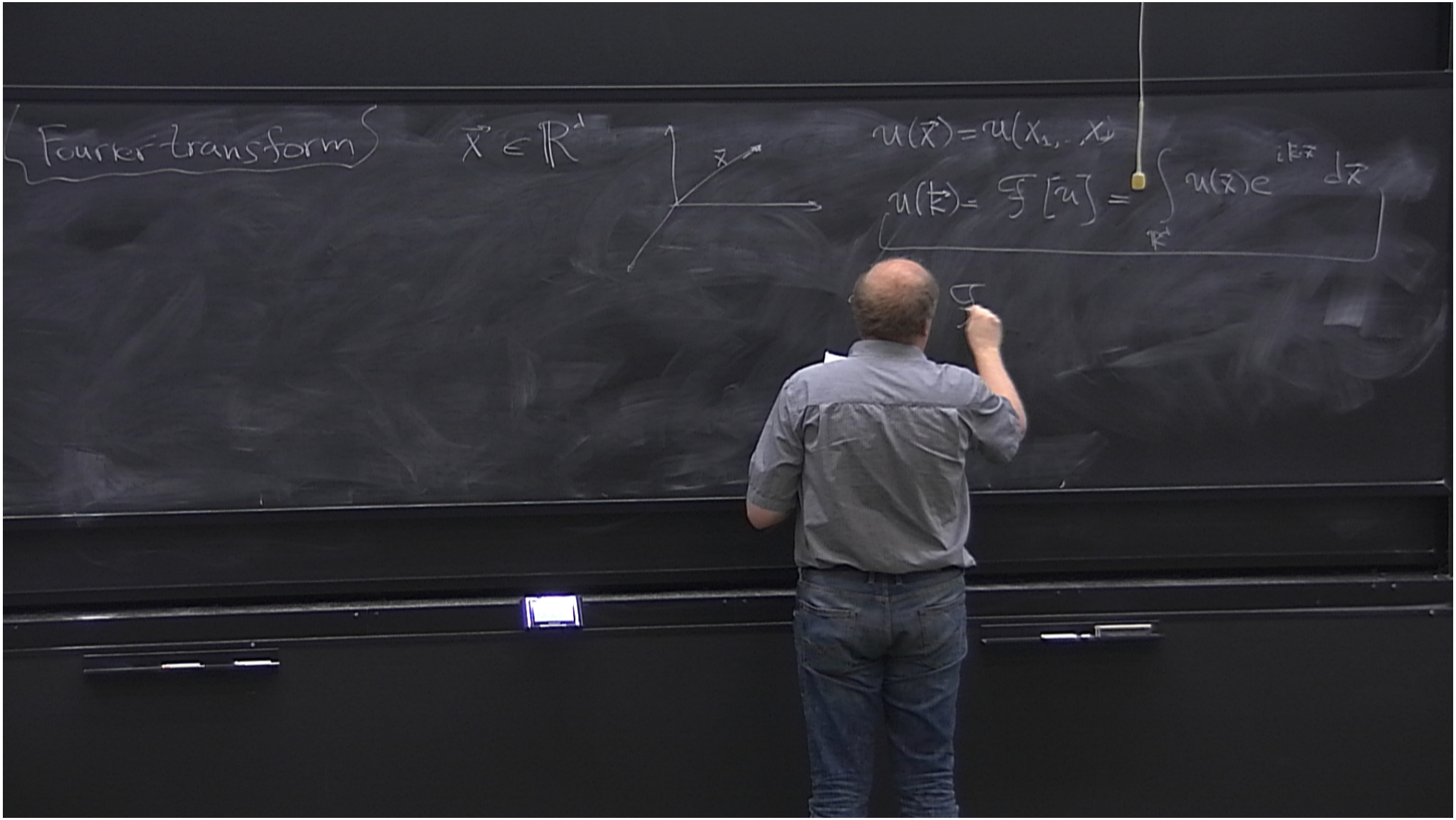
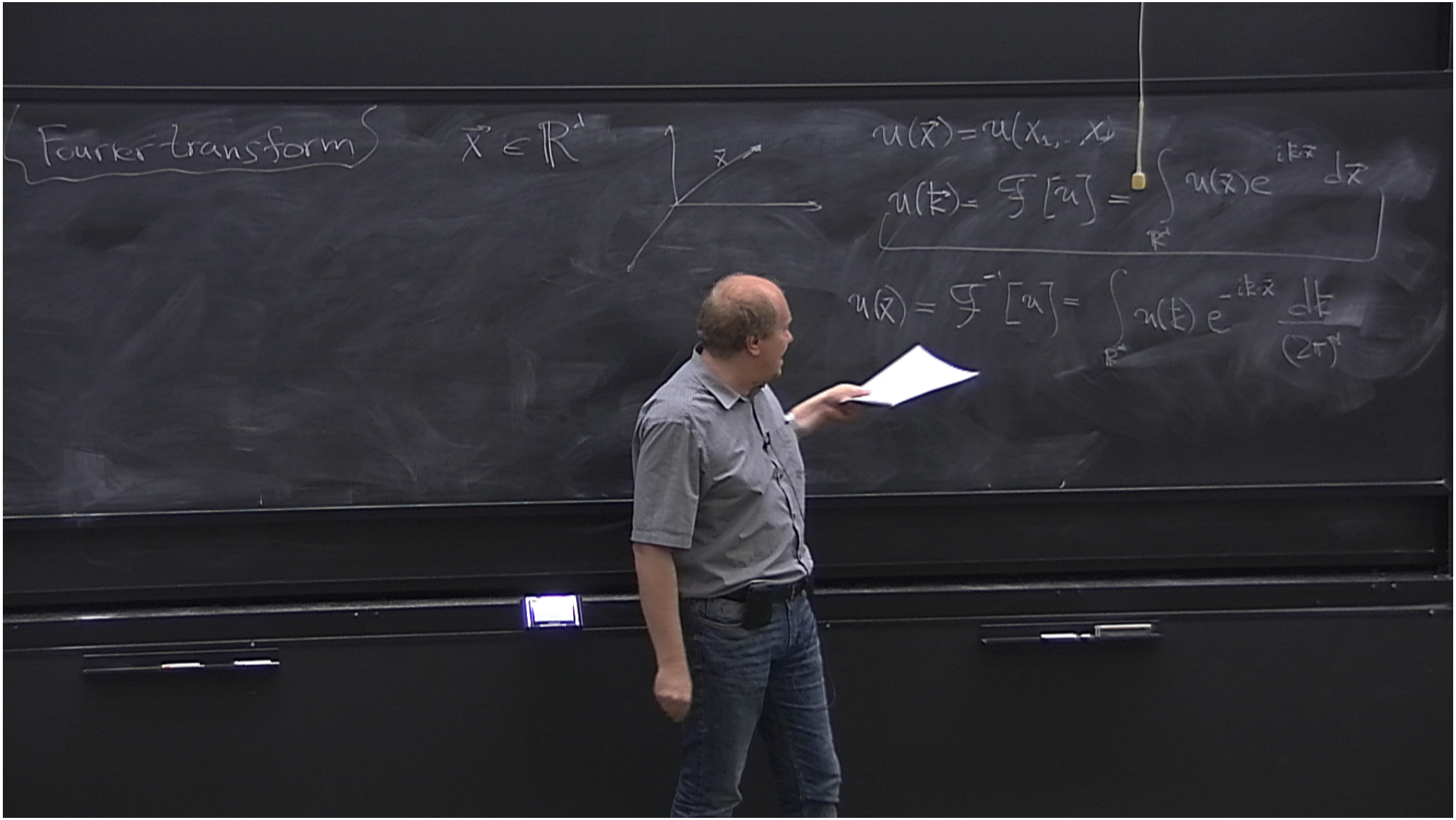
Title: PSI 2015/2016 Greens Functions - Denis Dalidovich - 1

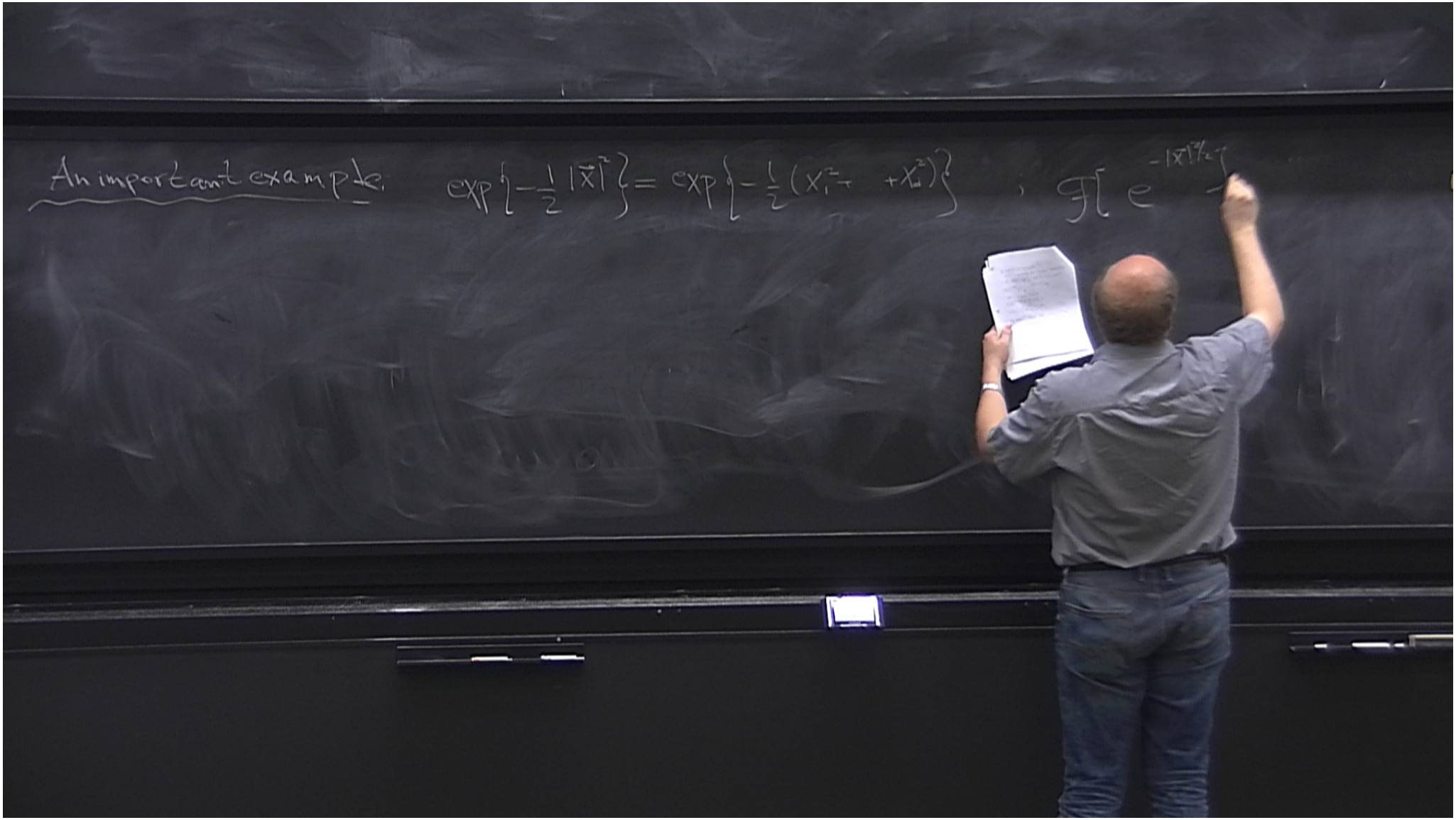
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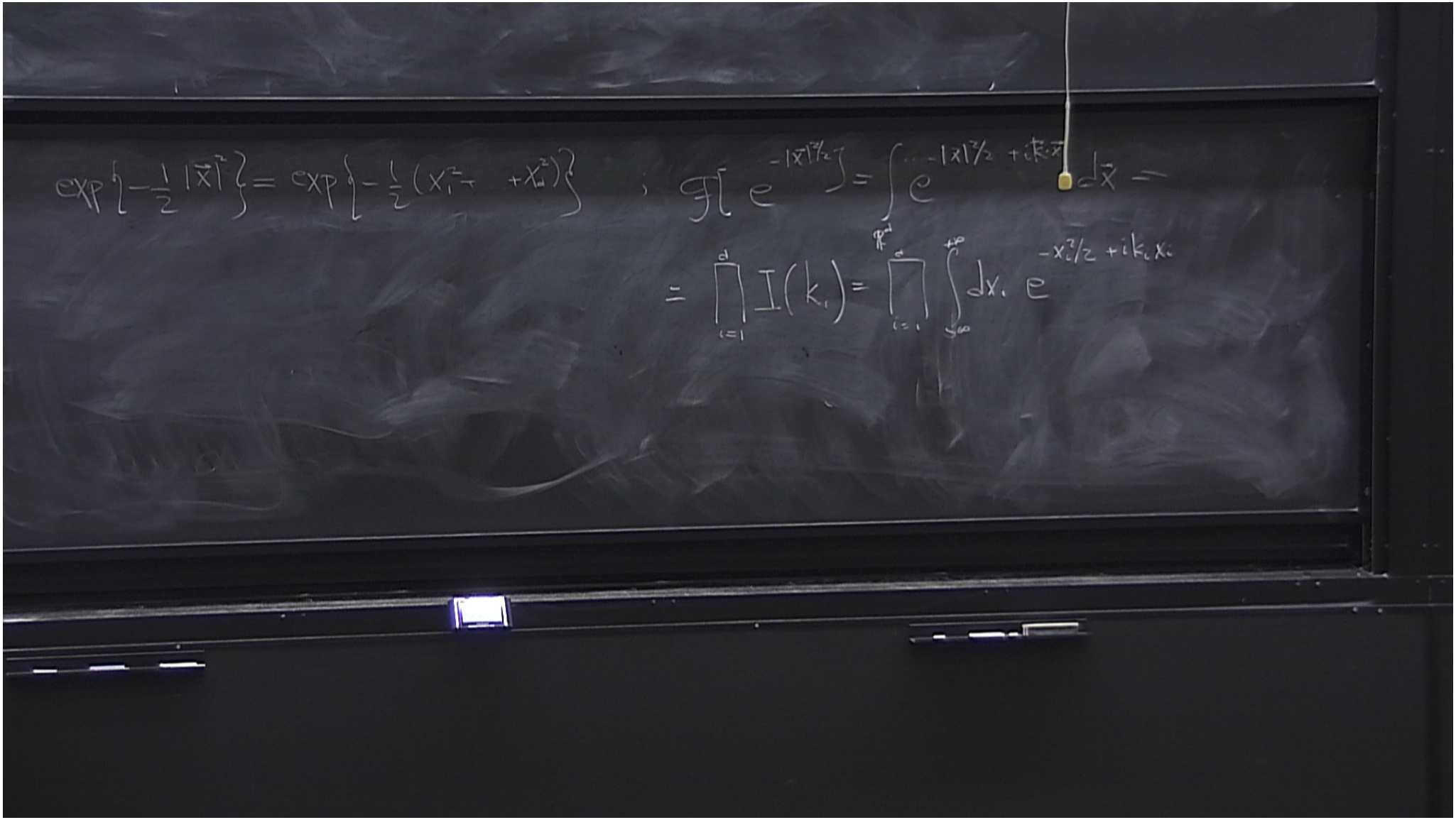
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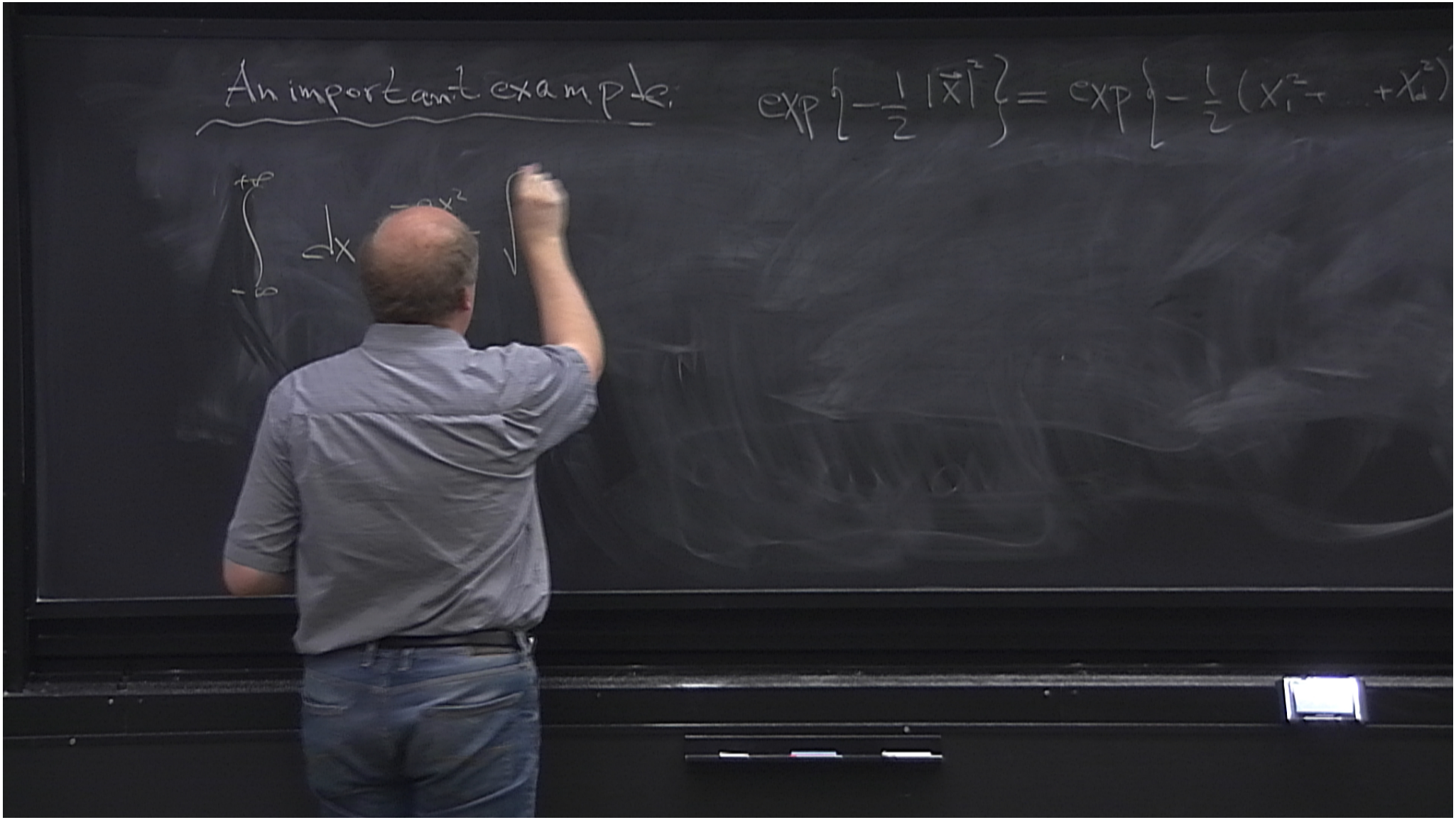
Abstract:

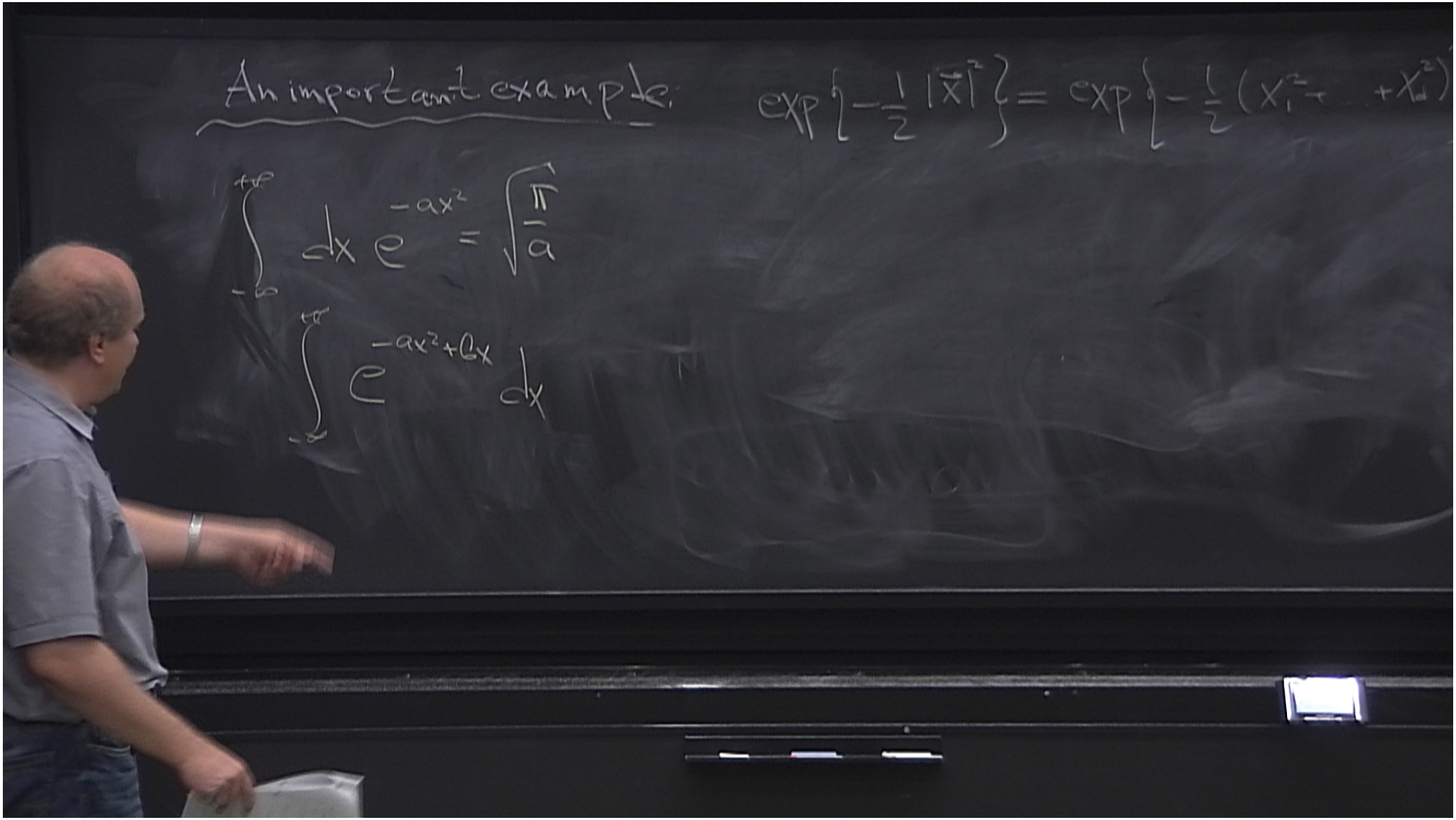

Fourier-transform
$\vec{x} \in \mathbb{R}^d$
$u(\vec{x}) = u(x_1, \ldots x_d)$
$u(\vec{k}) = \mathcal{F}[u] = \int_{\mathbb{R}^d} u(\vec{x}) e^{i\vec{k}\vec{x}} d\vec{x}$
$\mathcal{F}$

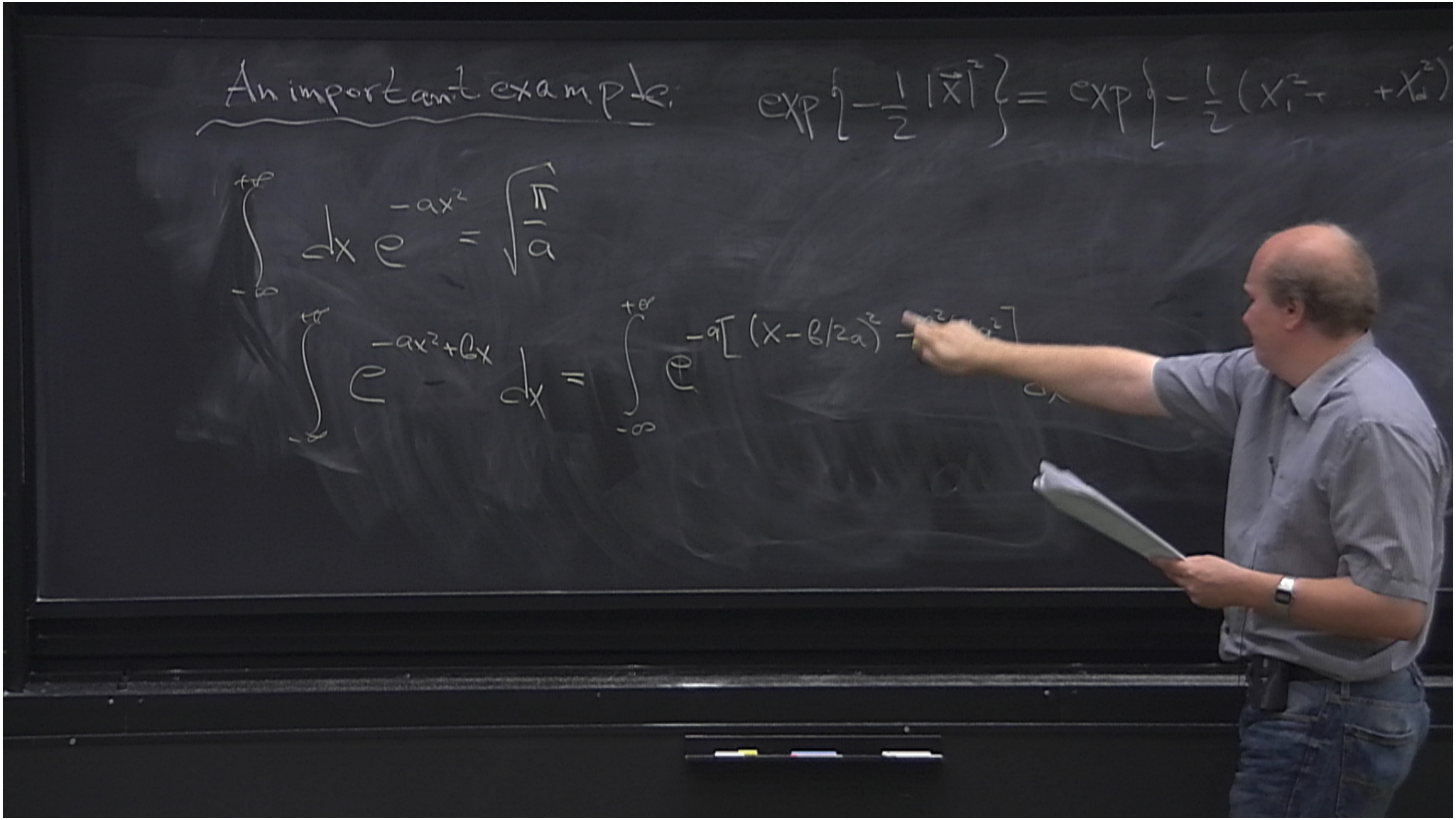

Fourier-transform
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$u(\vec{x}) = \mathcal{F}^{-1}[u] = \int_{\mathbb{R}^d} u(\vec{k}) e^{-i\vec{k}\vec{x}} \frac{d\vec{k}}{(2\pi)^d}$

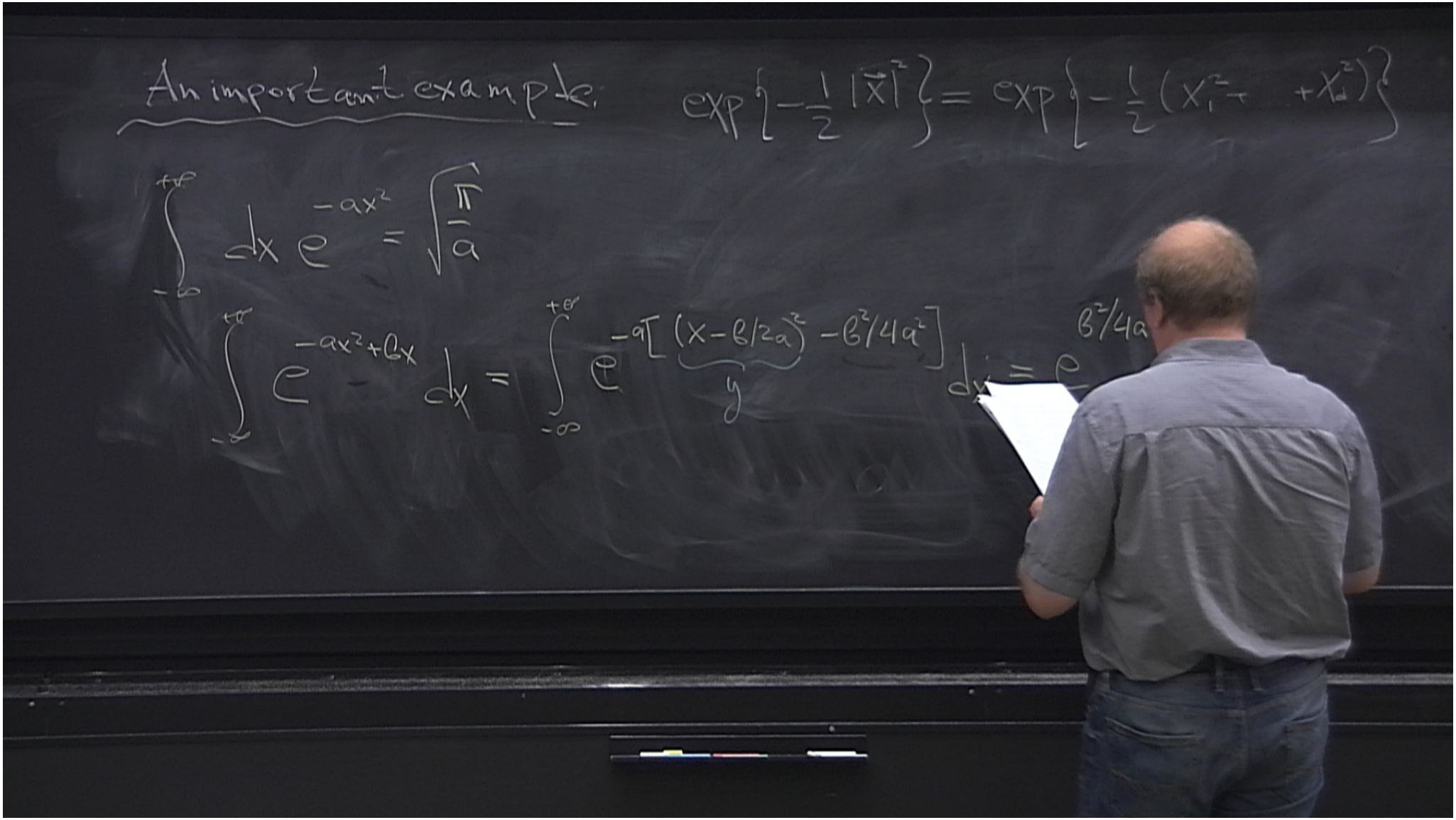

An important example:
exp{-½|x⃗|²} = exp{-½(x₁² + ... + x_d²)}
; F[e^{-|x|²/2}]

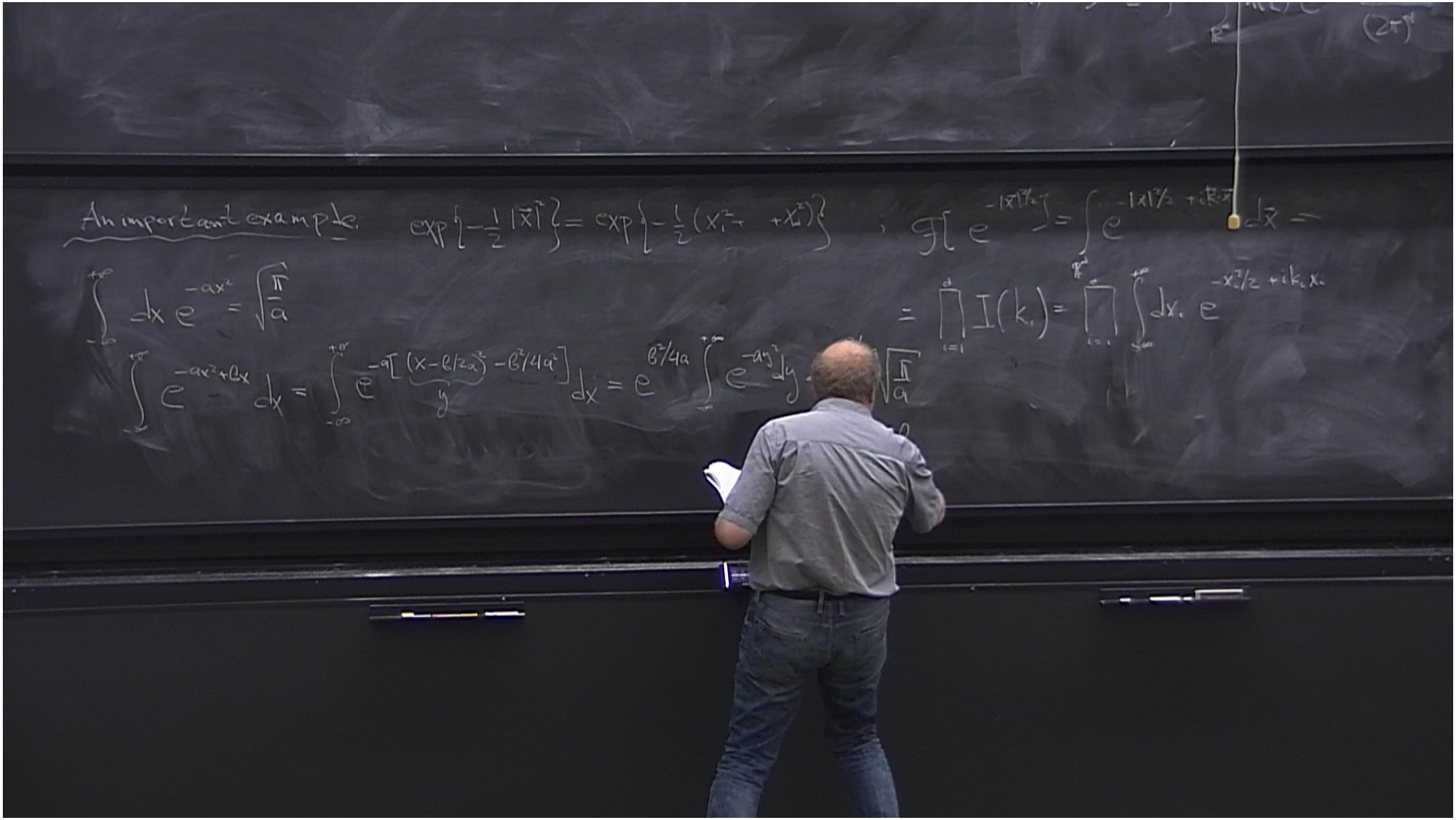

$\exp\left\{-\frac{1}{2}|\vec{x}|^2\right\} = \exp\left\{-\frac{1}{2}(x_1^2 + \dots + x_d^2)\right\}$
$\mathcal{F}\left[e^{-|x|^2/2}\right] = \int_{\mathbb{R}^d} e^{-|x|^2/2 + i\vec{k}\cdot\vec{x}}\, d\vec{x} =$
$= \prod_{i=1}^{d} I(k_i) = \prod_{i=1}^{d} \int_{-\infty}^{+\infty} dx_i\, e^{-x_i^2/2 + i k_i x_i}$


An important example:
exp{-1/2 |x⃗|²} = exp{-1/2 (x₁² + ... + x_d²)}
+∞
-∞
dx


An important example:
$\exp\{-\frac{1}{2}|\vec{x}|^2\} = \exp\{-\frac{1}{2}(x_1^2 + \ldots + x_n^2)\}$
$\int_{-\infty}^{+\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$
$\int_{-\infty}^{+\infty} e^{-ax^2+bx}\, dx$


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$\int_{-\infty}^{+\infty} e^{-ax^2+bx}\, dx = \int_{-\infty}^{+\infty} e^{-a\left[(x-b/2a)^2 - \ldots\right]}$

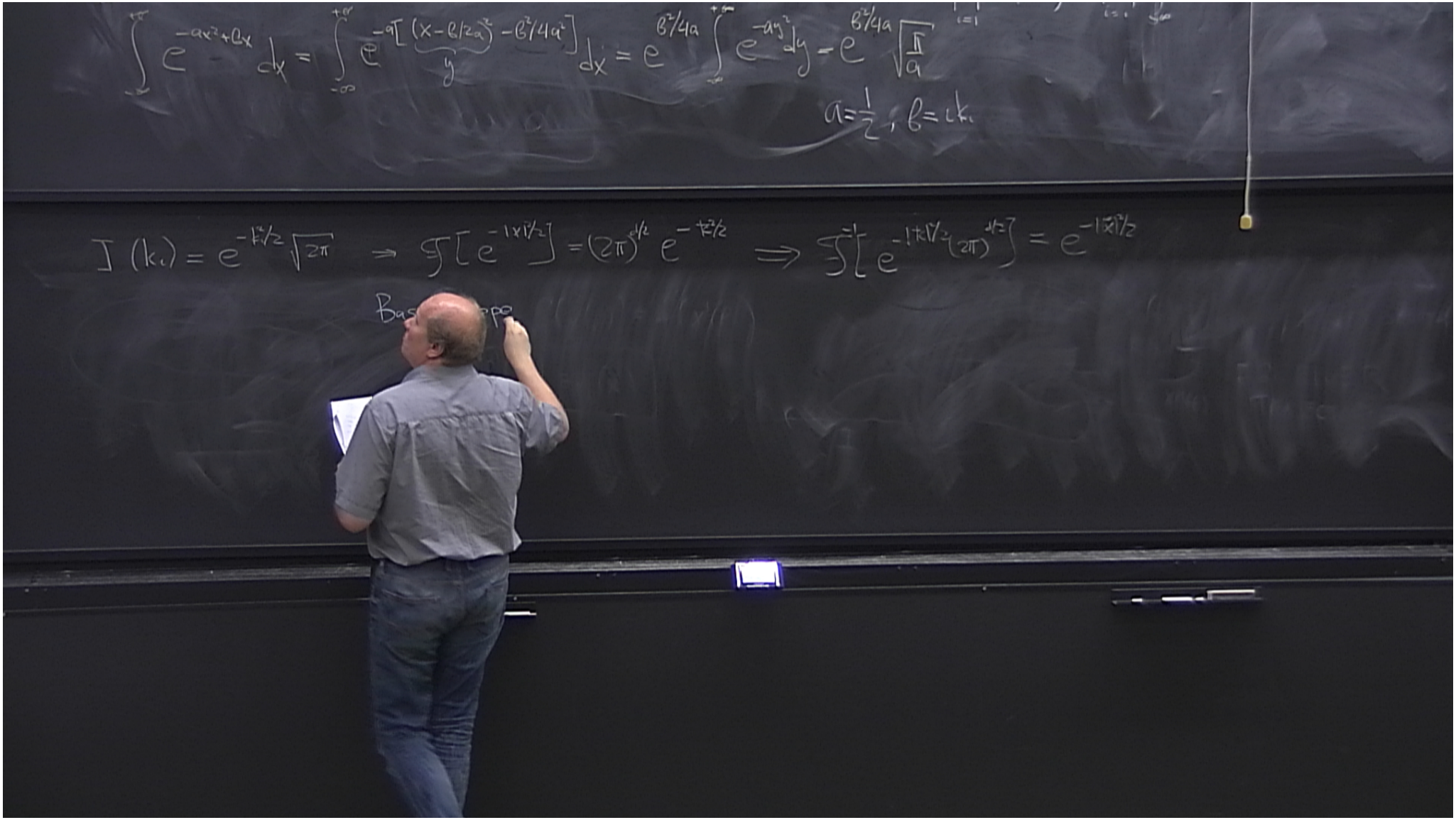

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$\int_{-\infty}^{+\infty} e^{-ax^2+bx}\, dx = \int_{-\infty}^{+\infty} e^{-a\left[\underbrace{(x-b/2a)}_{y}{}^2 - b^2/4a^2\right]} dx = e^{b^2/4a}$

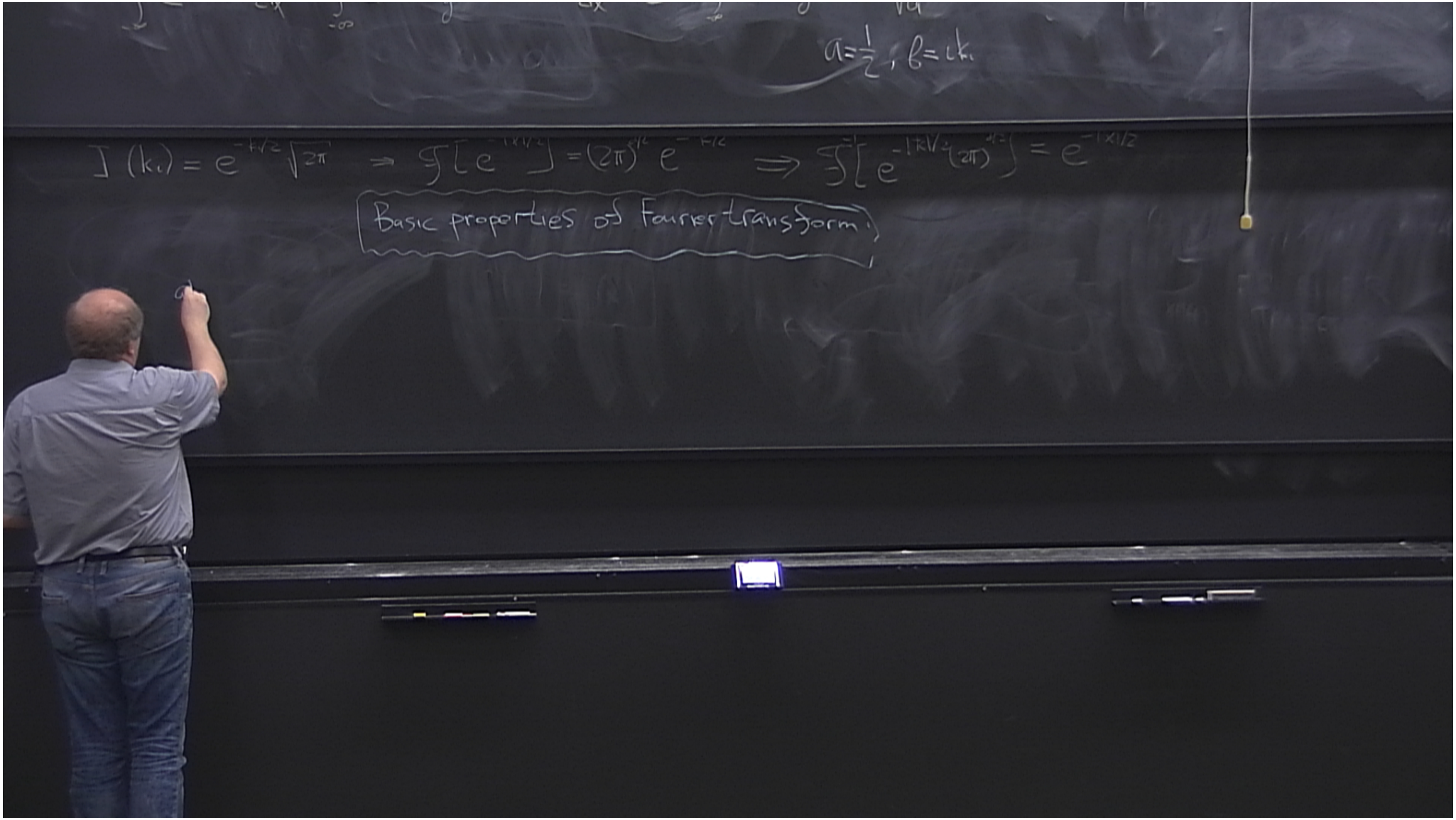

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$\mathcal{F}\left[e^{-|\vec{x}|^2/2}\right] = \int e^{-|\vec{x}|^2/2 + i\vec{k}\cdot\vec{x}} d\vec{x} =$
$= \prod_{i=1}^{d} I(k_i) = \prod_{i=1}^{d} \int_{-\infty}^{+\infty} dx_i\, e^{-x_i^2/2 + ik_i x_i}$
$(2\pi)^{d}$

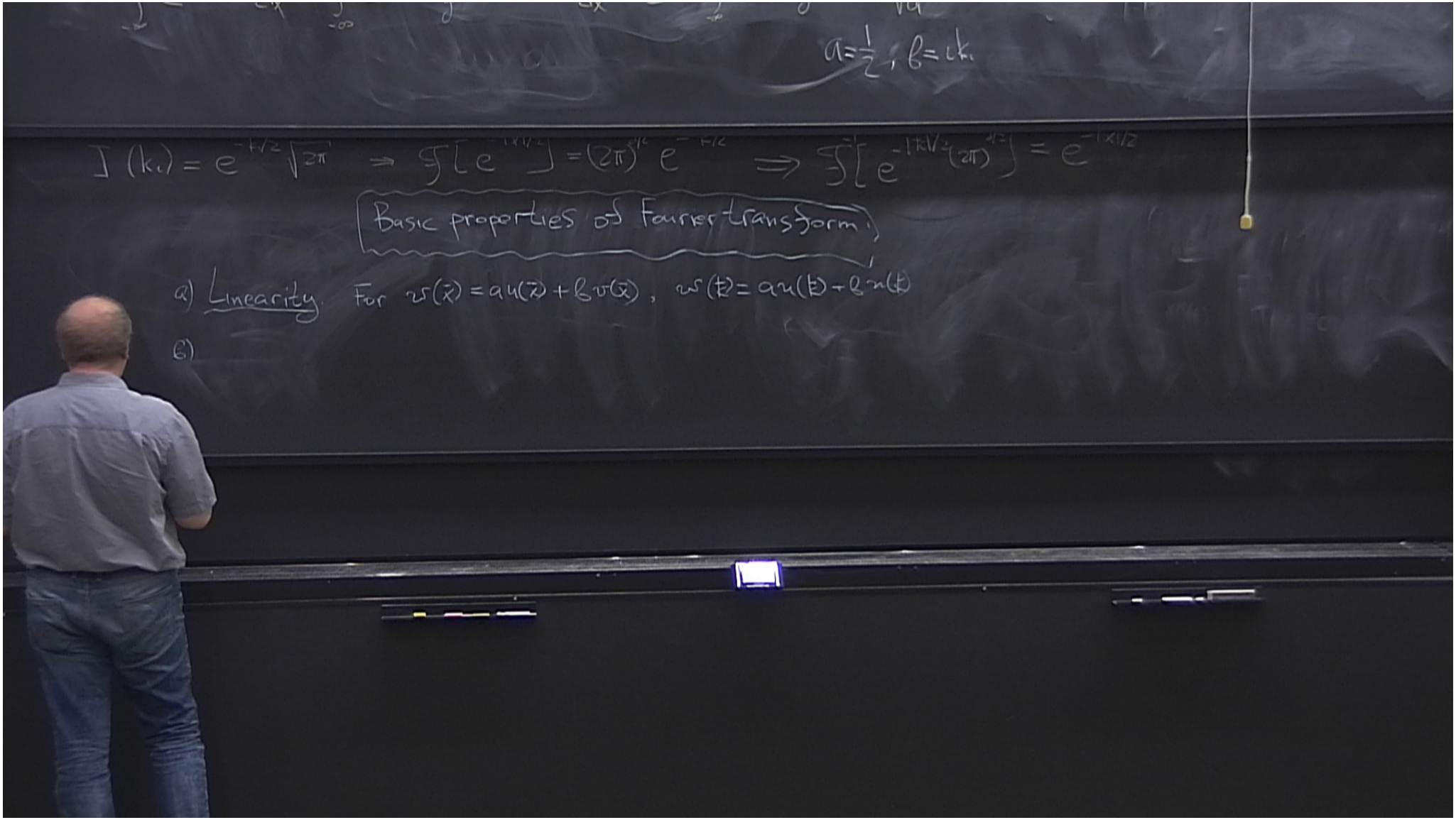
$$\int_{-\infty}^{+\infty} e^{-ax^2+bx}\,dx = \int_{-\infty}^{+\infty} e^{-a\left[\underbrace{(x-b/2a)}_{y}{}^2 - b^2/4a^2\right]}\,dx = e^{b^2/4a}\int_{-\infty}^{+\infty} e^{-ay^2}\,dy = e^{b^2/4a}\sqrt{\frac{\pi}{a}}$$

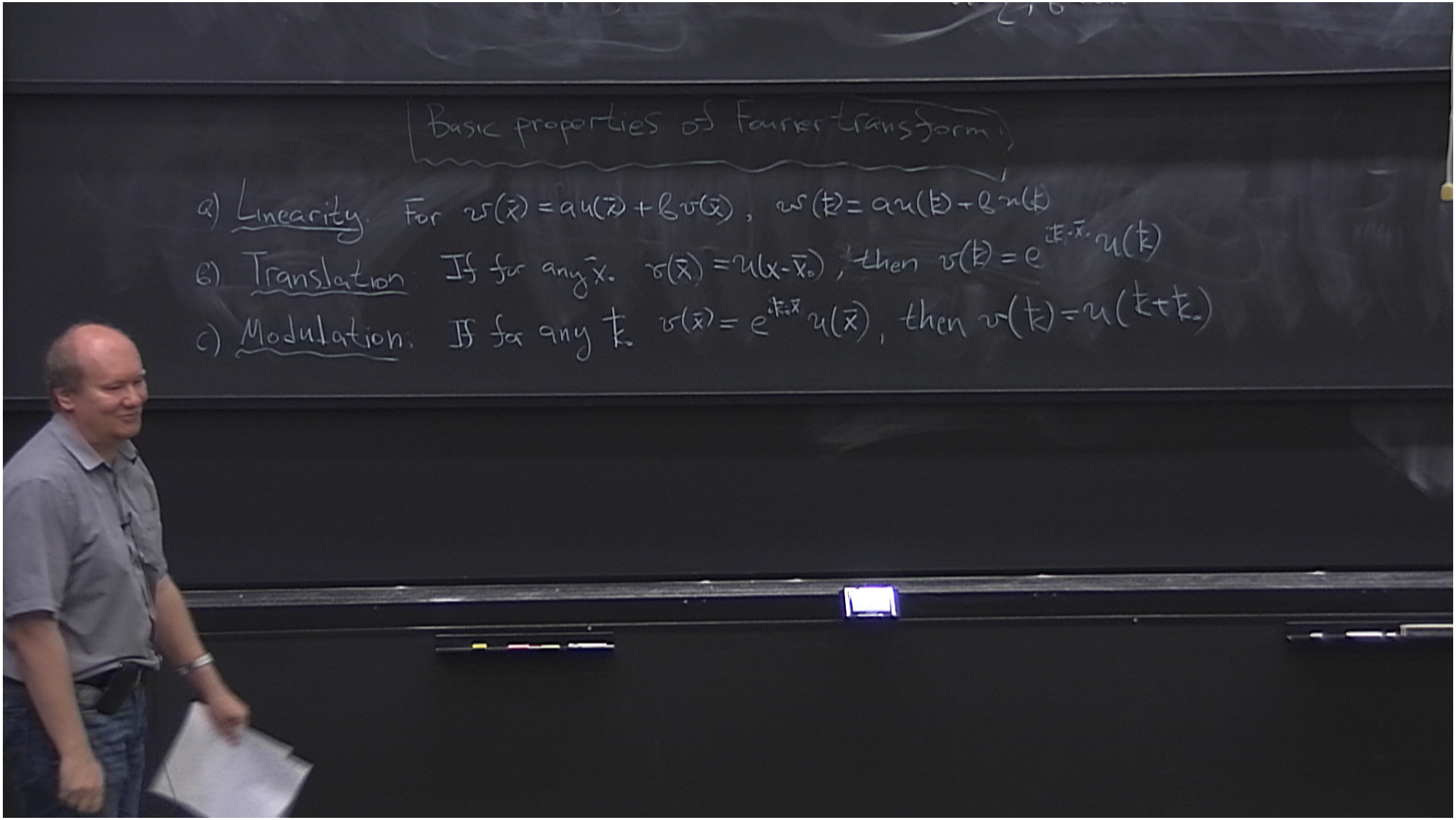
$$a=\frac{1}{2},\ b=ik_i$$

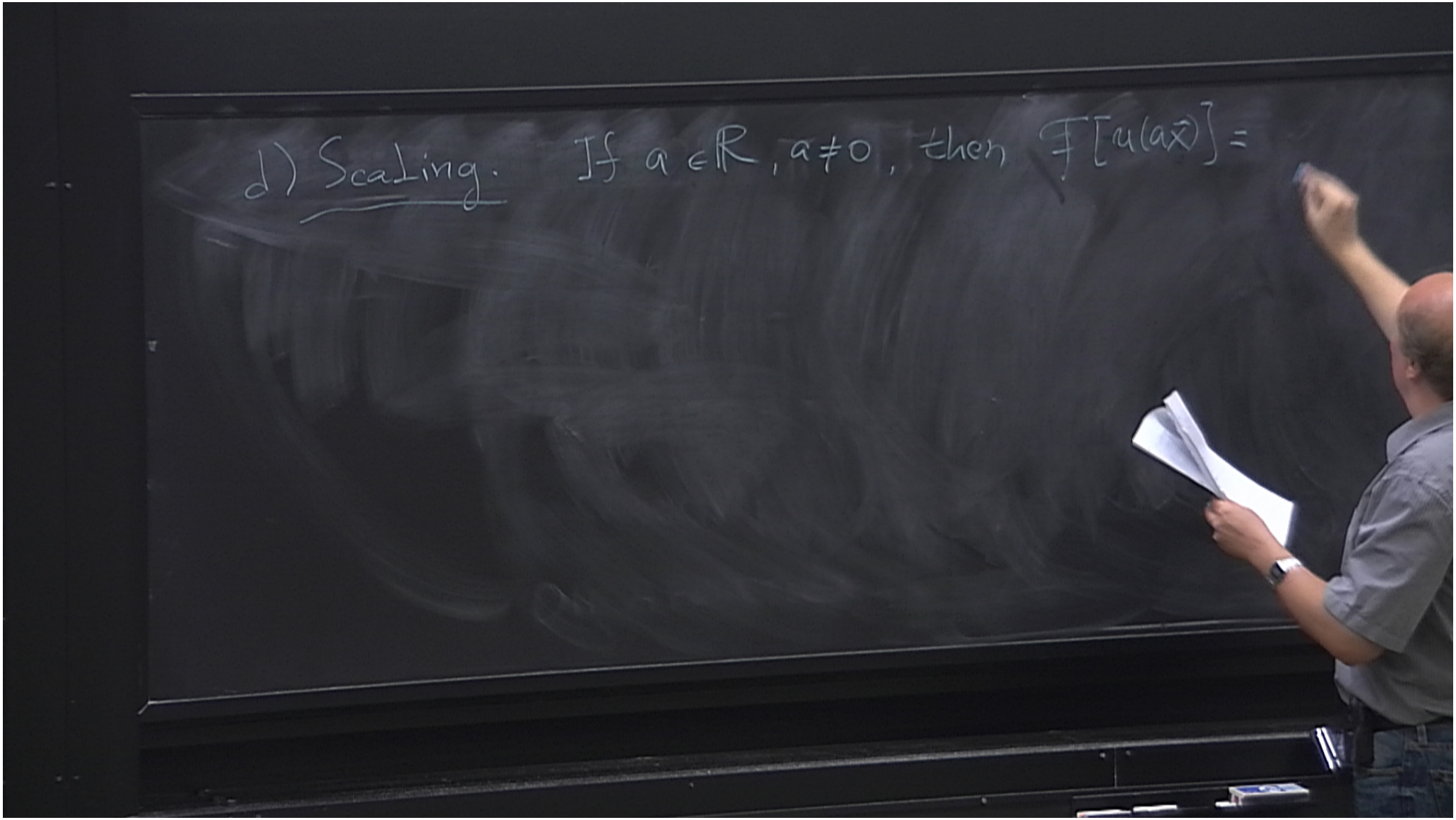
$$I(k_i) = e^{-k_i^2/2}\sqrt{2\pi} \Rightarrow \mathcal{F}\left[e^{-|x|^2/2}\right] = (2\pi)^{d/2} e^{-k^2/2} \Rightarrow \mathcal{F}^{-1}\left[e^{-|k|^2/2}(2\pi)^{d/2}\right] = e^{-|x|^2/2}$$

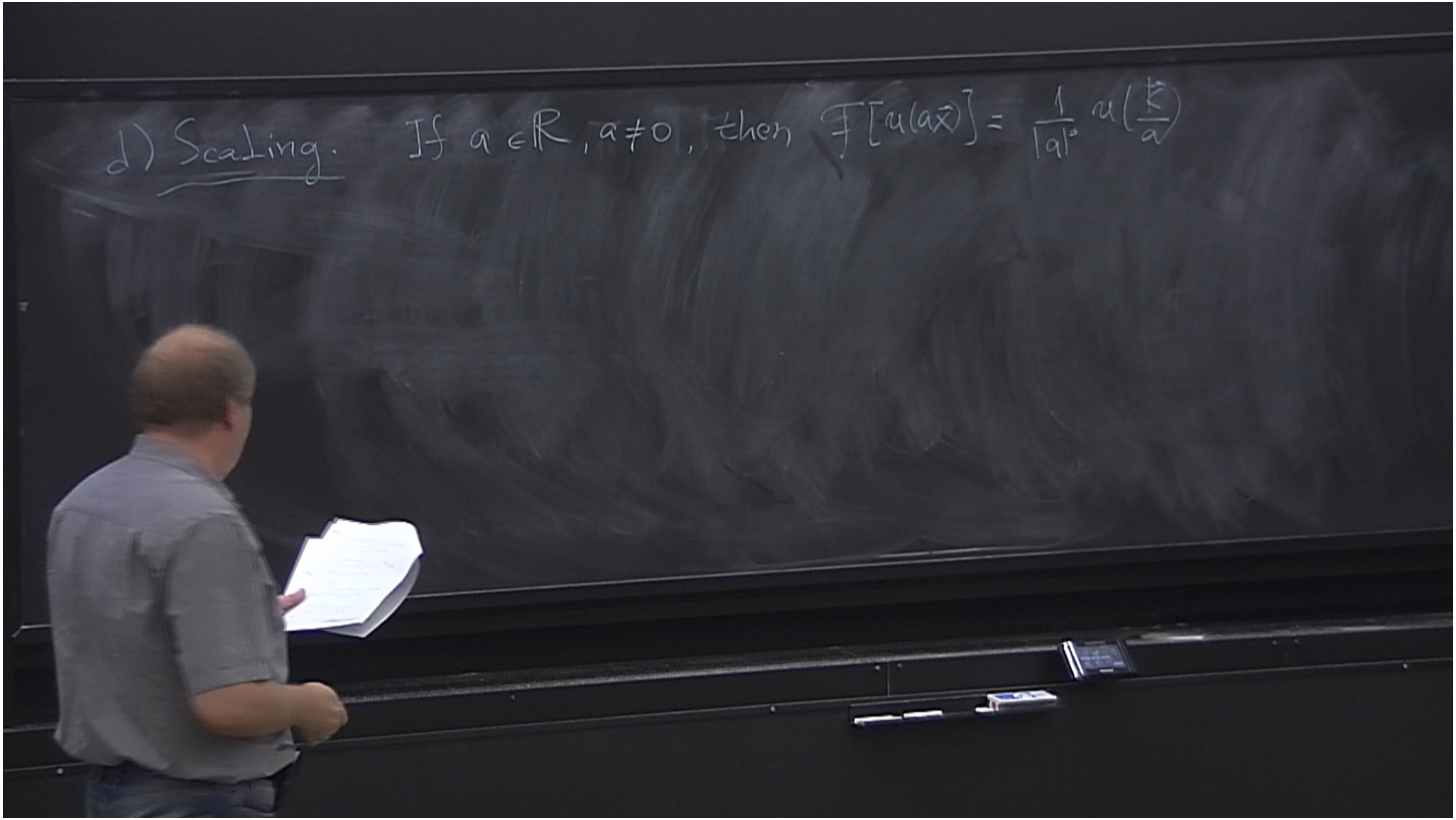

$\int_{-\infty}^{+\infty} e^{-ax^2+bx}\,dx = \int_{-\infty}^{+\infty} e^{-a\left[\underbrace{(x-b/2a)}_{y}{}^2 - b^2/4a^2\right]}\,dx = e^{b^2/4a}\int_{-\infty}^{+\infty} e^{-ay^2}dy = e^{b^2/4a}\sqrt{\frac{\pi}{a}}$
$a=\frac{1}{2}$, $b=ik_i$
$I(k_i) = e^{-k_i^2/2}\sqrt{2\pi} \Rightarrow \mathcal{F}\left[e^{-|x|^2/2}\right] = (2\pi)^{d/2} e^{-k^2/2} \Rightarrow \mathcal{F}^{-1}\left[e^{-|k|^2/2}(2\pi)^{d/2}\right] = e^{-|x|^2/2}$


$a=\frac{1}{2}$, $\beta=ik_1$
$I(k_1)=e^{-k_1^2/2}\sqrt{2\pi}$
$\Rightarrow \mathcal{F}[e^{-x^2/2}]=(2\pi)^{1/2}e^{-k^2/2}$
$\Rightarrow \mathcal{F}^{-1}[e^{-ik_1x}(2\pi)^{1/2}]=e^{-x^2/2}$
Basic properties of Fourier transform:


$a=\frac{1}{2}$, $b=ik$
$\mathcal{F}[e^{-|x|/2}] = (2\pi)^{1/2} e^{-k^2/2} \Rightarrow \mathcal{F}[e^{-|x|/2}(2\pi)^{1/2}] = e^{-k^2/2}$
Basic properties of Fourier transform:
a) Linearity For $z(x) = au(x) + bv(x)$, $w(k) = a\mathcal{u}(k) + b\mathcal{v}(k)$
b)

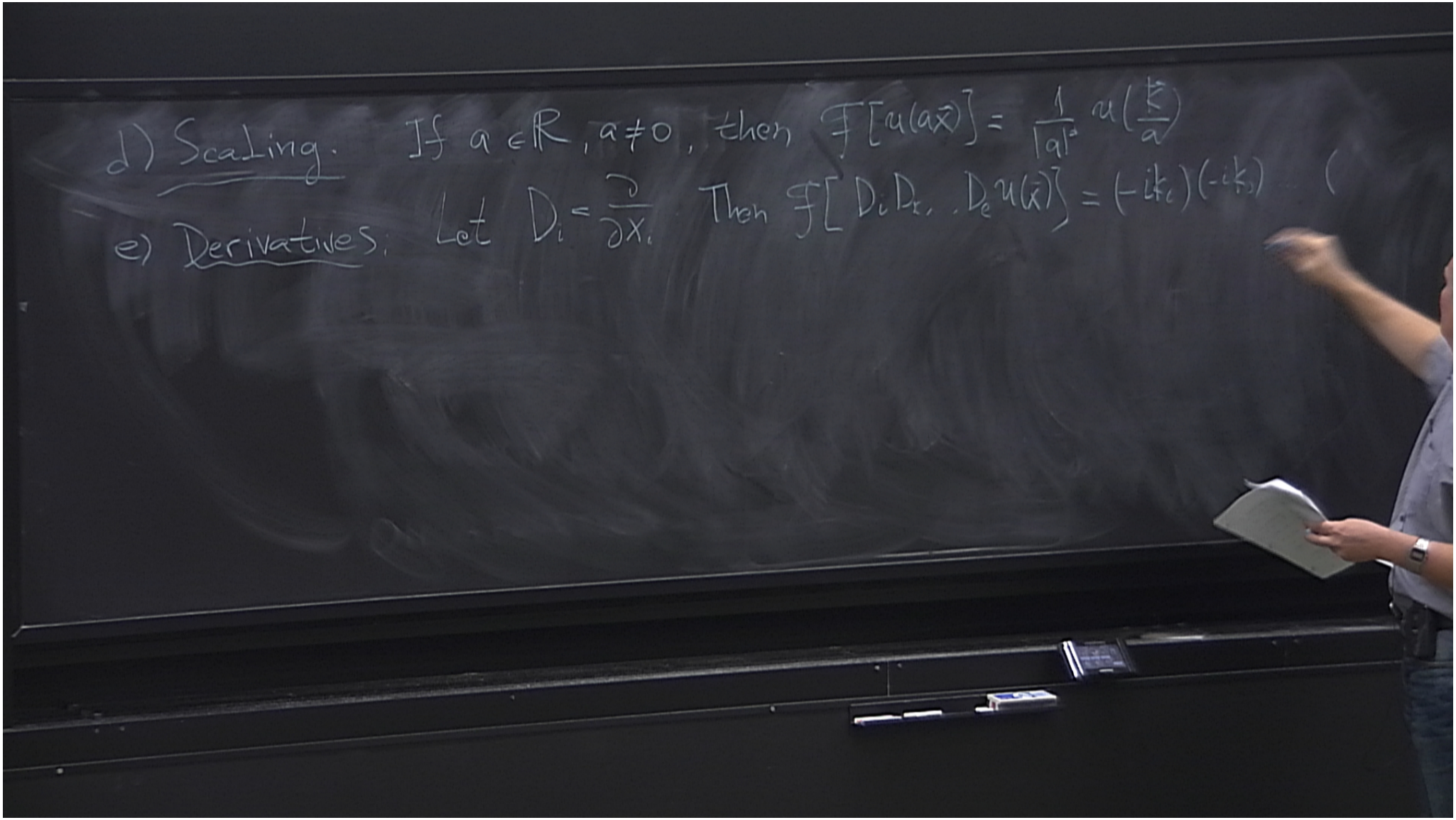

Basic properties of Fourier transform
a) Linearity. For $w(\vec{x}) = a u(\vec{x}) + b v(\vec{x})$, $w(\vec{k}) = a u(\vec{k}) + b v(\vec{k})$
b) Translation If for any $\vec{x}_0$ $v(\vec{x}) = u(\vec{x} - \vec{x}_0)$, then $v(\vec{k}) = e^{i\vec{k}\cdot\vec{x}_0} u(\vec{k})$
c) Modulation: If for any $\vec{k}_0$ $v(\vec{x}) = e^{i\vec{k}_0\cdot\vec{x}} u(\vec{x})$, then $v(\vec{k}) = u(\vec{k} + \vec{k}_0)$

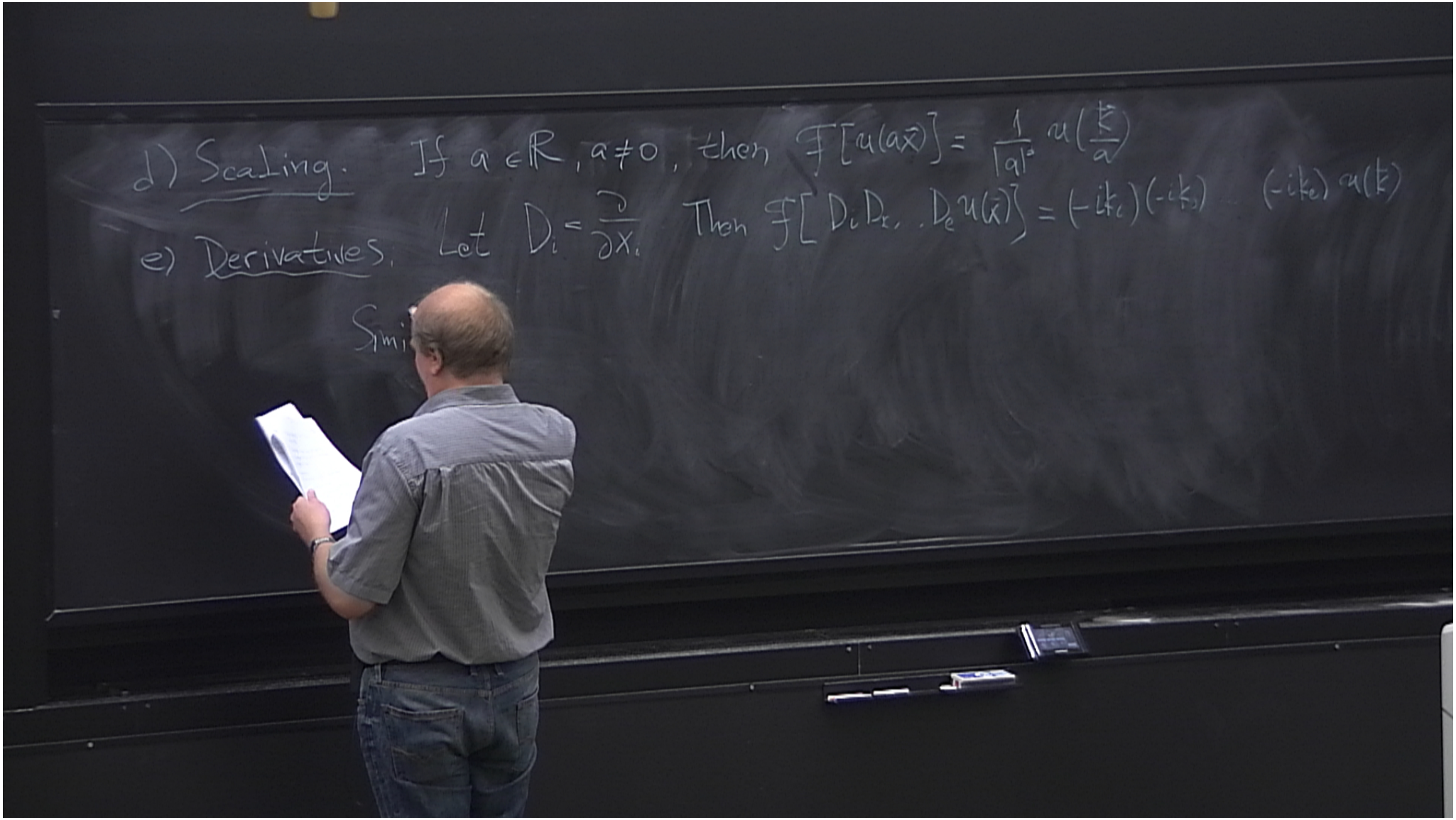

d) Scaling. If $a \in \mathbb{R}$, $a \neq 0$, then $\mathcal{F}[u(ax)] =$

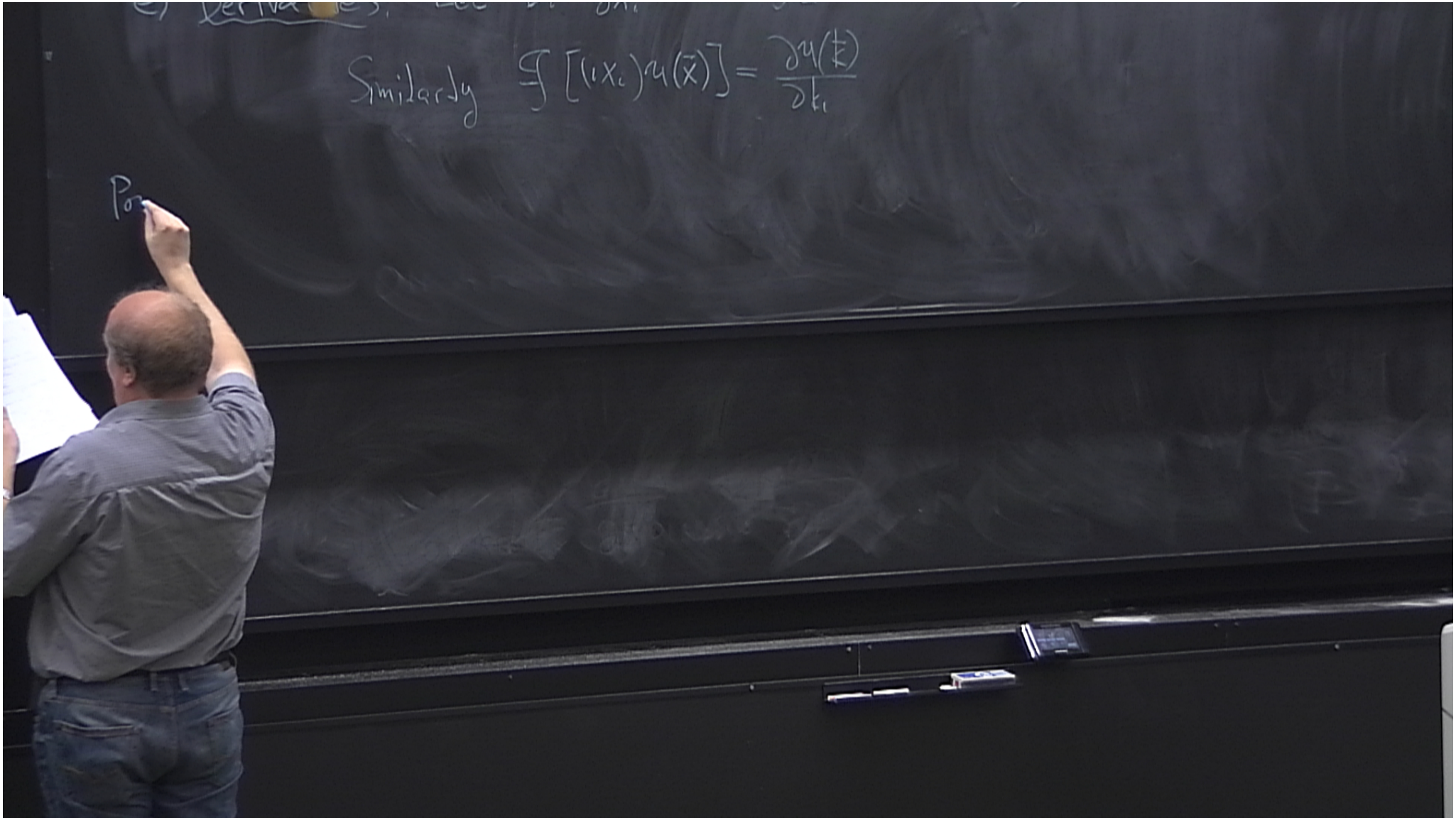

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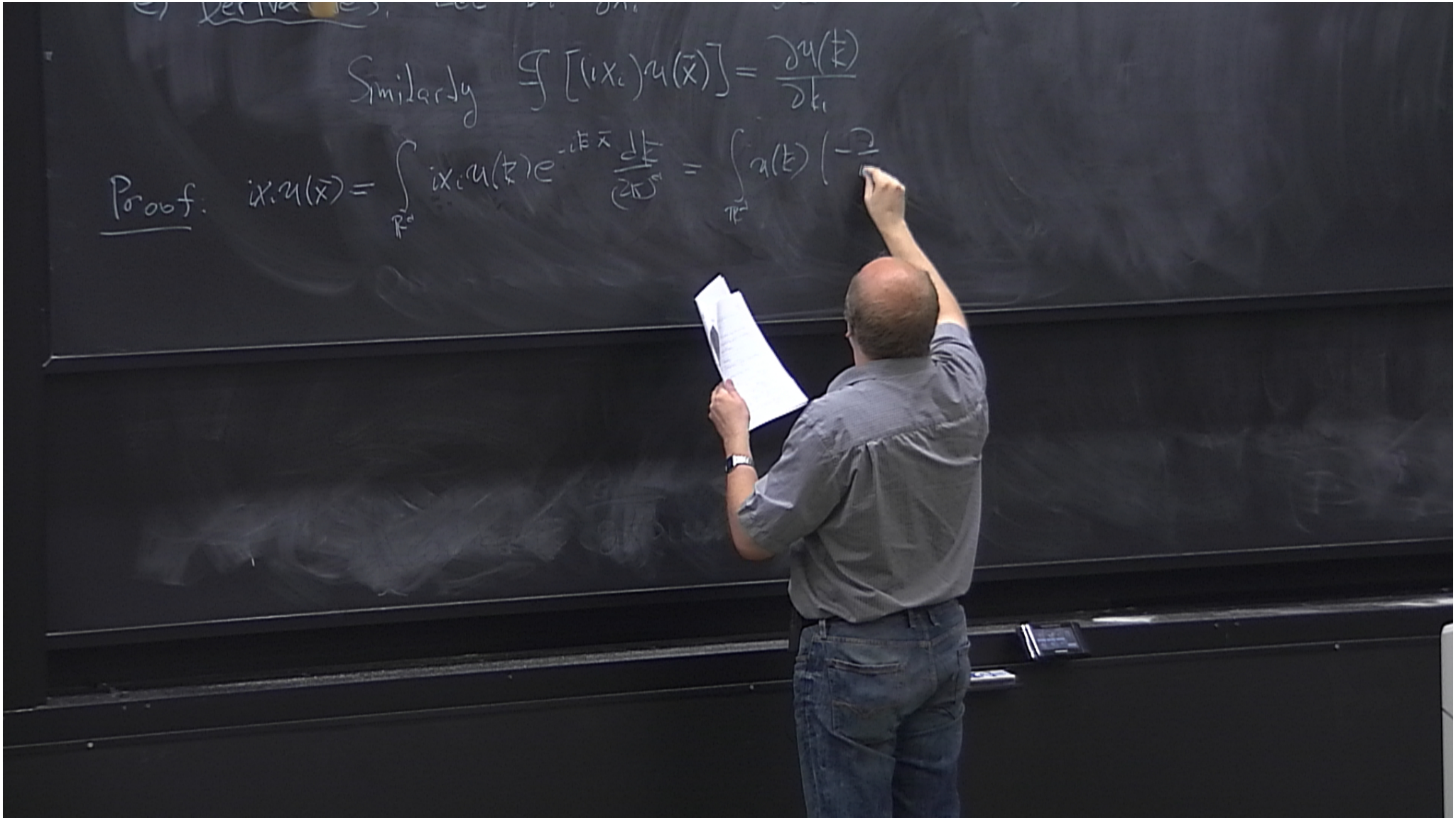
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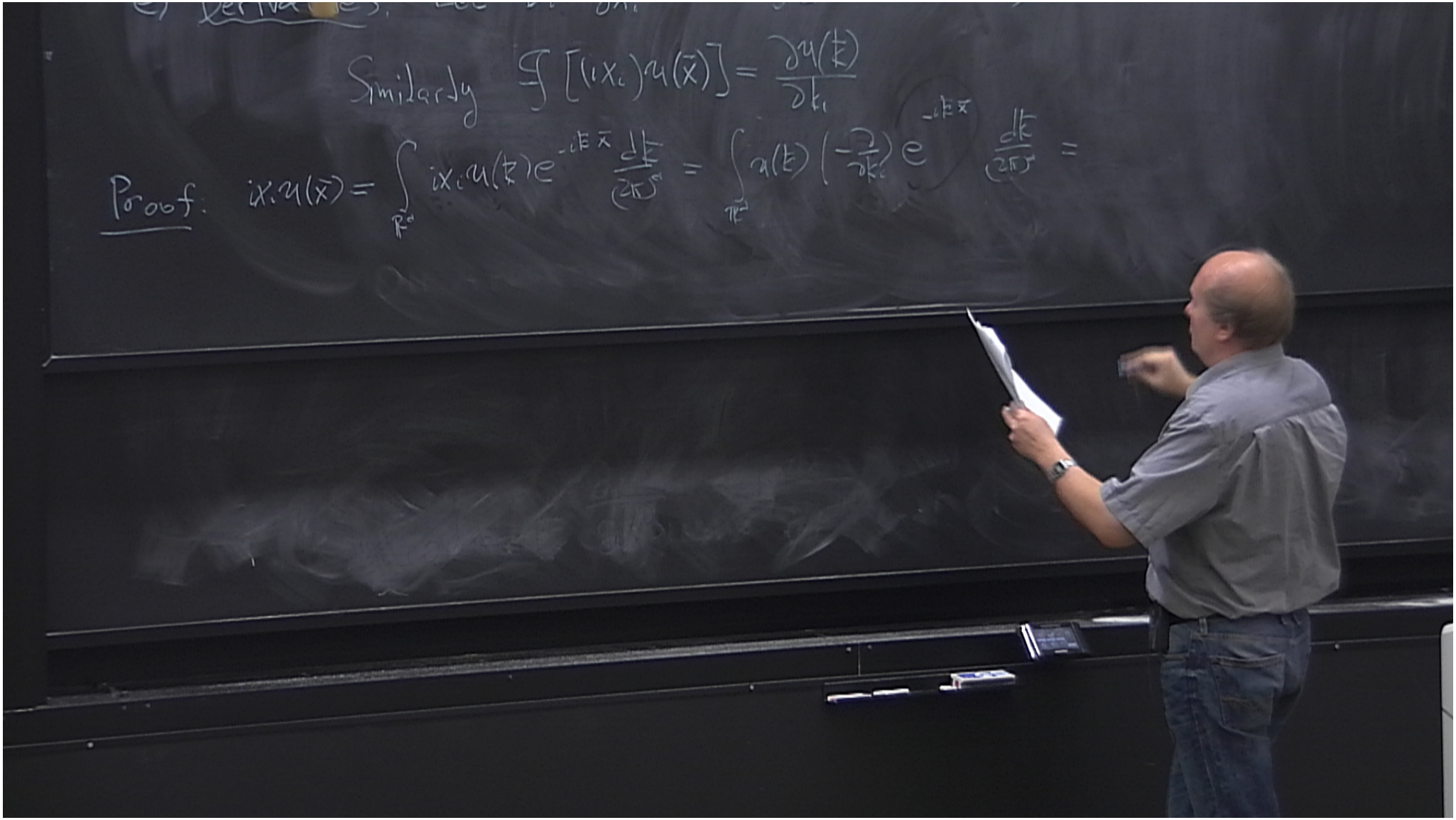
e) Derivatives. Let $D_i = \frac{\partial}{\partial x_i}$ Then $\mathcal{F}[D_i D_{x_\cdot} \cdots D_{\cdot} u(\vec{x})] = (-ik_i)(-ik_\cdot) \cdots$ (

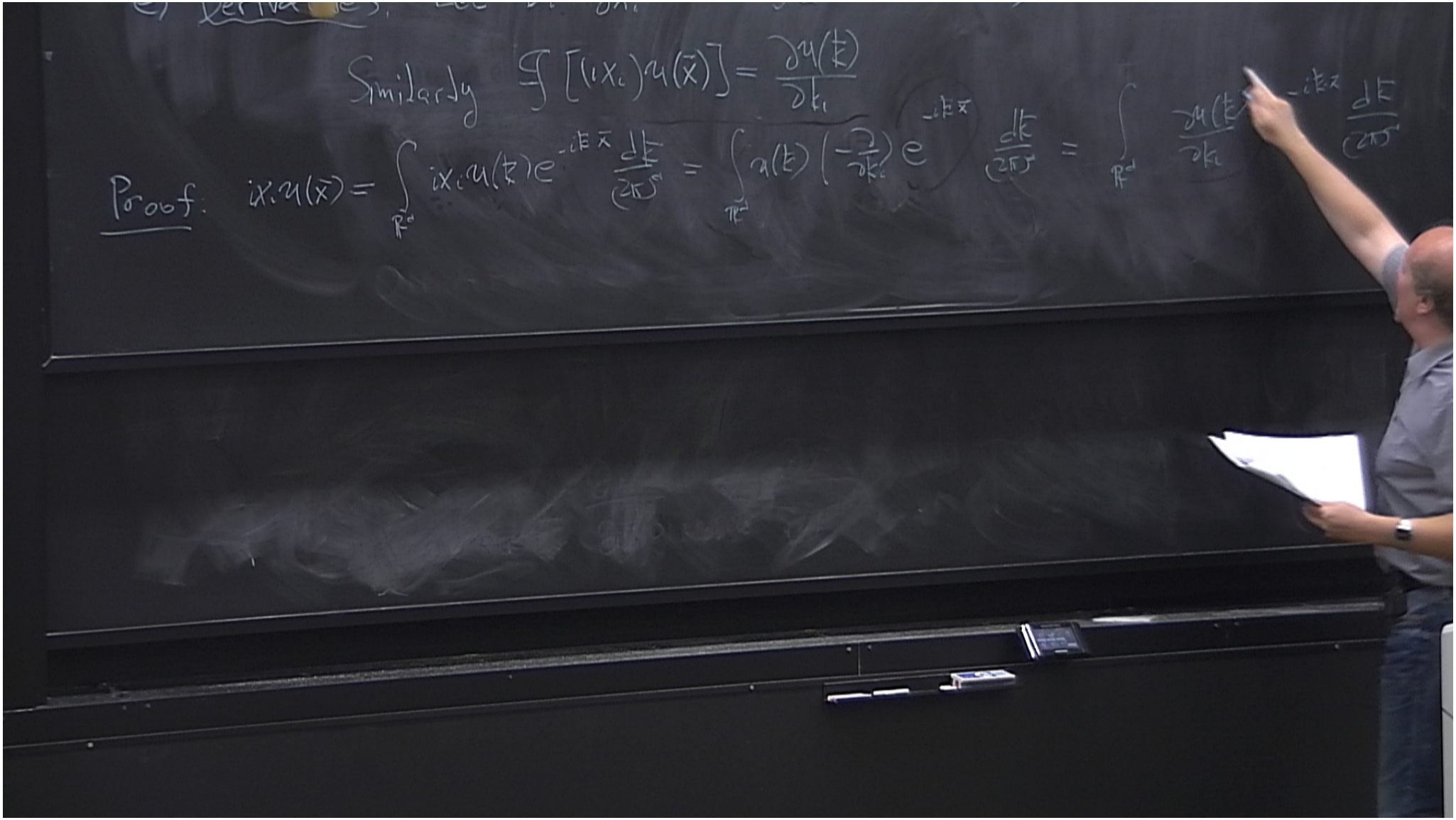


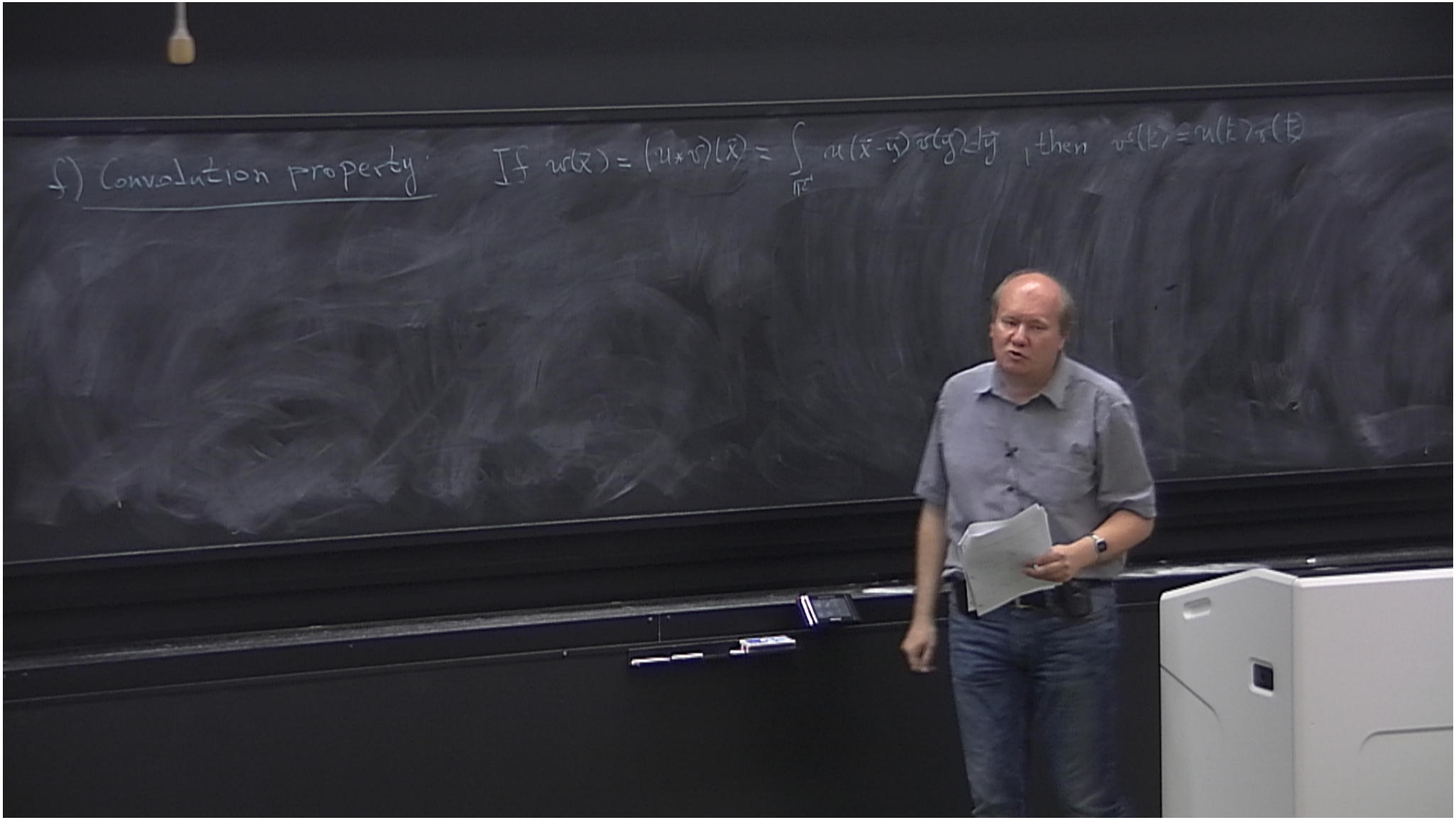

d) Scaling. If $a \in \mathbb{R}$, $a \neq 0$, then $\mathcal{F}[u(ax)] = \frac{1}{|a|^n} u\left(\frac{k}{a}\right)$
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Simi

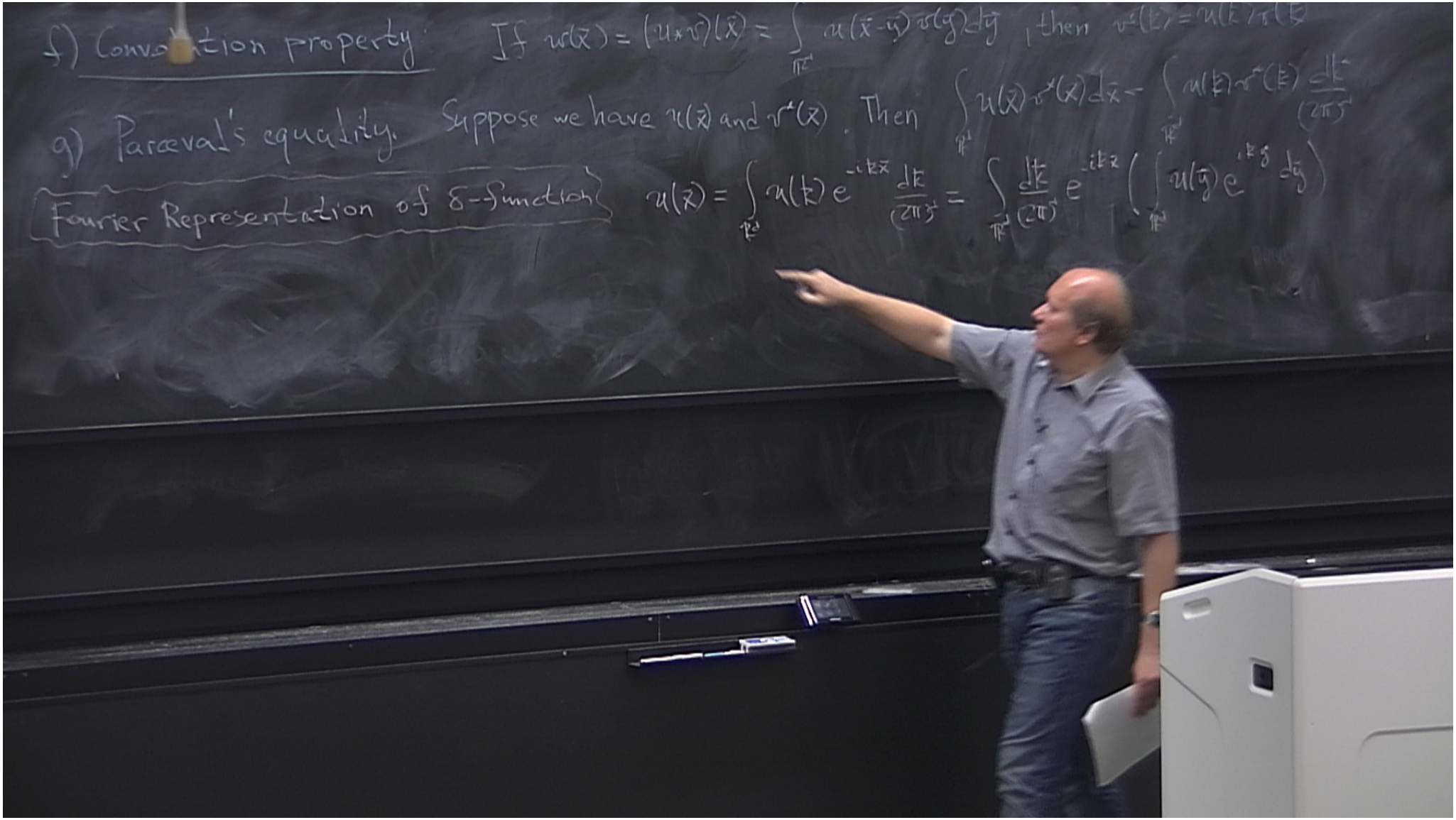

Similarly

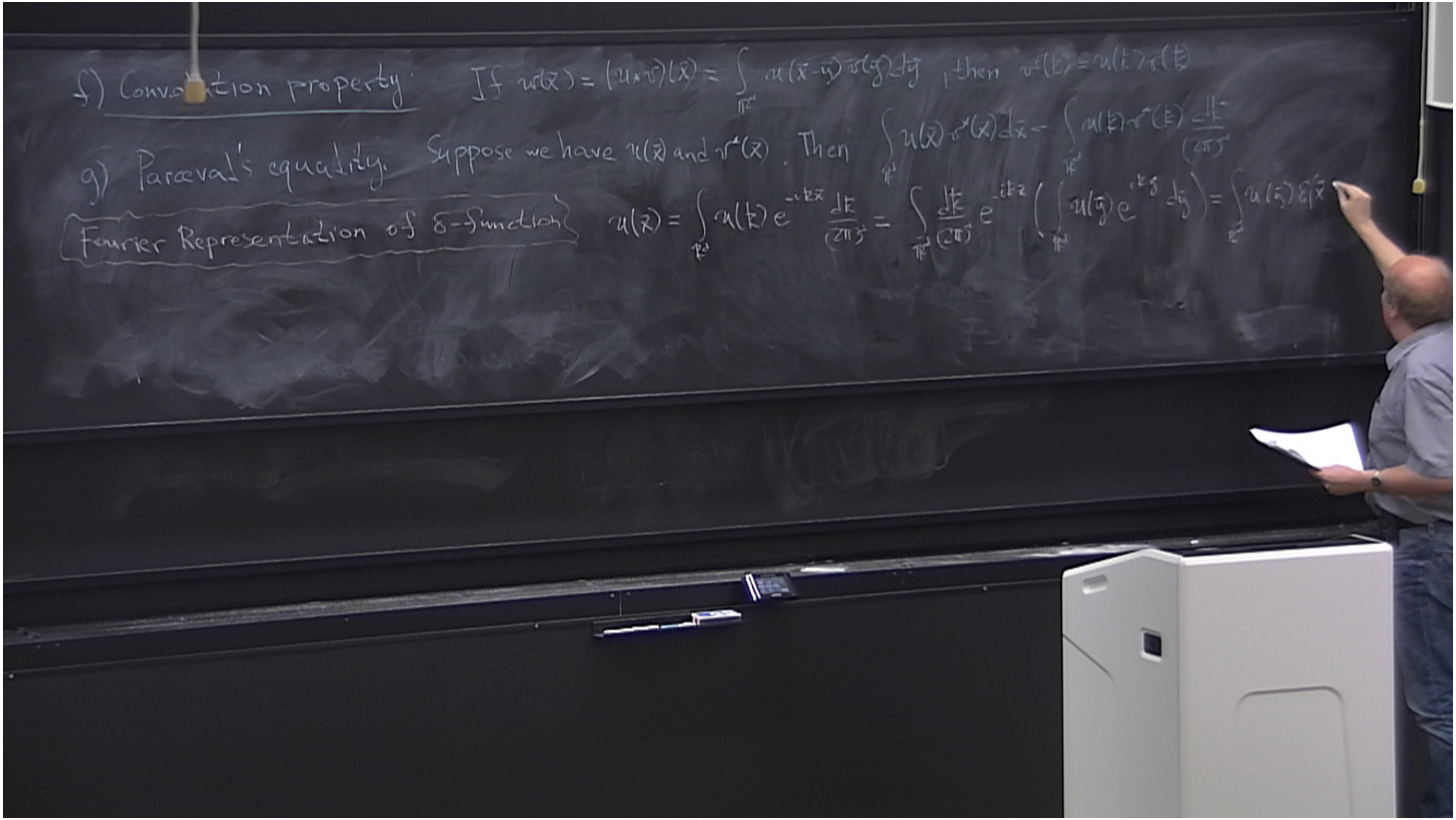

Similarly $\mathcal{F}[(ix_i)u(x)] = \frac{\partial \hat{u}(k)}{\partial k_i}$
Proof: $ix_i u(x) = \int_{\mathbb{R}^n} ix_i \hat{u}(k) e^{-ikx} \frac{dk}{(2\pi)^n} = \int_{\mathbb{R}^n} \hat{u}(k) \left(-\frac{\partial}{\partial}\right.$

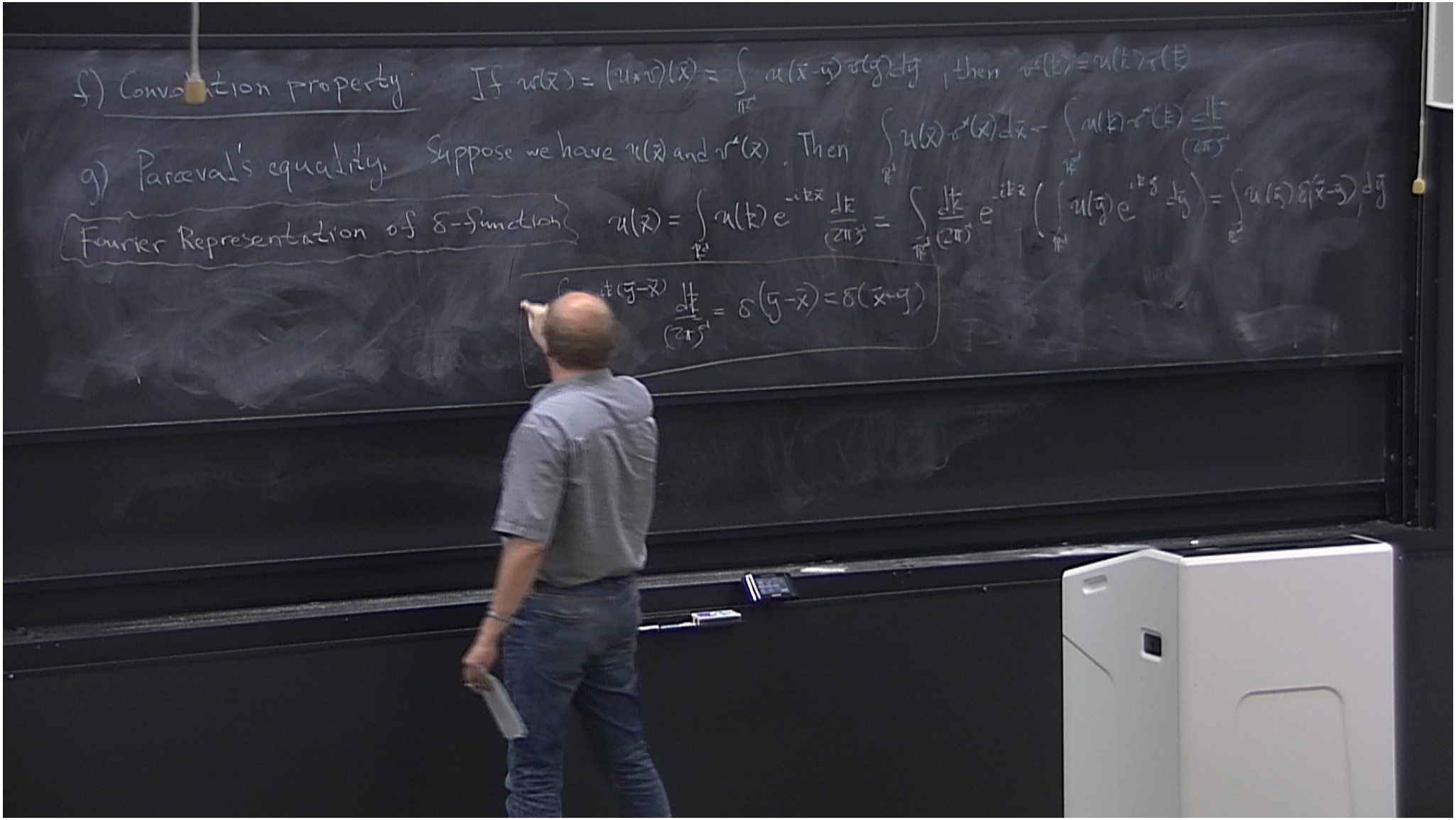

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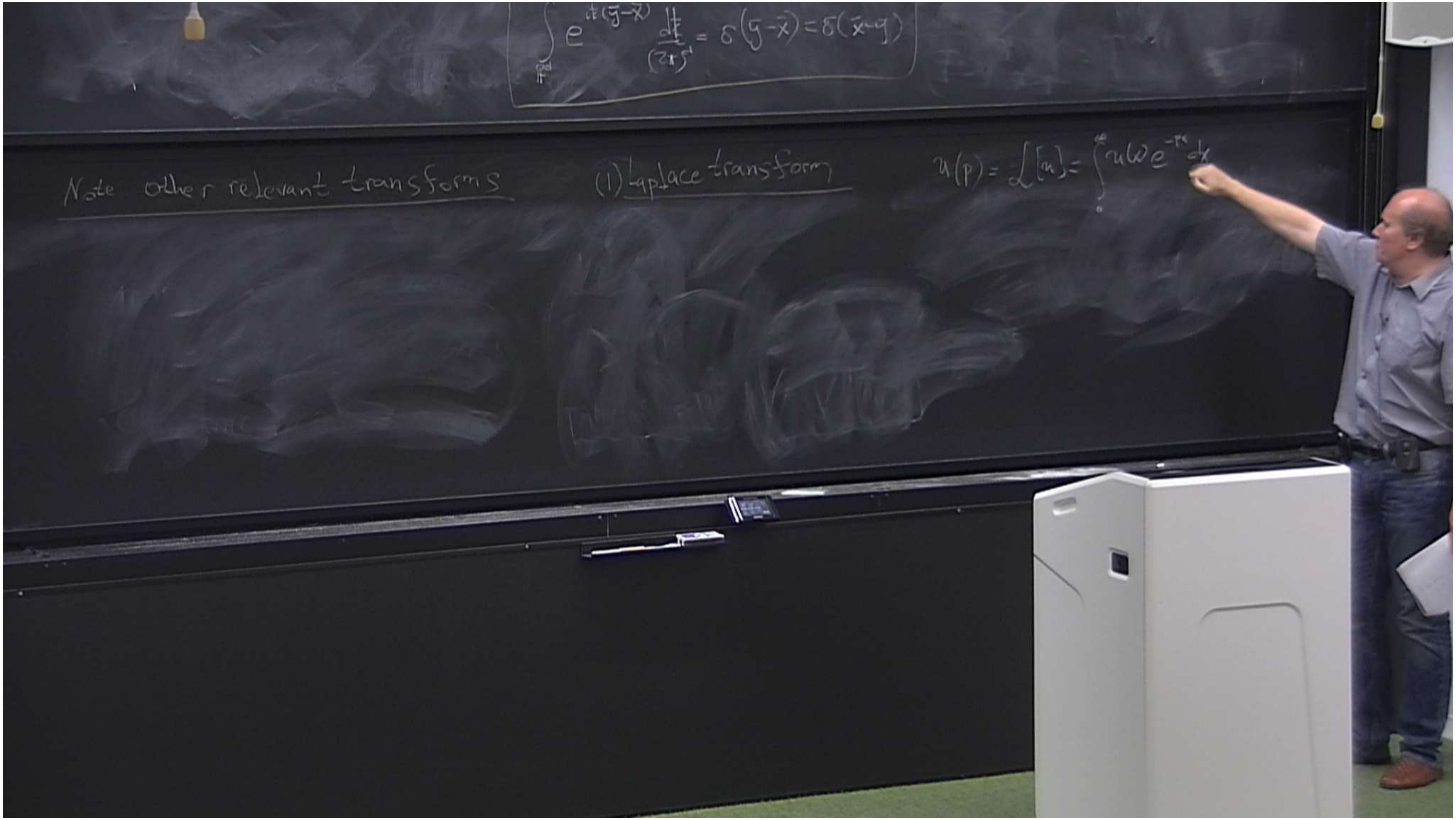

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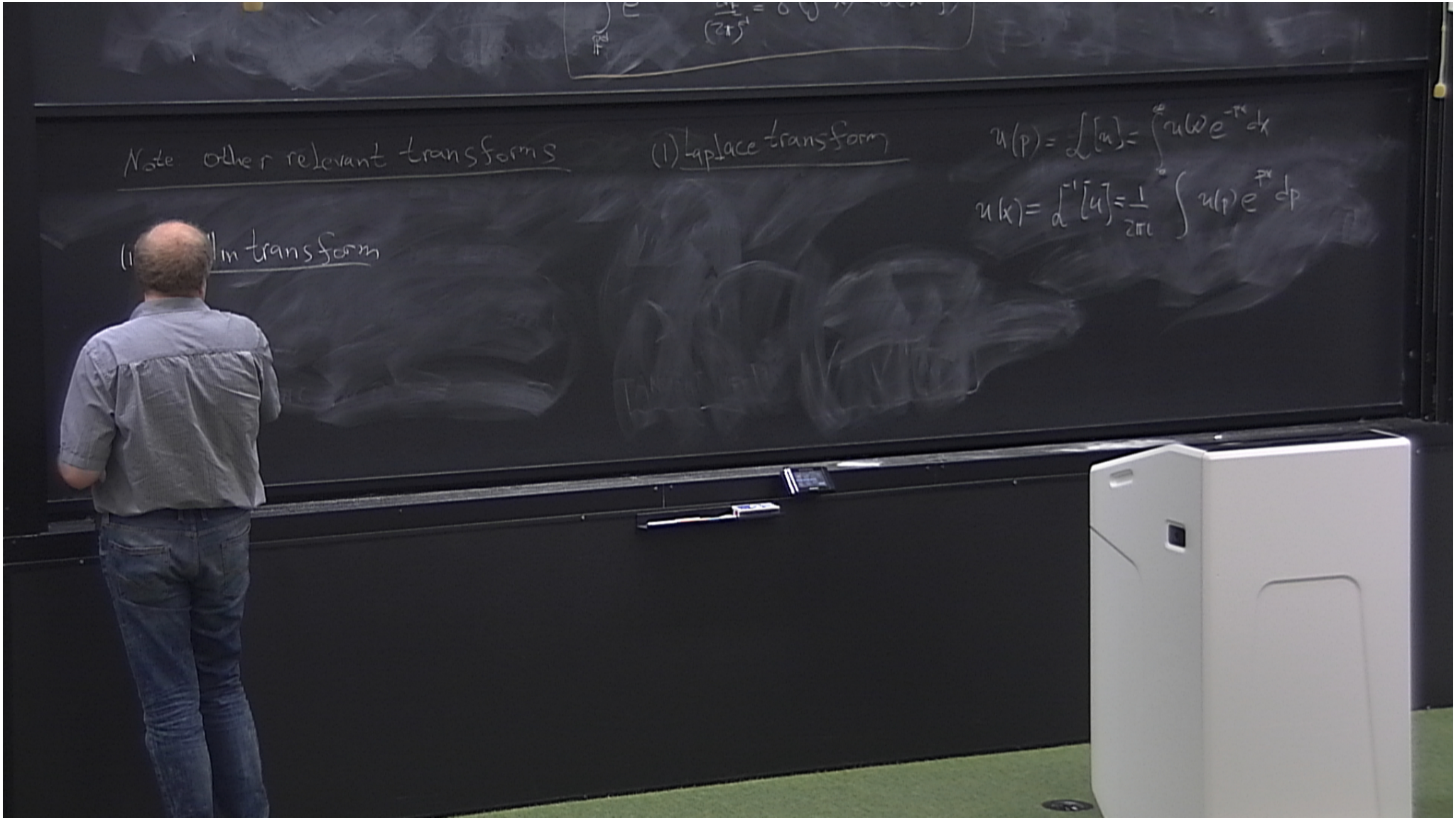

f) Convolution property
If $w(\vec{x}) = (u * v)(\vec{x}) = \int_{\mathbb{R}^n} u(\vec{x}-\vec{y})\, v(\vec{y})\, d\vec{y}$, then $\hat{w}(\vec{k}) = \hat{u}(\vec{k})\,\hat{v}(\vec{k})$

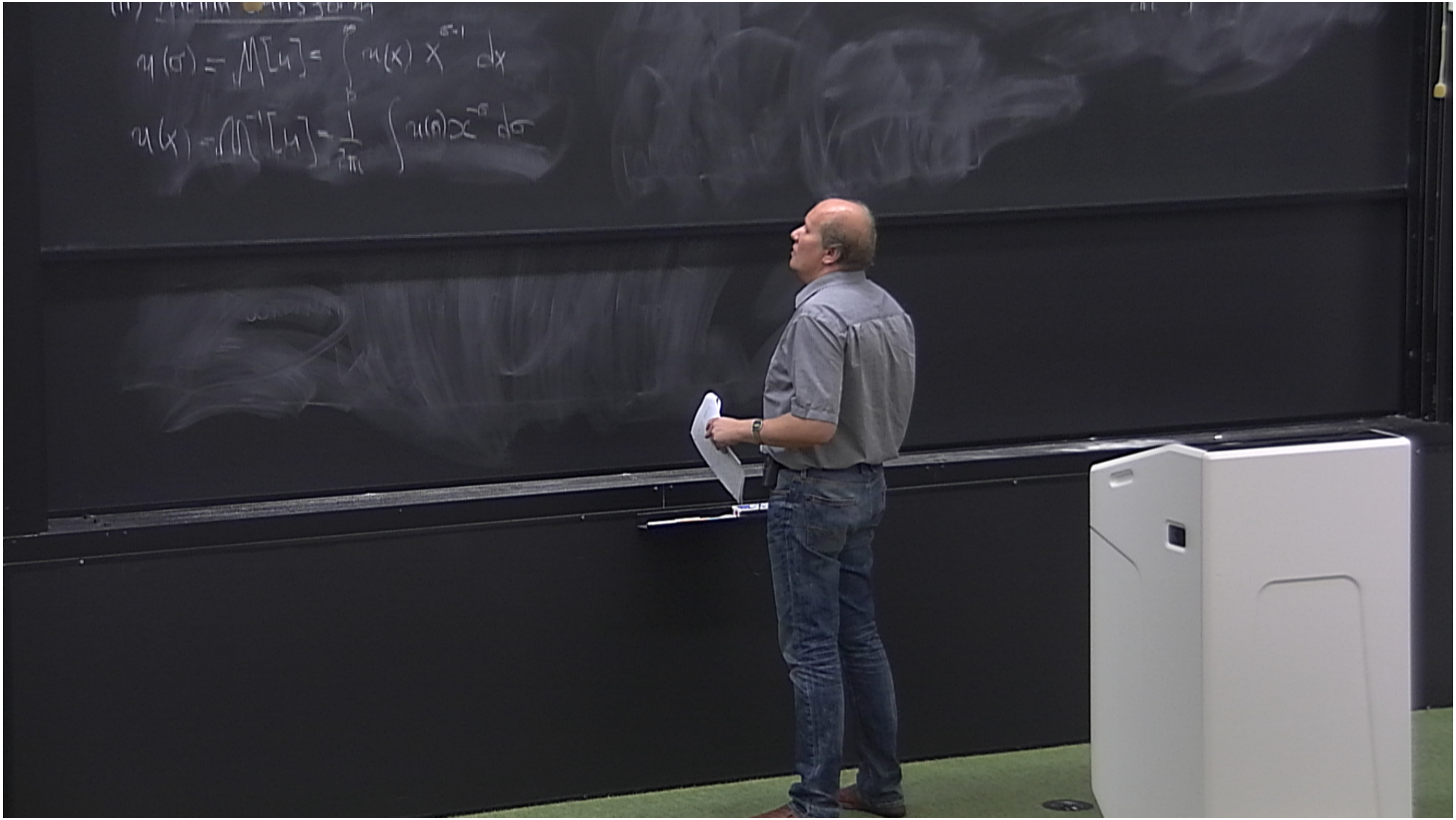

f) Convolution property
If $w(\vec{x}) = (u * v)(\vec{x}) = \int_{\mathbb{R}^d} u(\vec{x}-\vec{y})\, v(\vec{y})\, d\vec{y}$, then
g) Parseval's equality. Suppose we have $u(\vec{x})$ and $v^*(\vec{x})$. Then
Fourier Representation of δ-function
$u(\vec{x}) = \int_{\mathbb{R}^d} u(\vec{k})\, e^{-i\vec{k}\vec{x}}\, \frac{d\vec{k}}{(2\pi)^d} = \int_{\mathbb{R}^d} \frac{d\vec{k}}{(2\pi)^d}\, e^{-i\vec{k}\vec{x}} \left(\int_{\mathbb{R}^d} u(\vec{y})\, e^{i\vec{k}\vec{y}}\, d\vec{y} \right)$

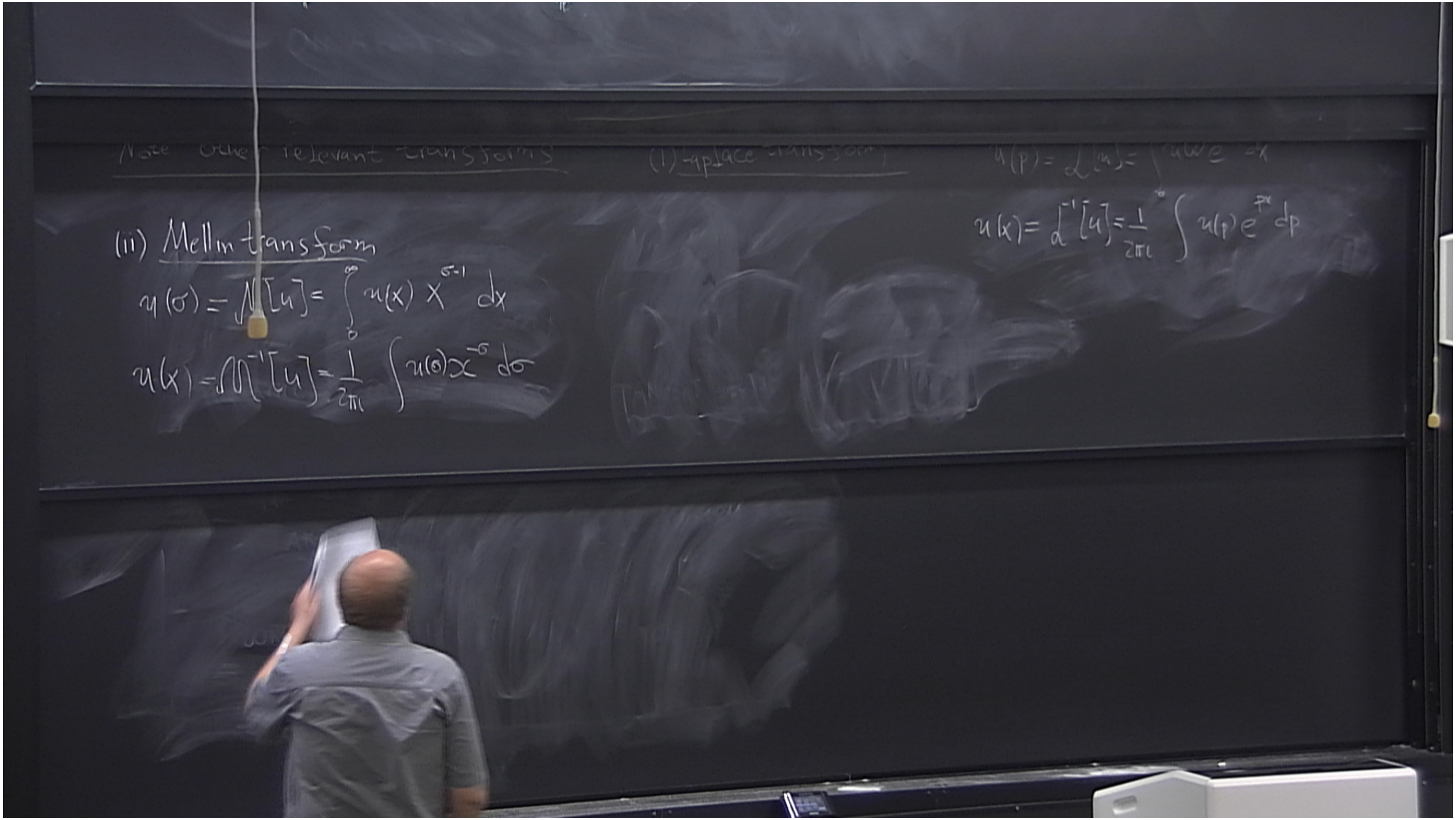

f) Convolution property
If $w(\vec{x}) = (u * v)(\vec{x}) = \int_{\mathbb{R}^3} u(\vec{x}-\vec{y})\, v(\vec{y})\, d\vec{y}$, then $w^F(\vec{k}) = u^F(\vec{k})\, v^F(\vec{k})$
g) Parseval's equality. Suppose we have $u(\vec{x})$ and $v^*(\vec{x})$. Then $\int_{\mathbb{R}^3} u(\vec{x})\, v^*(\vec{x})\, d\vec{x} = \int_{\mathbb{R}^3} u^F(\vec{k})\, v^{F*}(\vec{k})\, \frac{d\vec{k}}{(2\pi)^3}$
Fourier Representation of δ-function
$u(\vec{x}) = \int_{\mathbb{R}^3} u^F(\vec{k})\, e^{-i\vec{k}\vec{x}}\, \frac{d\vec{k}}{(2\pi)^3} = \int_{\mathbb{R}^3} \frac{d\vec{k}}{(2\pi)^3}\, e^{-i\vec{k}\vec{x}} \left(\int_{\mathbb{R}^3} u(\vec{y})\, e^{i\vec{k}\vec{y}}\, d\vec{y} \right) = \int_{\mathbb{R}^3} u(\vec{y})\, \delta(\vec{x}$


f) Convolution property
If $w(\vec{x}) = (u*v)(\vec{x}) = \int_{\mathbb{R}^d} u(\vec{x}-\vec{y})\, v(\vec{y})\, d\vec{y}$, then $\hat{w}(\vec{k}) = \hat{u}(\vec{k})\,\hat{v}(\vec{k})$
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Fourier Representation of δ-function $u(\vec{x}) = \int_{\mathbb{R}^d} \hat{u}(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}\, \frac{d\vec{k}}{(2\pi)^d} = \int_{\mathbb{R}^d} \frac{d\vec{k}}{(2\pi)^d}\, e^{-i\vec{k}\cdot\vec{x}} \left(\int_{\mathbb{R}^d} u(\vec{y})\, e^{i\vec{k}\cdot\vec{y}}\, d\vec{y} \right) = \int_{\mathbb{R}^d} u(\vec{y})\, \delta(\vec{x}-\vec{y})\, d\vec{y}$
$\int e^{i\vec{k}\cdot(\vec{y}-\vec{x})}\, \frac{d\vec{k}}{(2\pi)^d} = \delta(\vec{y}-\vec{x}) = \delta(\vec{x}-\vec{y})$


$\int e^{ik(\vec{y}-\vec{x})} \frac{d^n k}{(2\pi)^n} = \delta(\vec{y}-\vec{x}) = \delta(\vec{x}-\vec{y})$
Note other relevant transforms
(1) Laplace transform
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$\mathcal{M}[u] = \int_0^\infty u(x)\, x^{\sigma-1}\, dx$
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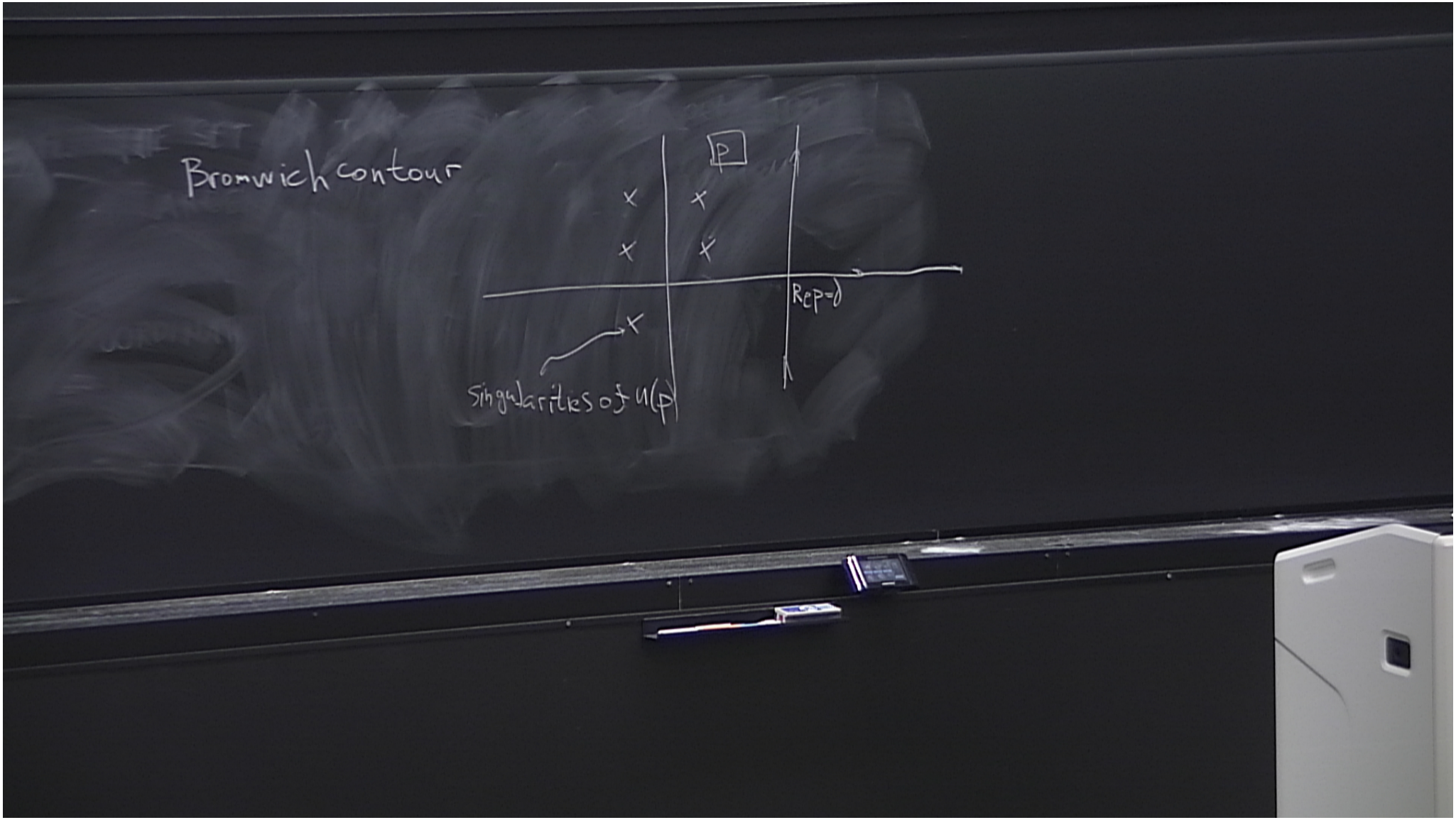

Note other relevant transforms
(ii) Mellin transform
$u(\sigma) = \mathcal{M}[u] = \int_0^\infty u(x)\, x^{\sigma-1}\, dx$
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(i) -place transform
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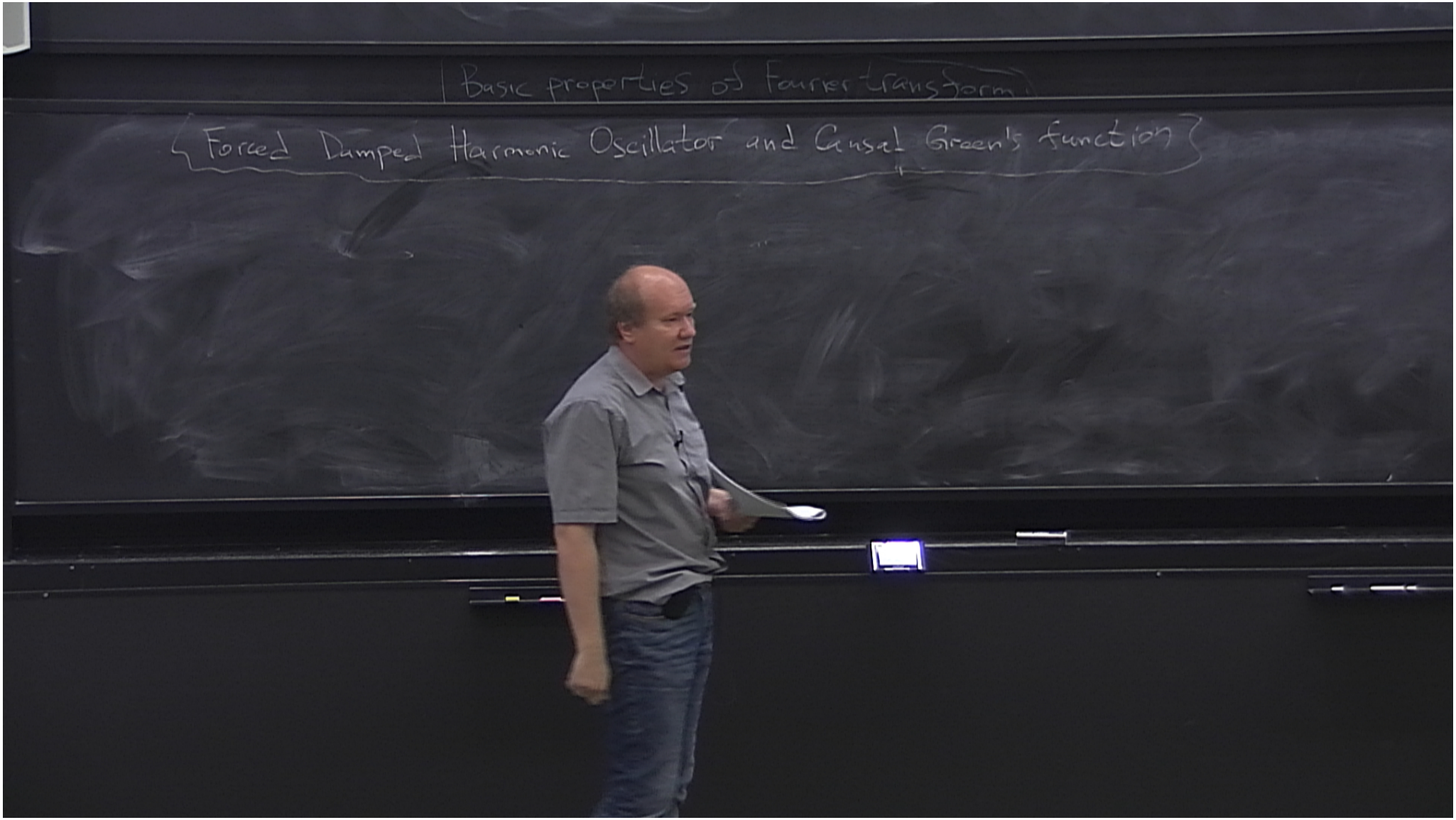
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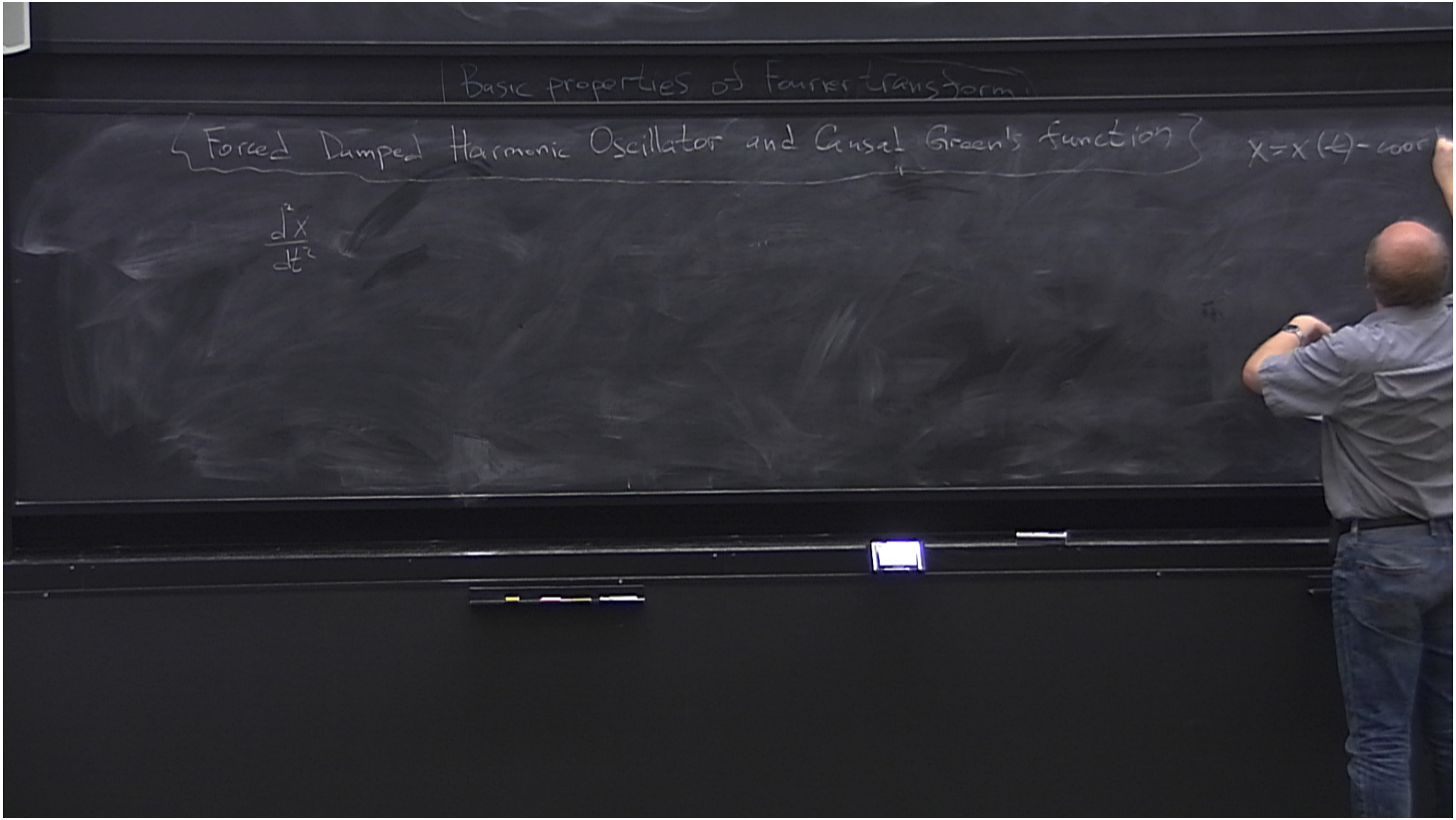
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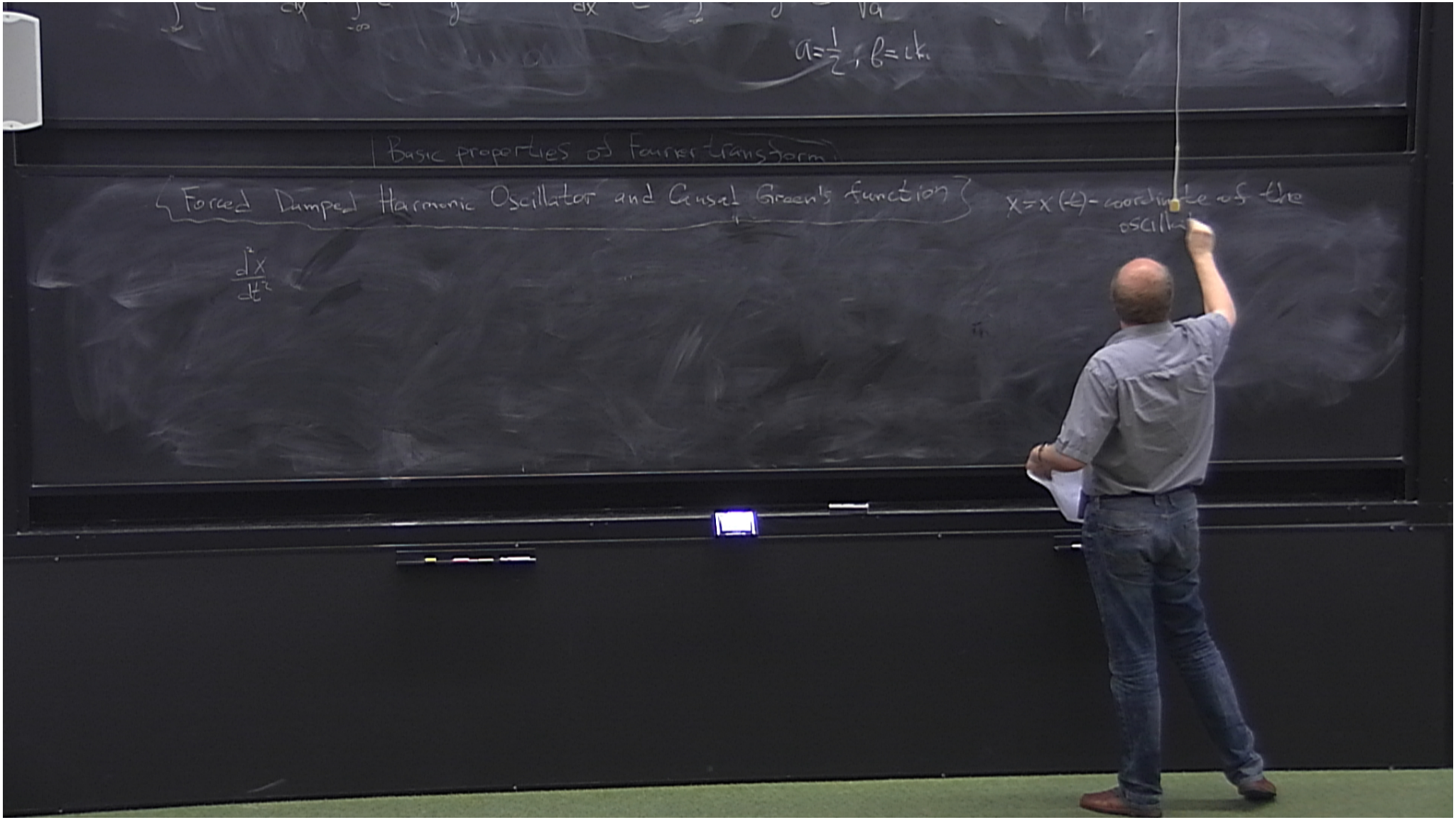
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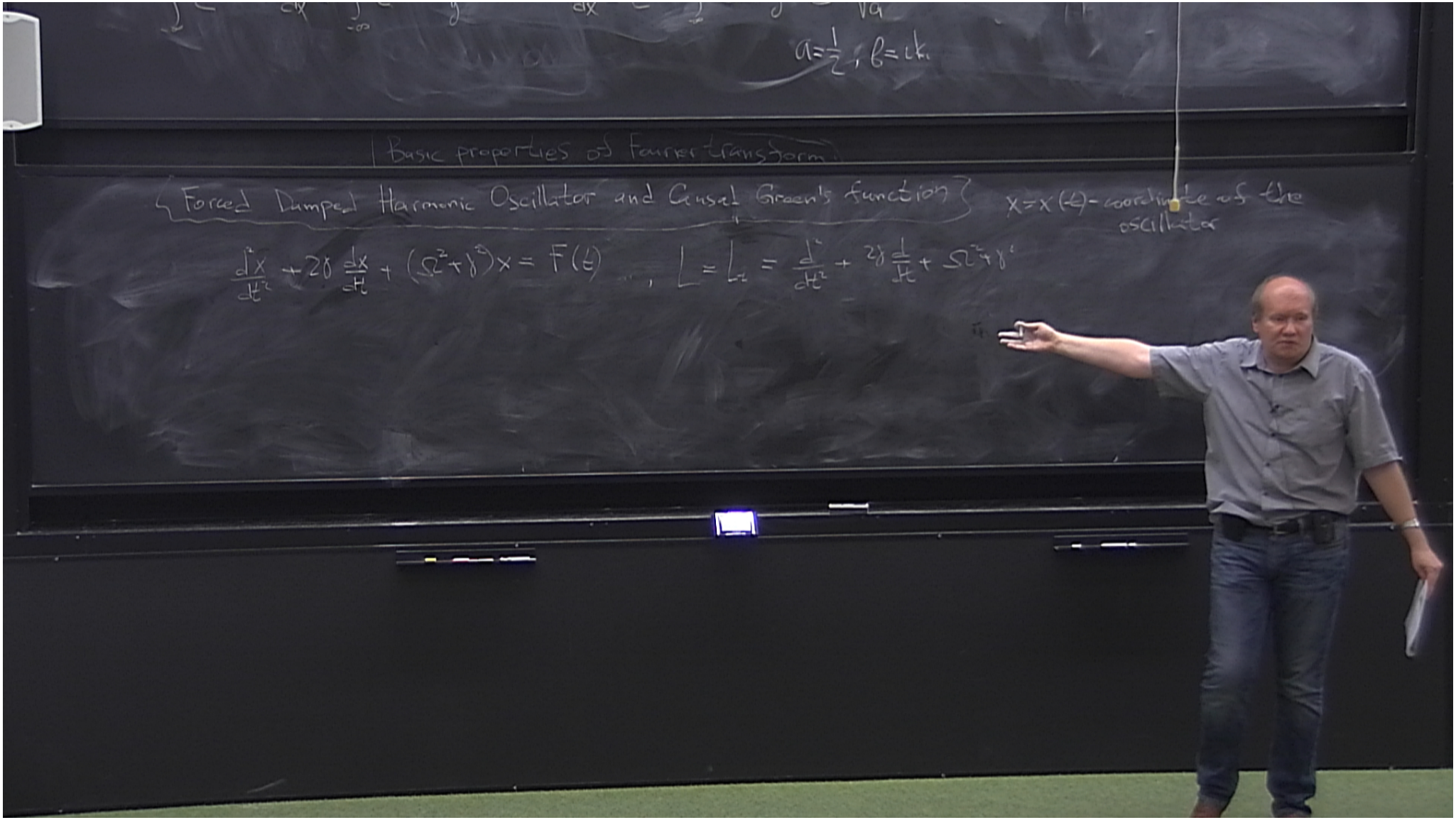
Bromwich contour

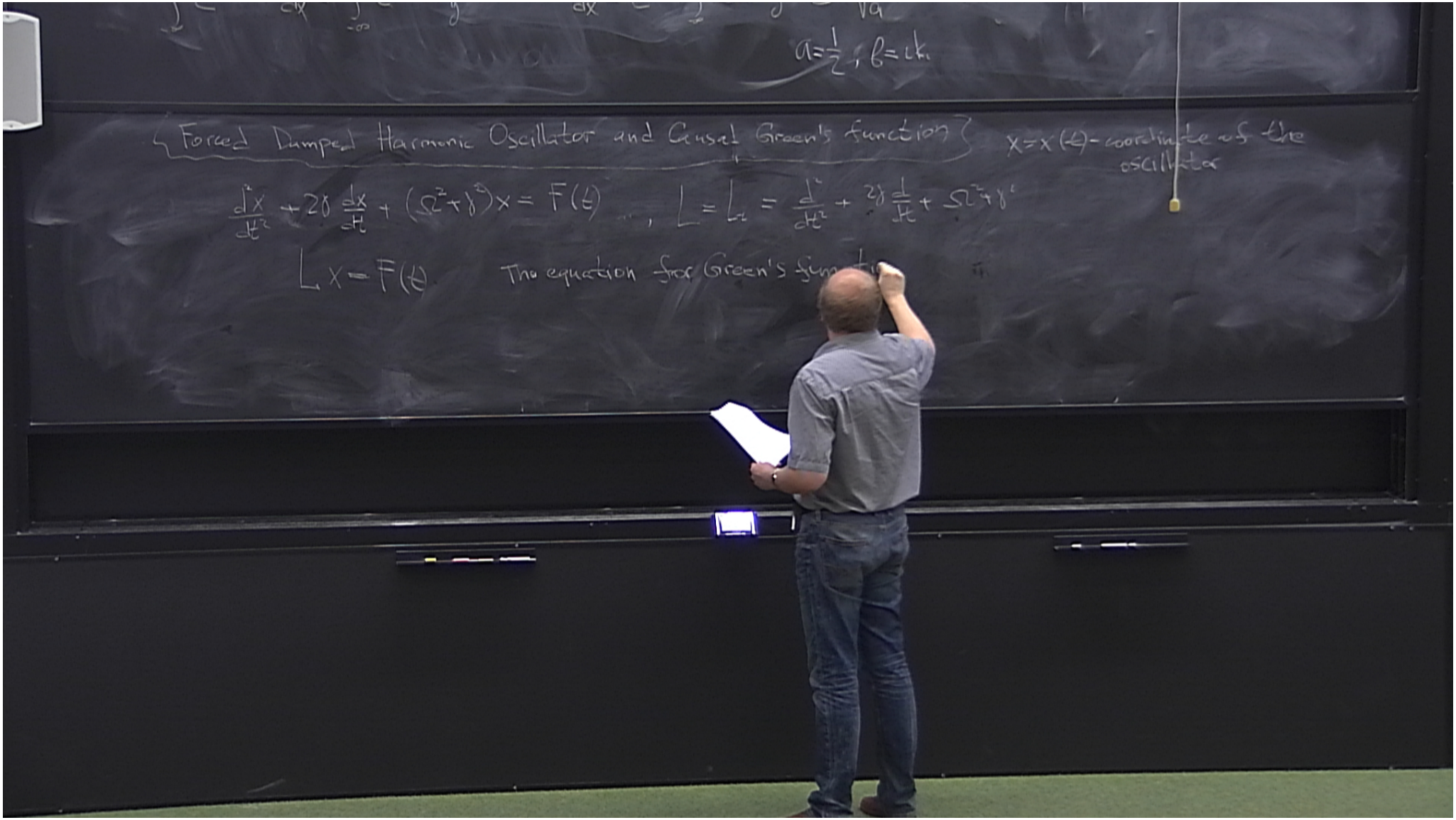

Bromwich contour
p
Re p = d
singularities of u(p)

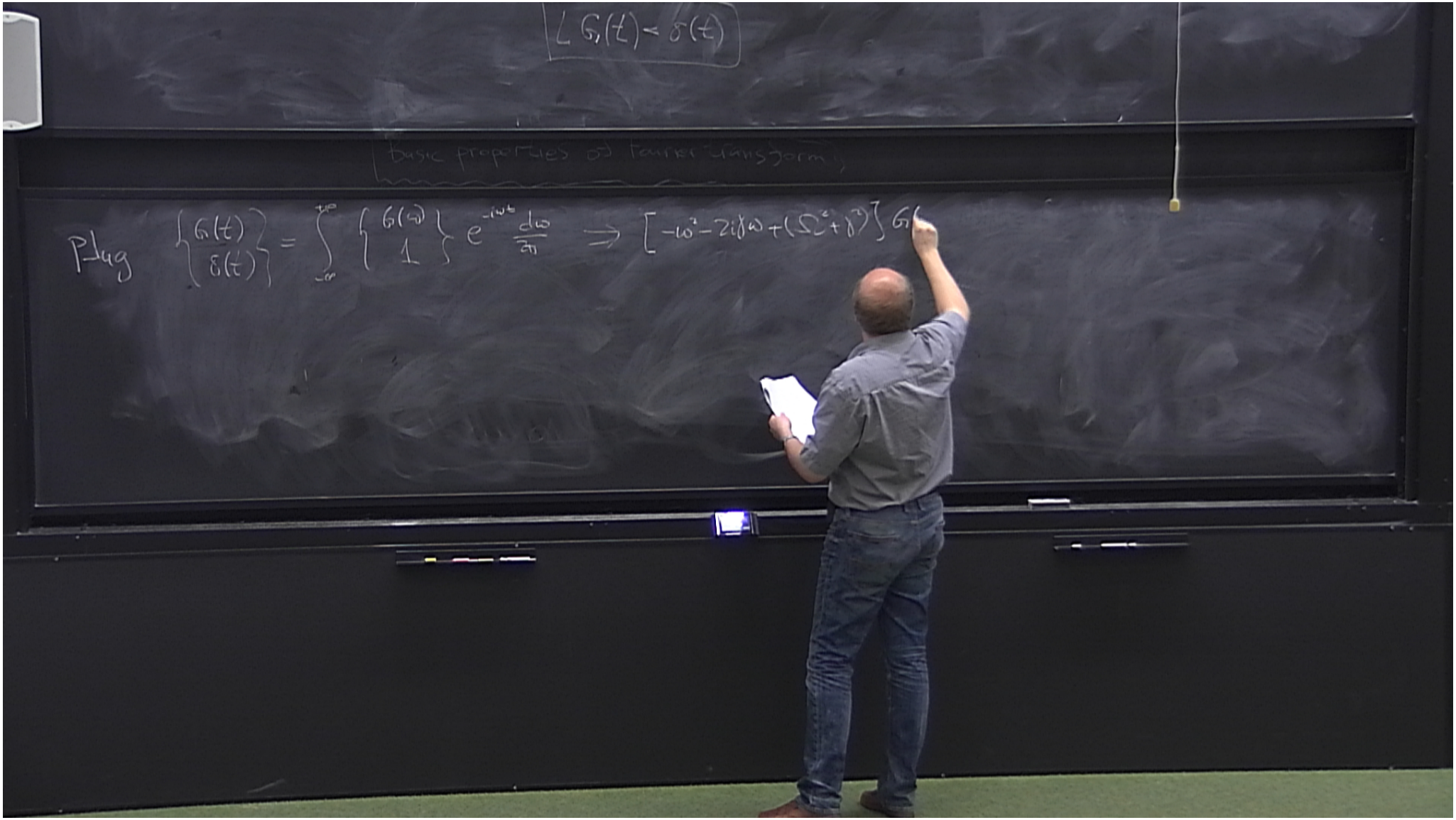

Basic properties of Fourier transform
Forced Damped Harmonic Oscillator and Causal Green's function

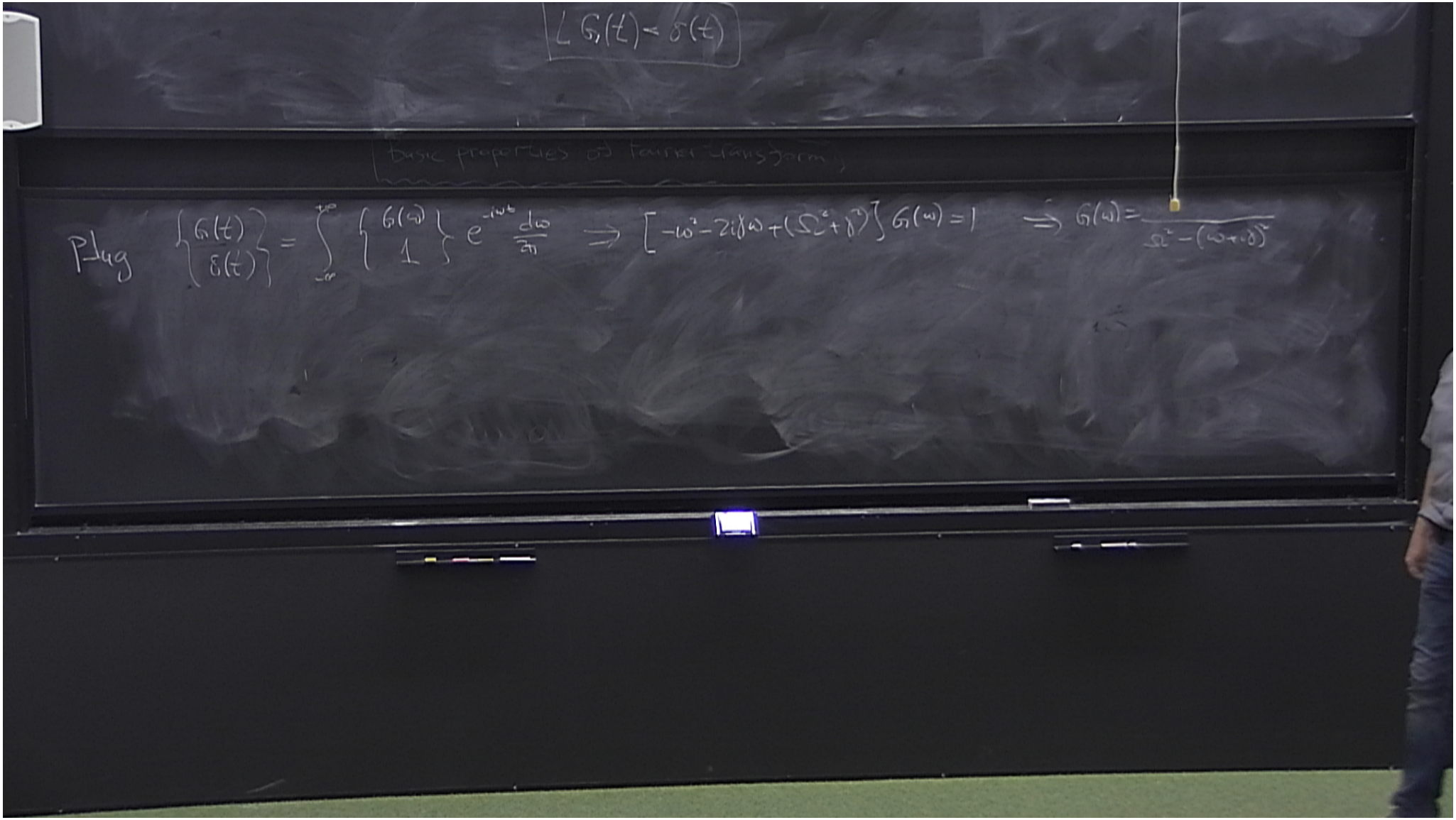

Basic properties of Fourier transform
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$\frac{d^2x}{dt^2}$


$a=\frac{1}{2}$, $\beta=$ ck.
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$x=x(t)$ - coordinate of the oscillator
$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+(\Omega^2+\gamma^2)x=F(t)$
$L=L_t=\frac{d^2}{dt^2}+2\gamma\frac{d}{dt}+\Omega^2+\gamma^2$

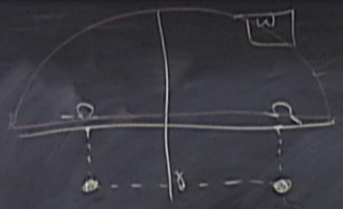

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$Lx=F(t)$
The equation for Green's fun

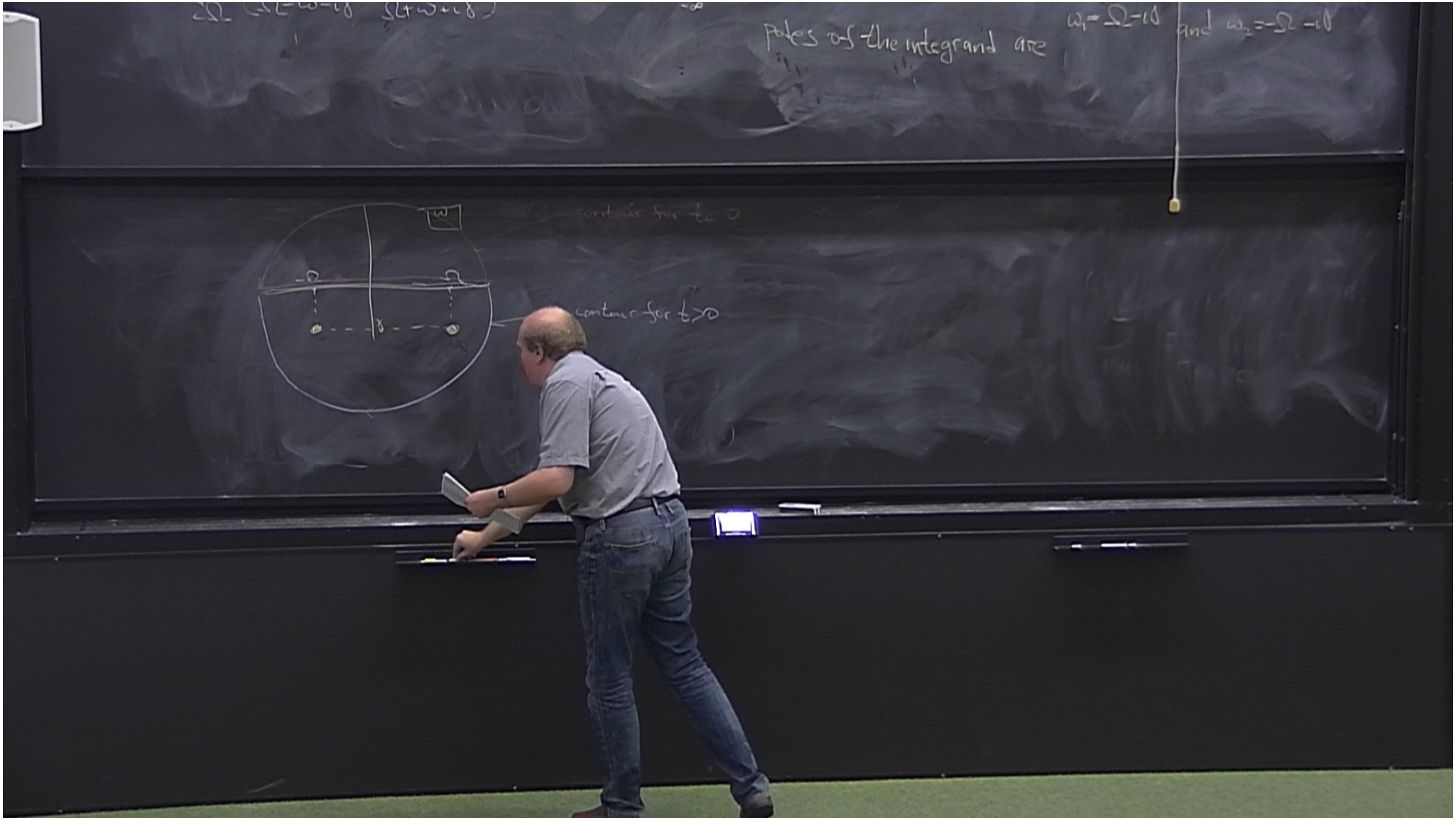

$L\,G(t) = \delta(t)$
basic properties of Fourier transform
Plug $\begin{Bmatrix} G(t) \\ \delta(t) \end{Bmatrix} = \int_{-\infty}^{+\infty} \begin{Bmatrix} G(\omega) \\ 1 \end{Bmatrix} e^{-i\omega t} \frac{d\omega}{2\pi} \Rightarrow \left[-\omega^2 - 2i\beta\omega + (\Omega^2 + \beta^2)\right] G($

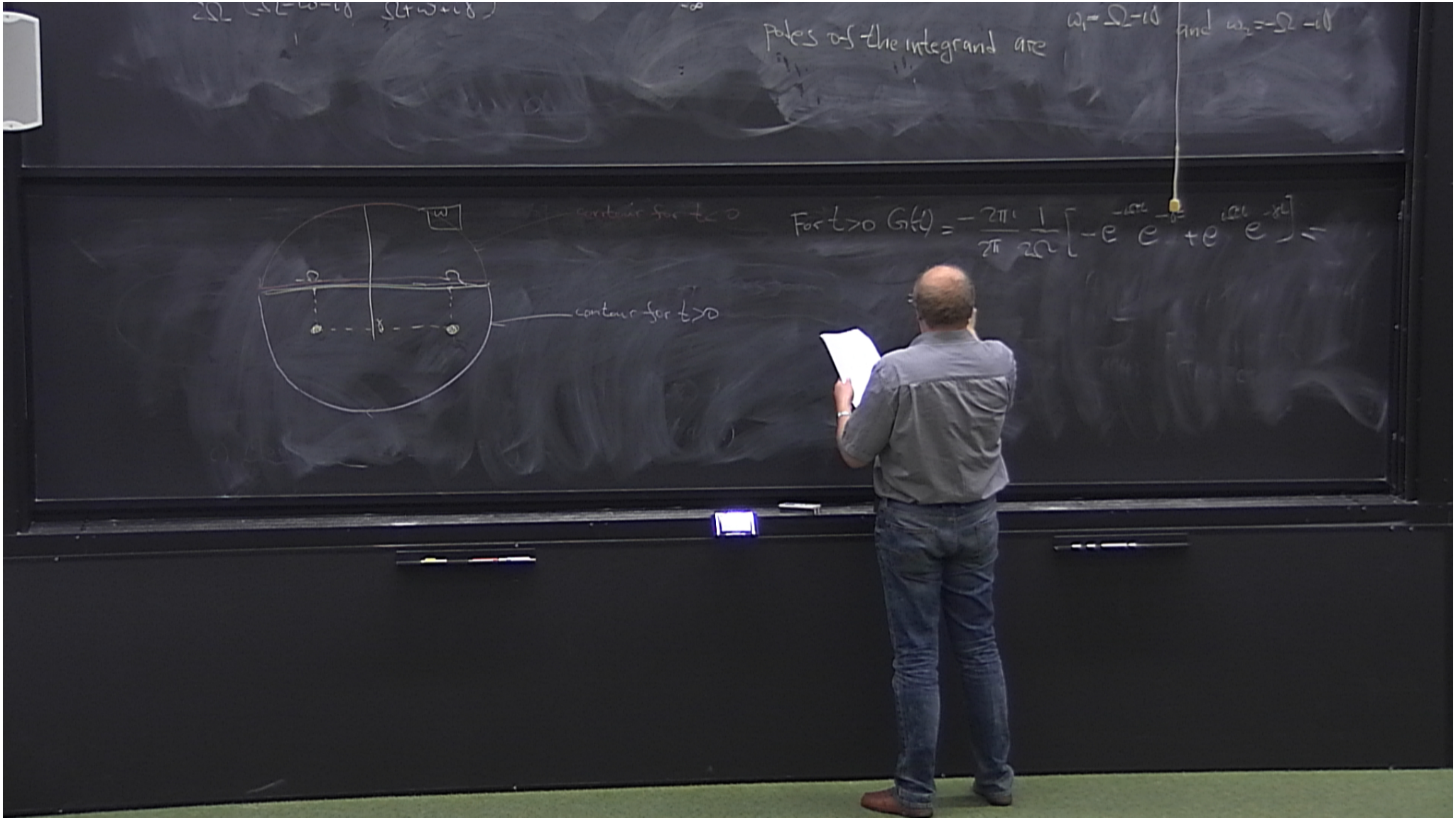

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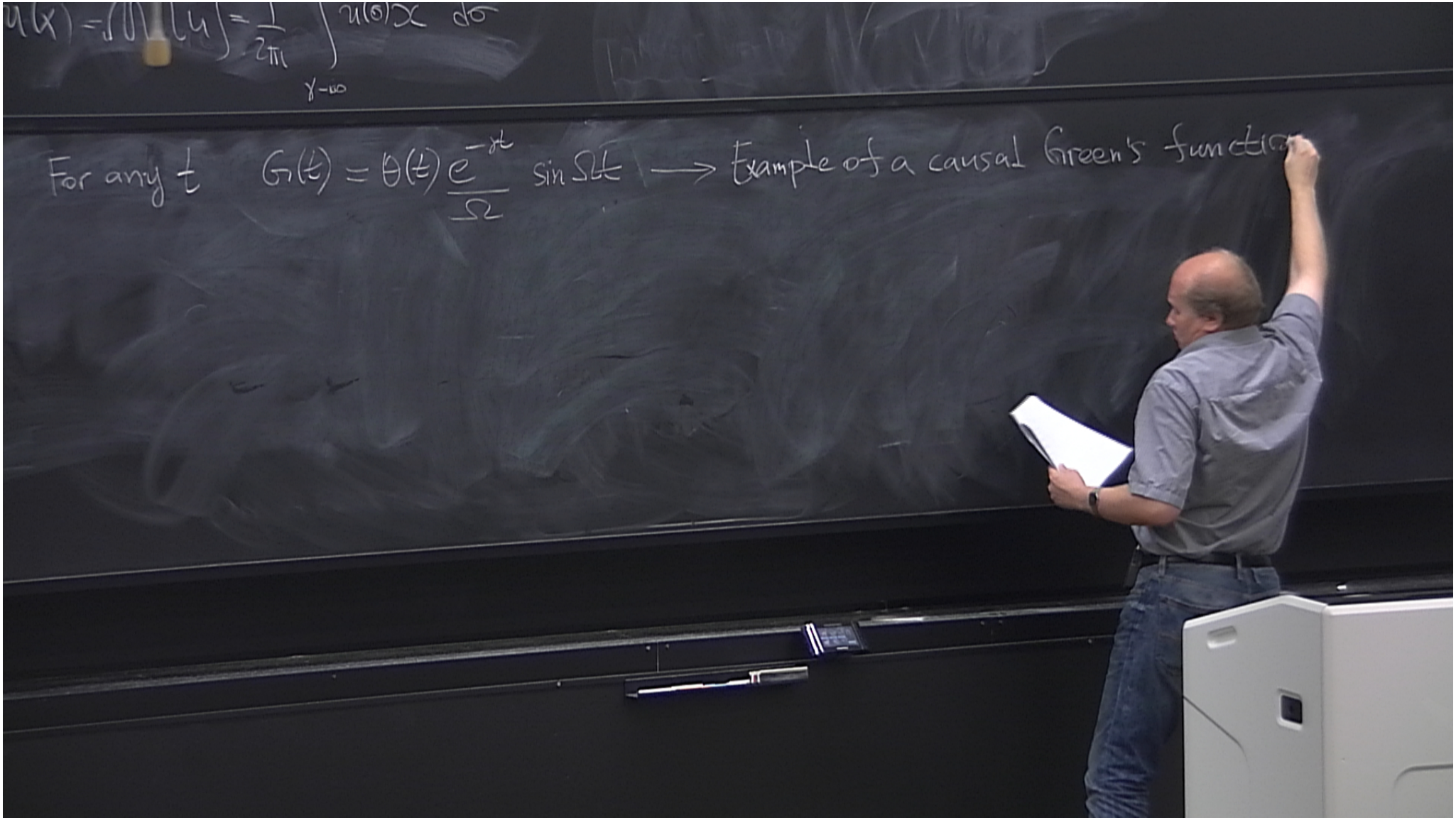

poles of the integrand are
and
contour for t<0

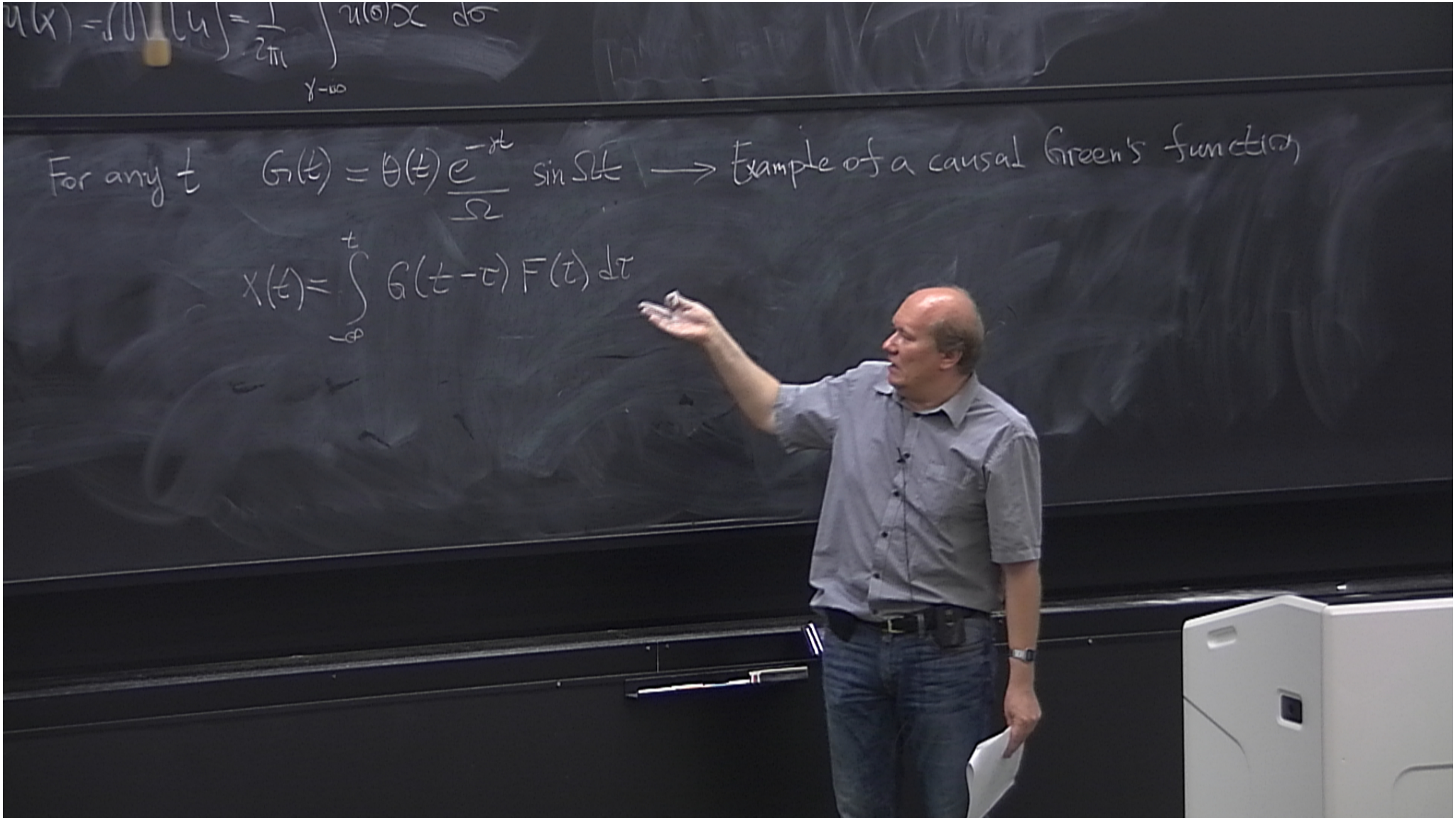
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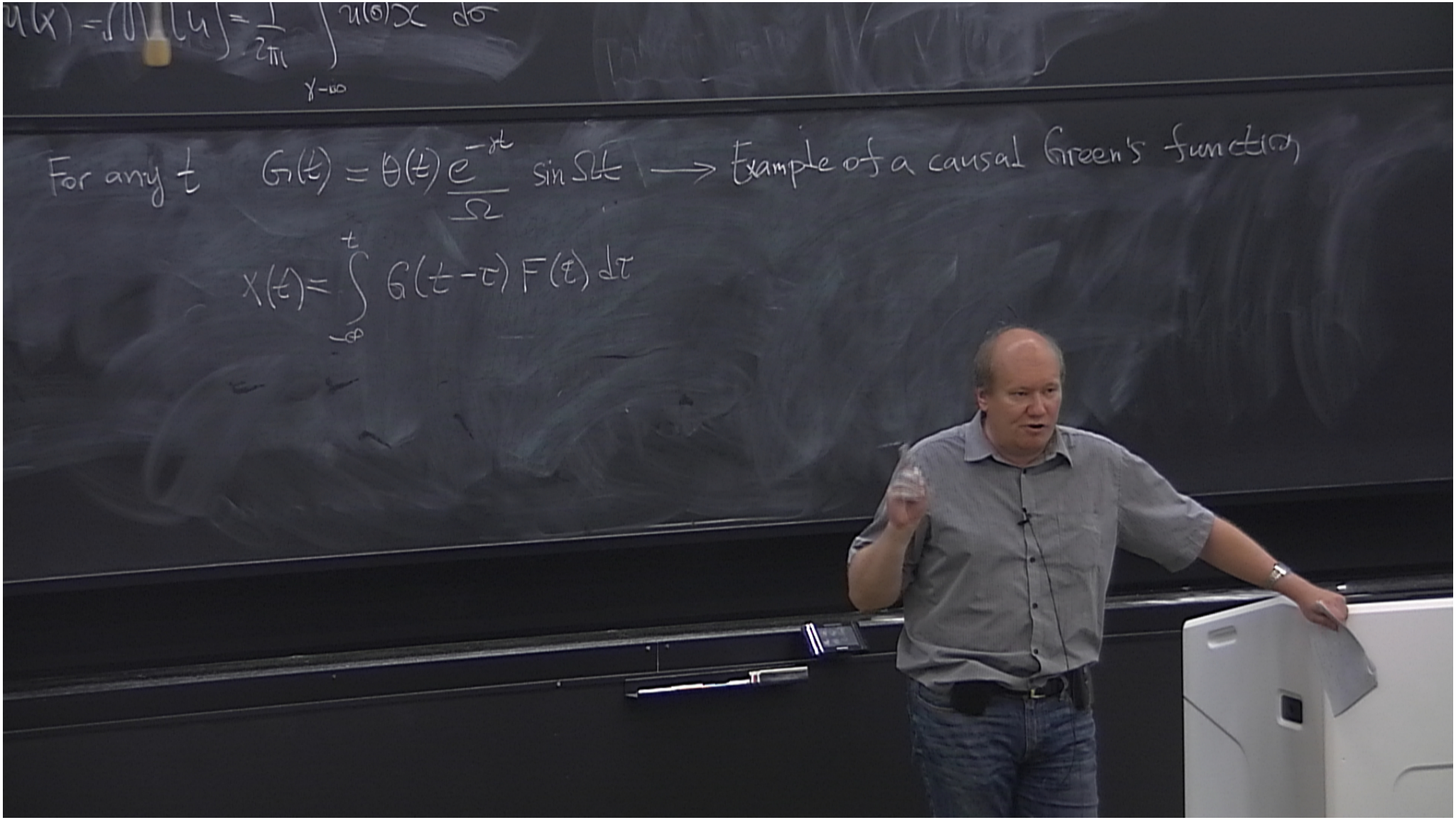


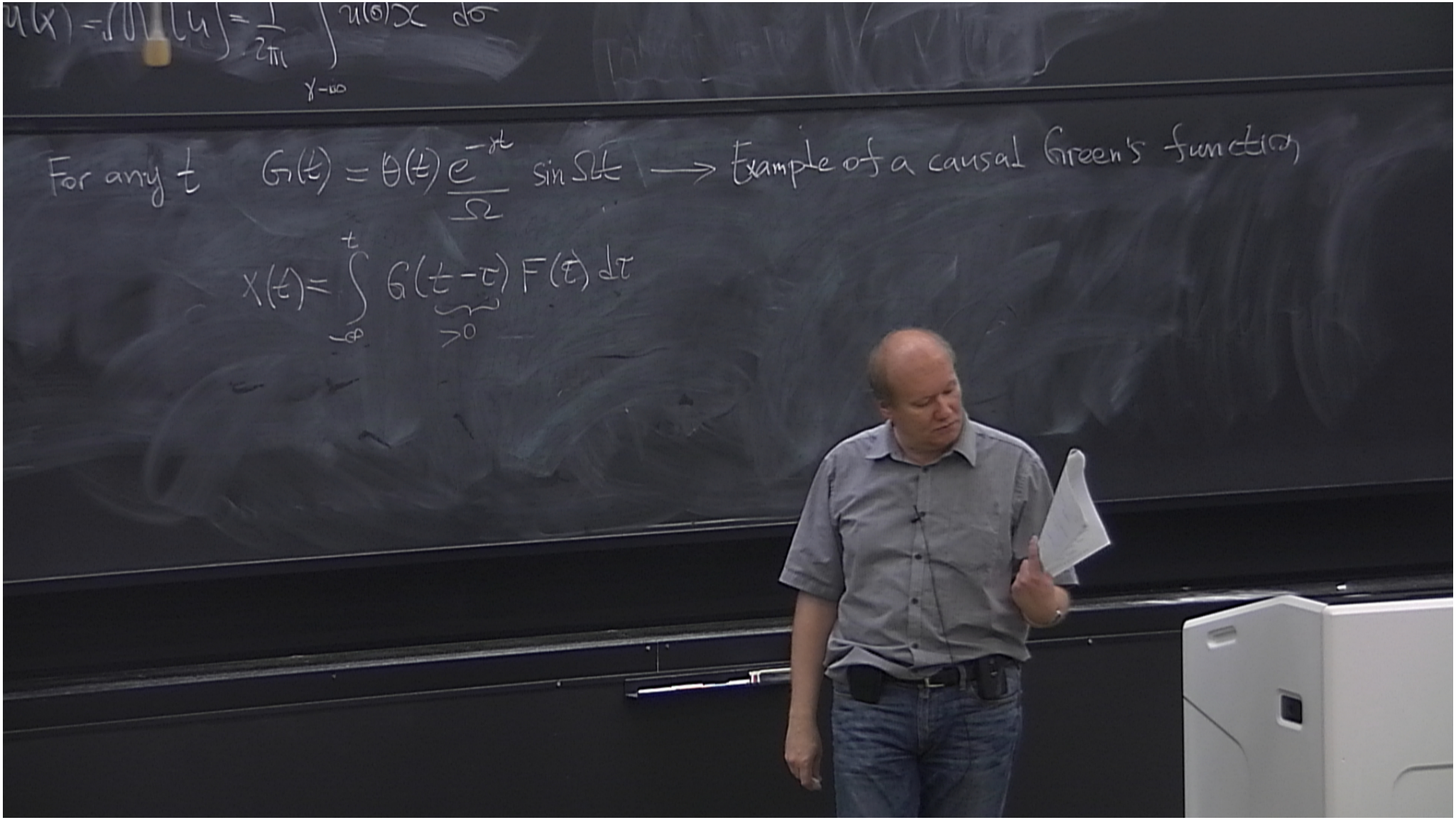

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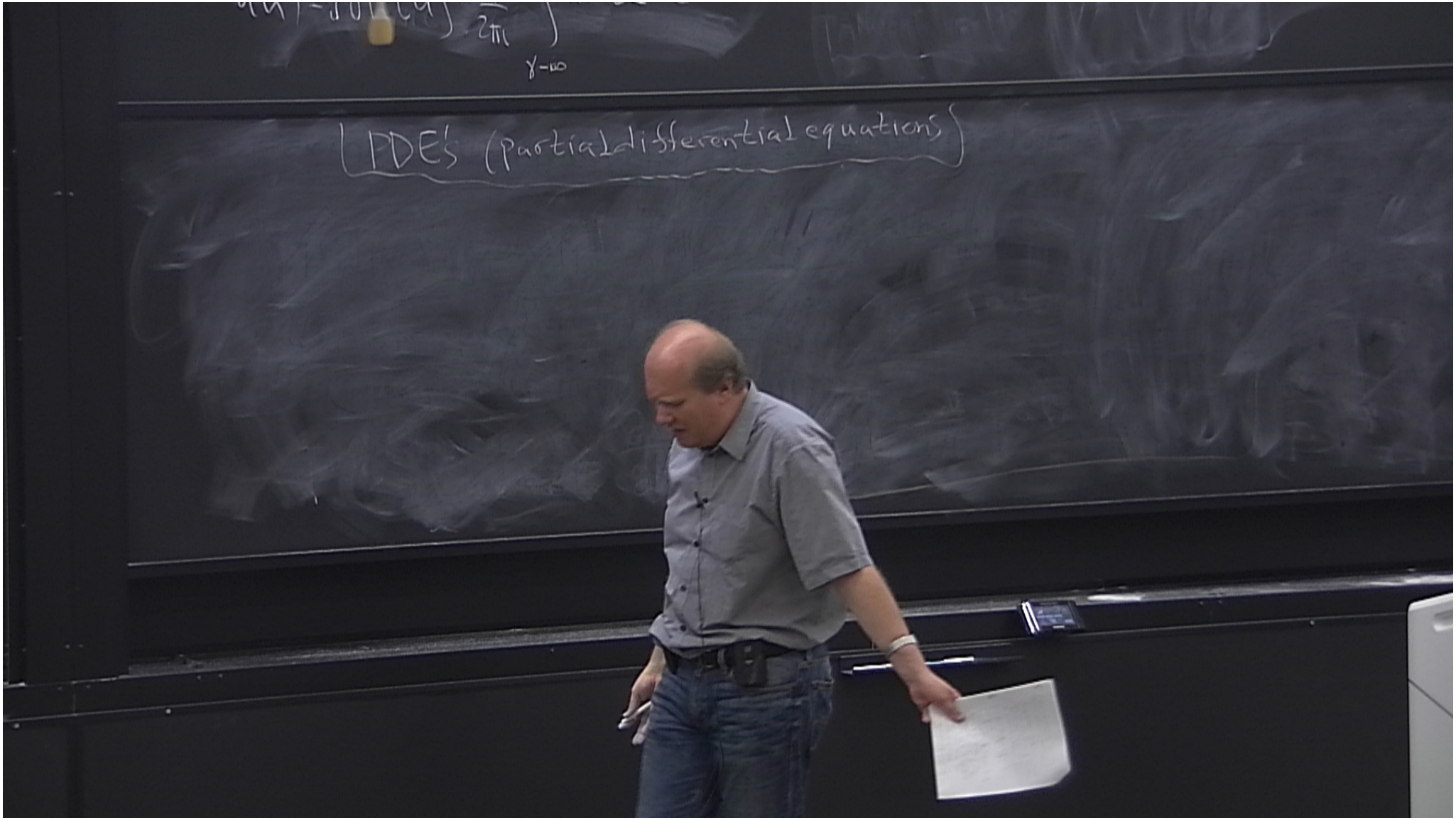

poles of the integrand are
$\omega_1 = \Omega - i\delta$ and $\omega_2 = -\Omega - i\delta$
contour for $t<0$
contour for $t>0$
For $t>0$ $G(t) = -\frac{2\pi i}{2\pi}\frac{1}{2\Omega i}\left[-e^{-i\Omega t}e^{-\delta t} + e^{i\Omega t}e^{-\delta t}\right] =$

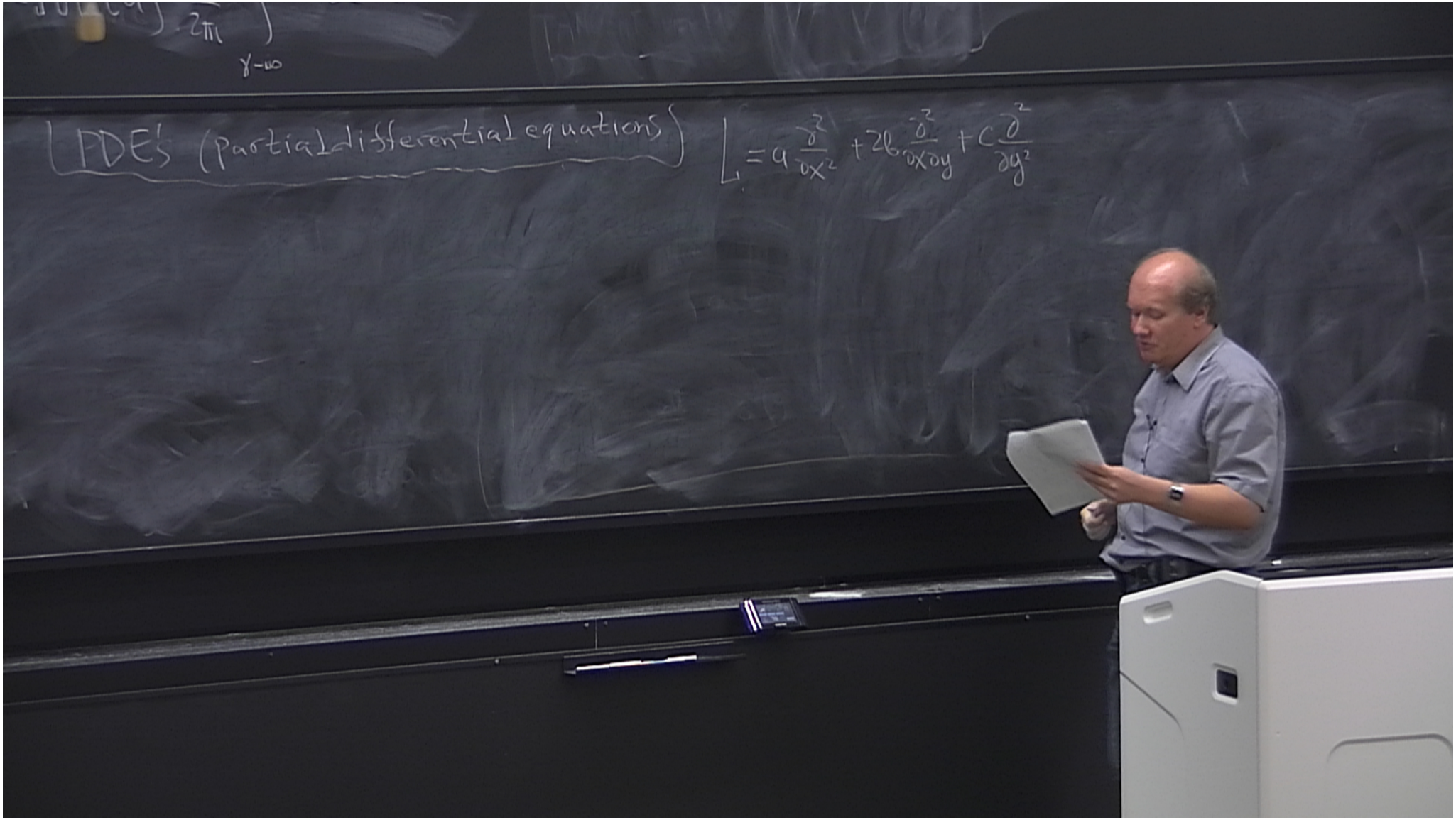

For any t $G(t) = \theta(t)\frac{e^{-\gamma t}}{\Omega}\sin\Omega t$ → Example of a causal Green's function

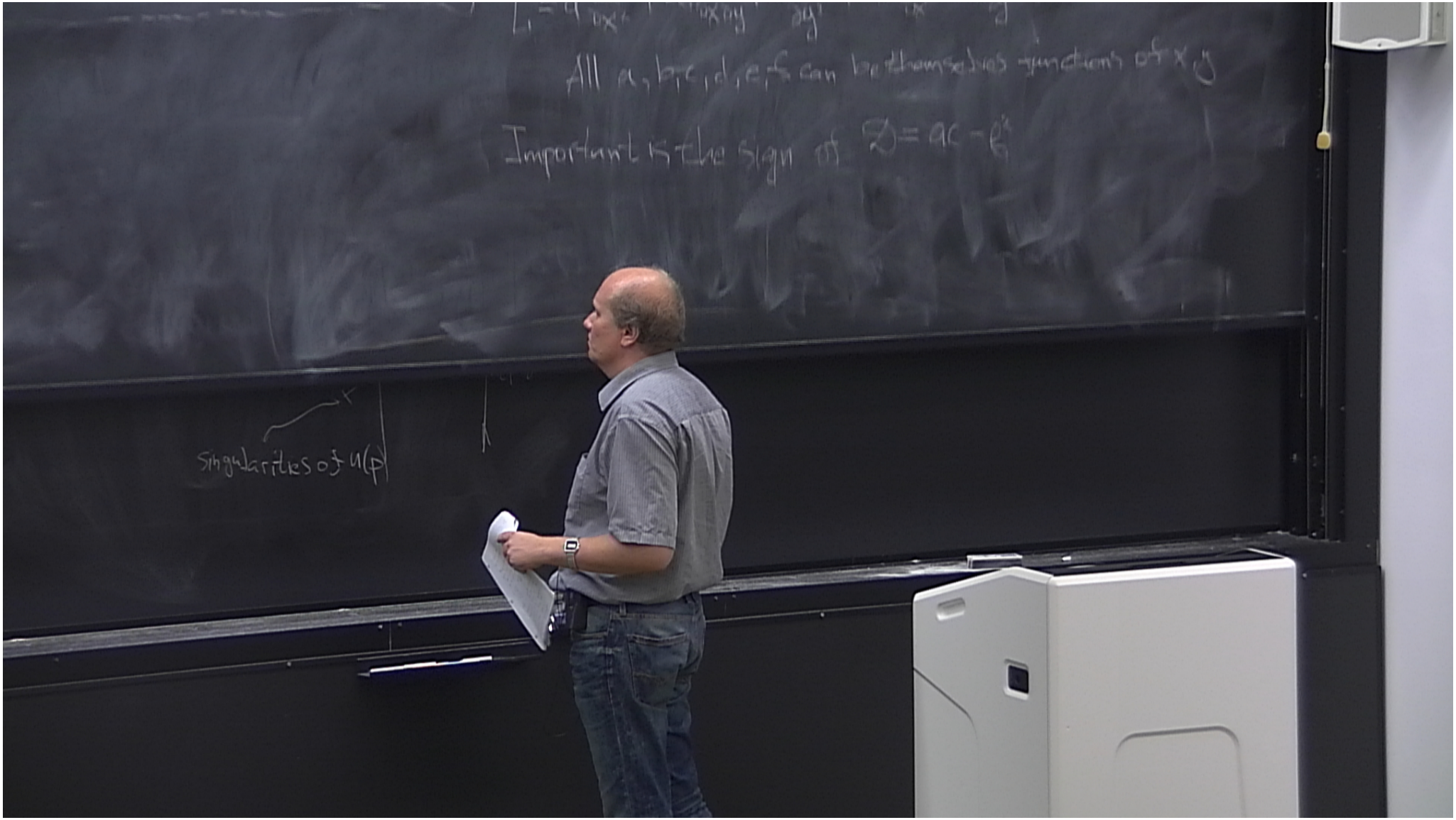

For any t $G(t) = \theta(t)\frac{e^{-\gamma t}}{\Omega}\sin\Omega t$ → Example of a causal Green's function
$x(t) = \int_{-\infty}^{t} G(t-\tau)F(\tau)\,d\tau$

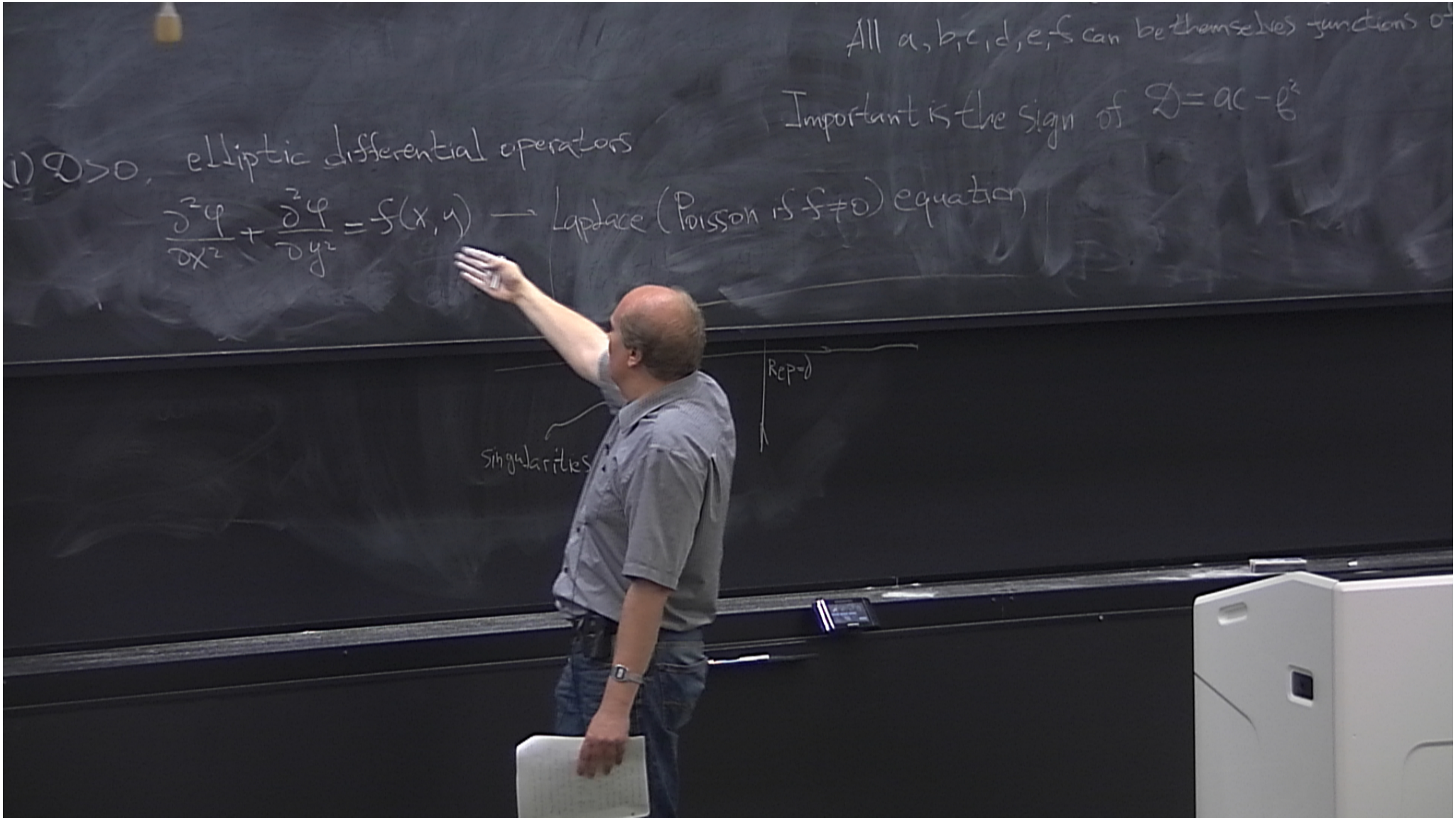

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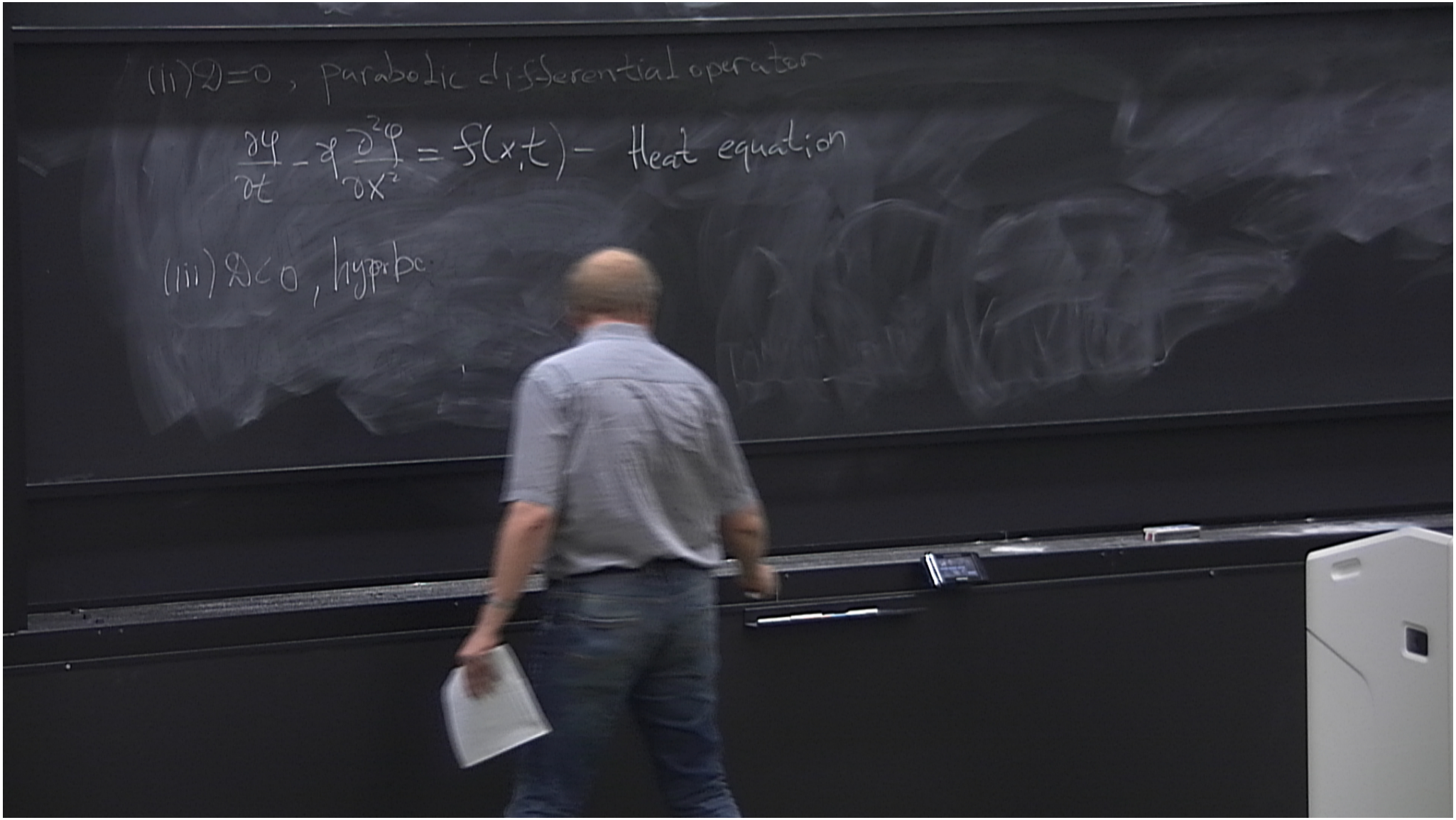

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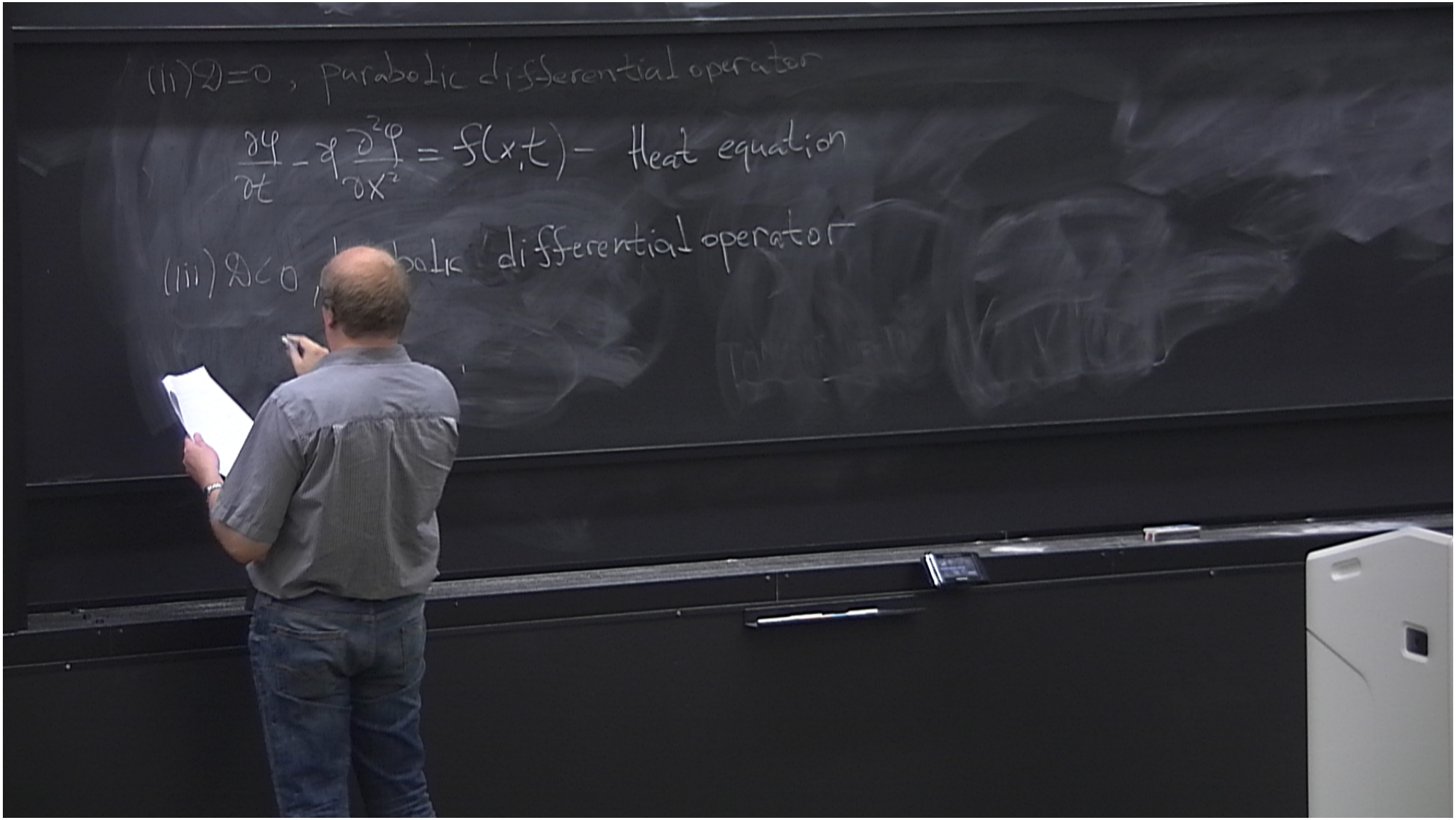

PDE's (partial differential equations)

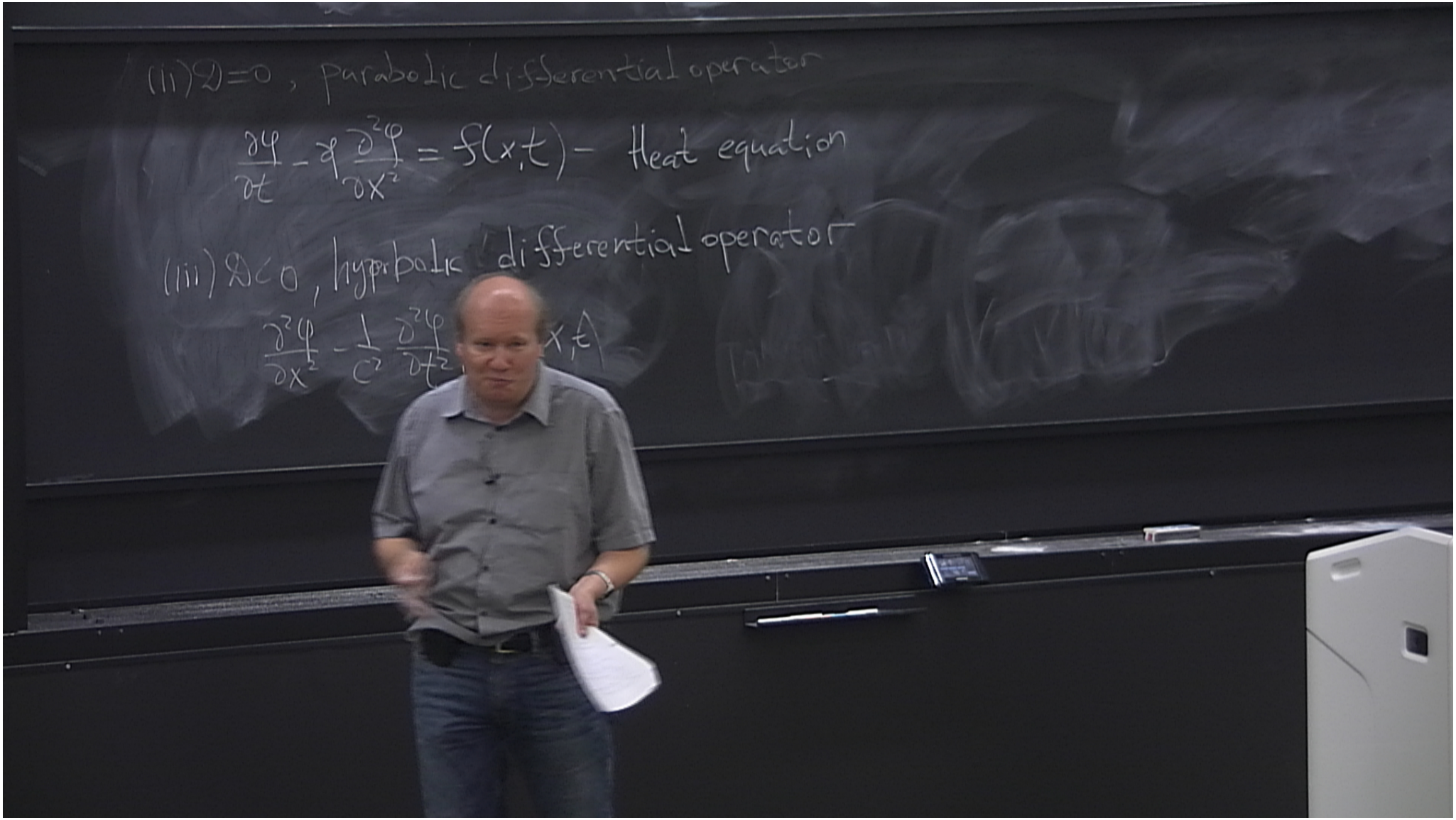

PDE's (partial differential equations)
$L = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2}$

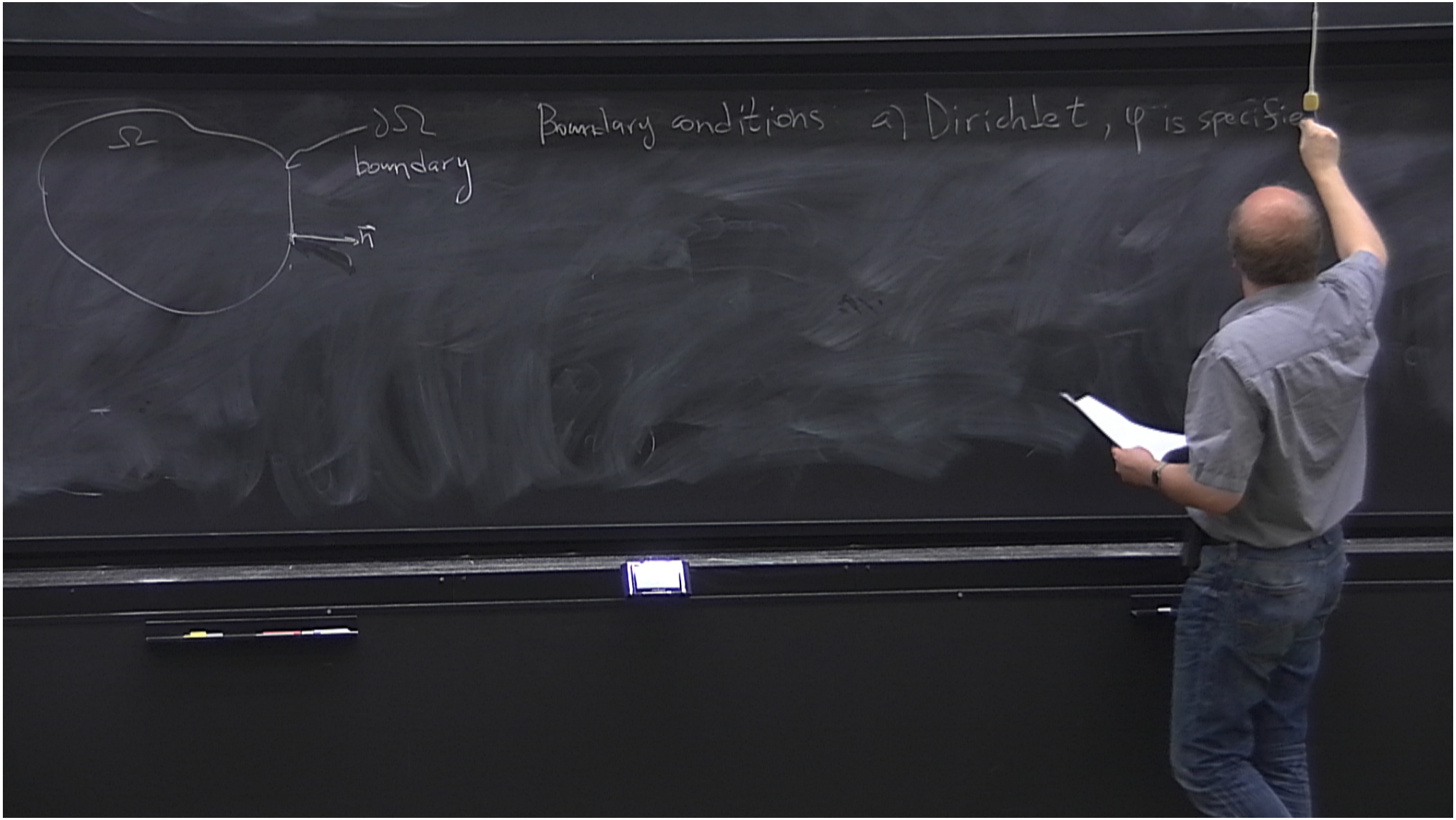

All a, b, c, d, e, f can be themselves functions of x, y
Important is the sign of $\mathcal{D} = ac - b^2$
Singularities of u(p)

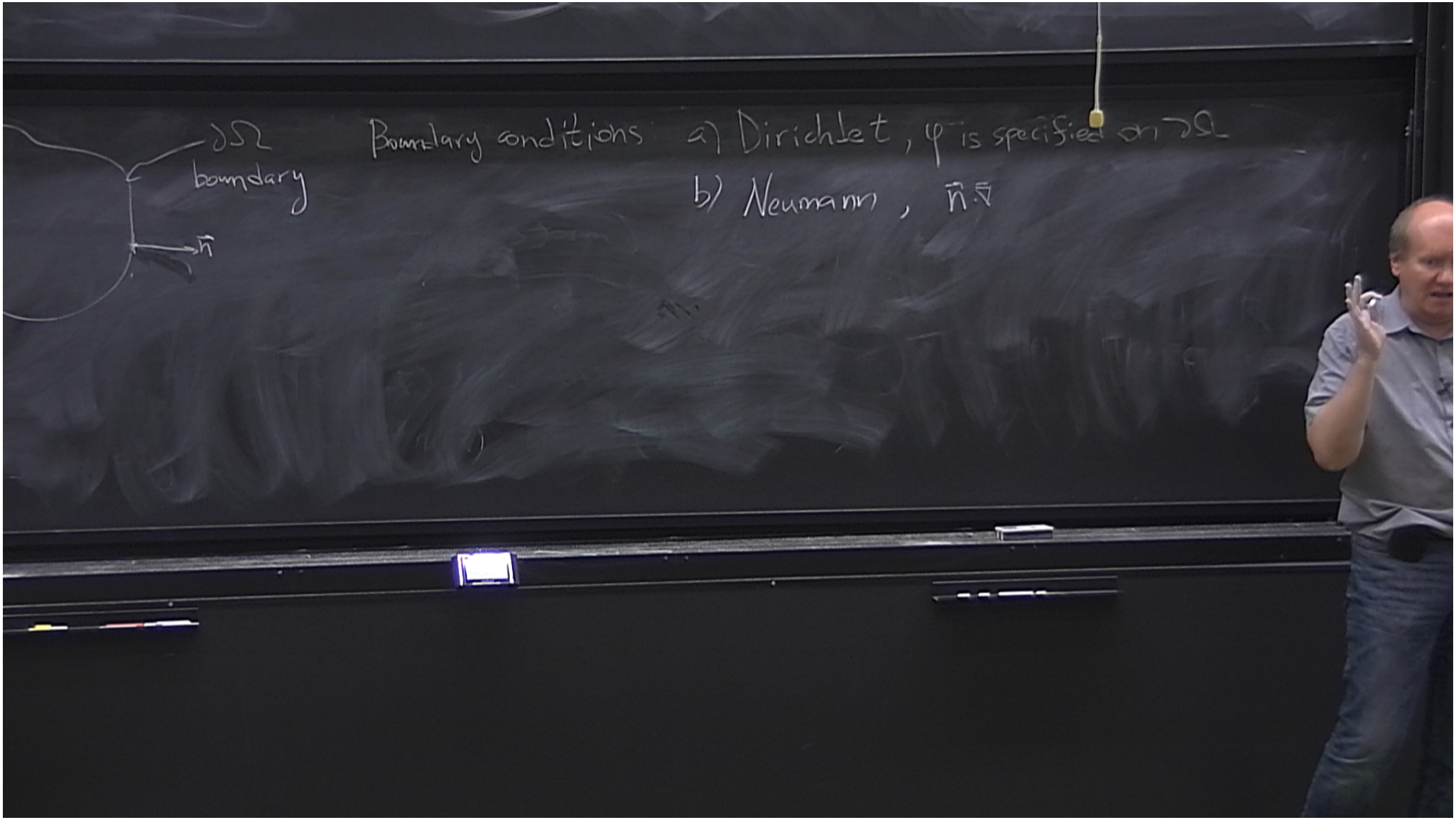

All a, b, c, d, e, f can be themselves functions of
Important is the sign of $D = ac - b^2$
(1) $D > 0$, elliptic differential operators
$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = f(x, y)$ → Laplace (Poisson if $f \neq 0$) equation
Rep=0
Singularities

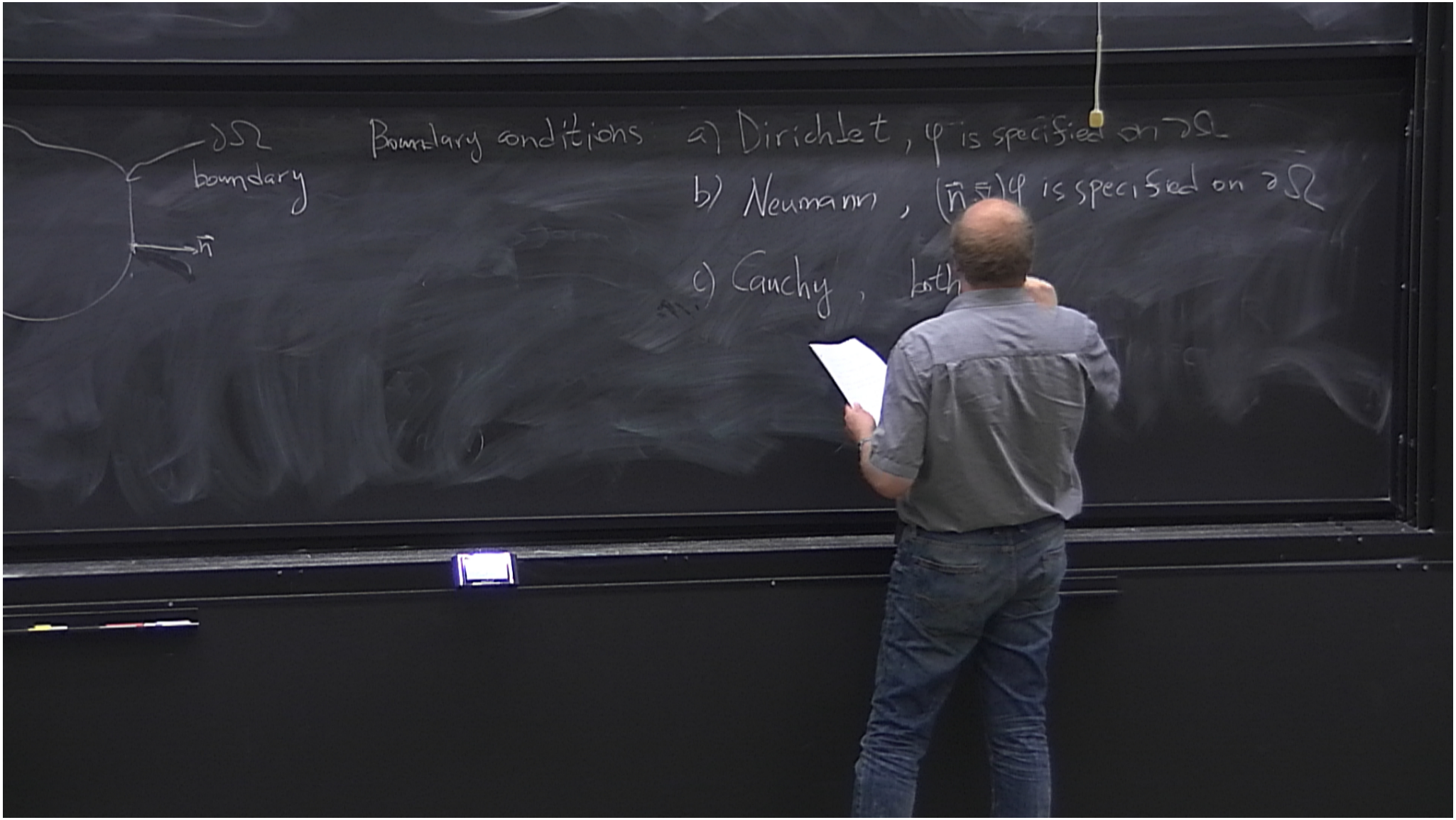

(ii) Δ=0, parabolic differential operator
$\frac{\partial \varphi}{\partial t} - \alpha \frac{\partial^2 \varphi}{\partial x^2} = f(x,t)$ - Heat equation
(iii) Δ<0, hyperbo


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$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2}$ (x,t)


Ω
∂Ω
boundary
n
Boundary conditions a) Dirichlet, φ is specifie


∂Ω
boundary
$\vec{n}$
Boundary conditions a) Dirichlet, φ is specified on $\partial\Omega$
b) Neumann, $\vec{n}\cdot\nabla$


∂Ω
boundary
n
Boundary conditions
a) Dirichlet, φ is specified on ∂Ω
b) Neumann, (n·∇)φ is specified on ∂Ω
c) Cauchy, both