\omega = \omega' + i\omega''$
$a > 0$

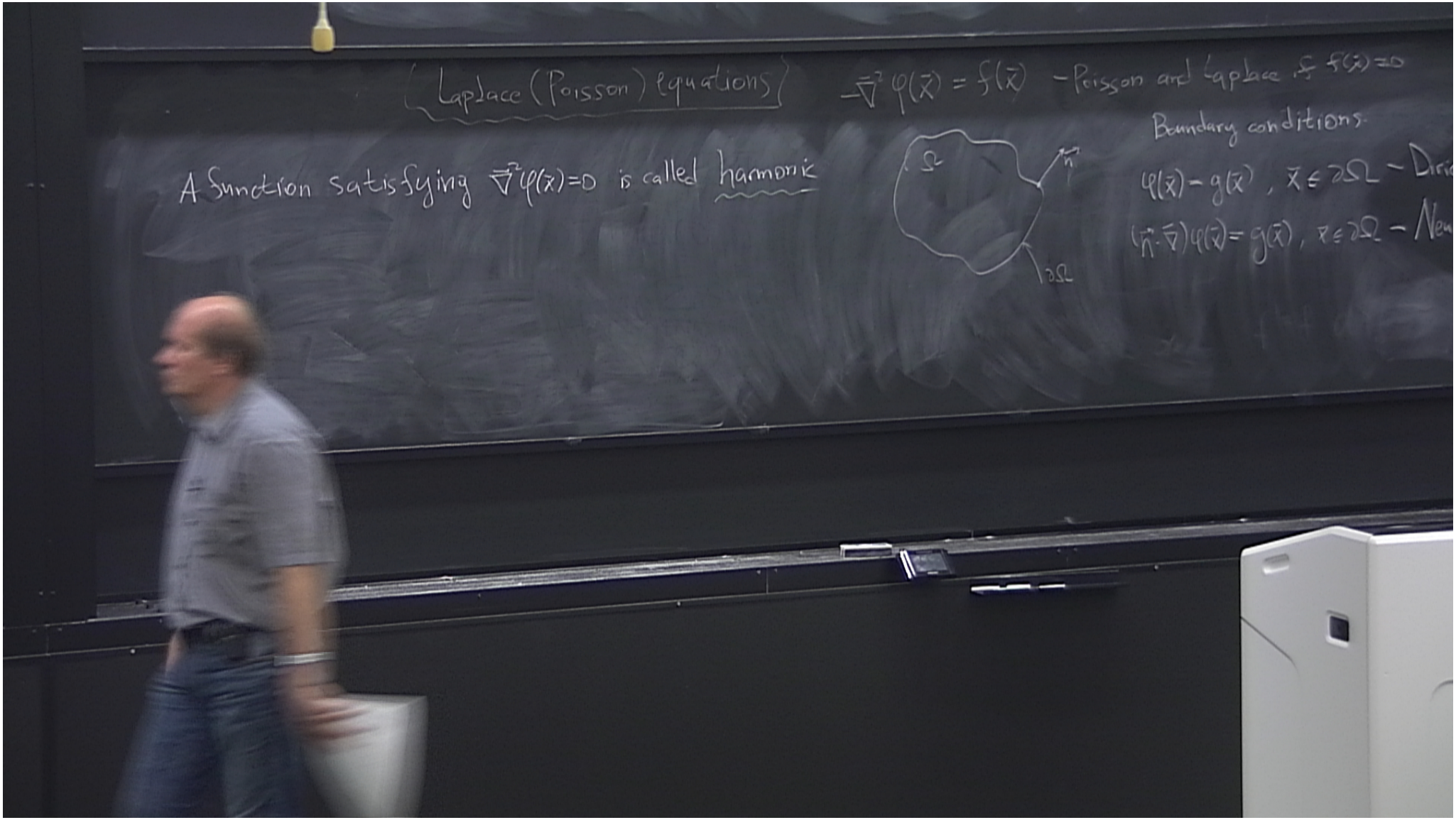

$G(x-\xi, t-\tau) = \theta(t-\tau)\, K(x-\xi, t-\tau)$; Note. If $\varphi(x,t)$ is subject to initial condition $u_0($

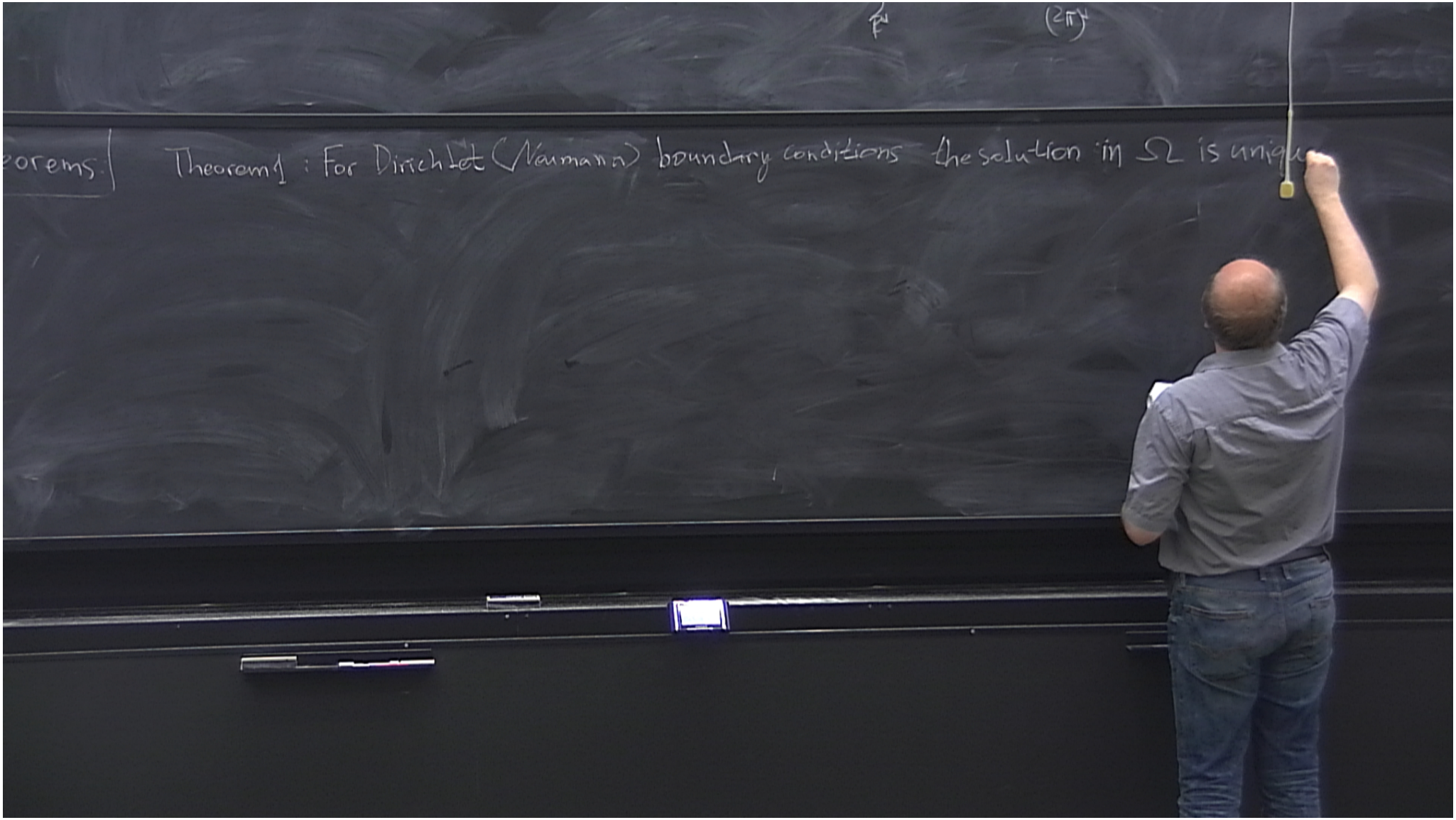

$G(\vec{x}-\vec{\xi}, t-\tau) = \theta(t-\tau)\, K(\vec{x}-\vec{\xi}, t-\tau)$, Note. If $\varphi(\vec{x},t)$ is subject to initial condition $u_0(\vec{x})$ at $t=0$ and the
source term $f(\vec{x},t)$ such that $f(\vec{x},t)=0$ for $t<0$

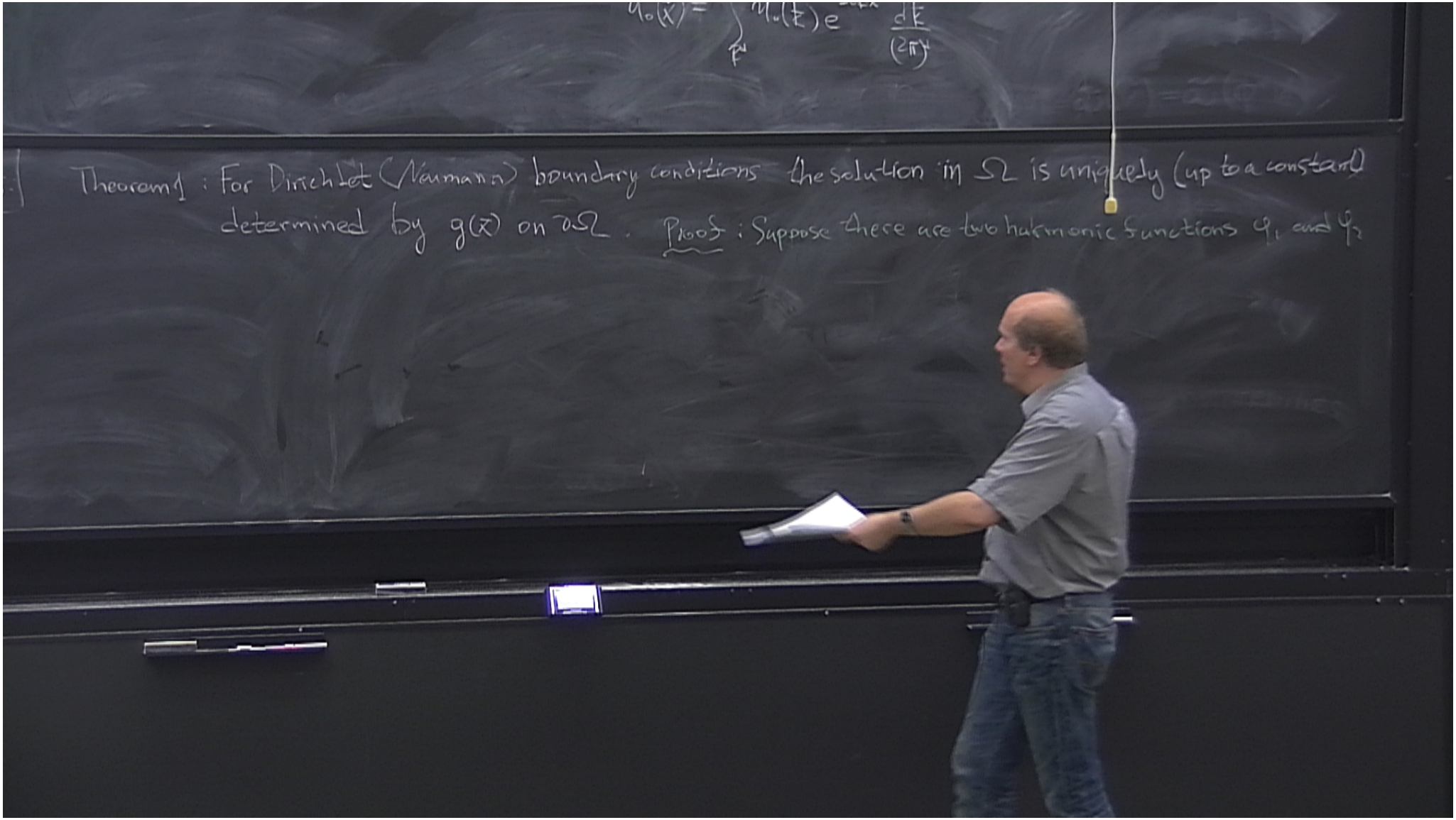

$G(\vec{x}-\vec{\xi}, t-\tau) = \theta(t-\tau)\, K(\vec{x}-\vec{\xi}, t-\tau)$, Note: If $\varphi(\vec{x},t)$ is subject to initial condition $u_0(\vec{x})$ at $t=0$ and the source term $f(\vec{x},t)$ such that $f(\vec{x},t)=0$ for $t<0$
General
tion to
then
$\varphi(\vec{x},t) = \int K(\vec{x}-\vec{\xi}, t)\, u_0(\vec{\xi})\, d\vec{\xi} +$
$+ \int d\vec{\xi} \int_0^t d\tau\, K(\vec{x}-\vec{\xi}, t-\tau)\, f(\vec{\xi}, \tau)$

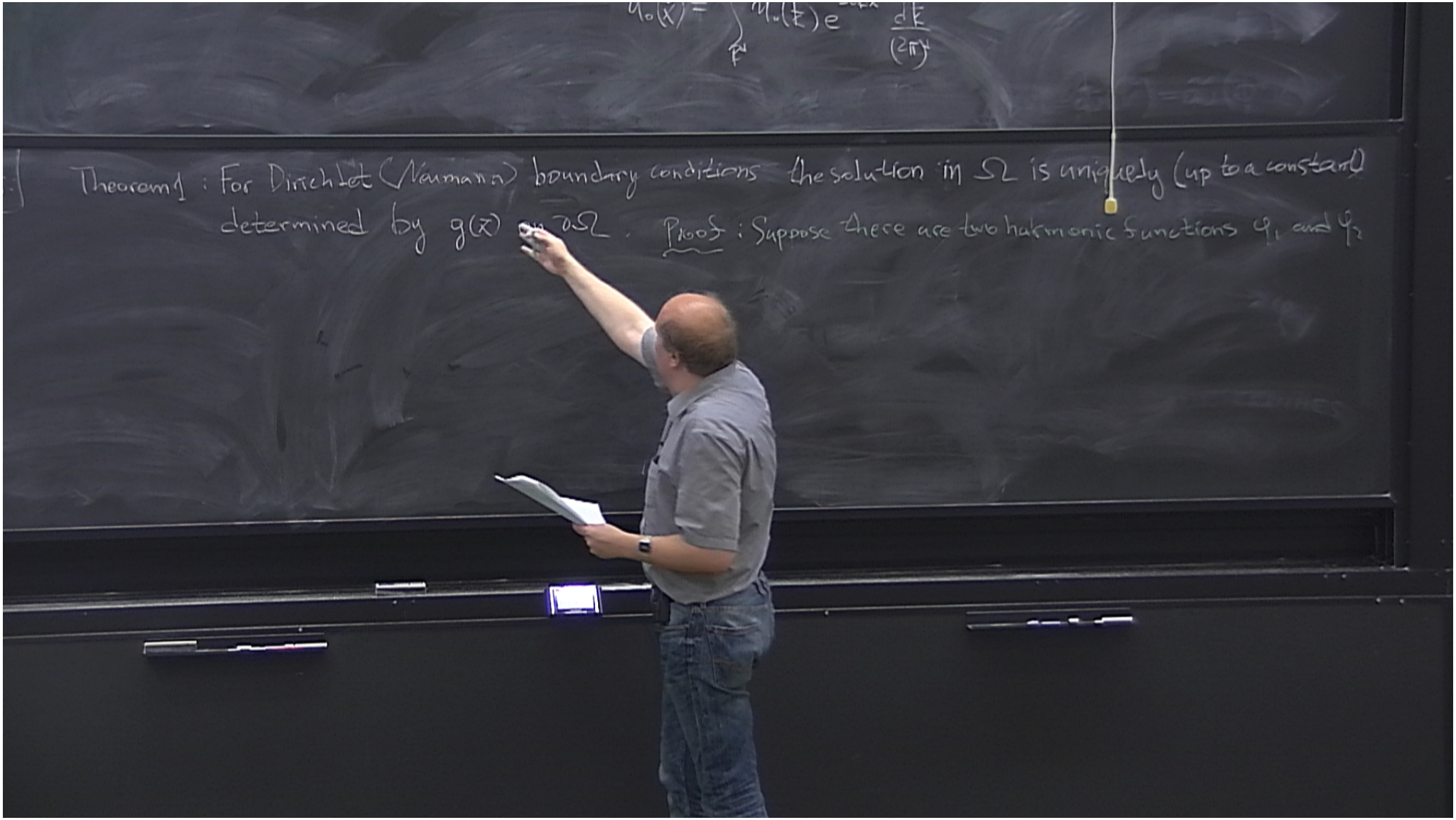

Laplace (Poisson) equations
$-\vec{\nabla}^2 \varphi(\vec{x}) = f(\vec{x})$ - Poisson and Laplace if $f(\vec{x}) = 0$


Laplace (Poisson) equations
$-\vec{\nabla}^2 \varphi(\vec{x}) = f(\vec{x})$ - Poisson and Laplace if $f(\vec{x}) = 0$
Boundary conditions.
Ω
$\vec{n}$
$\partial\Omega$


Laplace (Poisson) equations
$-\vec{\nabla}^2\varphi(\vec{x}) = f(\vec{x})$ - Poisson and Laplace if $f(\vec{x})=0$
A function satisfying $\vec{\nabla}^2\varphi(\vec{x})=0$ is called harmonic
Boundary conditions.
$\varphi(\vec{x}) = g(\vec{x})$, $\vec{x} \in \partial\Omega$ - Dirichlet
$(\vec{n}\cdot\vec{\nabla})\varphi(\vec{x}) = g(\vec{x})$, $\vec{x} \in \partial\Omega$ - Neumann
Ω
$\vec{n}$
$\partial\Omega$


orems:
Theorem1 : For Dirichlet (Naumann) boundary conditions the solution in Ω is


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determined by g(x) on ∂Ω. Proof : Suppose there are two harmonic functions φ1 and φ2


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Proof : Suppose there are two harmonic functions

ndary conditions the solution in Ω is uniquely (up to a constant)

Ω. Proof: Suppose there are two harmonic functions φ_1 and φ_2

For Neumann $(\vec{n}\cdot\vec{\nabla})\varphi_1 = (\vec{n}\cdot\vec{\nabla})\varphi_2 = g$ on $\partial\Omega$

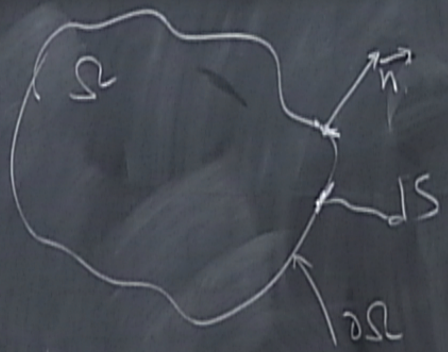
$\chi = \varphi_1 - \varphi_2$ is harmonic and satisfies $(\vec{n}\cdot\vec{\nabla})\chi = 0$ on $\partial\Omega$

$$\int_\Omega (\vec{\nabla}\chi)^2 dx = \int_\Omega \vec{\nabla}\cdot(\chi\vec{\nabla}\chi)dx - \int_\Omega \chi\nabla^2\chi\, dx = \int_{\partial\Omega} \chi(\vec{n}\cdot\vec{\nabla})\chi\, ds$$

Equations

$-\vec{\nabla}^2 \varphi(\vec{x}) = f(\vec{x})$ - Poisson and Laplace if $f(\vec{x}) = 0$

called harmonic



Boundary conditions:

$\varphi(\vec{x}) = g(\vec{x})$, $\vec{x} \in \partial\Omega$ - Dirichlet

$(\vec{n} \cdot \vec{\nabla})\varphi(\vec{x}) = g(\vec{x})$, $\vec{x} \in \partial\Omega$ - Neumann

$g(\vec{x})$ must satisfy the Fredholm constraint

Proof: $\int_\Omega \vec{\nabla}^2\varphi \, d\vec{x} = \int_{\partial\Omega} (\vec{n}\cdot\vec{\nabla})\varphi \, ds = \int_{\partial\Omega} g(\vec{x}) \, ds$

$\vec{\nabla}^2\varphi = -f(\vec{x})$

the Fredholm constraint

$$\int_{\partial\Omega} g(\bar{x})\,ds + \int_{\Omega} f\,d\bar{x} = 0$$

$$(\bar{n}\cdot\bar{\bar{v}})\,\varphi\,ds = \int_{\partial\Omega} g(\bar{x})\,ds$$

$-\nabla^2$ is self-adjoint $\quad \langle \phi, L\chi \rangle - \langle L^\dagger \phi, \chi \rangle$

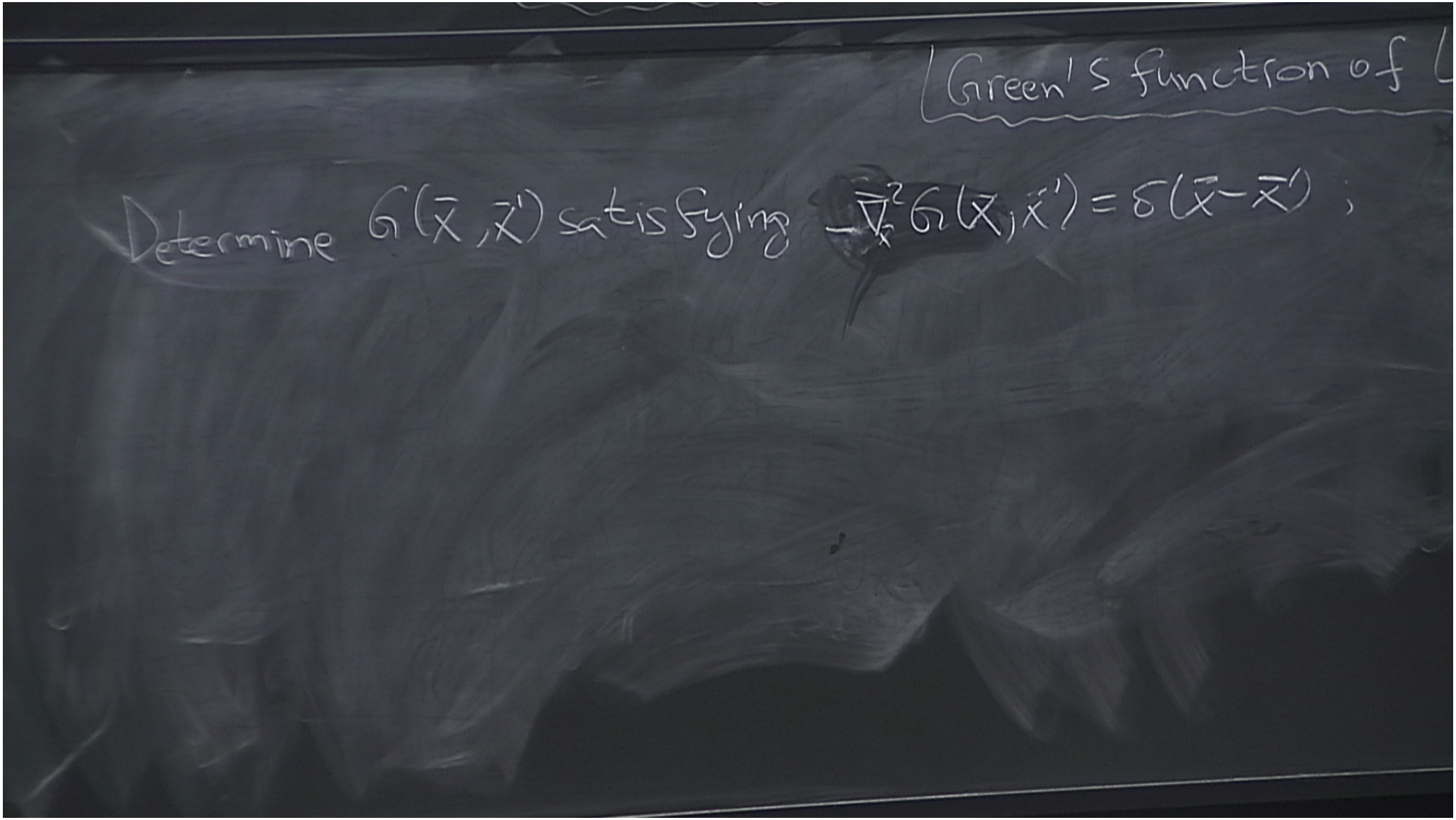
Identity $\int_\Omega \left[\phi^* (-\nabla^2 \chi) - \chi (-\nabla^2 \phi^*)\right] dx =$

$$= -\int_\Omega \nabla \left[\phi^* (\nabla \chi) - \chi (\nabla \phi^*)\right] dx = -\int_{\partial\Omega} \left[\phi^* \nabla \chi - \chi \nabla \phi^*\right] ds$$

perator $L=-\nabla^2$ is self-adjoint $\quad \langle \phi, L\chi \rangle - \langle L\phi, \chi \rangle$

om Green's identity $\int_\Omega \left[\phi^*(-\nabla^2\chi) - \chi(-\nabla^2\phi^*)\right] d\vec{x} =$

$$= -\int_\Omega \nabla\left[\phi^*(\nabla\chi) - \chi(\nabla\phi^*)\right] d\vec{x} = -\int_{\partial\Omega} \left[\phi^* \nabla\chi - \chi\nabla\phi^*\right] ds$$


Green's function of
Determine $G(\vec{x},\vec{x}')$ satisfying $-\vec{\nabla}_x^2 G(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}')$;

$-\nabla^2 \varphi(x) = f(x)$ — Poisson and Laplace

...on of Laplace operator $L = -\nabla^2$

...; Via eigenfunctions expansion $G(x, \bar{x}') = \sum_n \frac{\varphi_n(\bar{x}) \varphi_n^*(\bar{x}')}{\lambda_n}$

$\mathbb{R}^d$, $\lambda_{\vec{k}} = \vec{k}^2$, $\varphi_{\vec{k}} = e^{i\vec{k}\cdot\vec{x}}$

Green's function of Laplace

Determine $G(\vec{x},\vec{x}')$ satisfying $-\vec{\nabla}_x^2 G(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}')$; Dirichlet

For entire space $\mathbb{R}^d$, $f_{\vec{k}} = \vec{k}^2$,

$$G(\vec{x}-\vec{x}') = \int \frac{d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\vec{k}^2}$$

Green's function of Laplace

Determine $G(\vec{x},\vec{x}')$ satisfying $-\vec{\nabla}_x^2 G(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}')$; Using

For entire space $\mathbb{R}^d$, $\int_{\vec{k}} = \int \frac{d\vec{k}}{(2\pi)^d}$,

$$G(\vec{x}-\vec{x}') = \int \frac{d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\vec{k}^2}$$

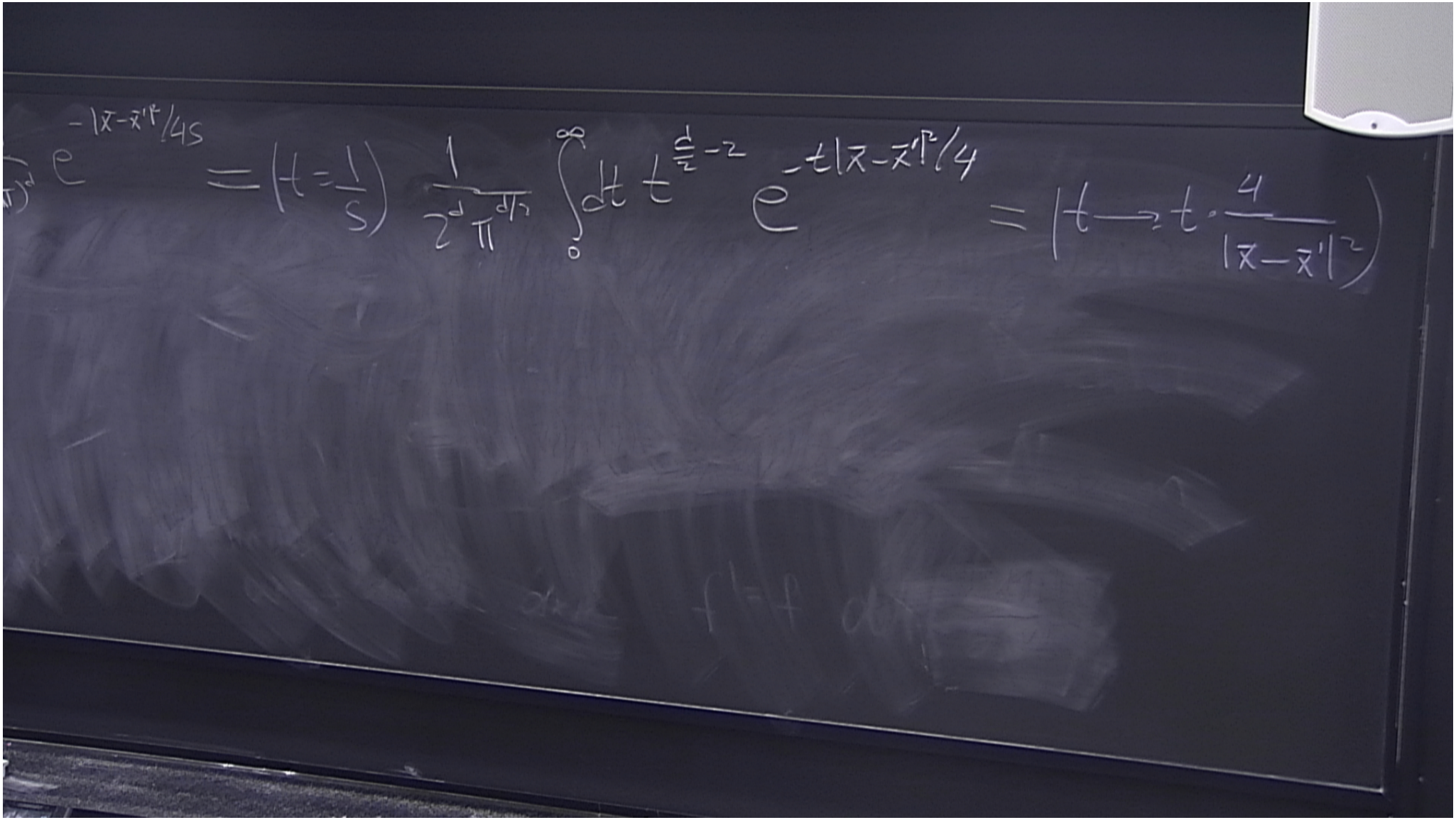
Calculate $G(\vec{x}-\vec{x}')$ for $d>2$. Use Schwinger's trick

function of Laplace operator $L = -\nabla^2$

$\delta(\vec{x}-\vec{x}')$, Via eigenfunctions expansion $G(\vec{x},\vec{x}') = \sum_n \frac{\varphi_n(\vec{x})\varphi_n^*(\vec{x}')}{f_n}$

space R^d, $f_{\vec{k}} = \vec{k}^2$, $\varphi_{\vec{k}}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}$

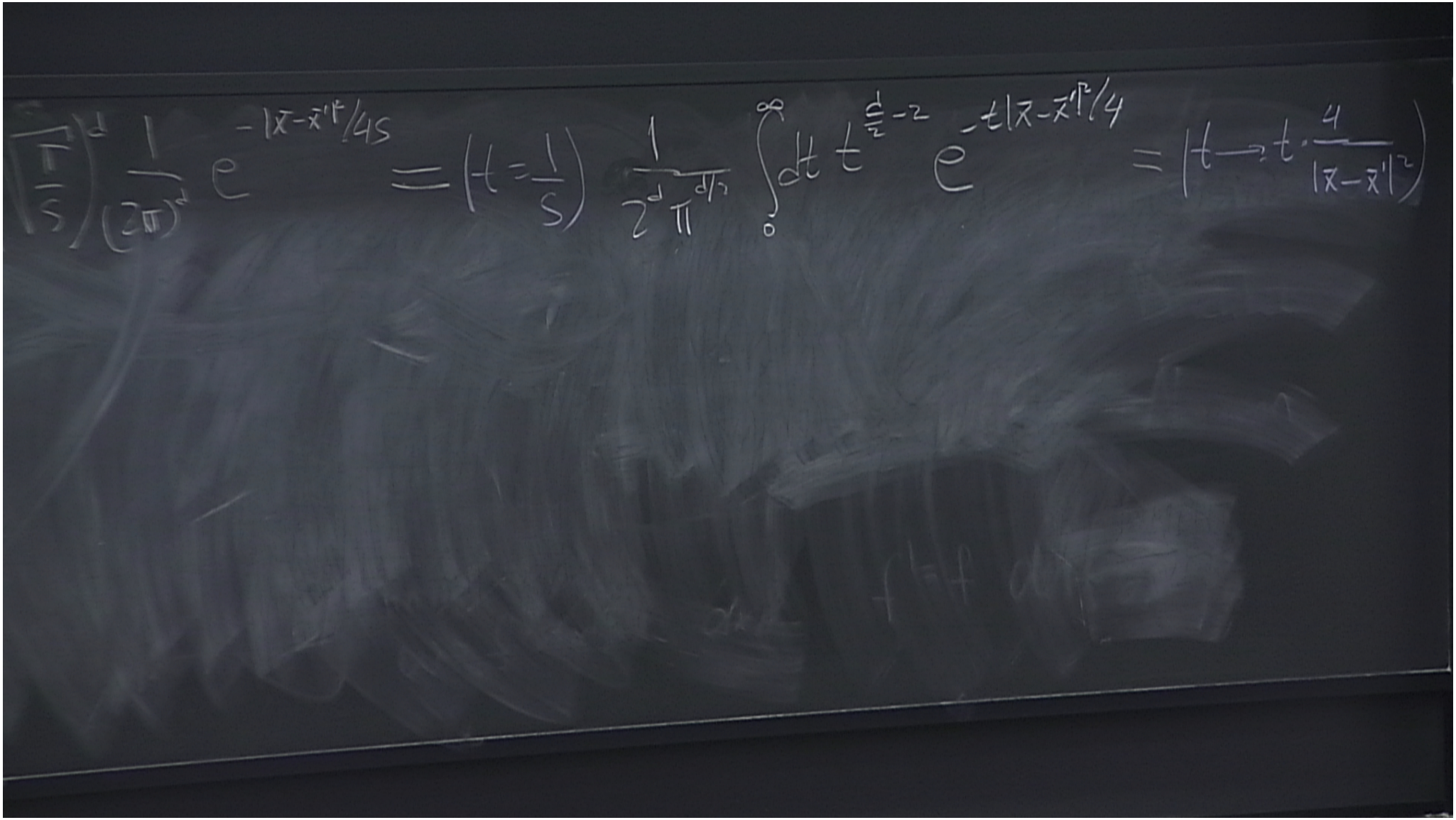
chwinger's trick

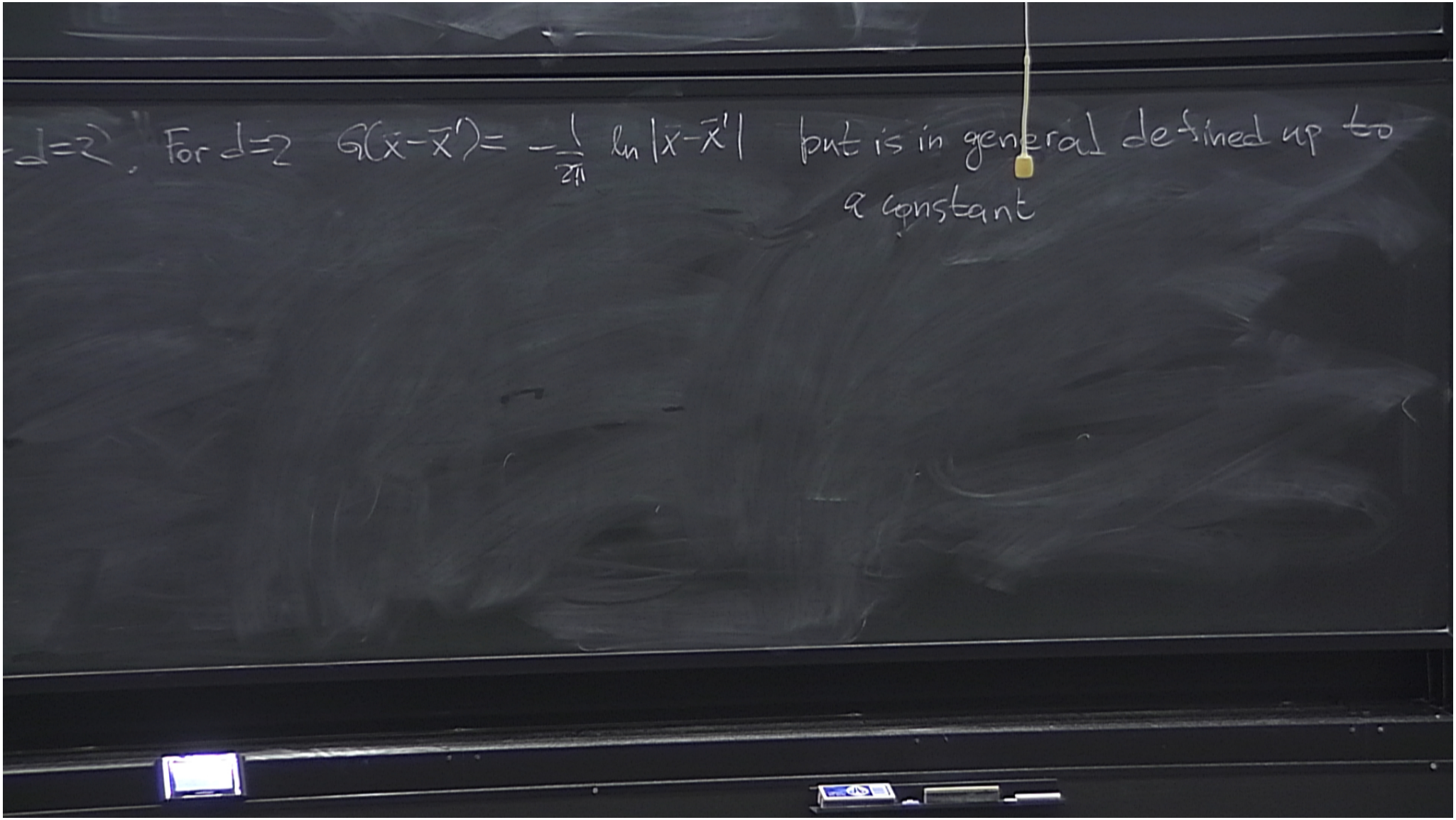

$e^{-|x-x'|^2/4s} = (t=\frac{1}{s})\ \frac{1}{2^d \pi^{d/2}} \int_0^\infty dt\, t^{\frac{d}{2}-2}\, e^{-t|x-x'|^2/4} = (t \to t \cdot \frac{4}{|x-x'|^2})$

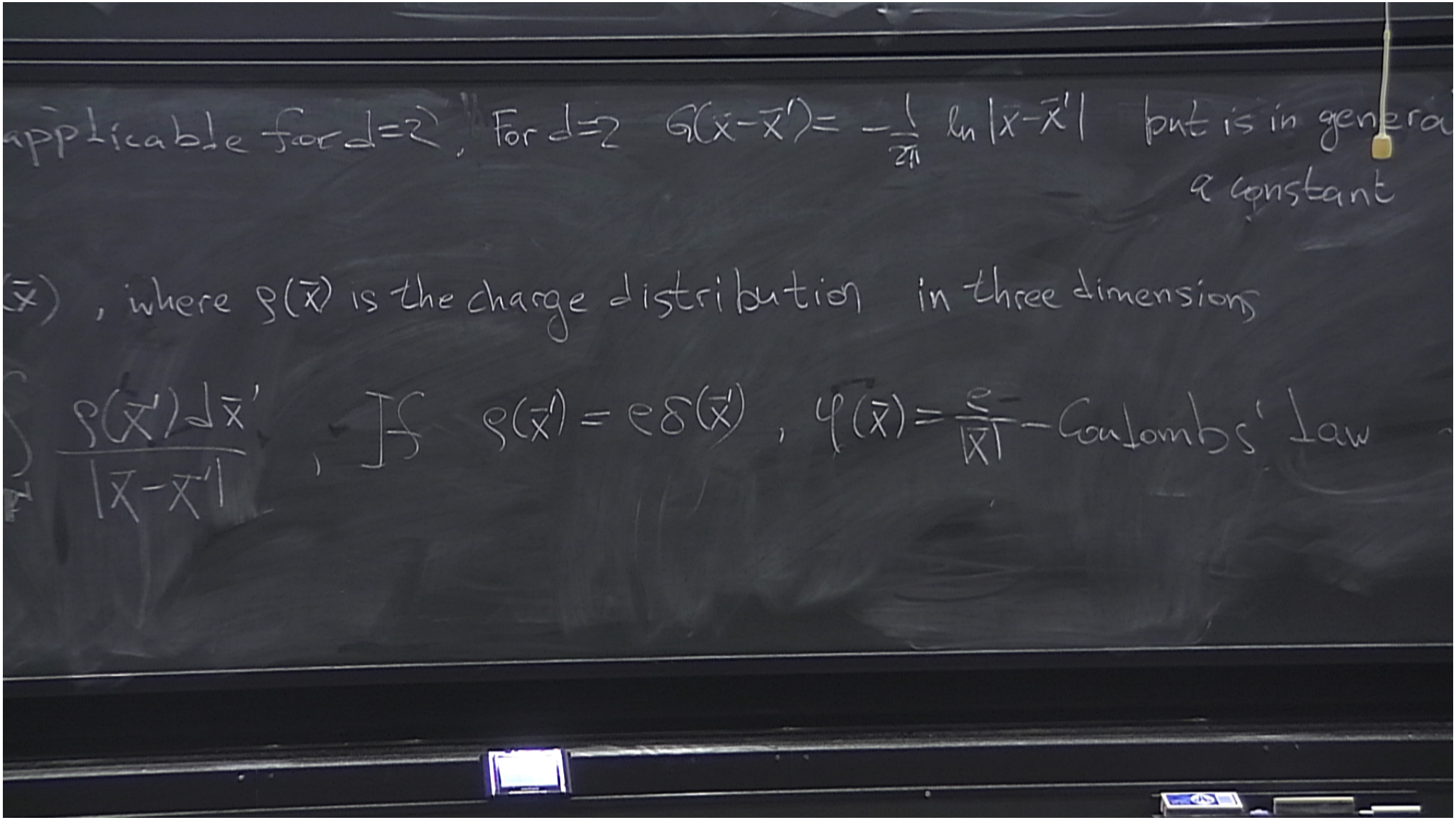
$$G(\vec{x}-\vec{x}') = \int_0^\infty ds \int \frac{d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{-sk^2} = \int_0^\infty ds \left(\sqrt{\frac{\pi}{s}}\right)^d \frac{1}{(2\pi)^d} e^{-|\vec{x}-\vec{x}'|}$$

$$= \frac{1}{2^d \pi^{d/2}} \Gamma\left(\frac{d}{2}-1\right) \left(\frac{4}{|\vec{x}-\vec{x}'|^2}\right)^{\frac{d}{2}-1}$$

For $d=3$: $G(\vec{x}-\vec{x}') = \frac{1}{4\pi|\vec{x}-\vec{x}'|}$


$\left(\frac{\pi}{s}\right)^{\frac{d}{2}} \frac{1}{(2\pi)^d} e^{-|x-x'|^2/4s} = (t=\frac{1}{s}) \frac{1}{2^d \pi^{d/2}} \int_0^\infty dt\, t^{\frac{d}{2}-2} e^{-t|x-x'|^2/4} = (t \to t \cdot \frac{4}{|x-x'|^2})$

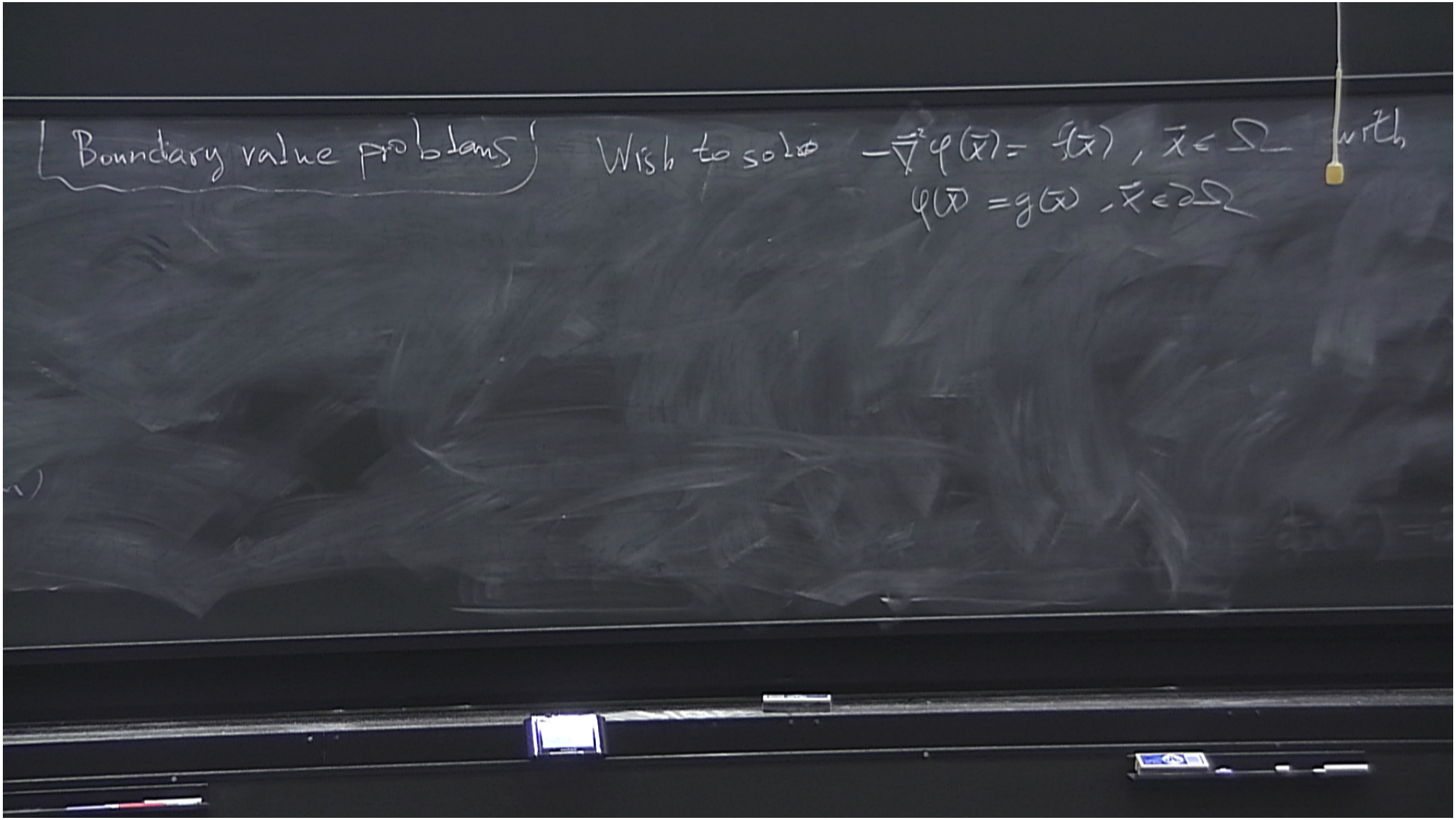

d=2. For d=2 $G(\vec{x}-\vec{x}') = -\frac{1}{2\pi} \ln|\vec{x}-\vec{x}'|$ but is in general defined up to
a constant


applicable for d=2. For d=2 $G(\bar{x}-\bar{x}') = -\frac{1}{2\pi}\ln|\bar{x}-\bar{x}'|$ but is in general a constant
$(\bar{x})$, where $\rho(\bar{x})$ is the charge distribution in three dimensions
$\frac{\rho(\bar{x}')d\bar{x}'}{|\bar{x}-\bar{x}'|}$, If $\rho(\bar{x}') = e\delta(\bar{x}')$, $\varphi(\bar{x}) = \frac{e}{|\bar{x}|}$ - Coulomb's law

Note 1: The result above is inapplicable for $d=2$. For $d=2$ G

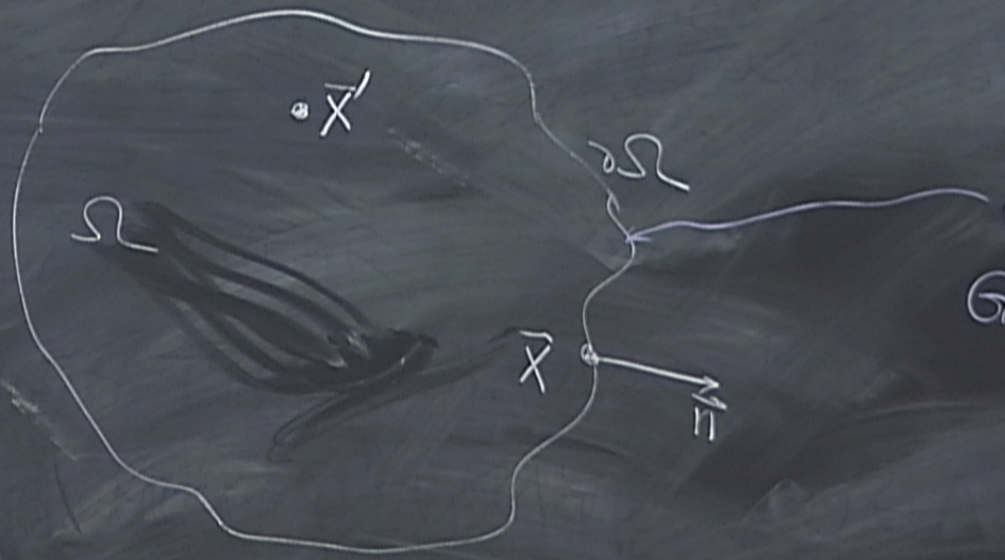
Note 2: To solve $-\nabla^2\varphi(\vec{x}) = 4\pi\rho(\vec{x})$, where $\rho(\vec{x})$ is the charge distr

$$\varphi(\vec{x}) = \int_{\mathbb{R}^3} G(\vec{x}-\vec{x}')\, 4\pi\rho(\vec{x}')\, d\vec{x}' = \int_{\mathbb{R}^3} \frac{\rho(\vec{x}')\, d\vec{x}'}{|\vec{x}-\vec{x}'|}, \quad \text{If } \rho(\vec{x}') = e\delta($$


Boundary value problems
Wish to solve $-\vec{\nabla}^2\varphi(\vec{x}) = f(\vec{x})$, $\vec{x} \in \Omega$ with
$\varphi(\vec{x}) = g(\vec{x})$, $\vec{x} \in \partial\Omega$

Suppose we have found $G(\vec{x}, \vec{x}') = G(\vec{x}', \vec{x})$ that obeys $-\vec{\nabla}_x$

$=$



and $G(\vec{x}, \vec{x}$

at $\partial\Omega$

$G(\vec{x}, \vec{x}') = 0$

$\varphi(\vec{x}) = g(\vec{x})$, $\vec{x} \in \partial\Omega$

$\vec{x}$) that obeys $-\vec{\nabla}_x^2 G(\vec{x}, \vec{x}') =$

$= \delta(\vec{x} - \vec{x}')$, $\vec{x}, \vec{x}' \in \Omega$

and $G(\vec{x}, \vec{x}') = 0$, $\vec{x} \in \partial\Omega$

at $\partial\Omega$

$G(\vec{x}, \vec{x}') = 0$