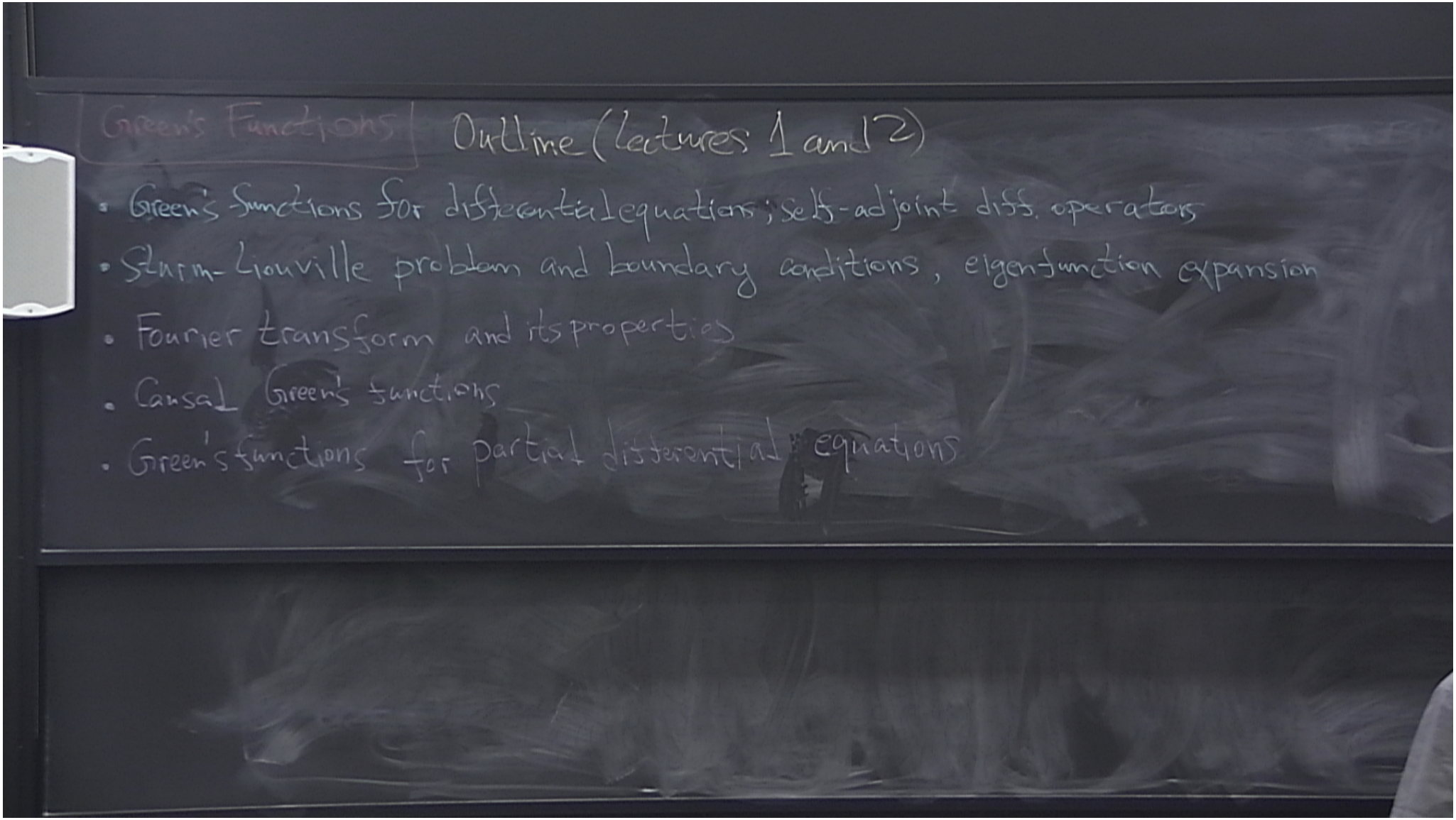
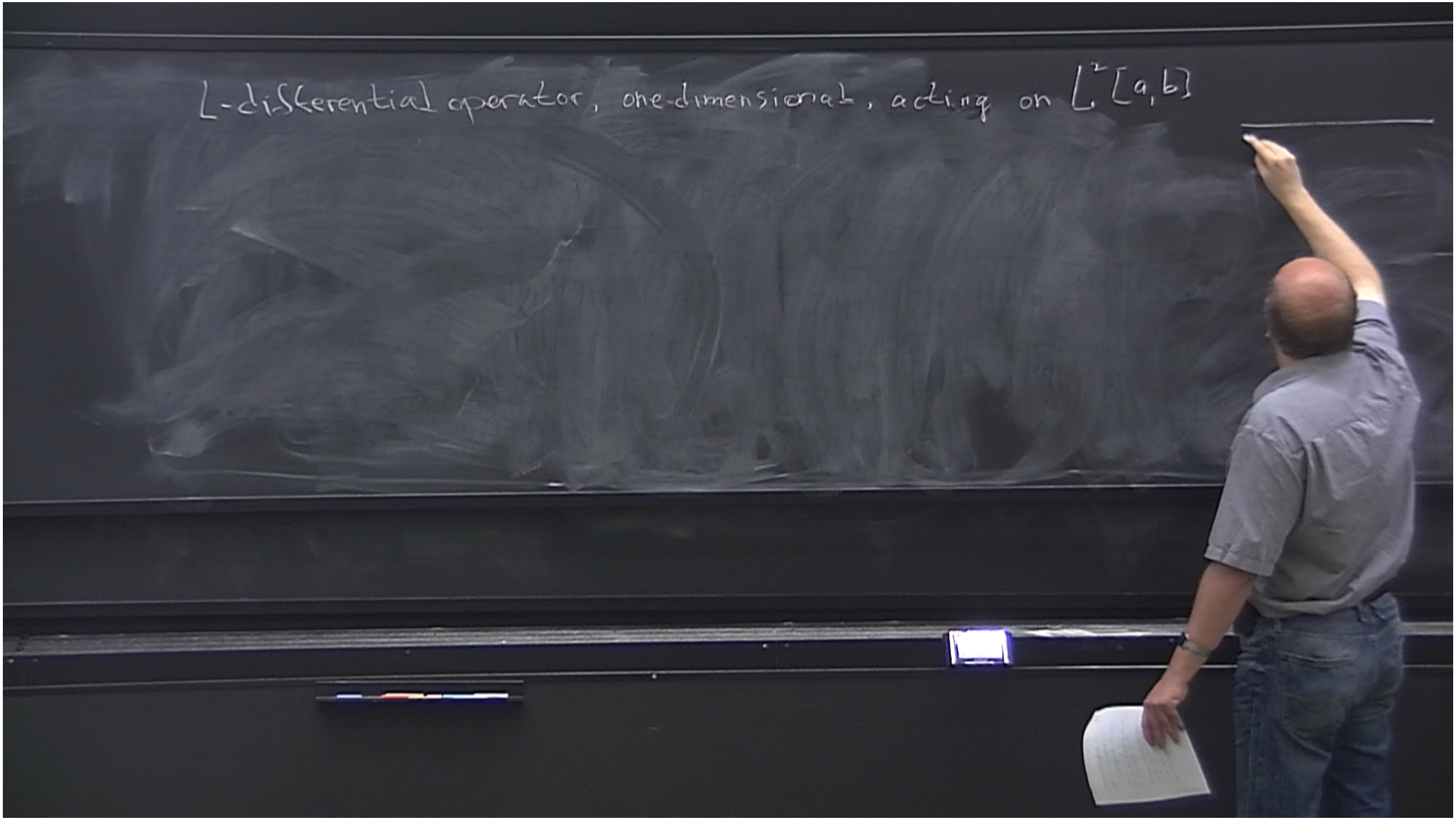
Title: Greens Functions

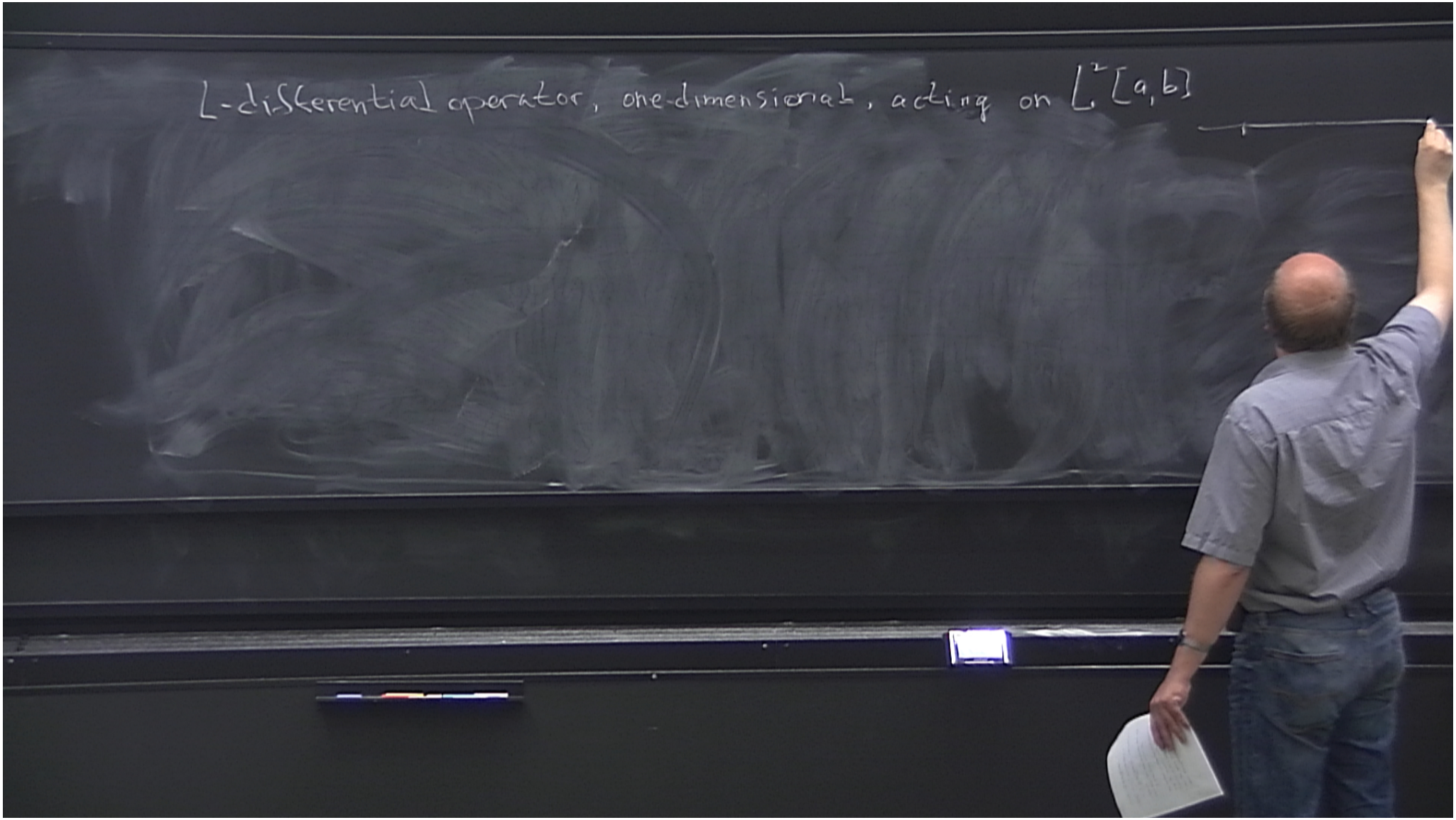
Date: Aug 18, 2015  10:30 AM

URL: http://pirsa.org/15080106

Abstract:


Green's Functions
Outline (lectures 1 and 2)
• Green's functions for differential equations; self-adjoint diff. operators
• Sturm-Liouville problem and boundary conditions, eigenfunction expansion
• Fourier transform and its properties
• Causal Green's functions
• Green's functions for partial differential equations


L-differential operator, one-dimensional, acting on $L^2[a,b]$


L-differential operator, one-dimensional, acting on L²[a,b]

L-differential operator, one-dimensional, acting on $L^2[a,b]$




$a \le x \le b$

$$L = p_0(x)\frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \quad + p_n(x)$$

$Ly$

L-differential operator, one-dimensional, acting on $L^2[a,b]$




$$L = p_0(x)\frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \quad + p_n(x)$$

$$L\,y(x) = f(x)$$

L - differential operator, one-dimensional, acting on $L^2[a,b]$

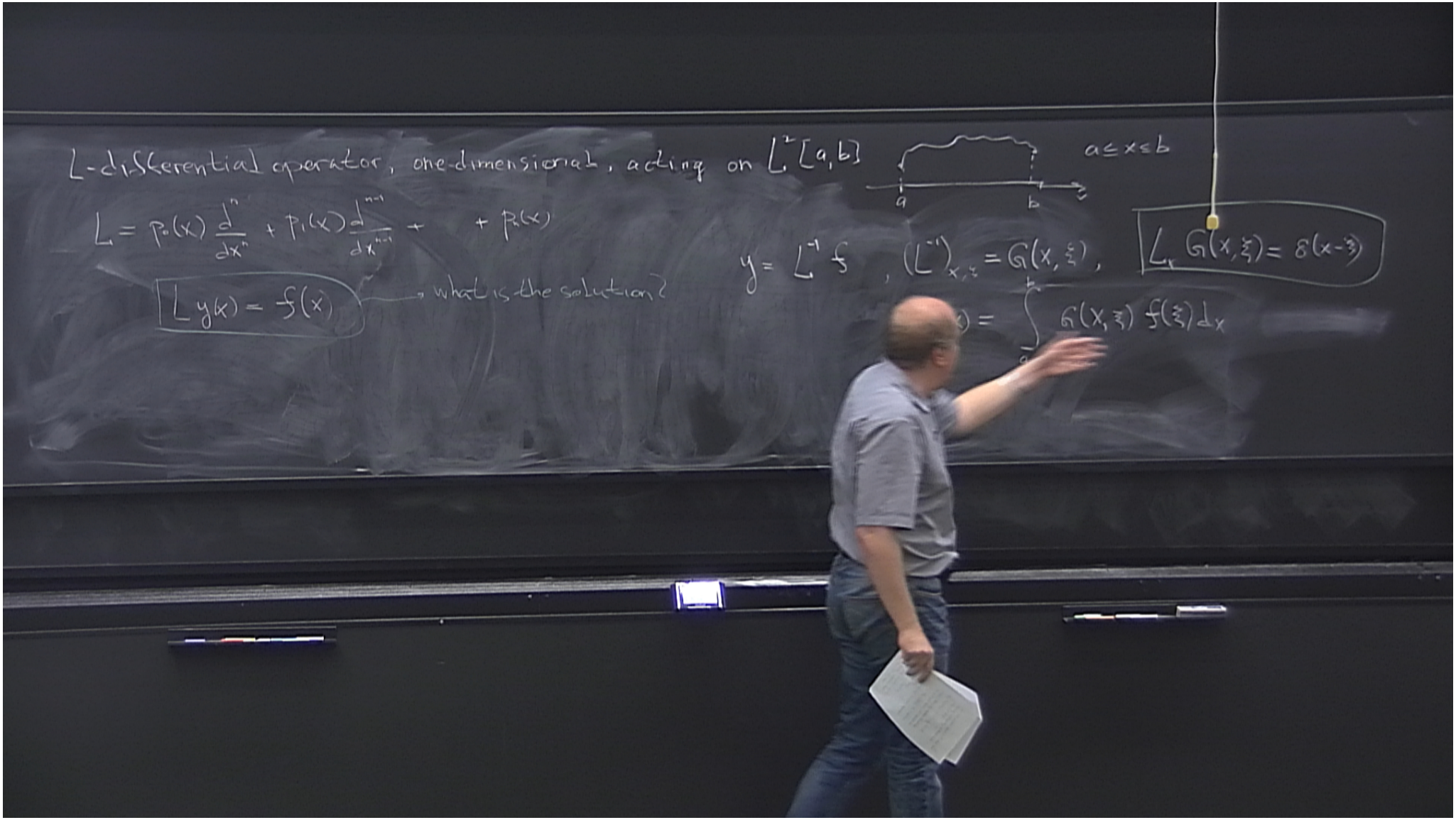


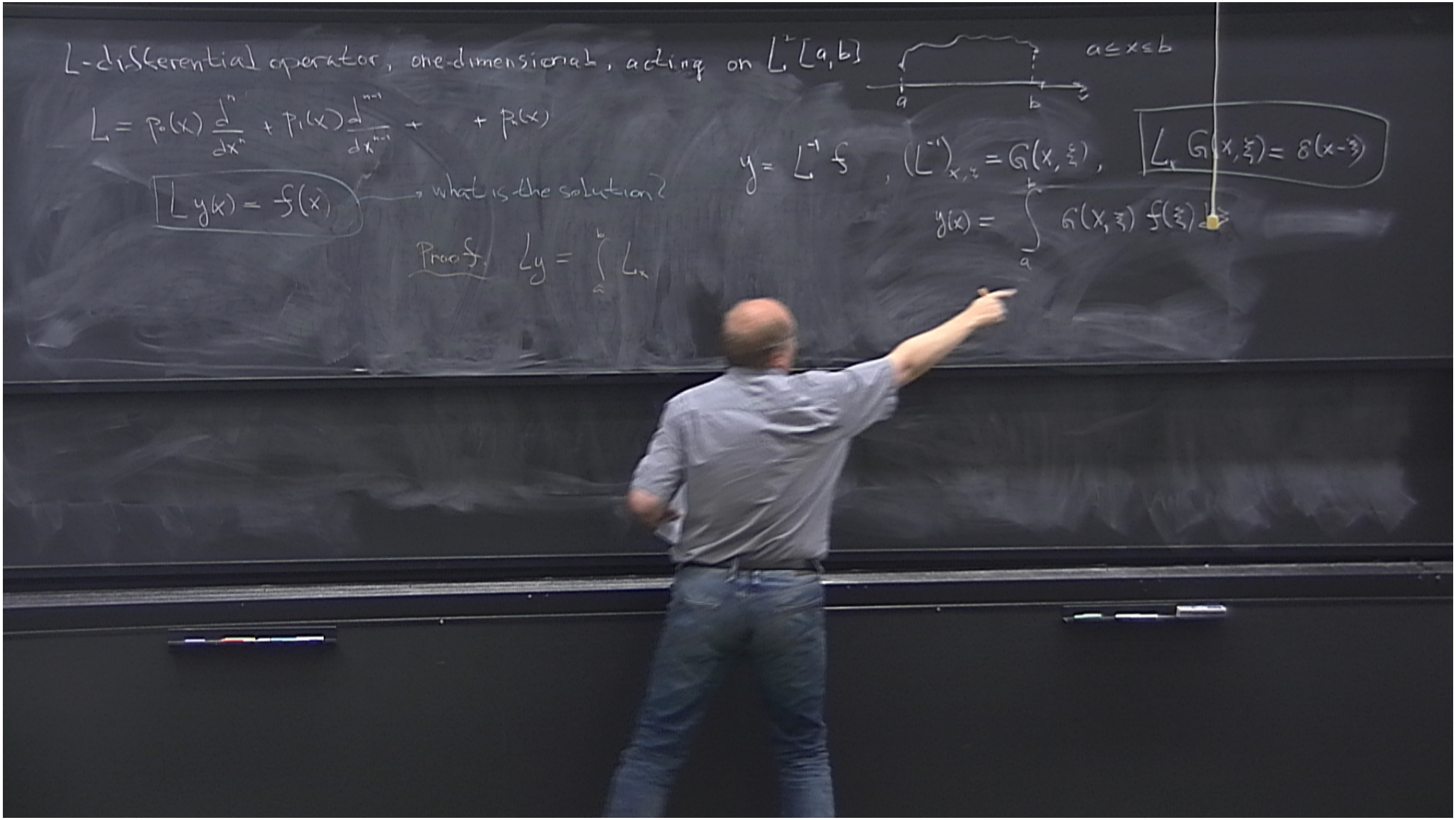

$a \le x \le b$

$$L = p_0(x)\frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x)$$

$$L\,y(x) = f(x) \rightarrow \text{what is the solution?}$$

$$y = L^{-1} f \quad , \quad (L^{-1})_{x,\xi} = G(x,\xi)$$


L-differential operator, one-dimensional, acting on $L^2[a,b]$
$a \le x \le b$
$L = p_n(x)\frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_0(x)$
$Ly(x) = f(x)$
what is the solution?
$y = L^{-1} f$, $(L^{-1})_{x,\xi} = G(x,\xi)$,
$L_x G(x,\xi) = \delta(x-\xi)$
$= \int_a^b G(x,\xi) f(\xi) d\xi$


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$y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi$
Proof $Ly = \int_a^b L_x$

$L\,y(x) = f(x)$ → what is the solution?

Proof: $Ly = \int_a^b L_x\, G(x,\xi)\, f(\xi)\, d\xi = f(x)$

$L_x\, G(x,\xi) = \delta(x-\xi)$

$y(x) = \int_a^b G(x,\xi)\, f(\xi)\, d\xi$

{Self-adjoint differential operators}

Suppose we have $L_x$, and weight function $w(x)$

$L\,y(x) = f(x)$ → what is the solution?

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{Self-adjoint differential operators} Suppose we have $L$, and weight function $w(x)$ on $[a,b]$

We can find another $L^\dagger$ such that for $u(x)$ and $v(x)$

$$w\left(u^* L v - v L^\dagger u^*\right) = \frac{d}{dx} Q[u,v]$$

depends bi-linearly on $u$ and $v$

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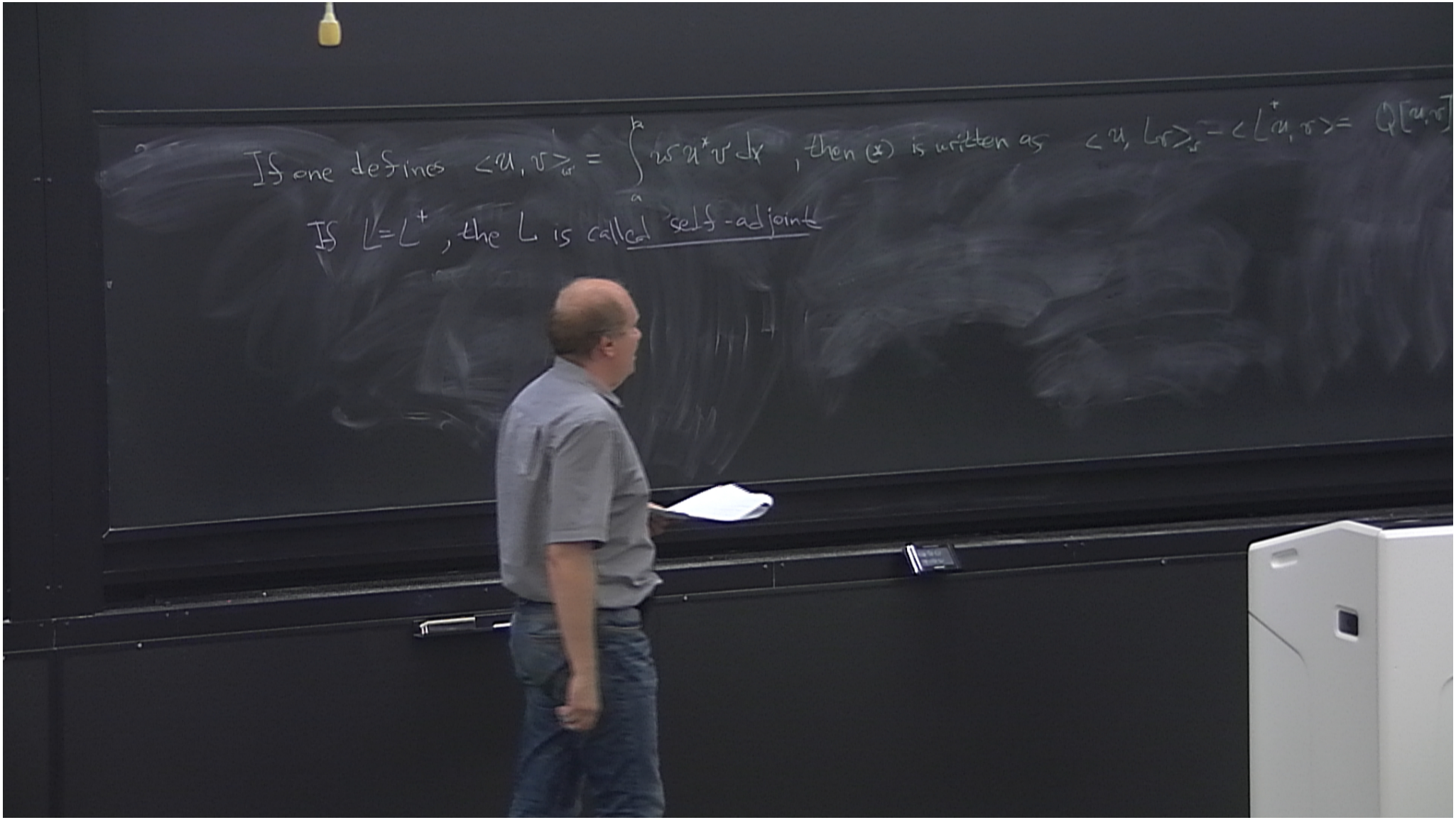
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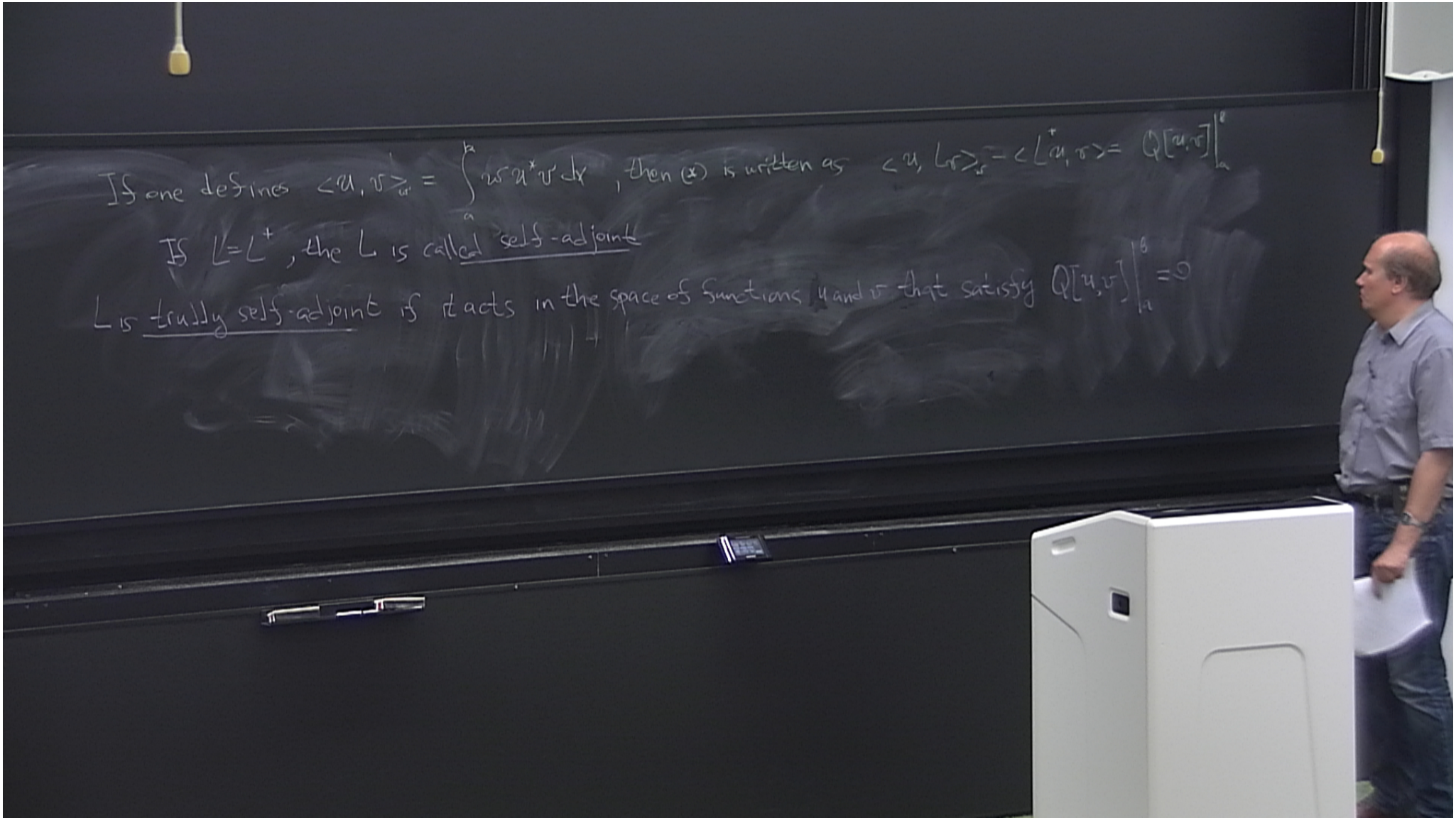
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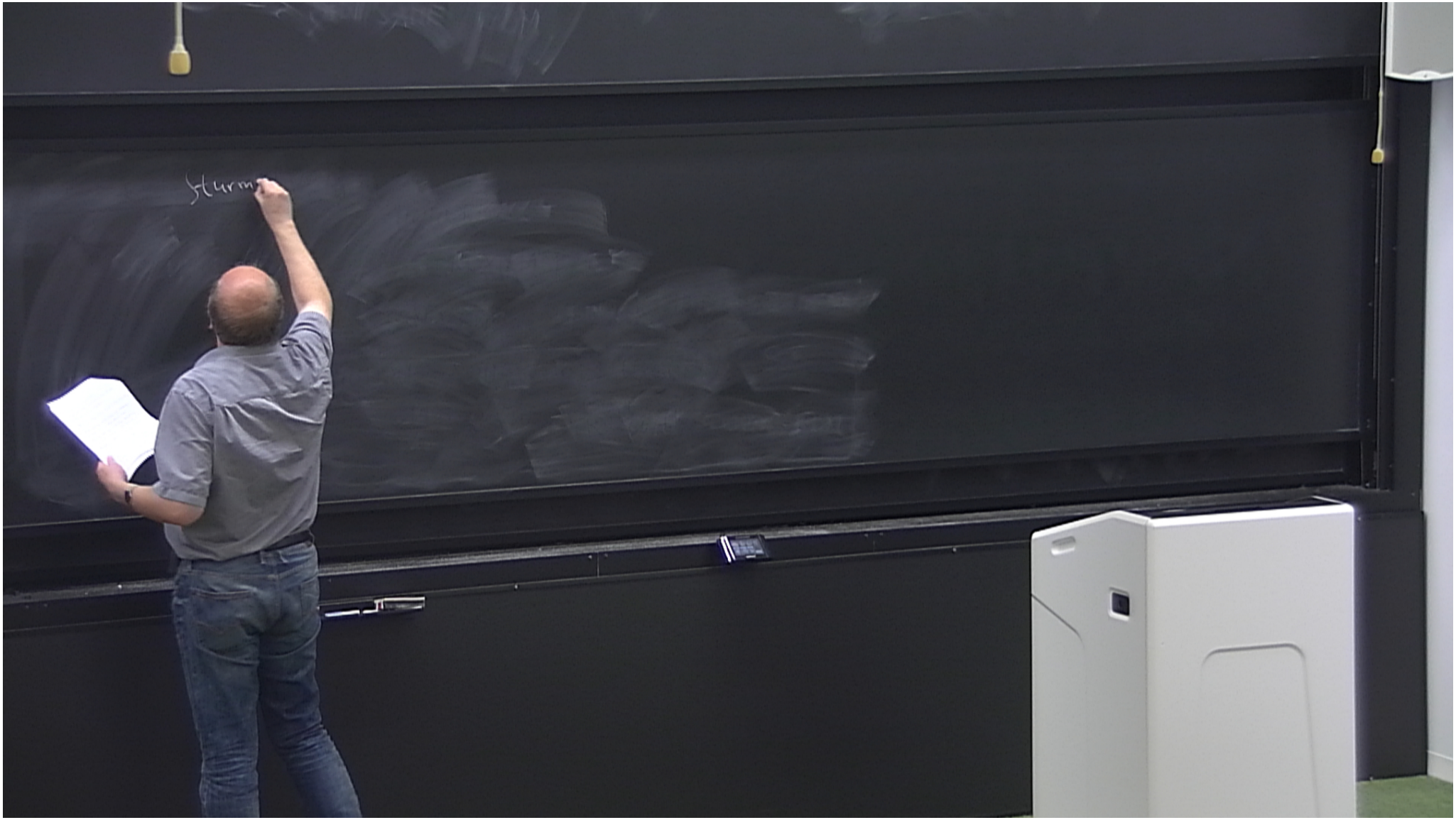
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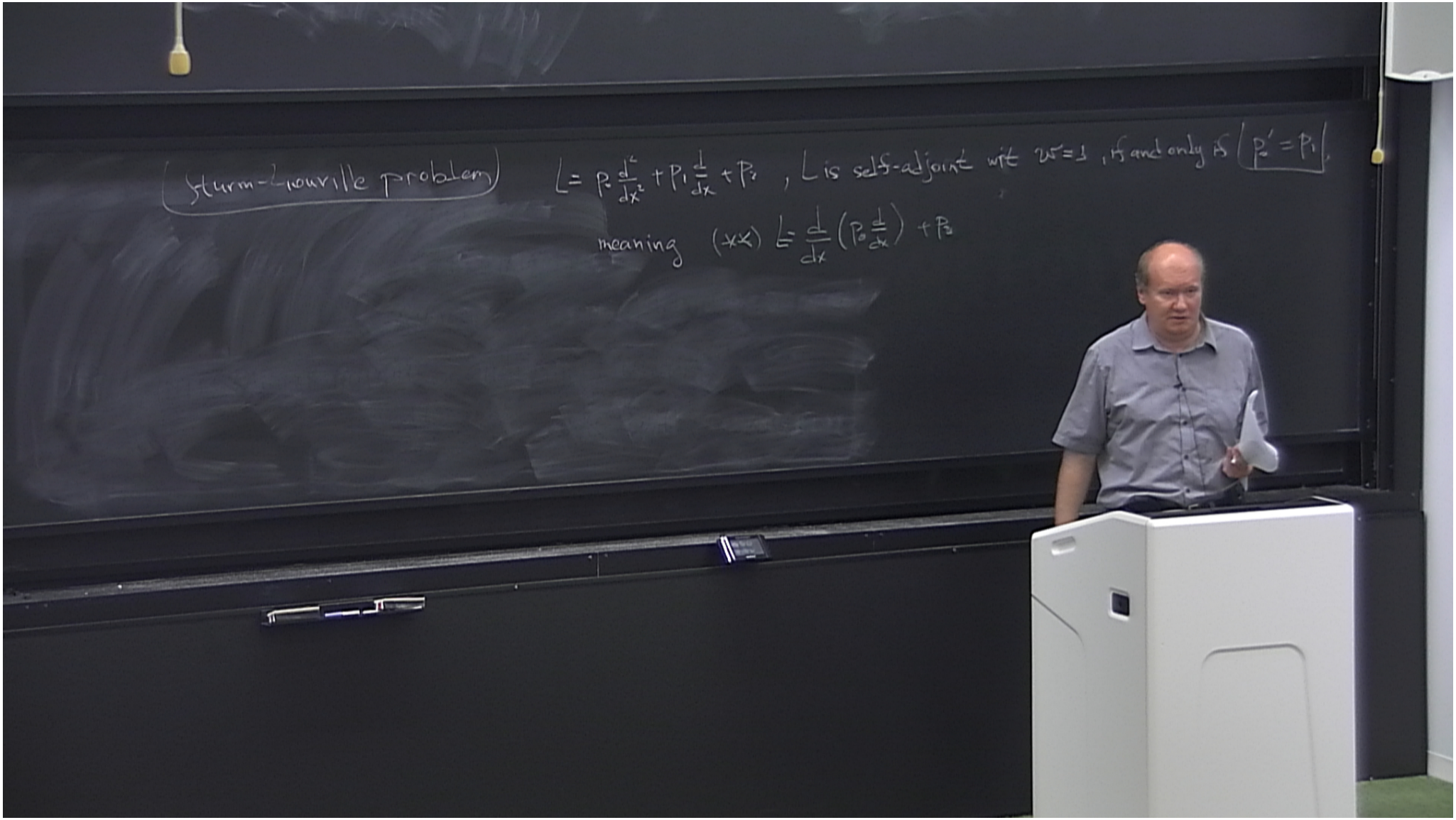
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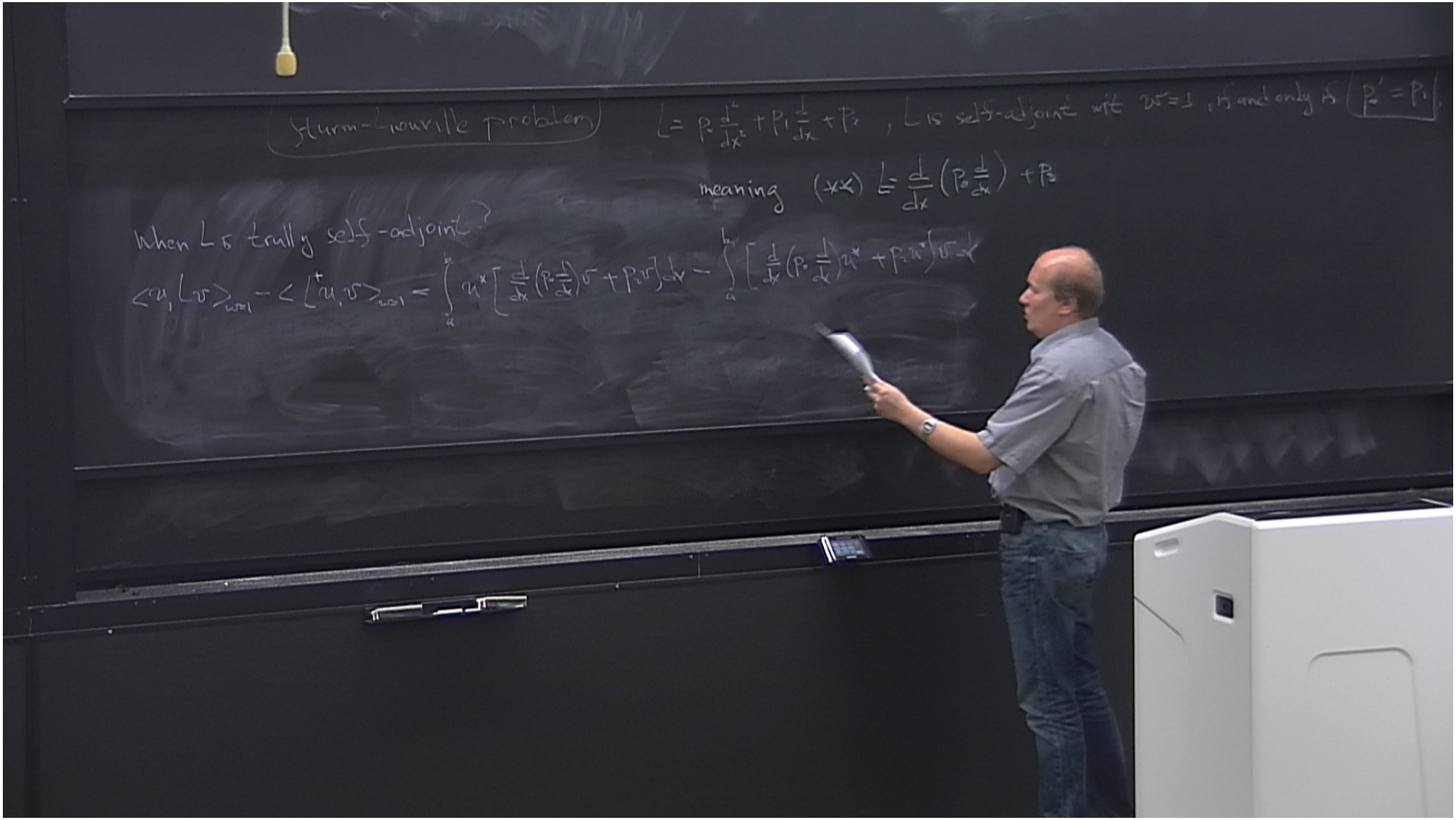
$L^\dagger$ is the adjoint to $L$

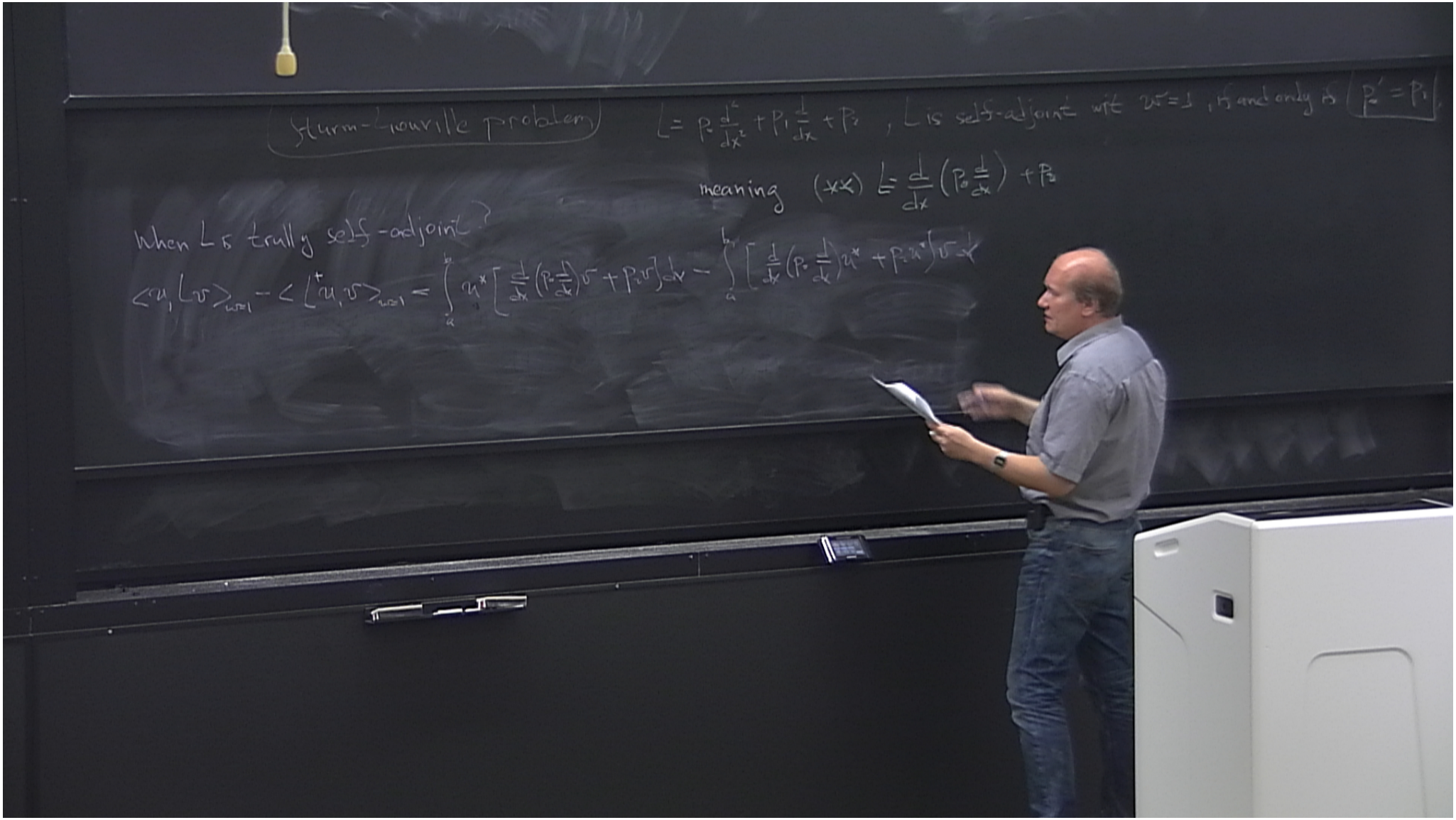

If one defines $\langle u, v \rangle_w = \int_a^b w u^* v \, dx$, then (*) is written as $\langle u, Lv \rangle_w = \langle L^+ u, v \rangle$
If $L = L^+$, the L is called self-adjoint

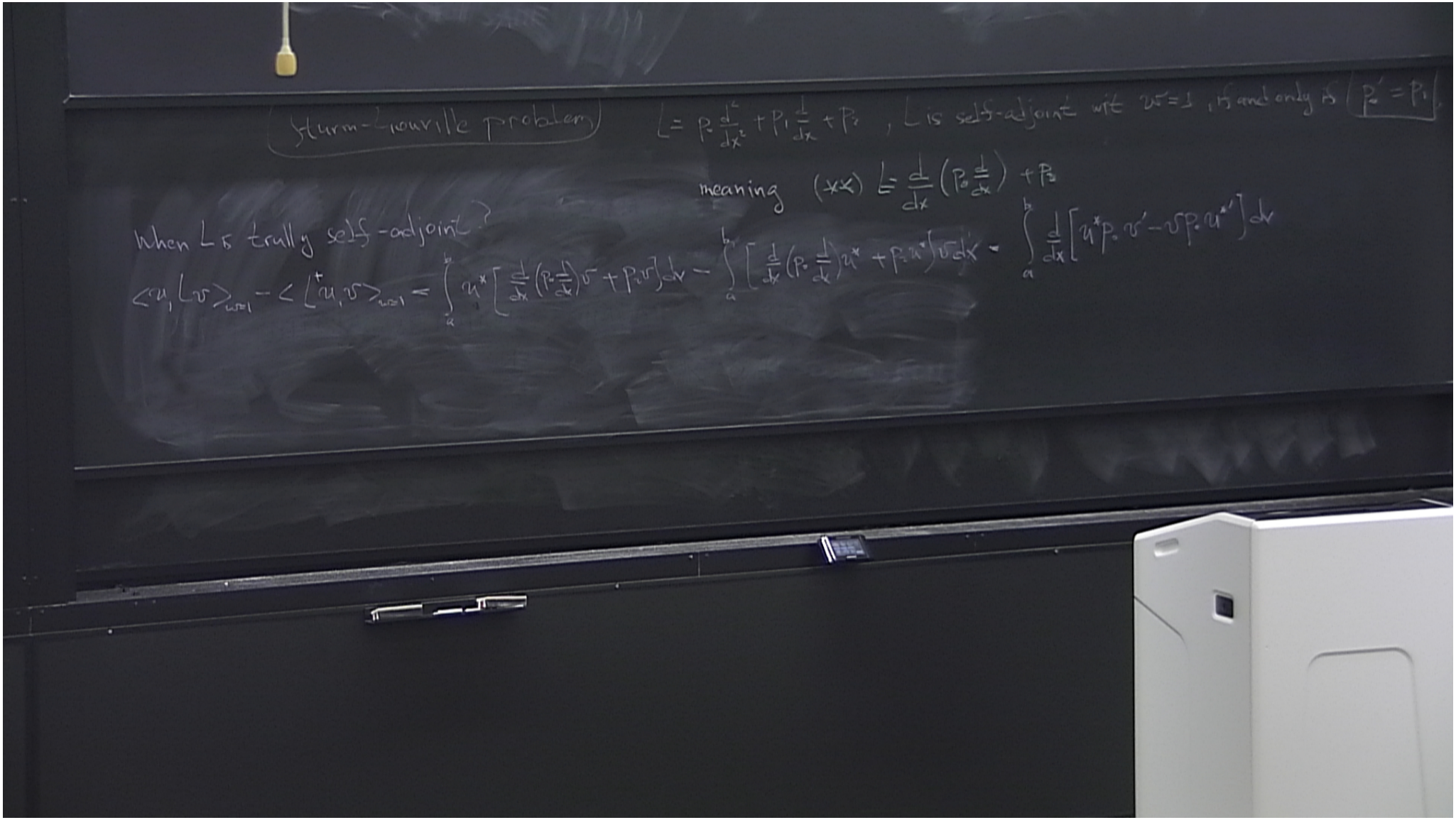

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If $L = L^+$, the $L$ is called self-adjoint
$L$ is truly self-adjoint if it acts in the space of functions $u$ and $v$ that satisfy $Q[u,v]\Big|_a^b = 0$

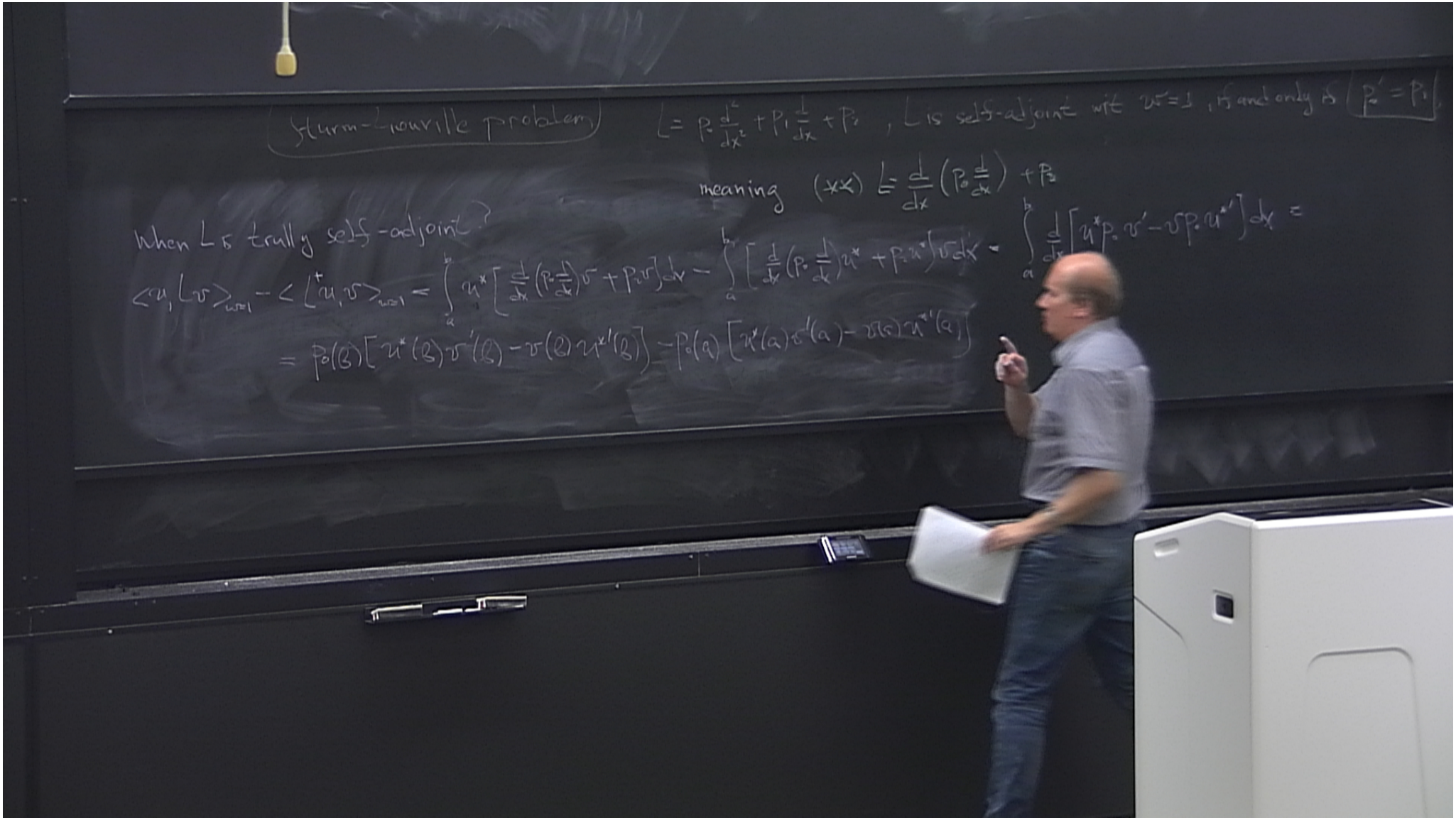


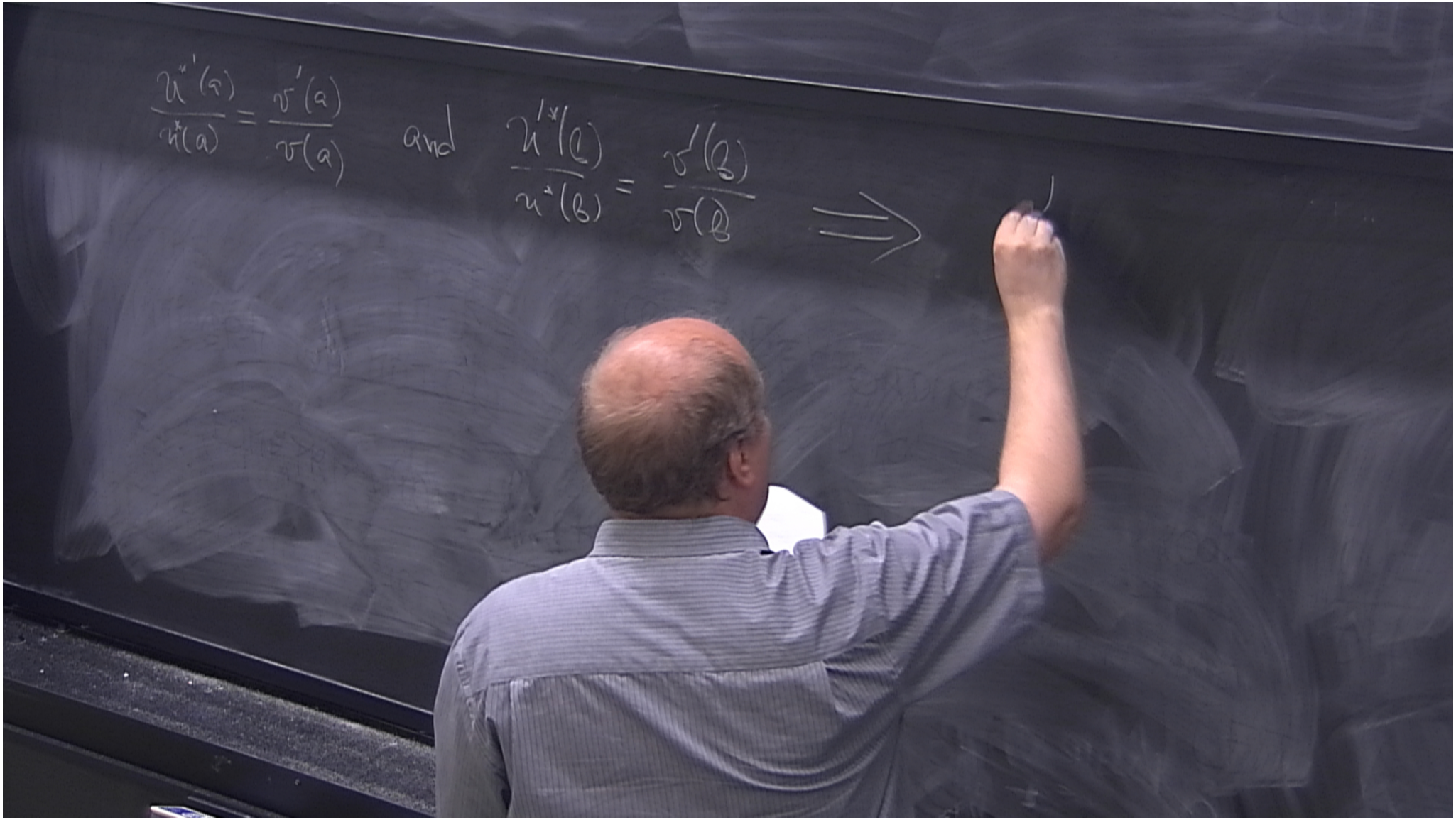

Sturm-Liouville problem
$L = p_0 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_2$ , L is self-adjoint wrt $\langle \cdot, \cdot \rangle = 1$, if and only if $p_0' = p_1$.
meaning $(*x)$ $L = \frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_2$

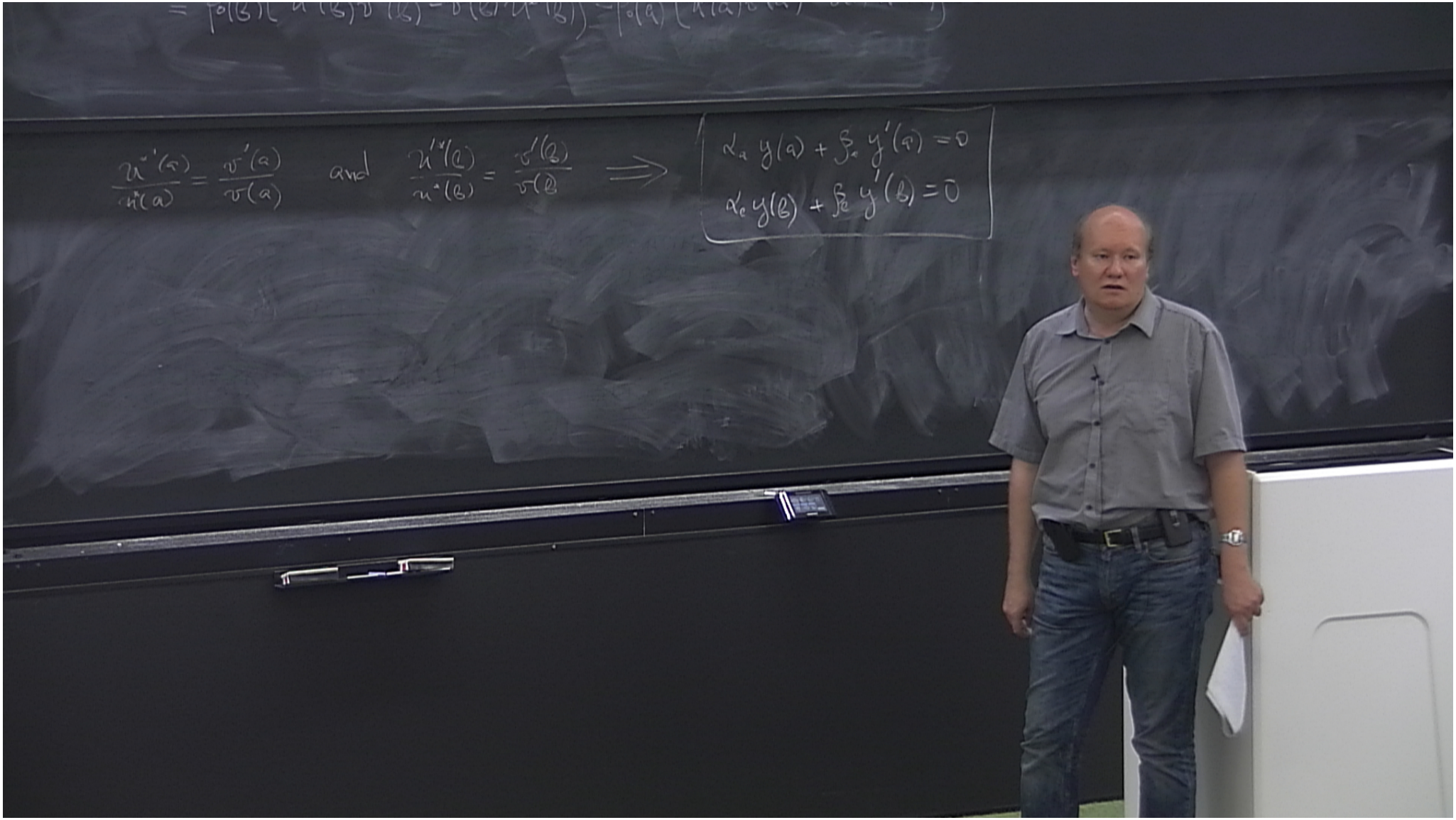

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When $L v$ truly self-adjoint?
$\langle u, Lv \rangle_{w=1} - \langle L^{+} u, v \rangle_{w=1} = \int_a^b u^* \left[ \frac{d}{dx}\left(p_0 \frac{dv}{dx}\right) + p_2 v \right] dx - \int_a^b \left[ \frac{d}{dx}\left(p_0 \frac{du^*}{dx}\right) + p_2 u^* \right] v \, dx$

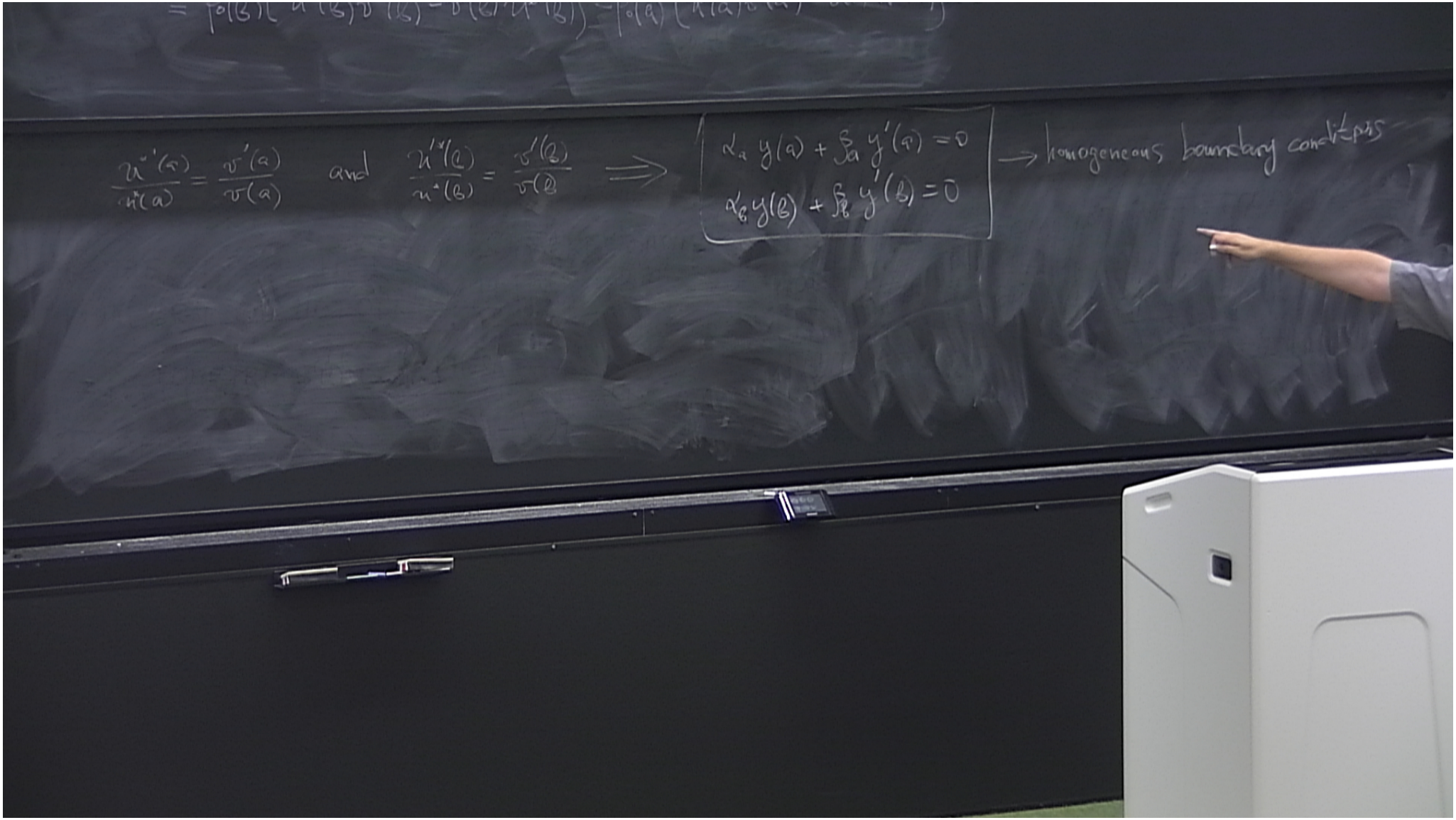

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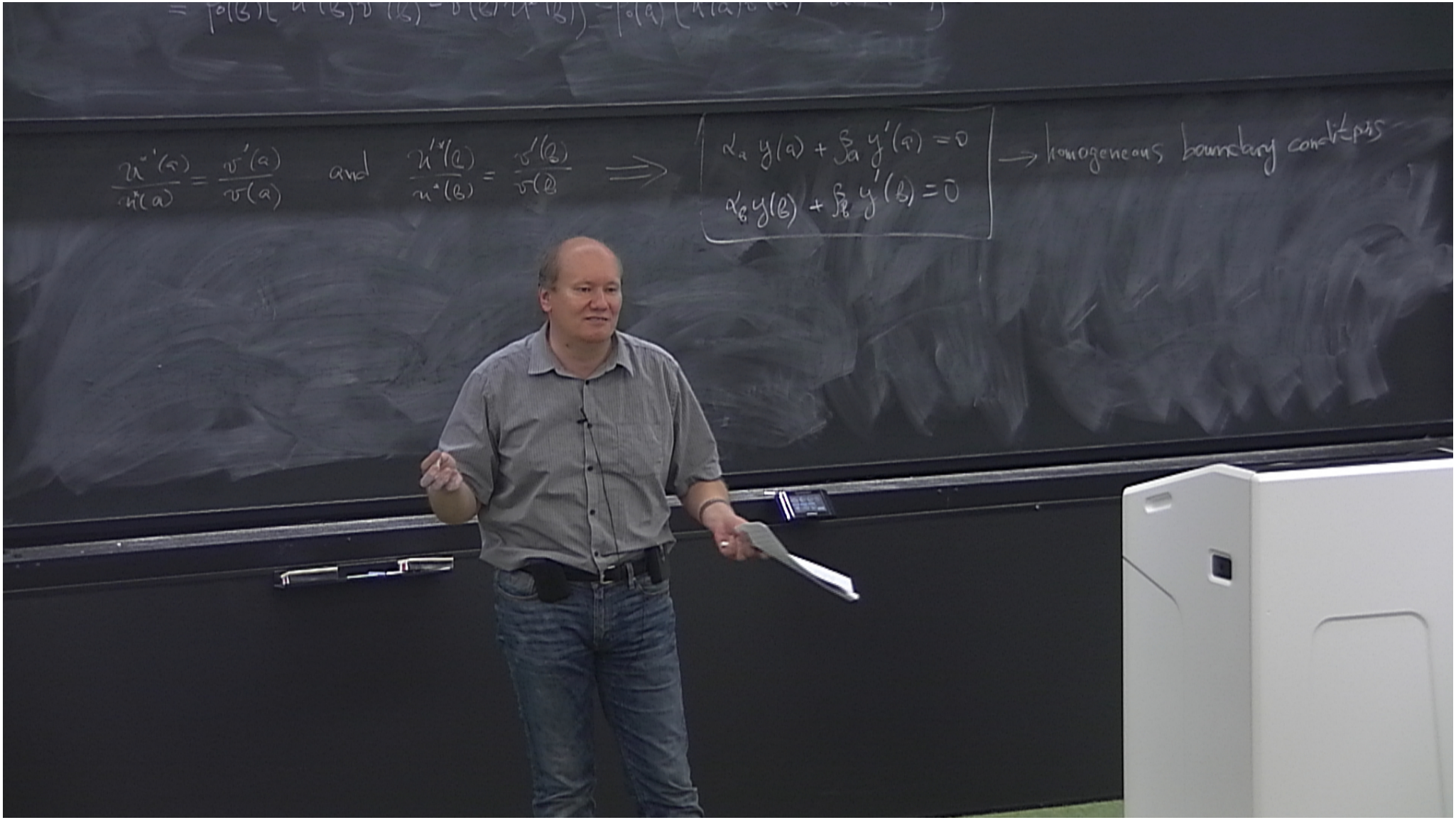

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$\int_a^b \frac{d}{dx}\left[ u^* p_0 v' - v p_0 u^{*\prime} \right] dx$

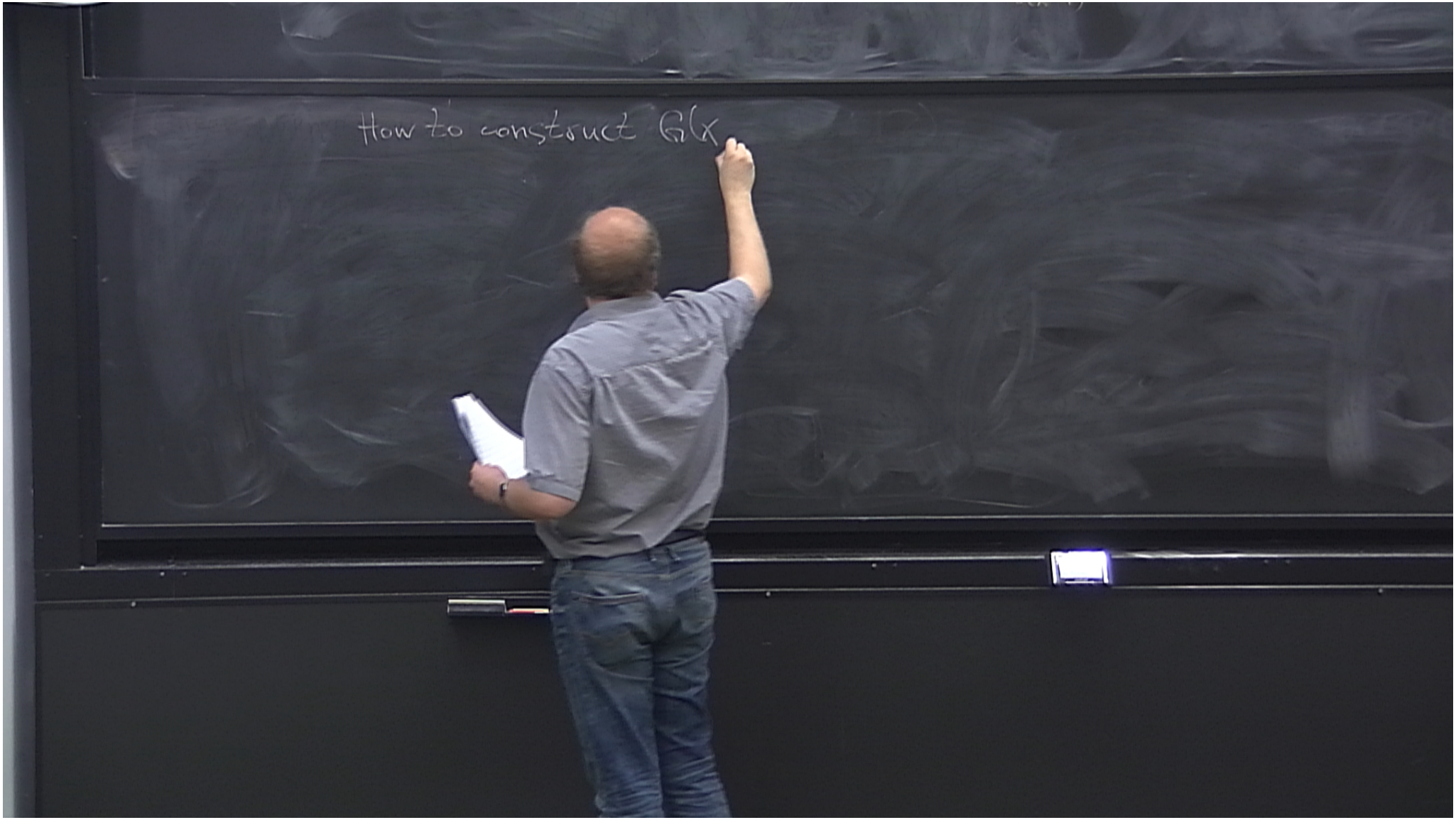

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$\int_a^b \frac{d}{dx}\left[ u^* p_0 v' - v p_0 u^{*\prime} \right] dx =$
$= p_0(b)\left[ u^*(b) v'(b) - v(b) u^{*\prime}(b) \right] - p_0(a)\left[ u^*(a) v'(a) - v(a) u^{*\prime}(a) \right]$

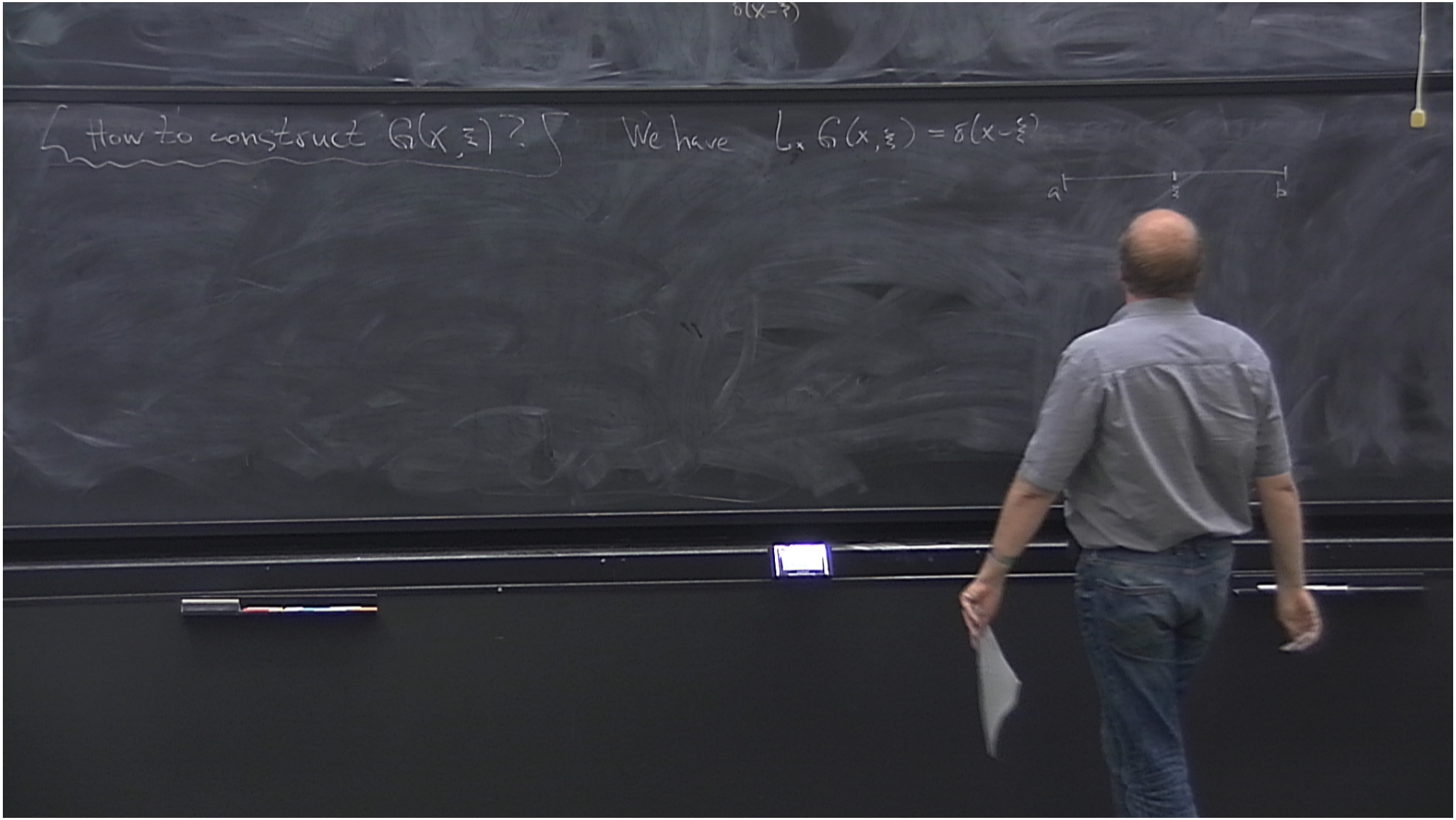

$\frac{u^{*\prime}(a)}{u^{*}(a)} = \frac{v'(a)}{v(a)}$ and $\frac{u^{*\prime}(b)}{u^{*}(b)} = \frac{v'(b)}{v(b)} \Rightarrow$

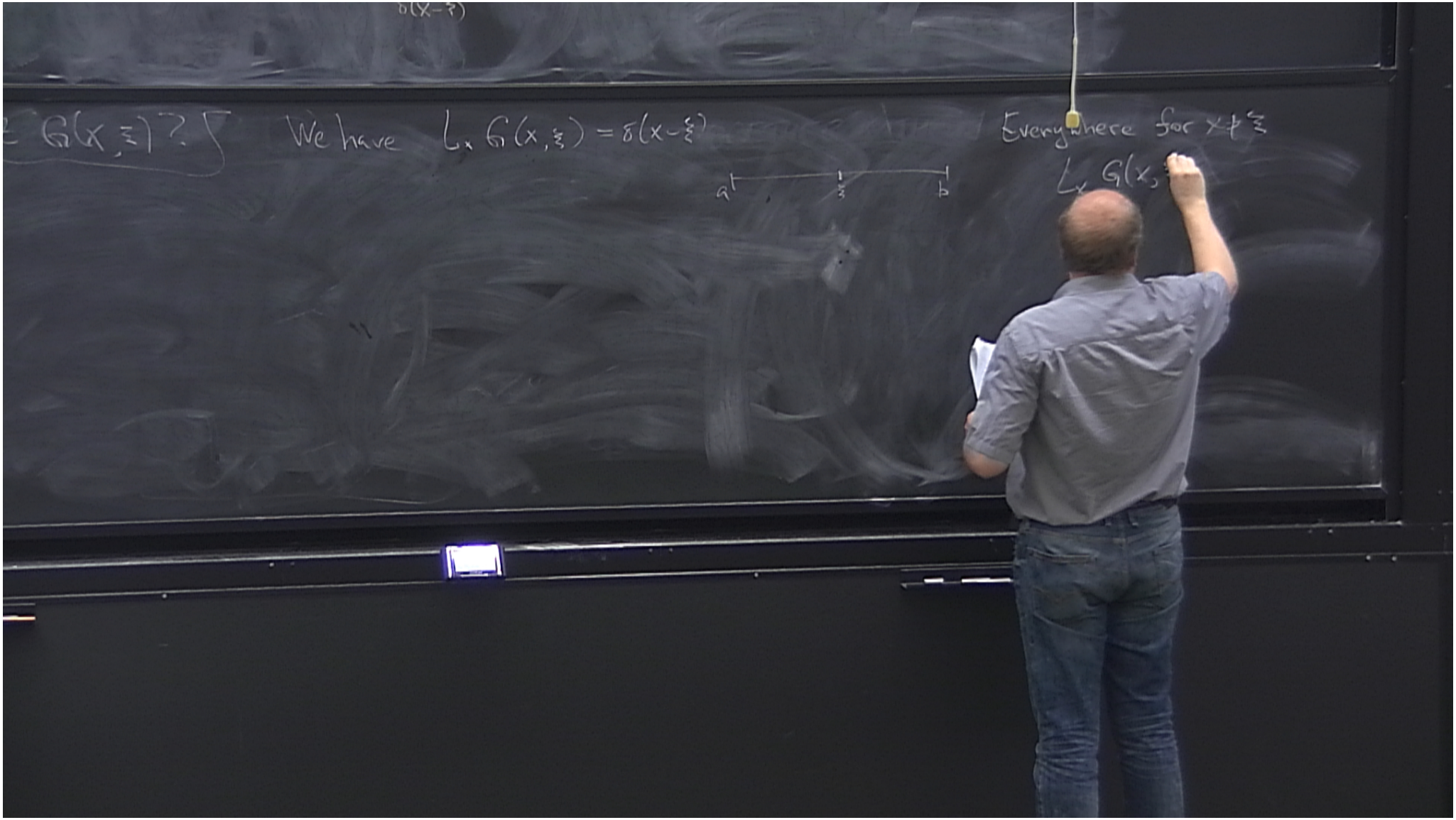

$\frac{u'(a)}{u(a)} = \frac{v'(a)}{v(a)}$ and $\frac{u'(b)}{u(b)} = \frac{v'(b)}{v(b)} \Rightarrow$
$\alpha_1 y(a) + \beta_1 y'(a) = 0$
$\alpha_2 y(b) + \beta_2 y'(b) = 0$

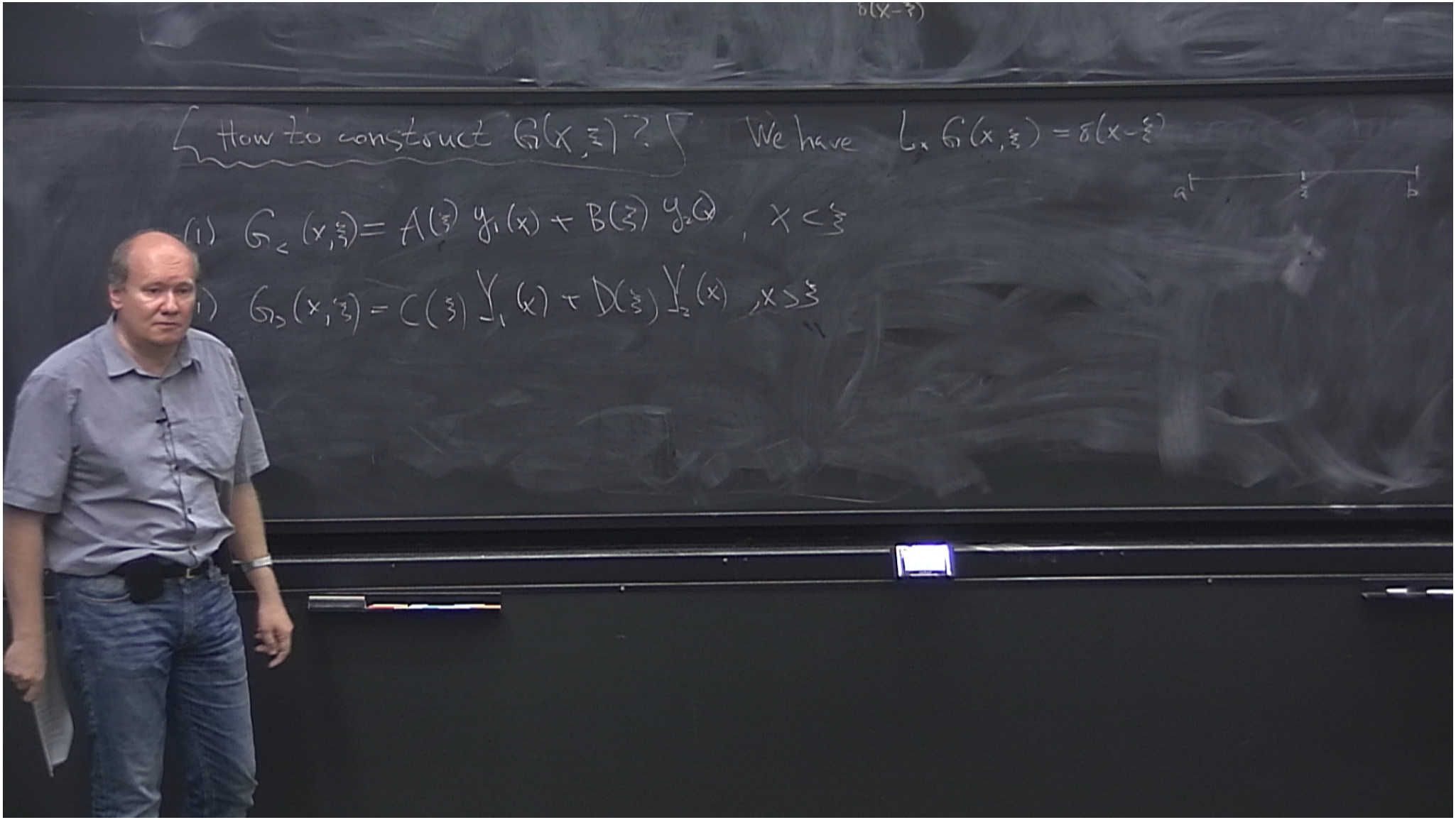

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How to construct G(x


δ(x-ξ)
How to construct G(x,ξ)?
We have L_x G(x,ξ) = δ(x-ξ)
a
ξ
b


We have
Everywhere for

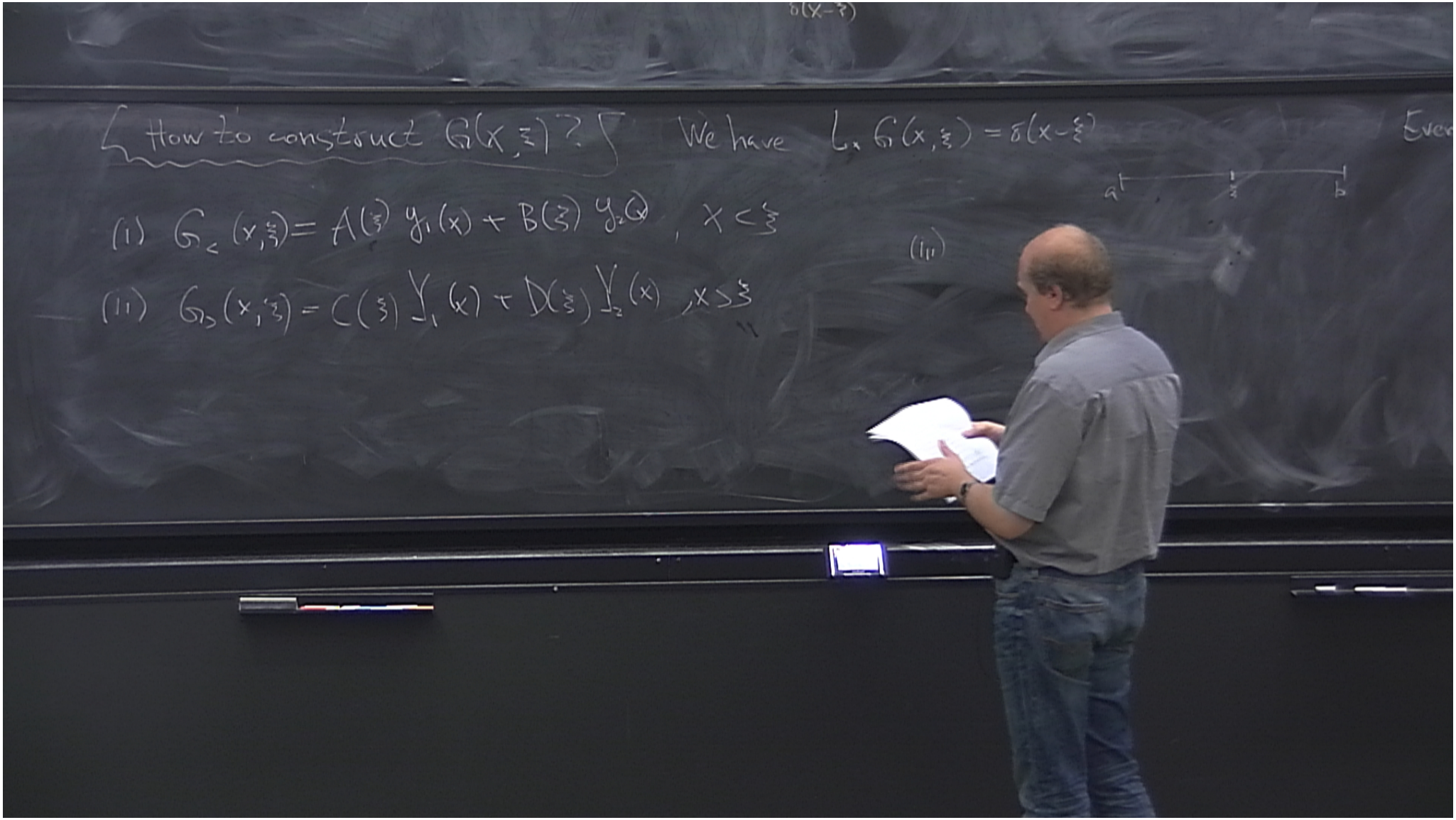

δ(x−ξ)
How to construct G(x,ξ)?
We have L_x G(x,ξ) = δ(x−ξ)
a
ξ
b
(1) G_<(x,ξ) = A(ξ) y_1(x) + B(ξ) y_2(x) , x < ξ
(1) G_>(x,ξ) = C(ξ) y_1(x) + D(ξ) y_2(x) , x > ξ

How to construct $G(x,\xi)$? We have $L_x G(x,\xi) = \delta(x-\xi)$




(i) $G_<(x,\xi) = A(\xi)\, y_1(x) + B(\xi)\, y_2(x)$, $x < \xi$

(ii) $G_>(x,\xi) = C(\xi)\, y_1(x) + D(\xi)\, y_2(x)$, $x > \xi$

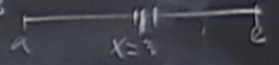

How to construct G(x,ξ)?
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(iii)
a
ξ
b

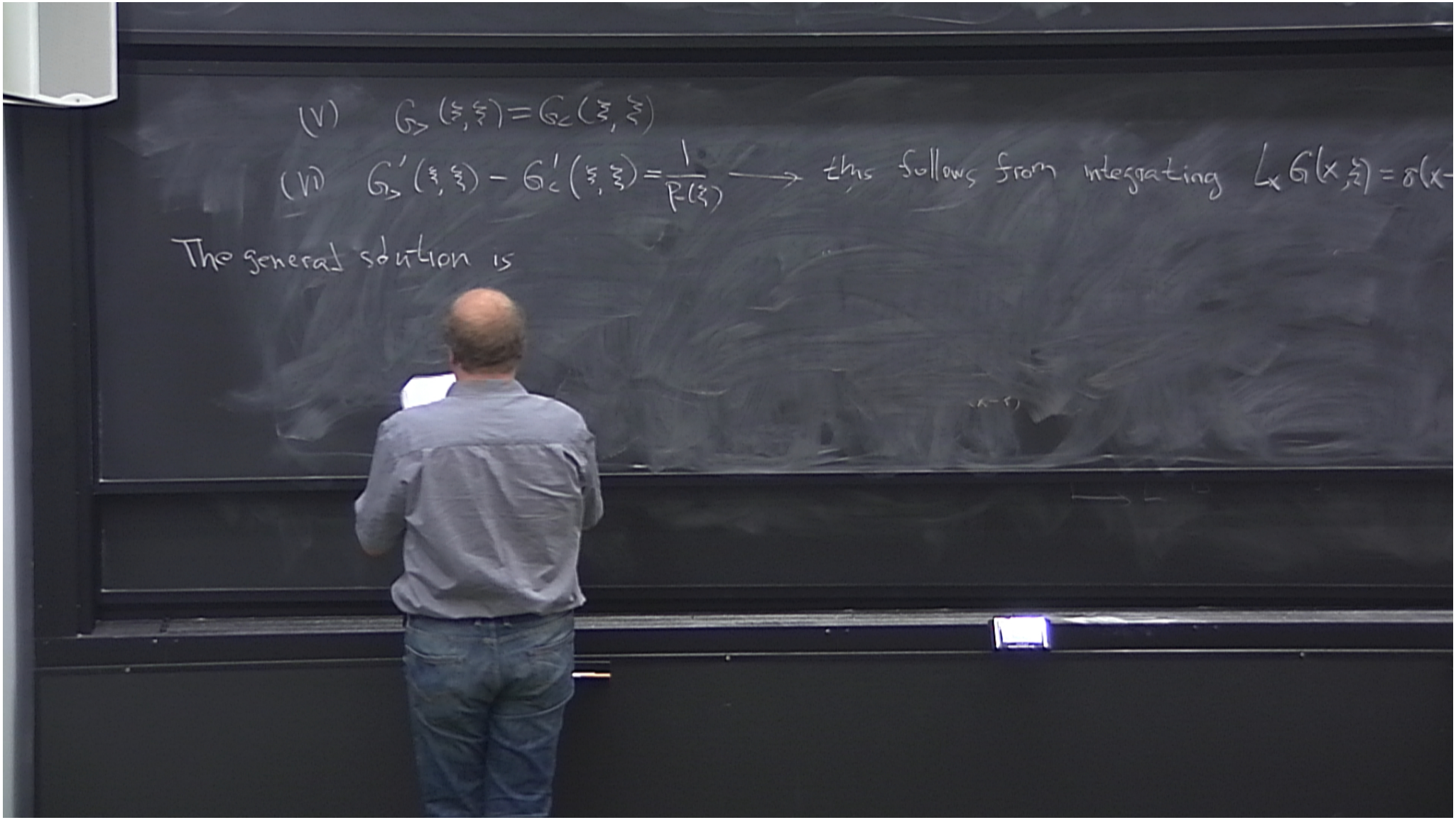
(V) $G_>(\xi,\xi) = G_<(\xi,\xi)$

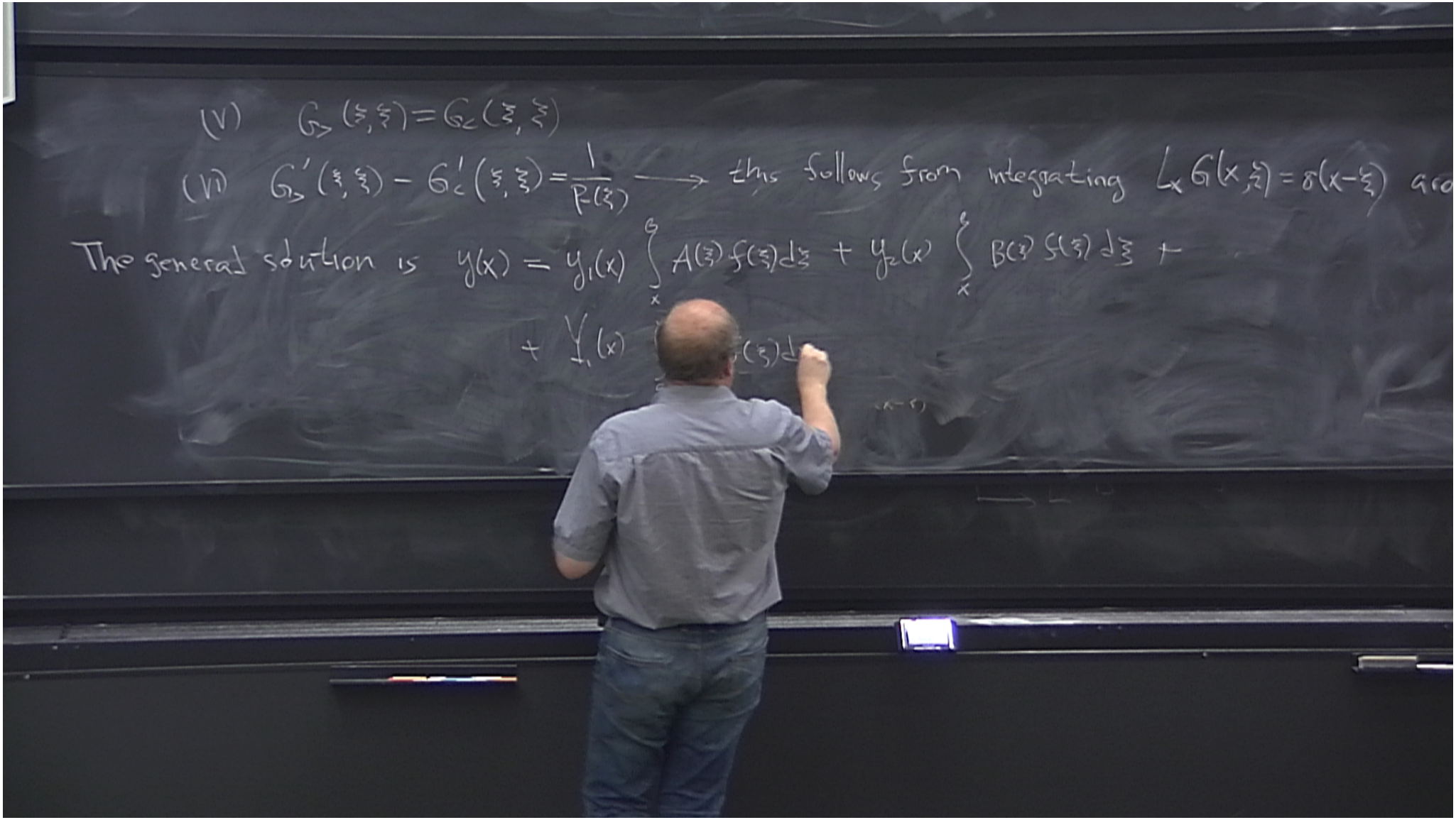
(VI) $G_>'(\xi,\xi) - G_<'(\xi,\xi) = \frac{1}{P(\xi)}$ ⟶ this follows from integrating $L_x G(x,\xi) = \delta(x-\xi)$ around

$G_2(\xi,\xi)$

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The general solution is

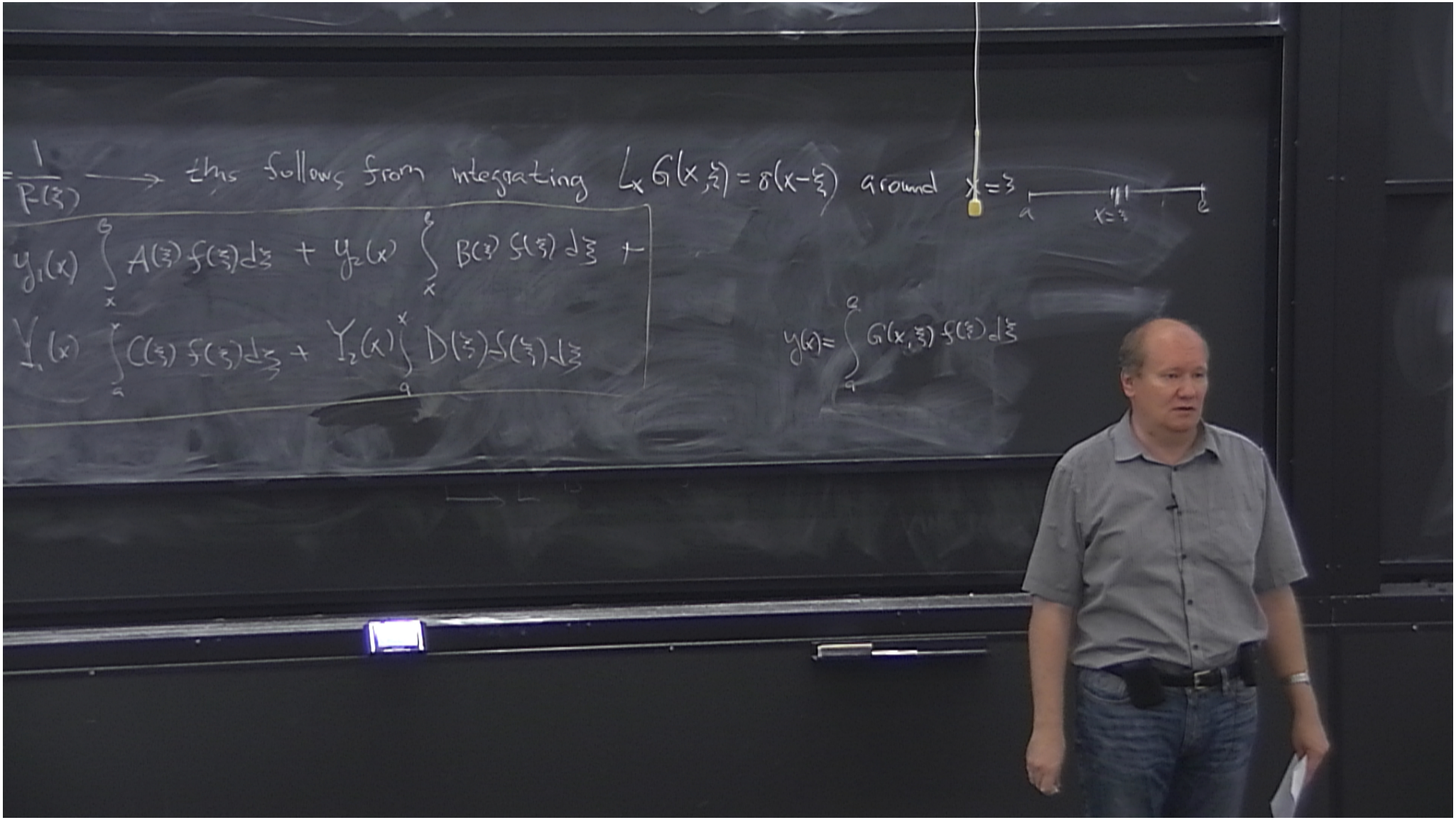

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(VI) $G_>'(\xi,\xi) - G_<'(\xi,\xi) = \frac{1}{P(\xi)}$ → this follows from integrating $L_x G(x,\xi) = \delta(x-\xi)$ aro
The general solution is $y(x) = y_1(x)\int_x^b A(\xi) f(\xi) d\xi + y_2(x)\int_x^b B(\xi) f(\xi) d\xi +$
$+ y_1(x)$ $(\xi) d\xi$

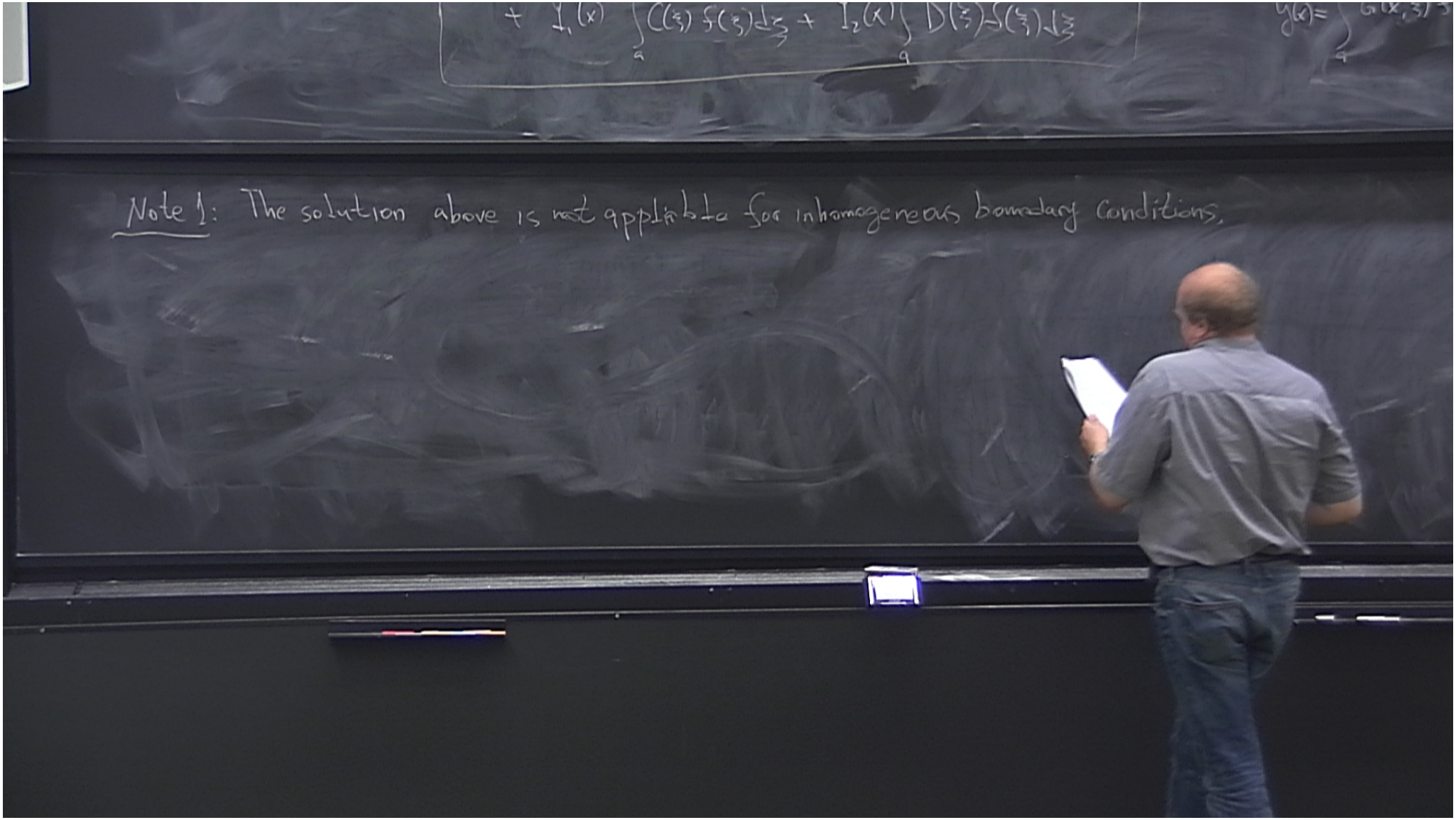
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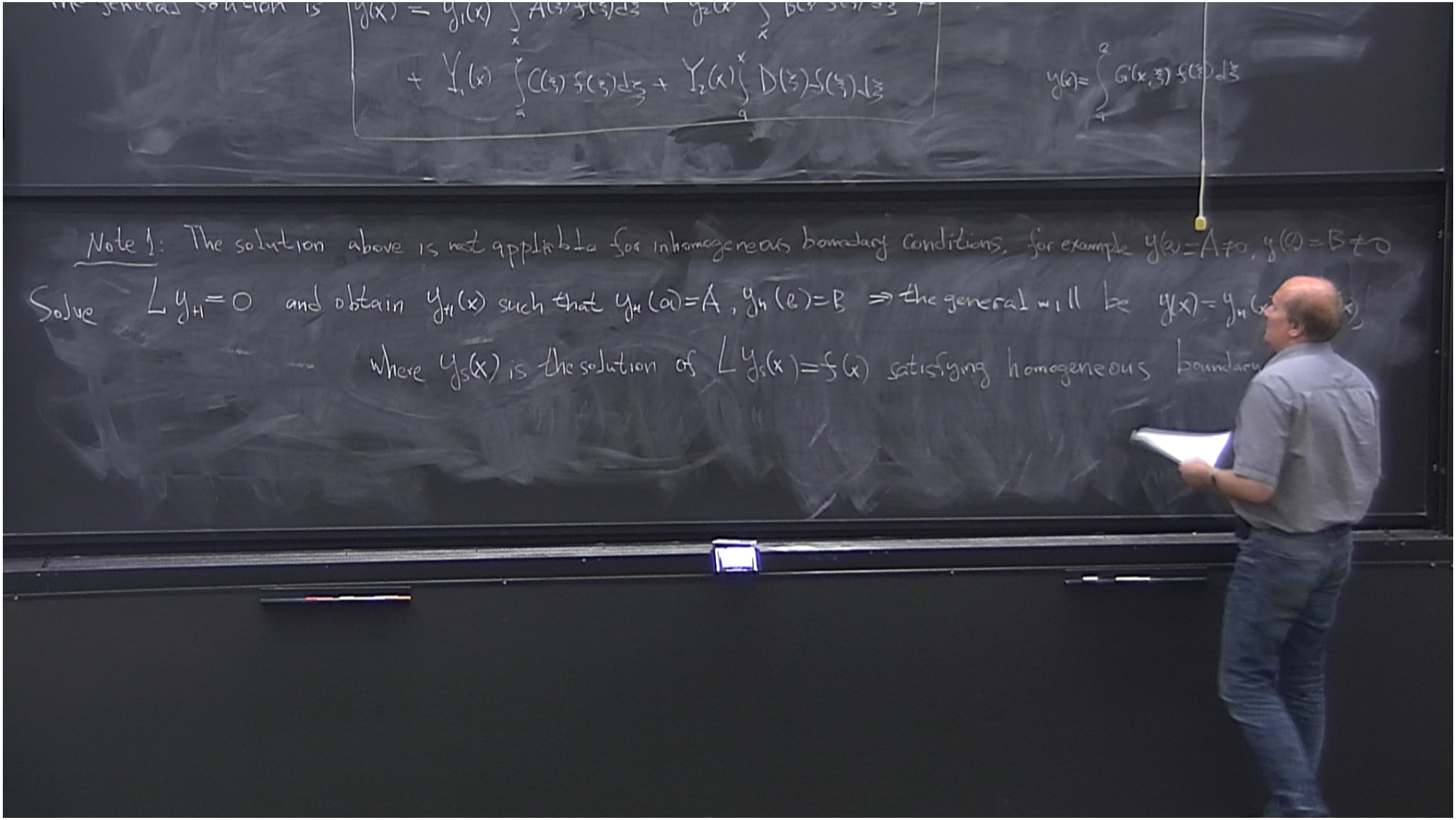
(VI) $G_>'(\xi,\xi) - G_<'(\xi,\xi) = \frac{1}{F(\xi)}$ ⟶ this follows from integrating $L_x G(x,\xi) = \delta(x-\xi)$ aro

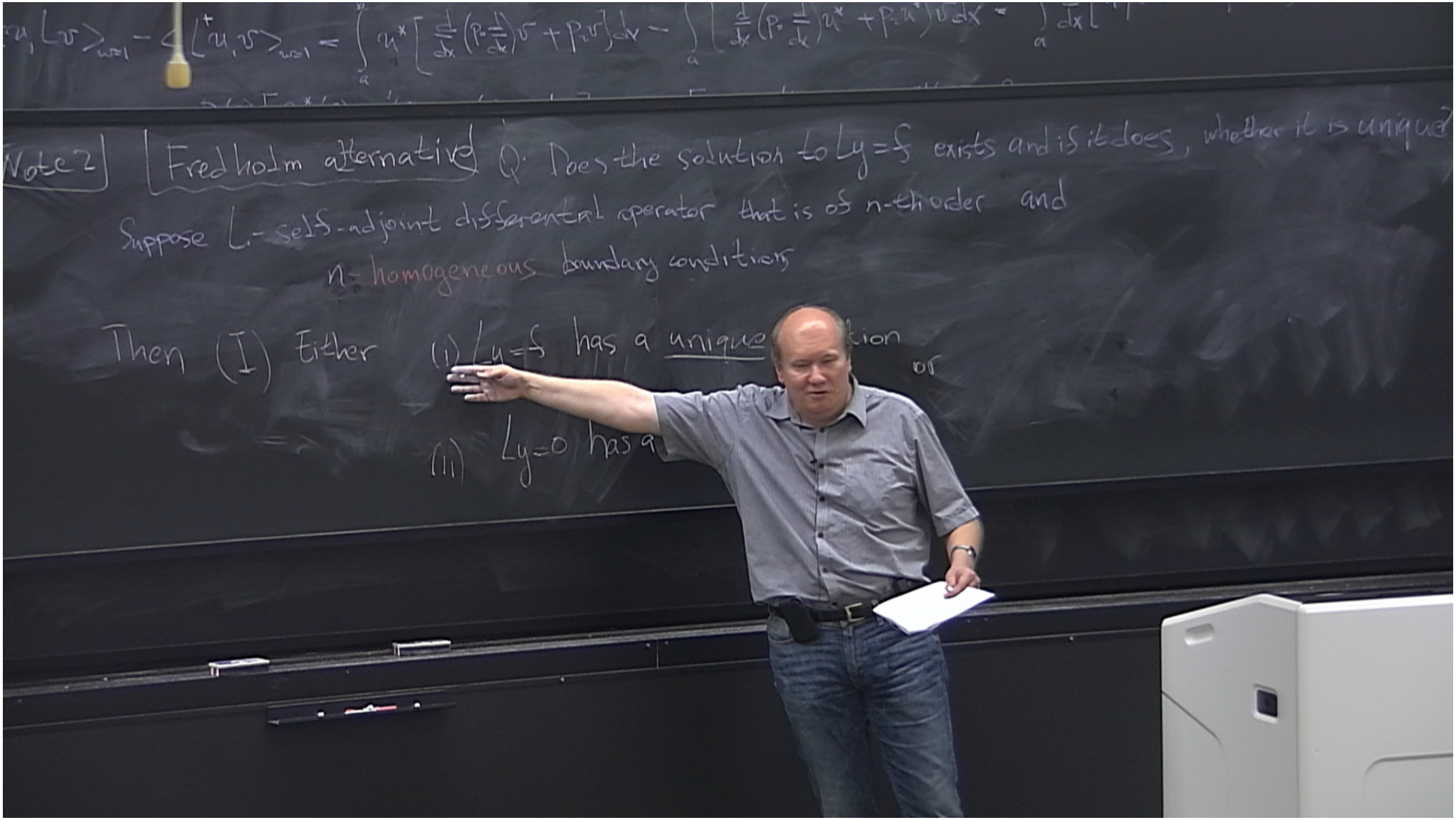
The general solution is

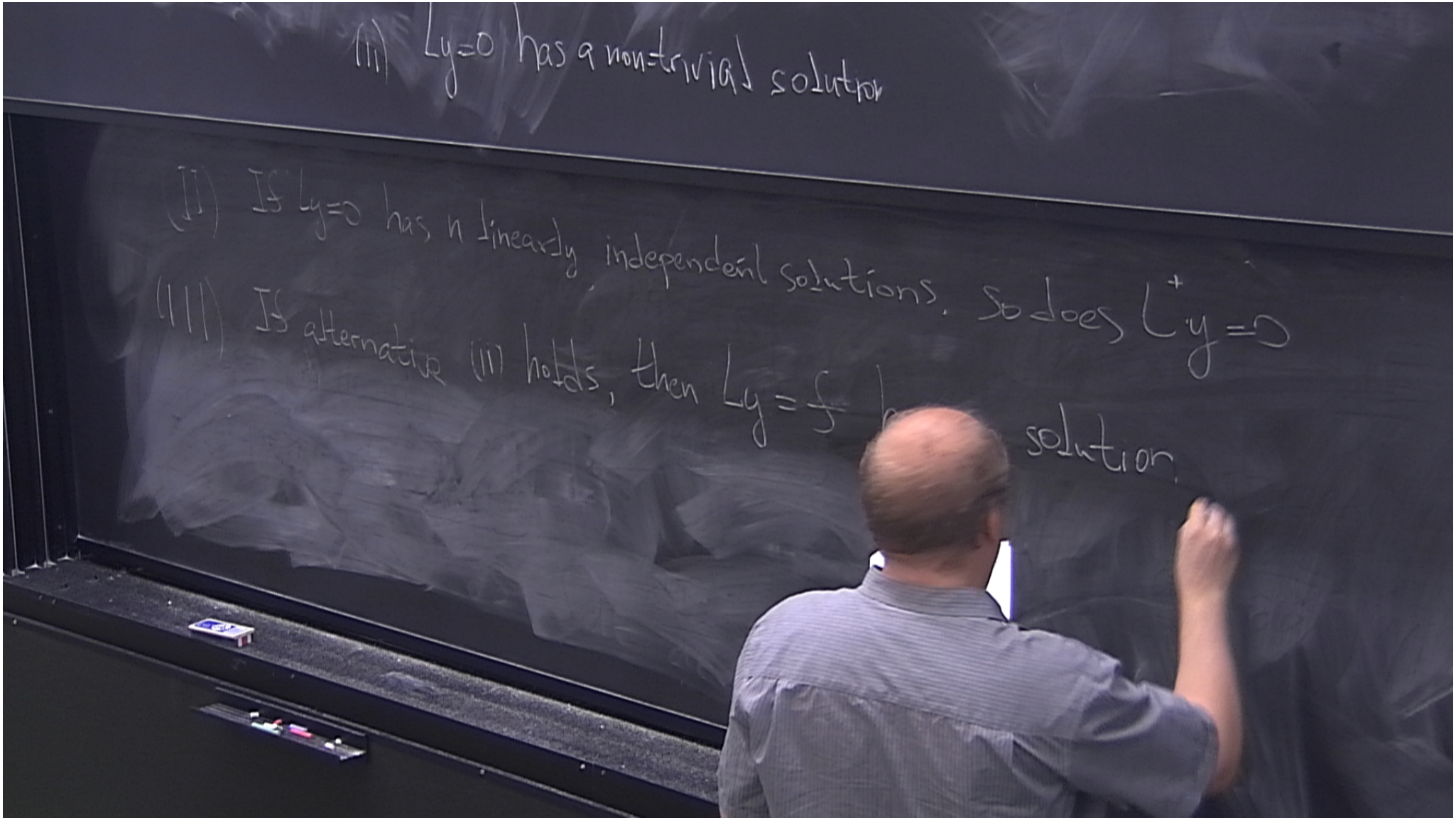
$$y(x) = y_1(x)\int_x^b A(\xi) f(\xi)\,d\xi + y_2(x)\int_x^b B(\xi) f(\xi)\,d\xi +$$
$$+ Y_1(x)\int_a^x C(\xi) f(\xi)\,d\xi + Y_2(x)\int_a^x D(\xi) f(\xi)\,d\xi$$

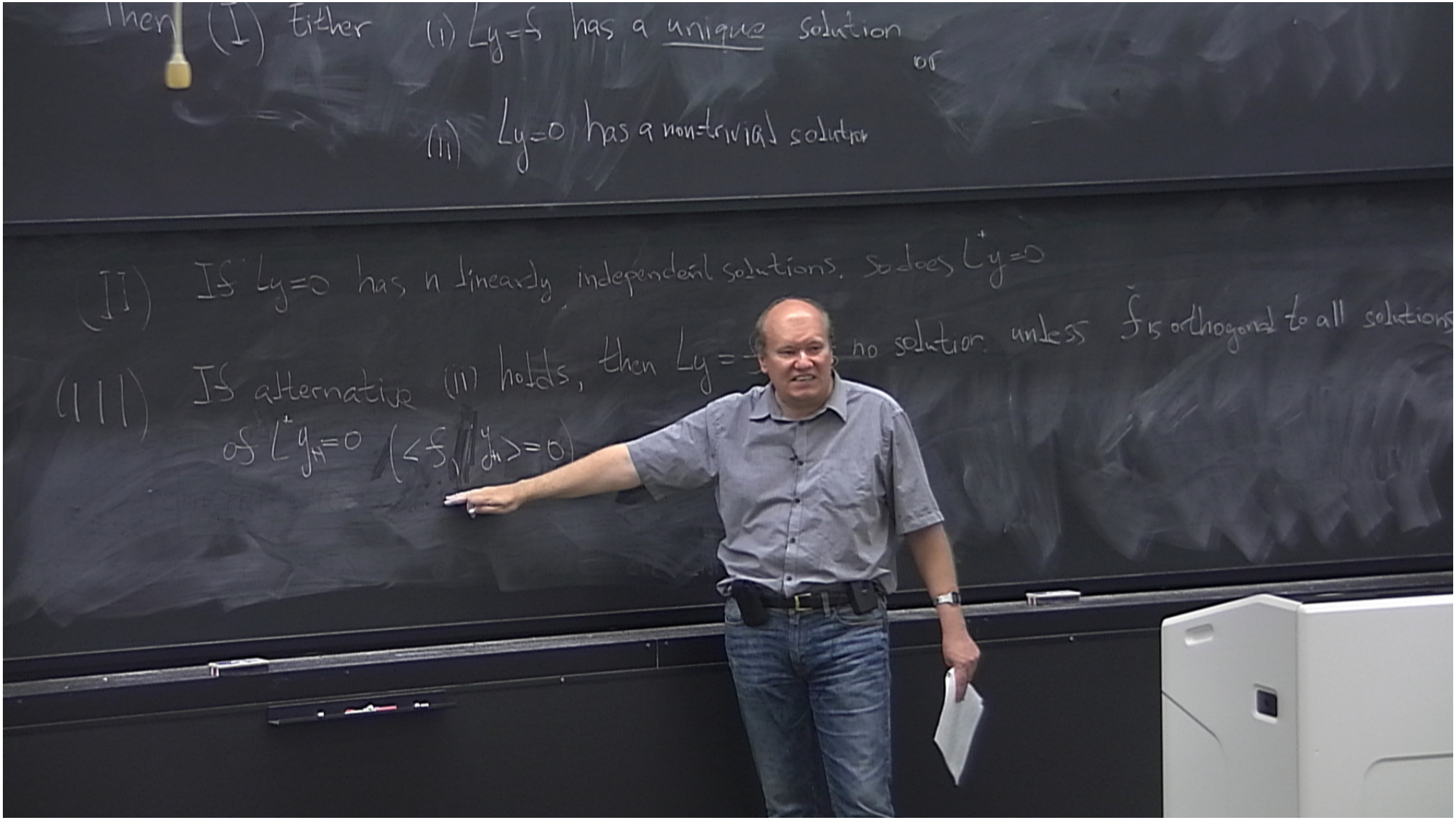

this follows from integrating $L_x G(x,\xi) = \delta(x-\xi)$ around $x=\xi$
$y_1(x) \int_x^b A(\xi) f(\xi)\, d\xi + y_2(x) \int_x^b B(\xi) f(\xi)\, d\xi +$
$Y_1(x) \int_a^x C(\xi) f(\xi)\, d\xi + Y_2(x) \int_a^x D(\xi) f(\xi)\, d\xi$
$y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi$
a
x=ξ
b

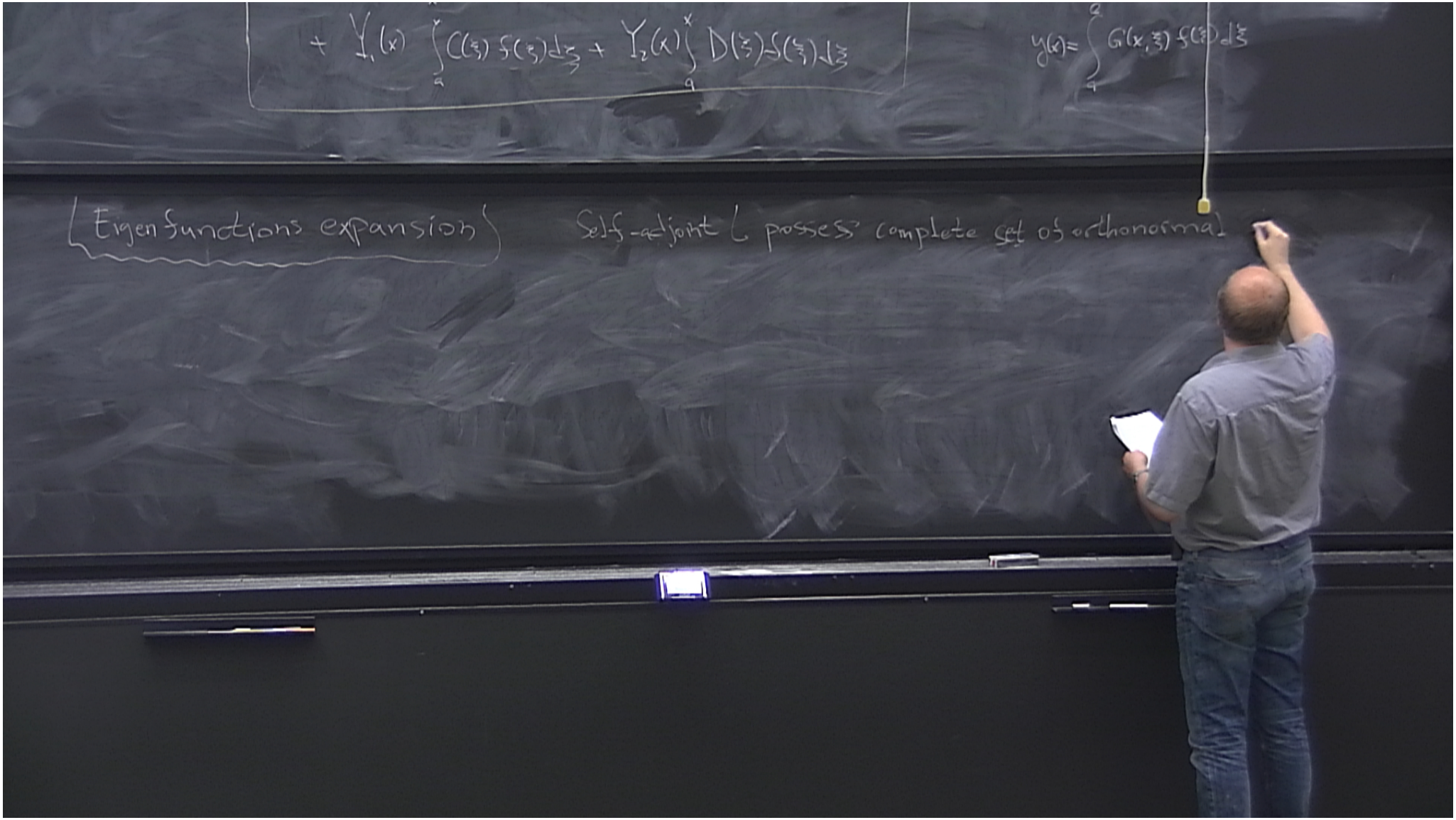

Note 1: The solution above is not applicable for inhomogeneous boundary conditions.

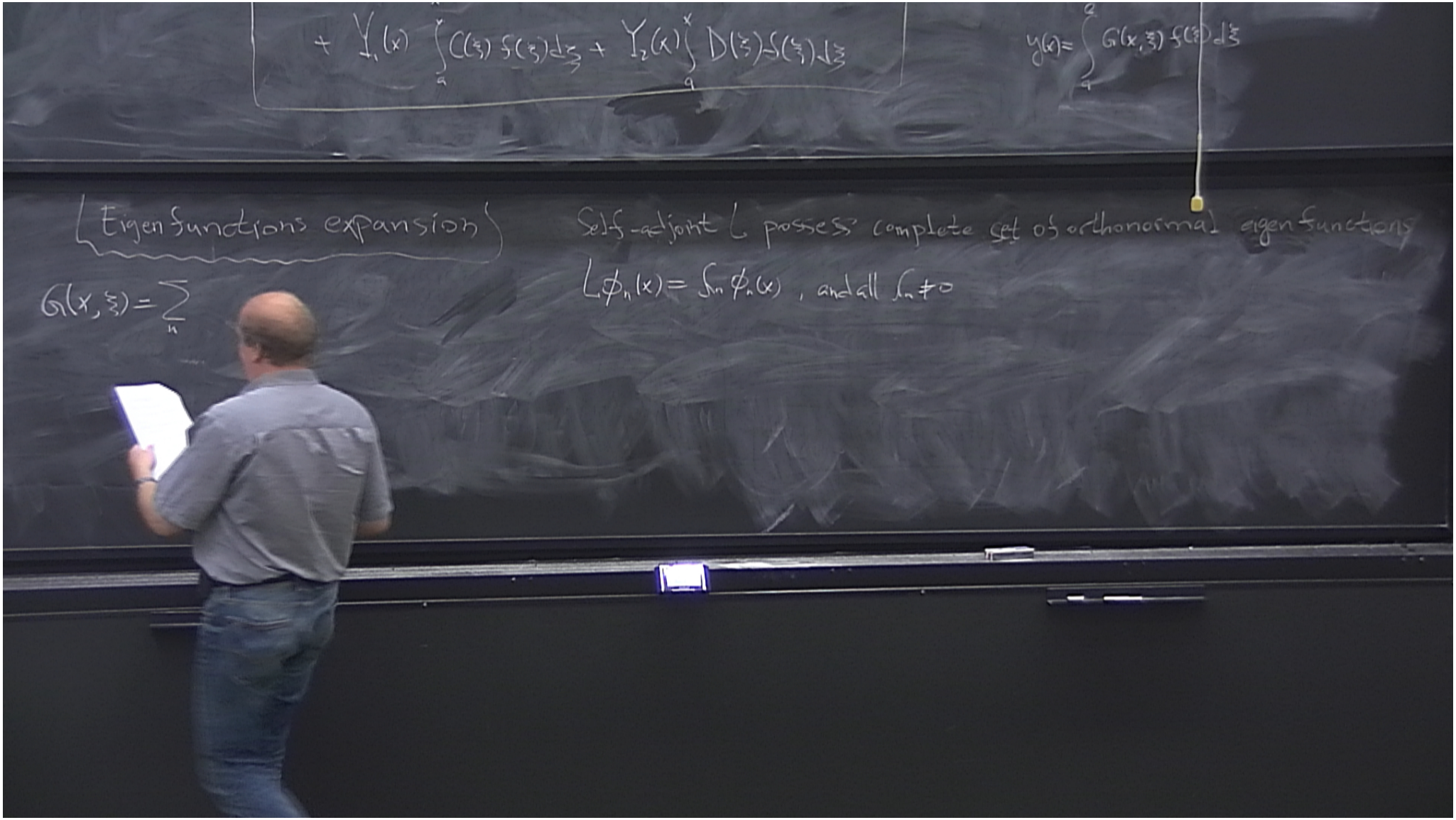

$y(x) = y_1(x) \int_x^{\cdot} A(\xi) f(\xi) d\xi + y_2(x) \int_{\cdot}^{x} B(\xi) f(\xi) d\xi +$
$+ Y_1(x) \int_a^{x} C(\xi) f(\xi) d\xi + Y_2(x) \int_a^{x} D(\xi) f(\xi) d\xi$
$y(x) = \int_a^{a} G(x,\xi) f(\xi) d\xi$
Note 1: The solution above is not applicable for inhomogeneous boundary conditions, for example $y(a) = A \neq 0$, $y(a) = B \neq 0$
Solve $L y_H = 0$ and obtain $y_H(x)$ such that $y_H(a) = A$, $y_H(e) = B$ $\Rightarrow$ the general will be $y(x) = y_H(x)$
where $y_S(x)$ is the solution of $L y_S(x) = f(x)$ satisfying homogeneous

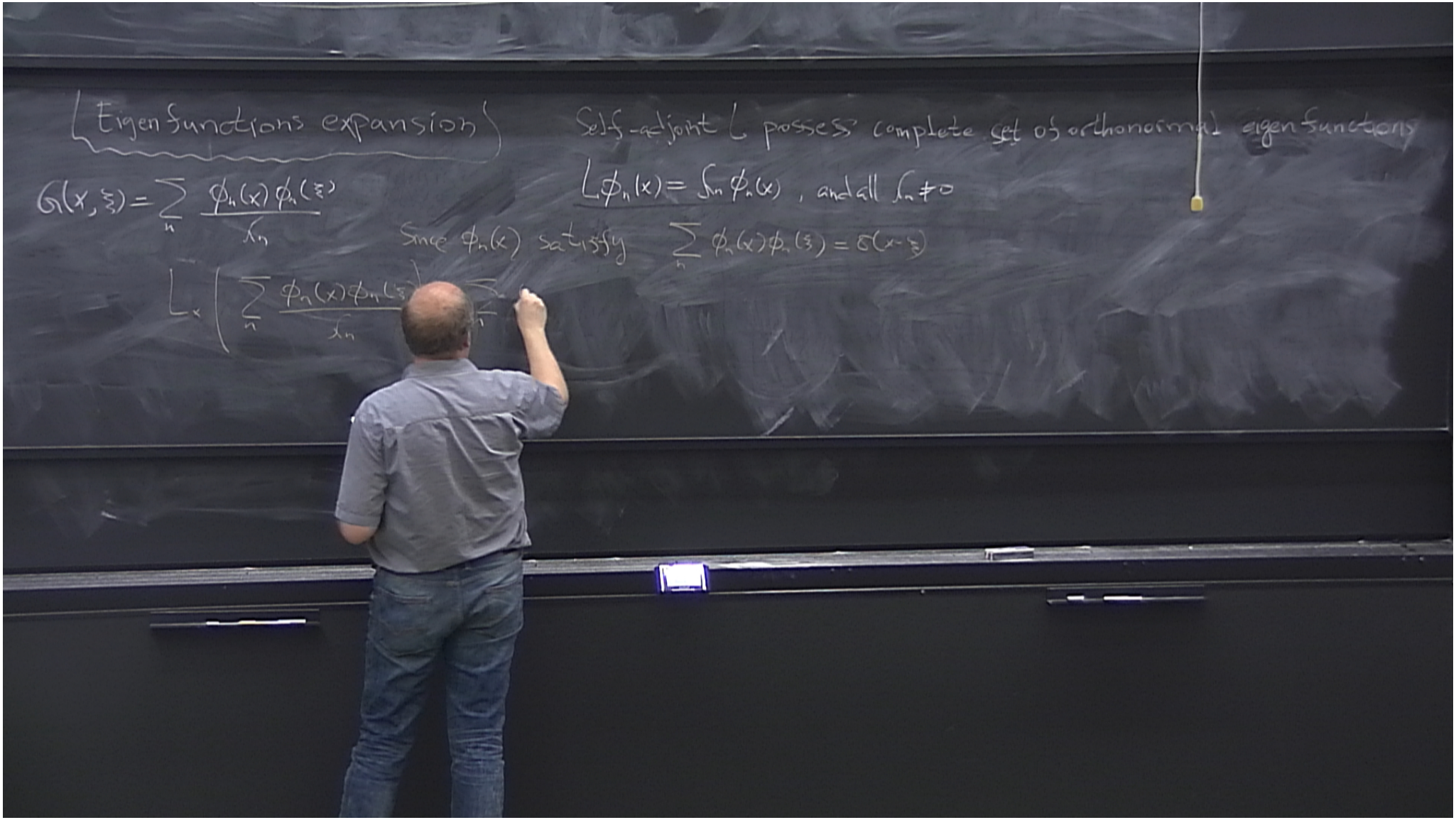

Note 2
Fredholm alternative
Q: Does the solution to Ly=f exists and if it does, whether it is unique
Suppose L - self-adjoint differential operator that is of n-th order and
n- homogeneous boundary conditions
Then (I) Either (i) Ly=f has a unique
or
(ii) Ly=0 has a

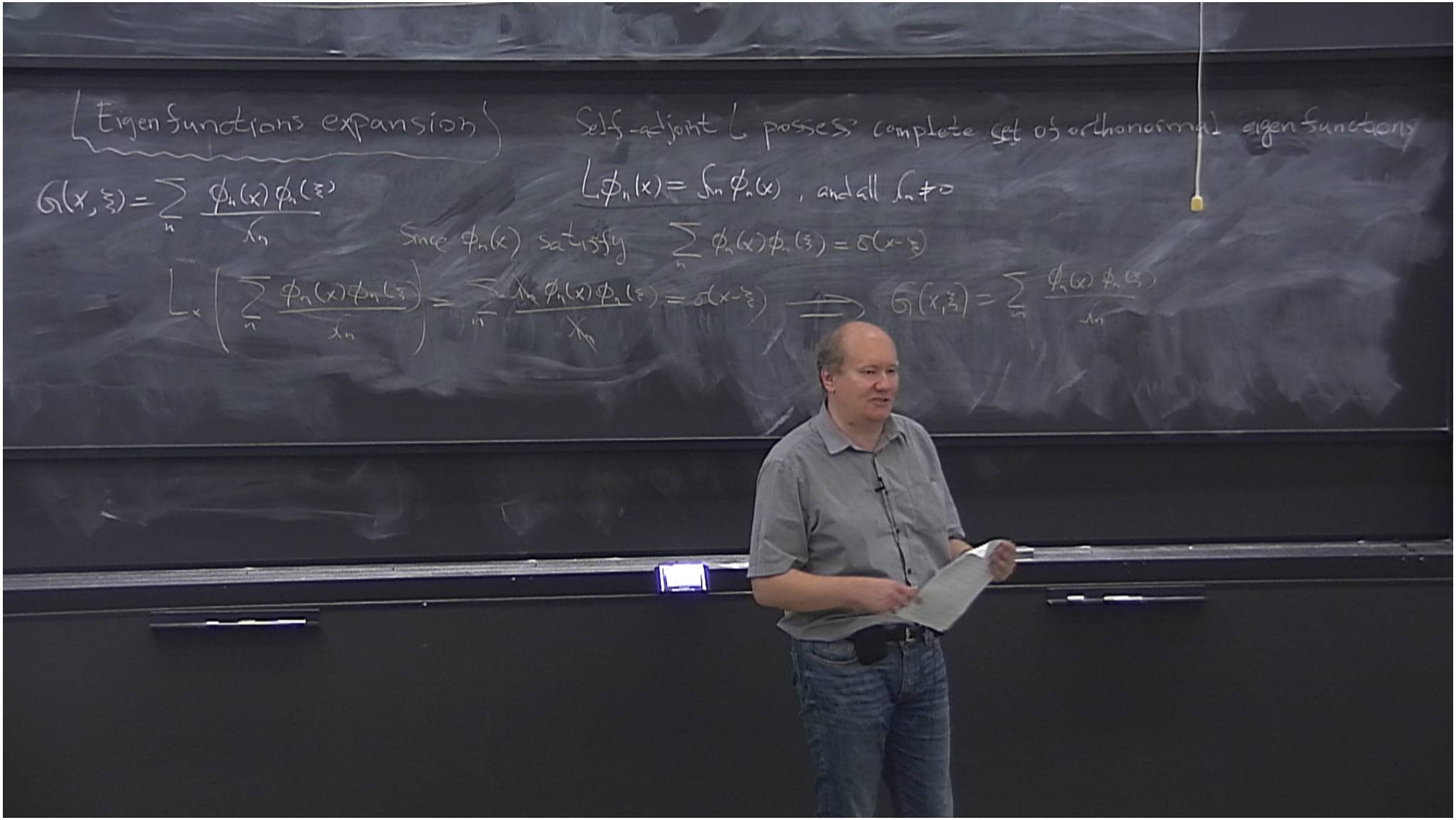

(II) $Ly=0$ has a non-trivial solution
(I) If $Ly=0$ has $n$ linearly independent solutions, so does $L^+y=0$
(III) If alternative (II) holds, then $Ly=f$
solution


Then (I) Either (i) $Ly=f$ has a unique solution
or
(ii) $Ly=0$ has a non-trivial solution
(II) If $Ly=0$ has $n$ linearly independent solutions. So does $L^+y=0$
(III) If alternative (II) holds, then $Ly=f$ has no solution unless $f$ is orthogonal to all solutions
of $L^+y_n=0$ $(\langle f, y_n \rangle = 0)$


+ Y1(x) ∫ C(ξ) f(ξ) dξ + Y2(x) ∫ D(ξ) f(ξ) dξ
y(x) = ∫ G(x,ξ) f(ξ) dξ
Eigenfunctions expansion
Self-adjoint ⟹ possess complete set of orthonormal


+ $Y_1(x)\int_a^x C(\xi) f(\xi) d\xi + Y_2(x)\int_a^x D(\xi) f(\xi) d\xi$
$y(x) = \int_a^a G(x,\xi) f(\xi) d\xi$
Eigenfunctions expansion
Self-adjoint $L$ possess complete set of orthonormal eigenfunctions
$G(x,\xi) = \sum_n$
$L\phi_n(x) = f_n \phi_n(x)$, and all $f_n \neq 0$


Eigenfunctions expansion
Self-adjoint L possess complete set of orthonormal eigenfunctions
$G(x,\xi)=\sum_n \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n}$
$L\phi_n(x)=\lambda_n\phi_n(x)$, and all $\lambda_n\neq 0$
Since $\phi_n(x)$ satisfy $\sum_n \phi_n(x)\phi_n(\xi)=\delta(x-\xi)$
$L_x\left(\sum_n \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n}\right)=\sum_n$


Eigenfunctions expansion
Self-adjoint L possess complete set of orthonormal eigenfunctions
$G(x,\xi)=\sum_n \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n}$
$L\phi_n(x)=\lambda_n\phi_n(x)$, and all $\lambda_n\neq 0$
Since $\phi_n(x)$ satisfy $\sum_n \phi_n(x)\phi_n(\xi)=\delta(x-\xi)$
$L_x\left(\sum_n \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n}\right)=\sum_n \phi_n(x)\phi_n(\xi)=\delta(x-\xi)$
$\Rightarrow G(x,\xi)=\sum_n \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n}$