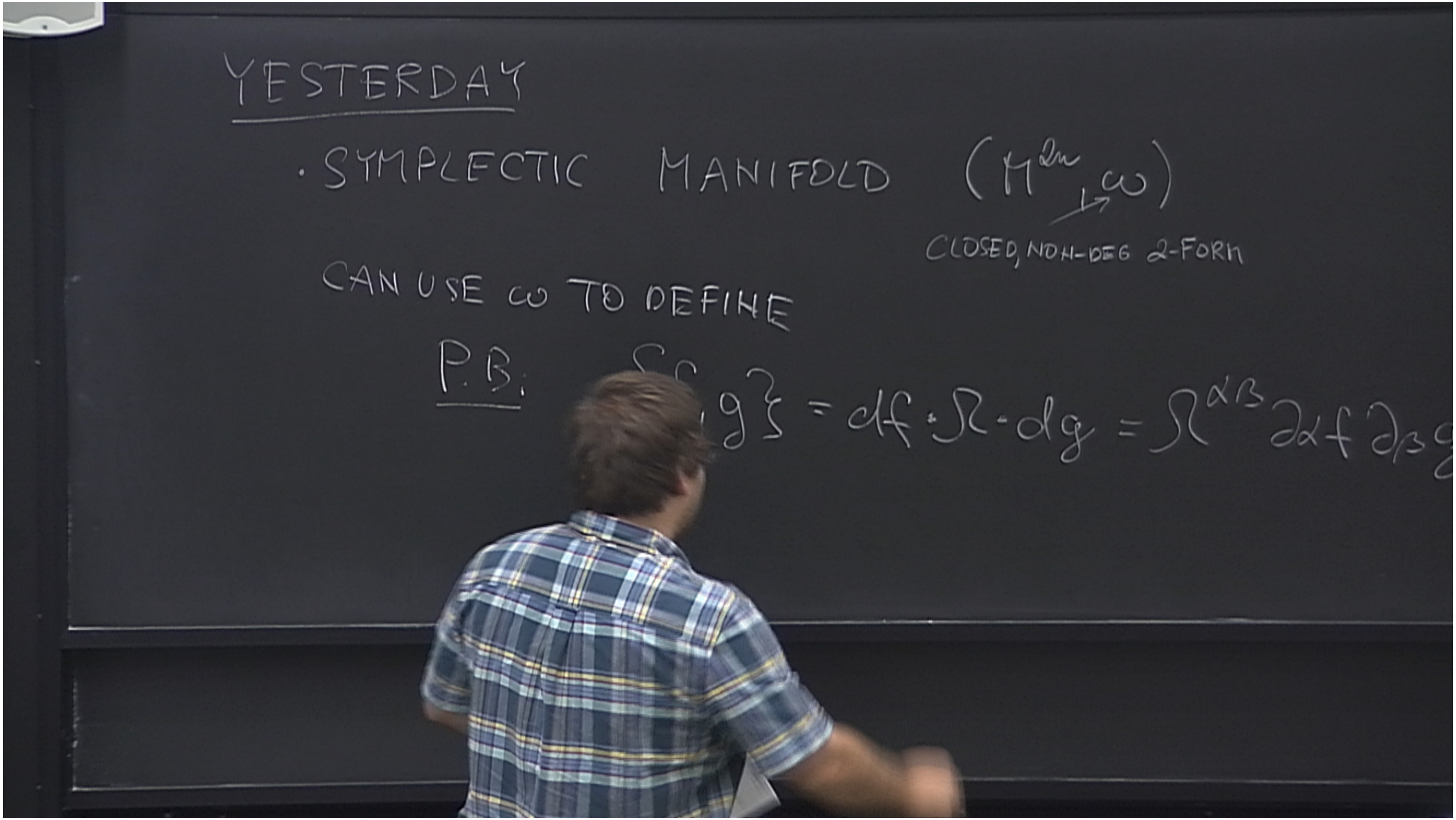
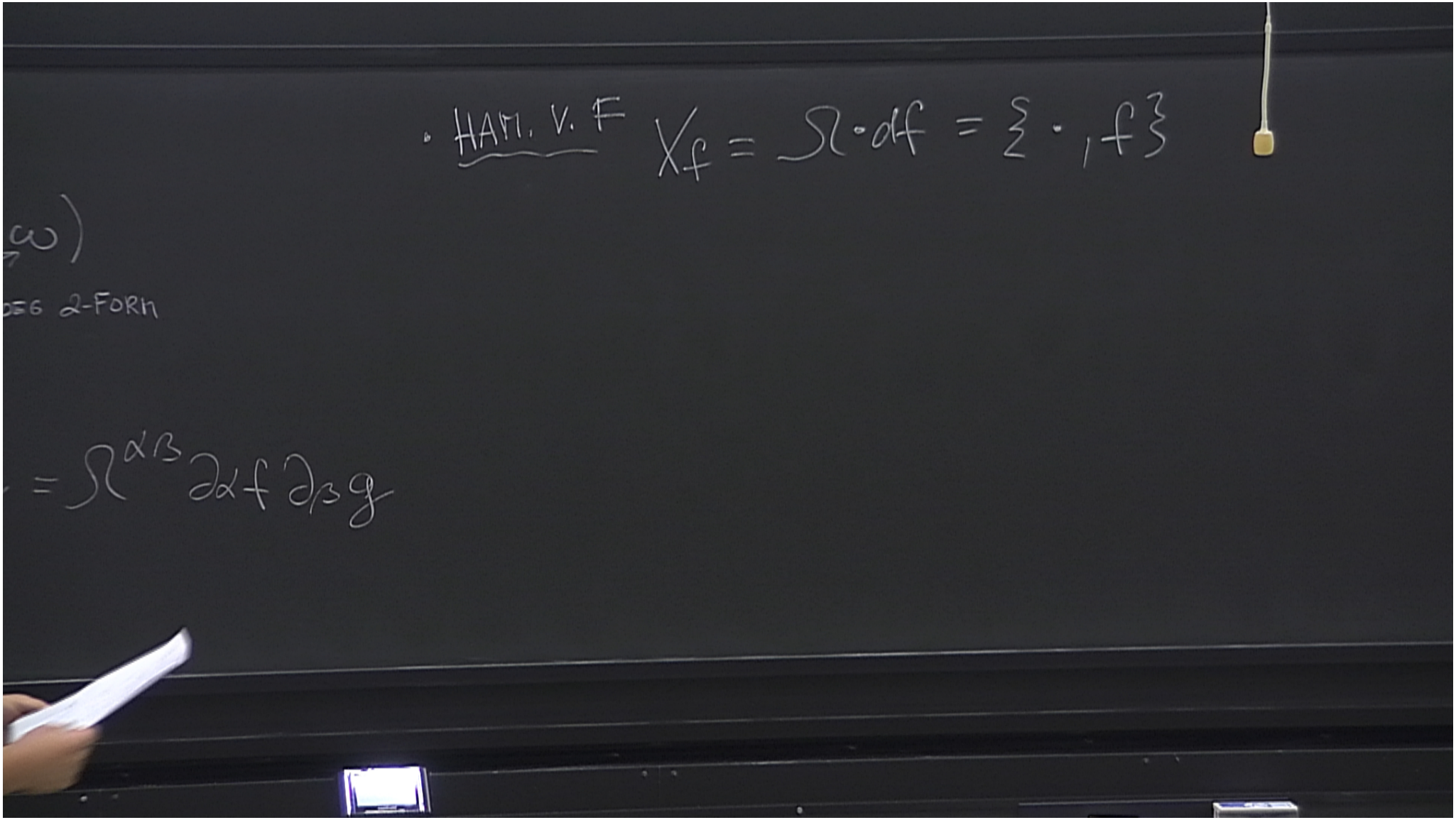
Title: Classical Mechanics - David Kubiznak

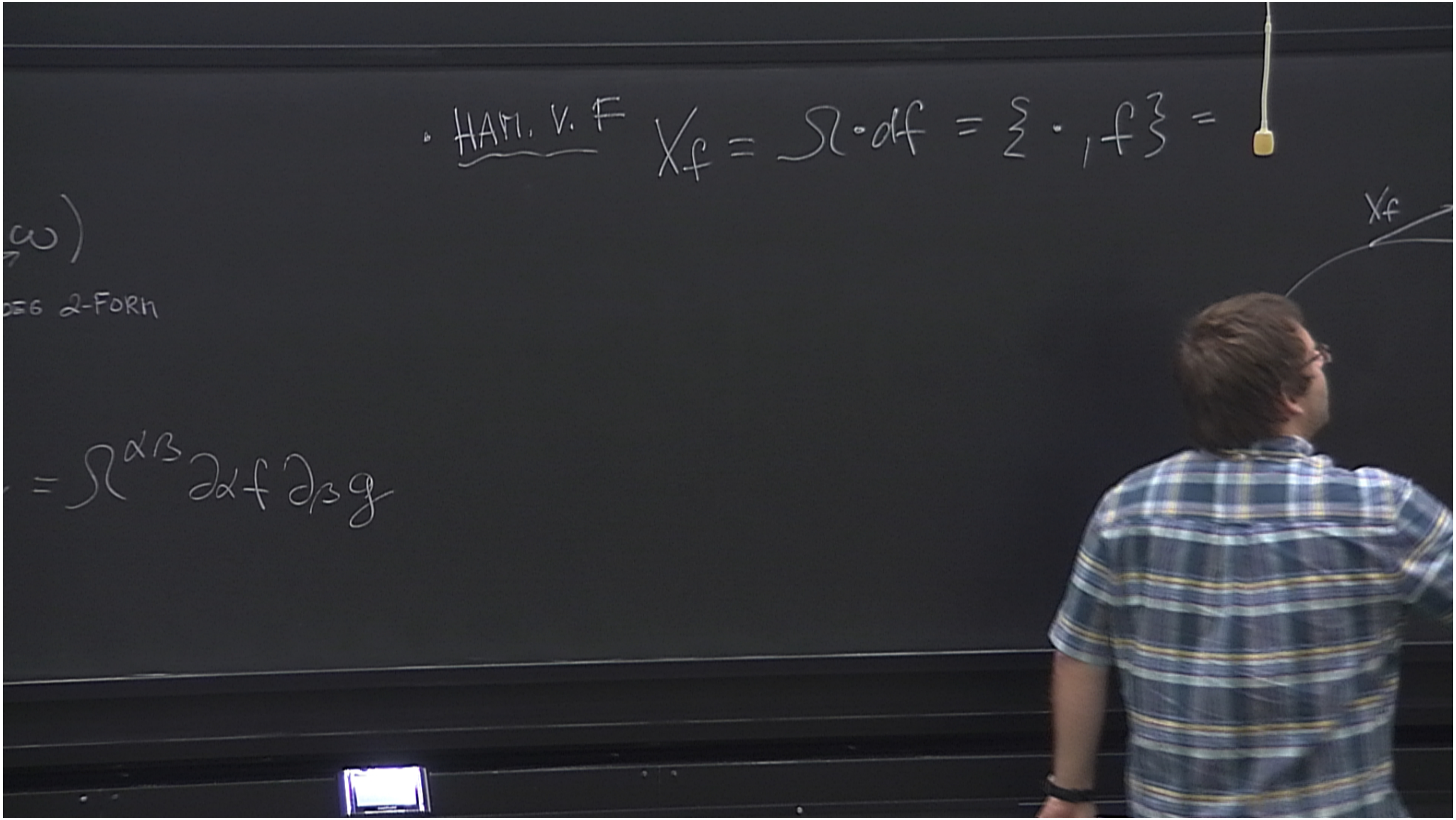
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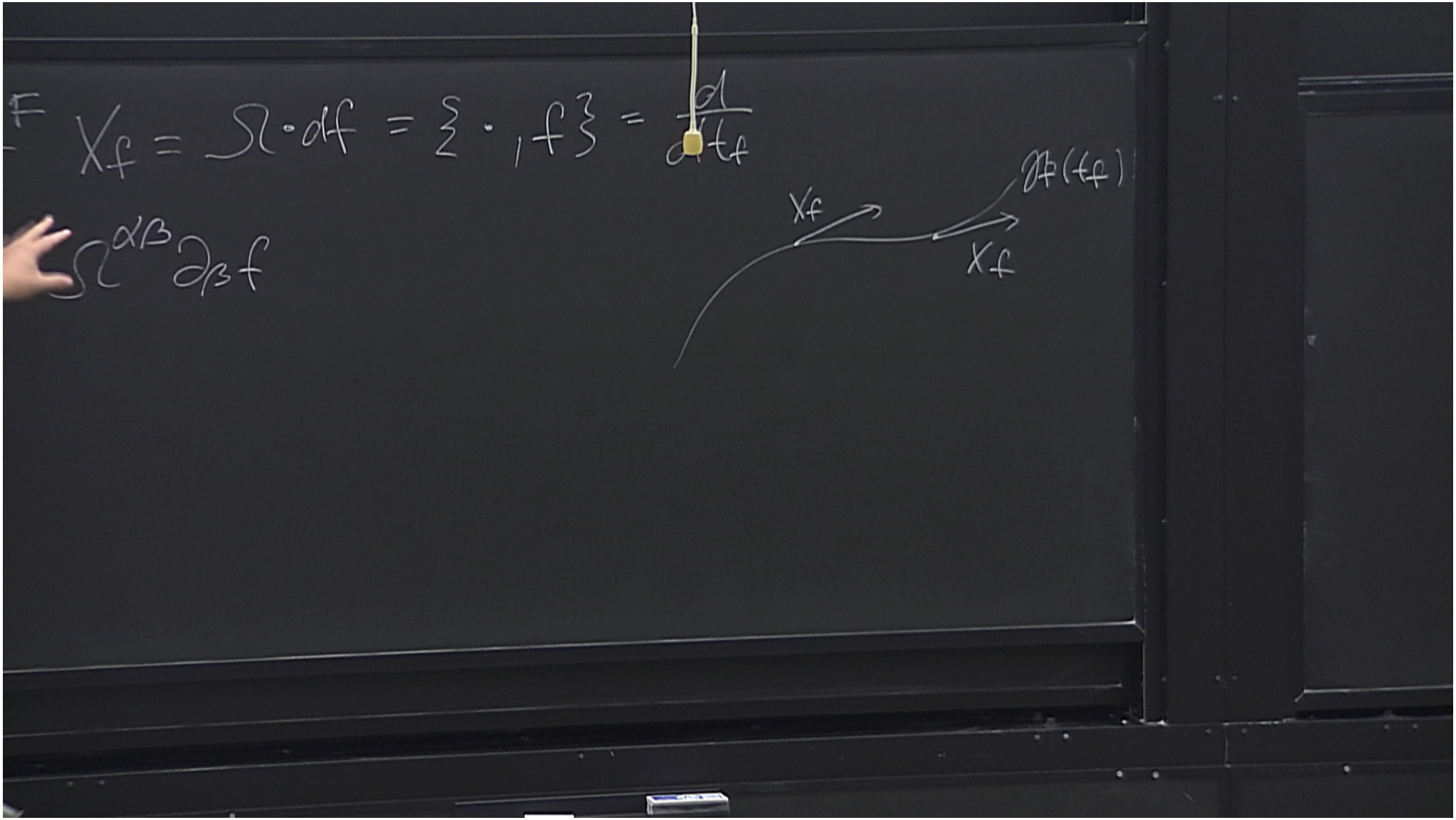
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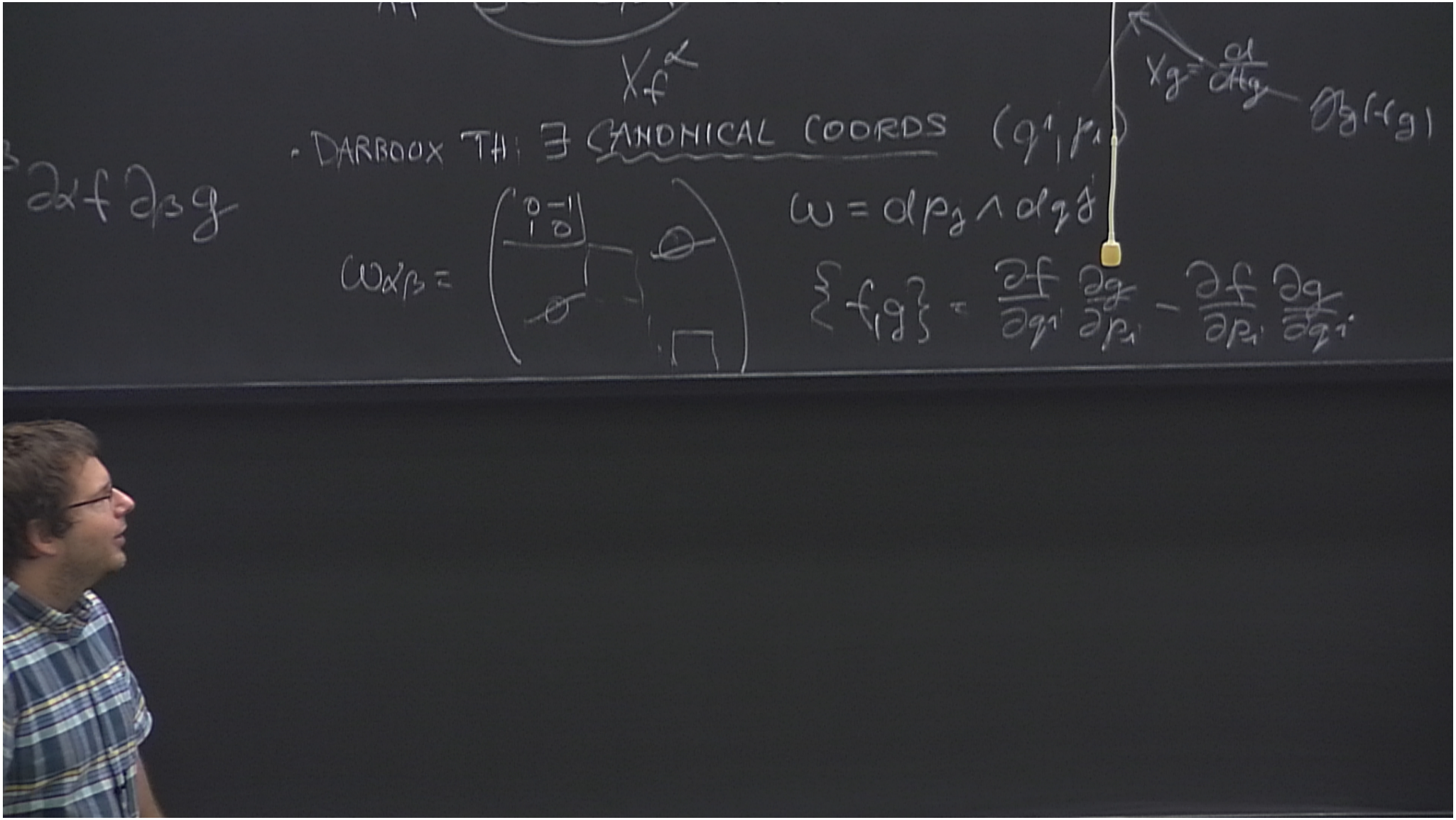
Abstract:


YESTERDAY
· SYMPLECTIC MANIFOLD $(M^{2n}, \omega)$
CLOSED, NON-DEG 2-FORM
CAN USE $\omega$ TO DEFINE
P.B.
$\{f, g\} = df \cdot \Omega \cdot dg = \Omega^{\alpha\beta} \partial_\alpha f \partial_\beta g$


· HAM. V. F $X_f = \Omega \cdot df = \{\cdot, f\}$
$\omega)$
DEG 2-FORM
$= \Omega^{\alpha\beta} \partial_\alpha f \partial_\beta g$


HAM. V. F
$X_f = \Omega \cdot df = \{\cdot, f\} =$
$X_f$
$\omega)$
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$= \Omega^{\alpha\beta} \partial_\alpha f \partial_\beta g$

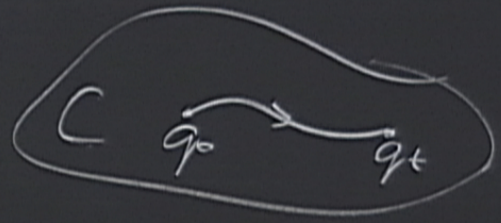

$X_f = \Omega \cdot df = \{\cdot, f\} = \frac{d}{dt_f}$
$\Omega^{\alpha\beta}\partial_\beta f$
$\phi_t(t_f)$
$X_f$
$X_f$


$X_f^\alpha$
$X_g = \frac{d}{dt_g}\Big|_{t_g=0}\varphi_{g,t_g}$
• DARBOUX TH: ∃ CANONICAL COORDS $(q^i, p_i)$
$\omega = dp_i \wedge dq^i$
$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$
$\omega_{\alpha\beta} =$
$\partial_\alpha f \partial_\beta g$

f) HAMILTONIAN MECHANICS $(H \neq H(t))$

## f) HAMILTONIAN MECHANICS $(H \neq H(t))$

- CONFIGURATION SPACE $C$.

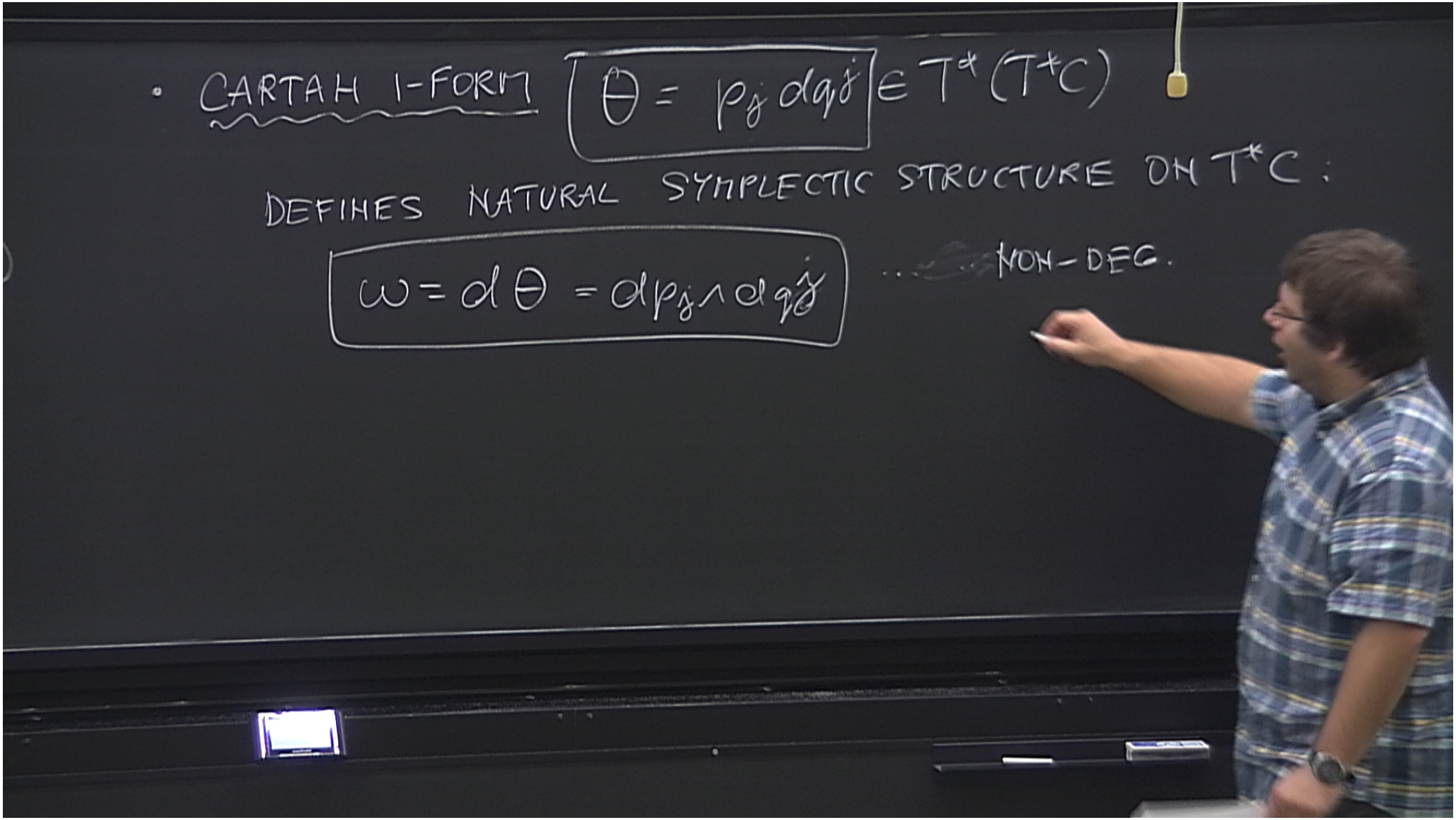


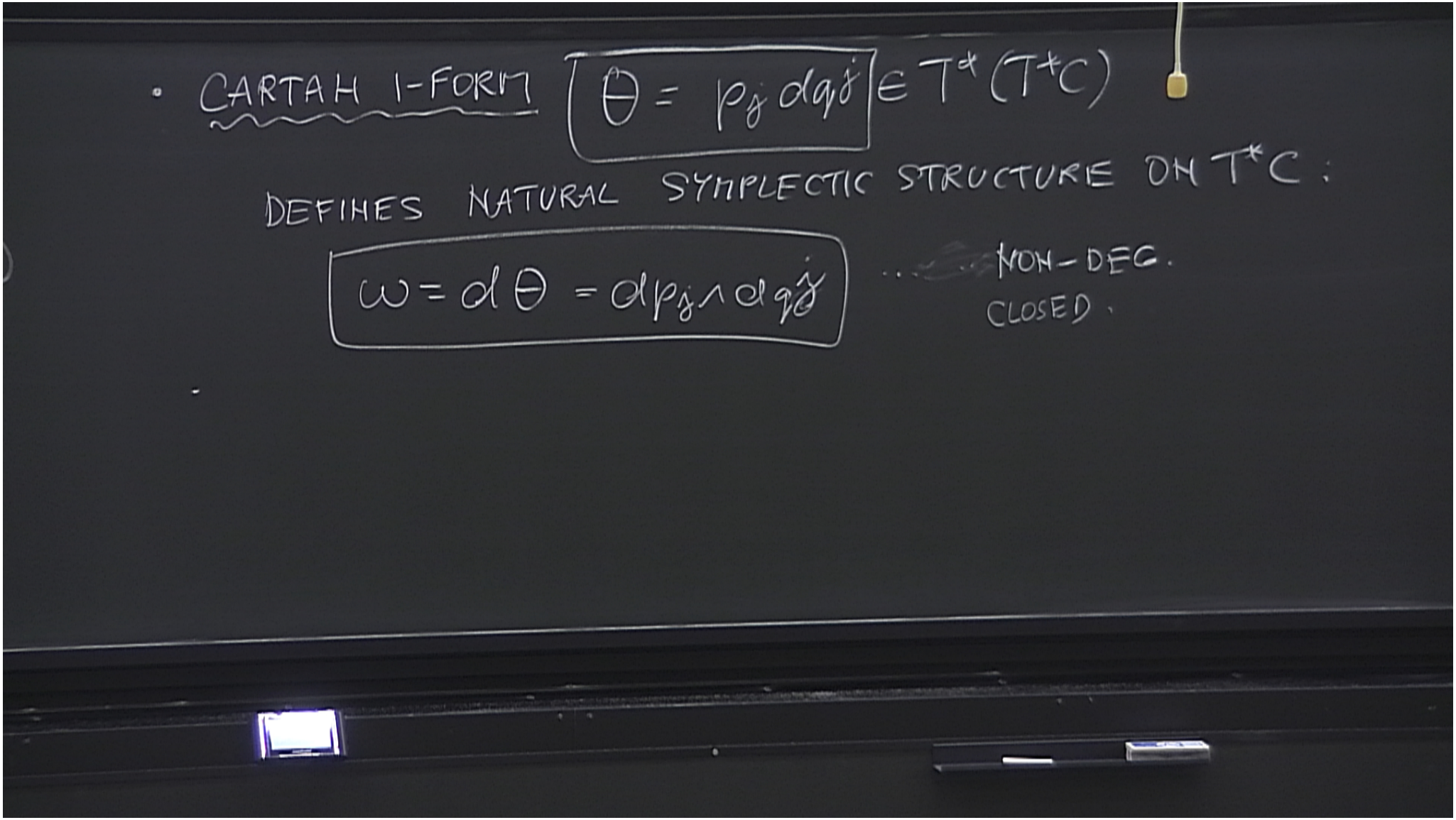

$\dim C = n$

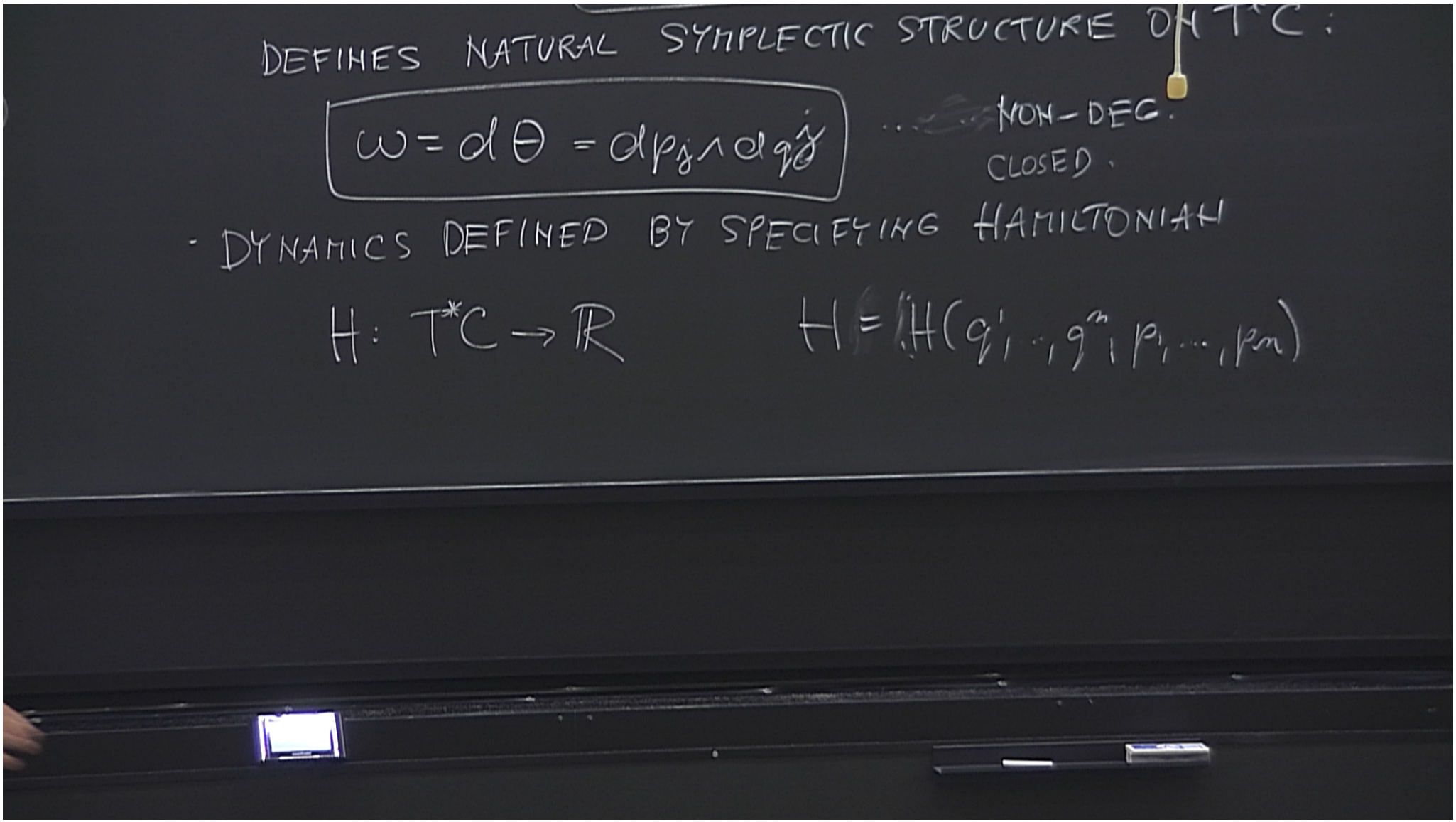
$(q^1, \dots, q^n)$

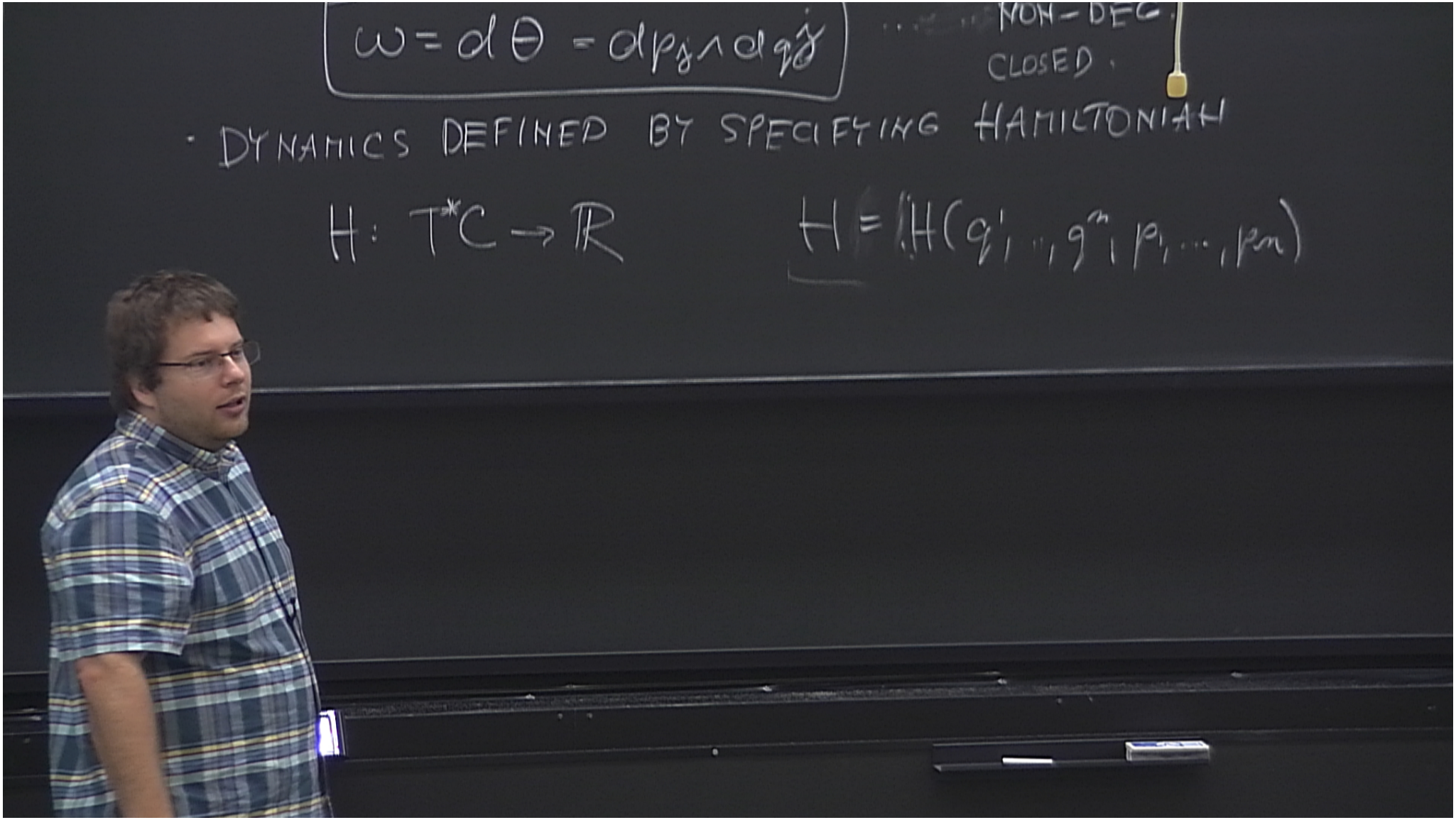
- PHASE SPACE $= T^*C$

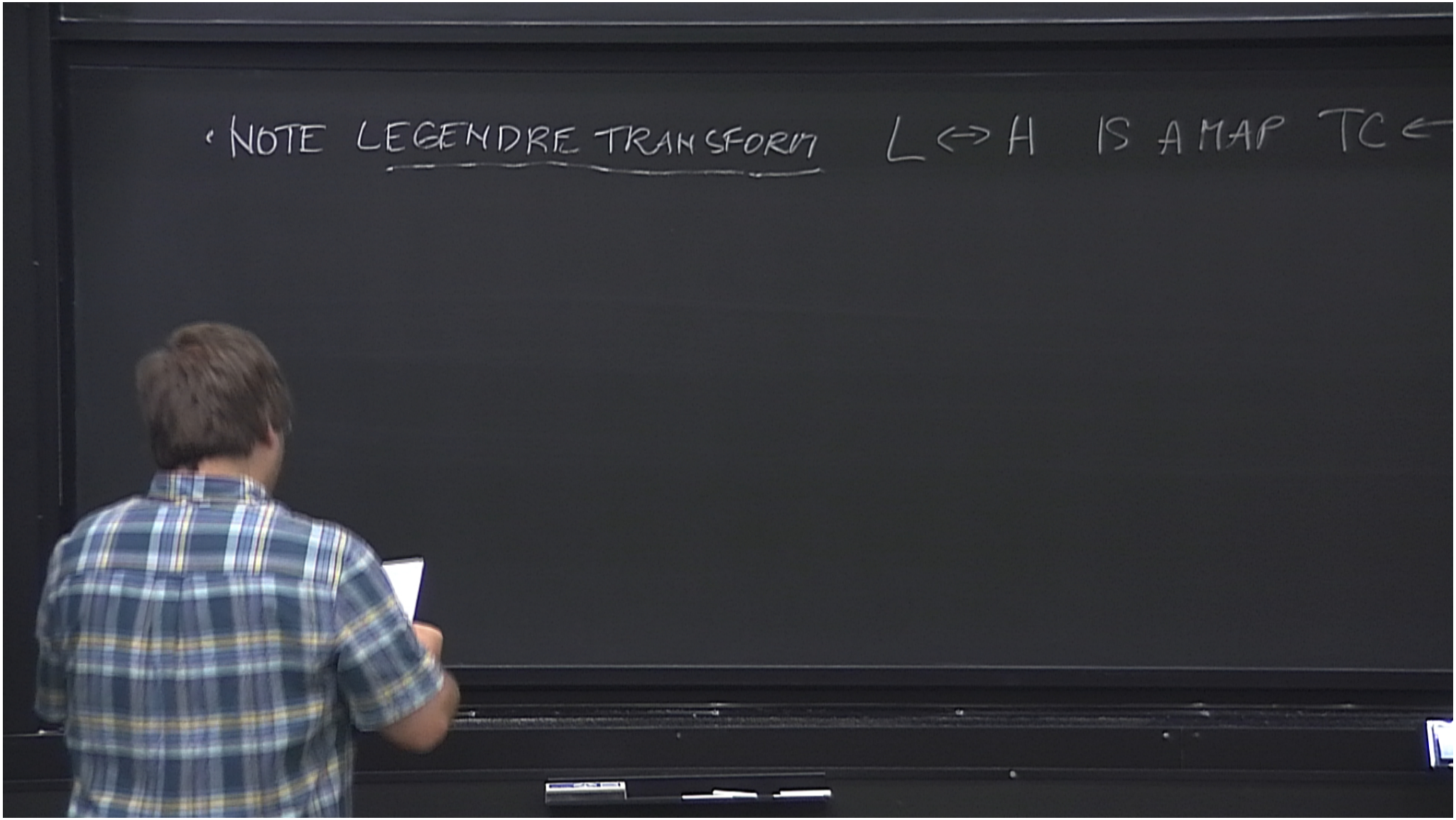
$\dim = 2n$ $\quad (p_1, \dots, p_n, q^1, \dots, q^n)$


• CARTAN 1-FORM $\Theta = p_j\, dq^j \in T^*(T^*C)$
DEFINES NATURAL SYMPLECTIC STRUCTURE ON $T^*C$:
$\omega = d\Theta = dp_j \wedge dq^j$
... NON-DEG.


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NON-DEG.
CLOSED.
· DYNAMICS DEFINED BY SPECIFYING HAMILTONIAN
$H: T^*C \to \mathbb{R}$
$H = H(q^1, \ldots, q^m, p_1, \ldots, p_m)$

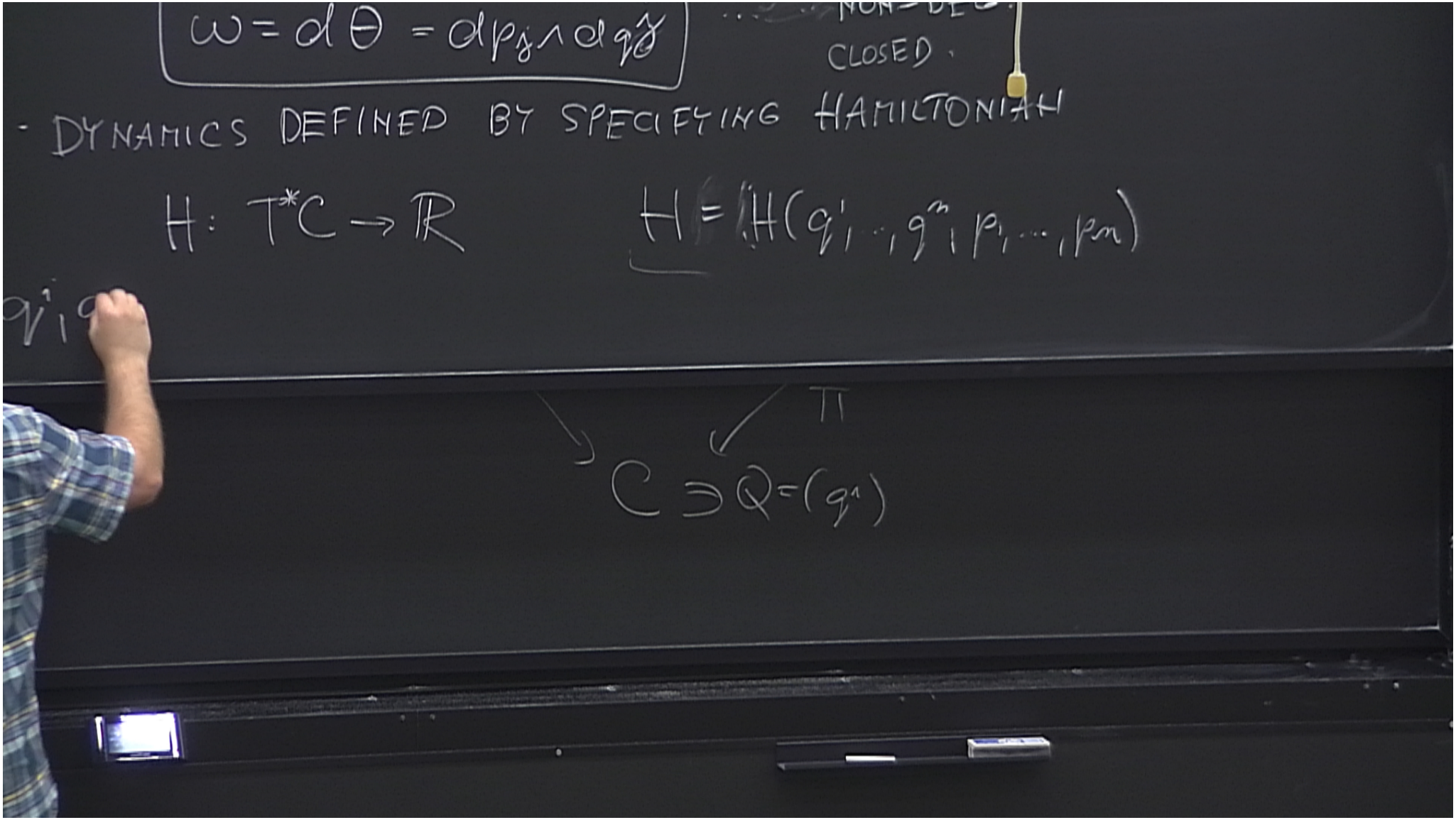

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· DYNAMICS DEFINED BY SPECIFYING HAMILTONIAN
$H: T^*C \to \mathbb{R}$
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• NOTE LEGENDRE TRANSFORM L ↔ H IS A MAP TC

• NOTE LEGENDRE TRANSFORM $L \leftrightarrow H$ IS A MAP TO THAT IDENTIFIES $\dot{q}_j \leftrightarrow p_j$ ACCORDING TO

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j}$$

$$H = \dot{q}_j p_j - L\Big|_{\dot{q}_j =}$$


$\omega = d\theta = dp_i \wedge dq^i$
CLOSED
- DYNAMICS DEFINED BY SPECIFYING HAMILTONIAN
$H: T^*C \to \mathbb{R}$
$H = H(q^1, \dots, q^n, p_1, \dots, p_n)$
$q^i, q$
$\pi$
$C \ni Q = (q^i)$

$\omega = d\theta = dp_\gamma \wedge dq^\gamma$

CLOSED.

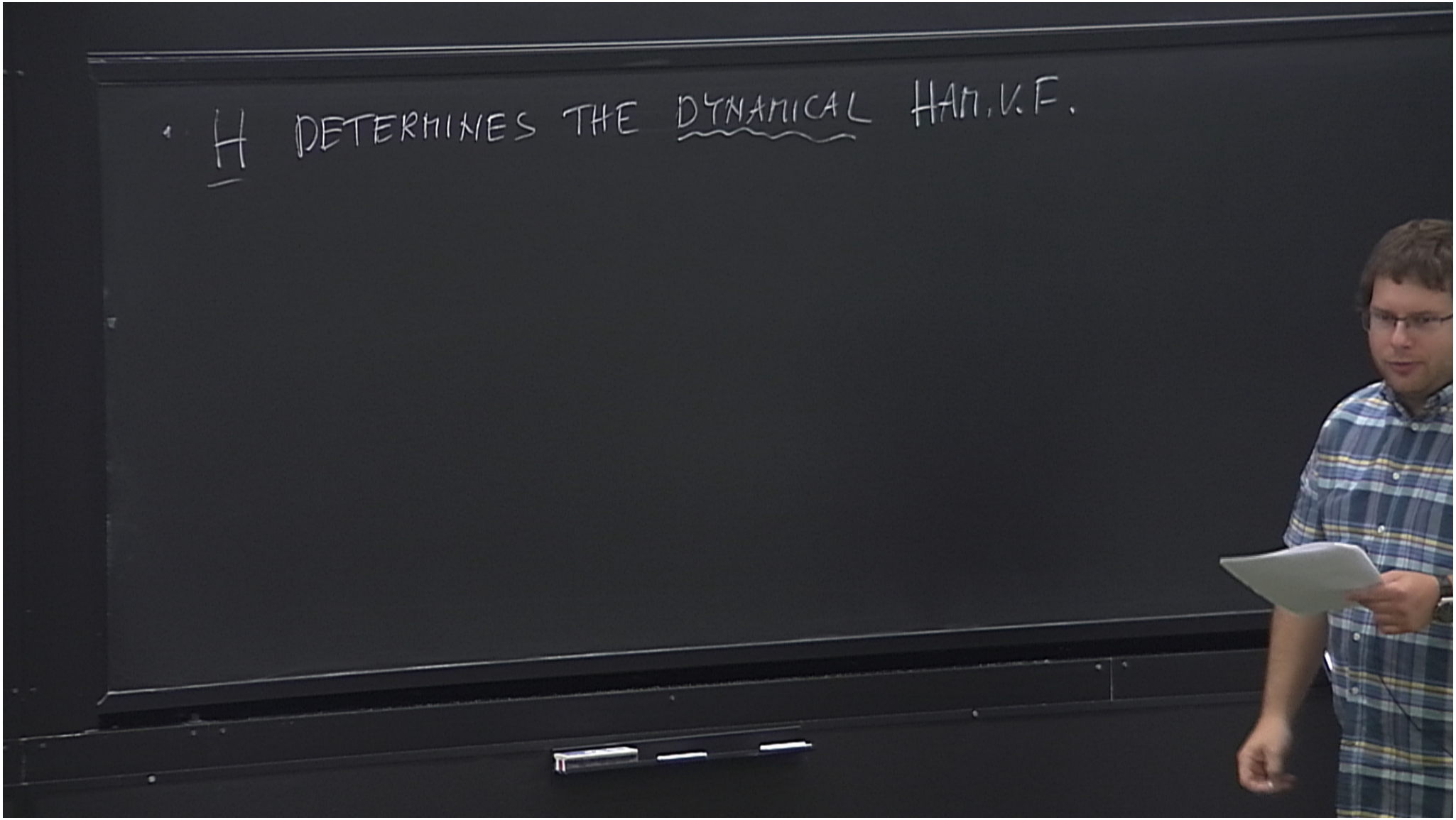
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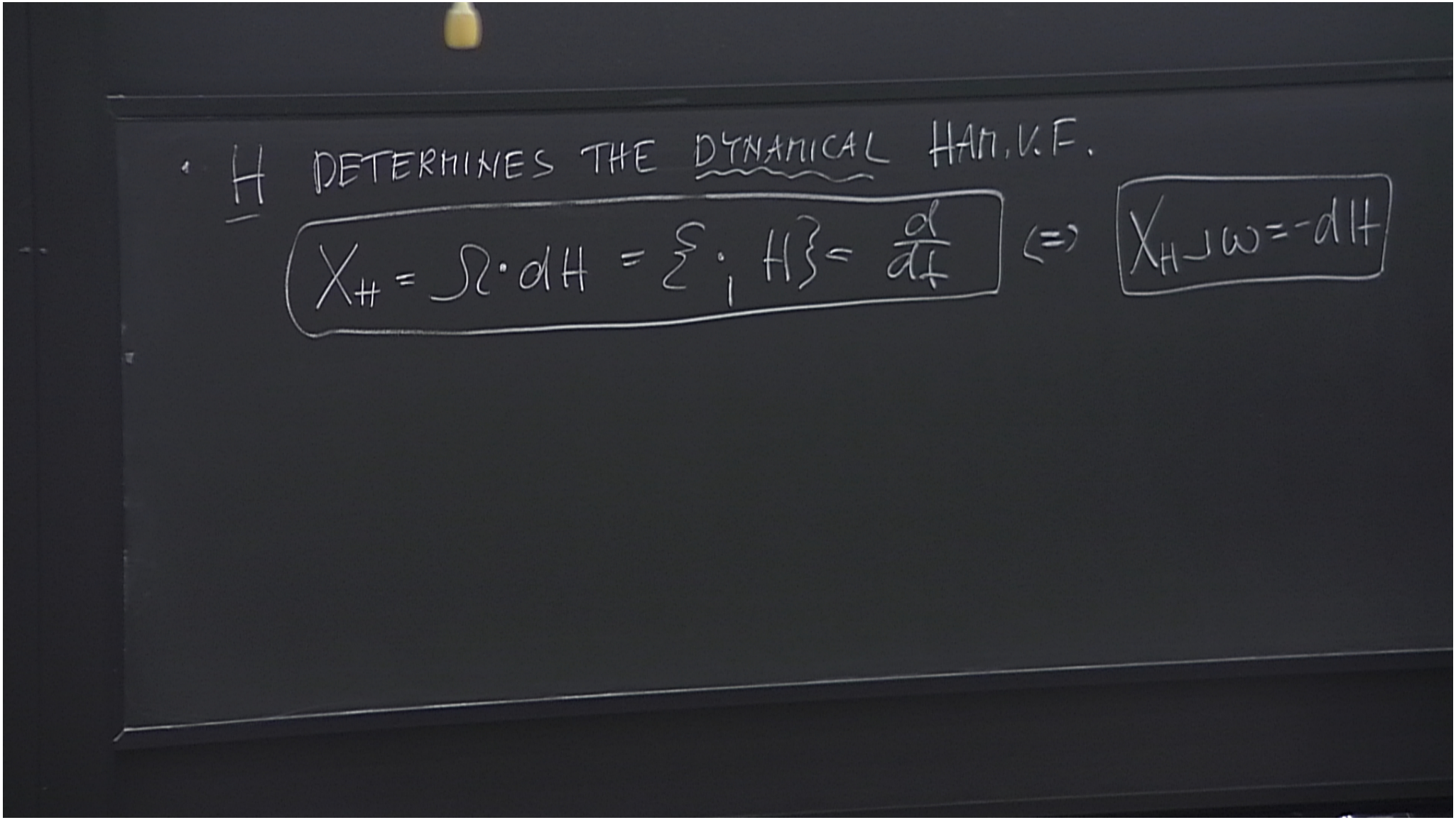
$H : T^*C \to \mathbb{R}$ $\qquad H = H(q^1, \dots, q^n, p_1, \dots, p_n)$

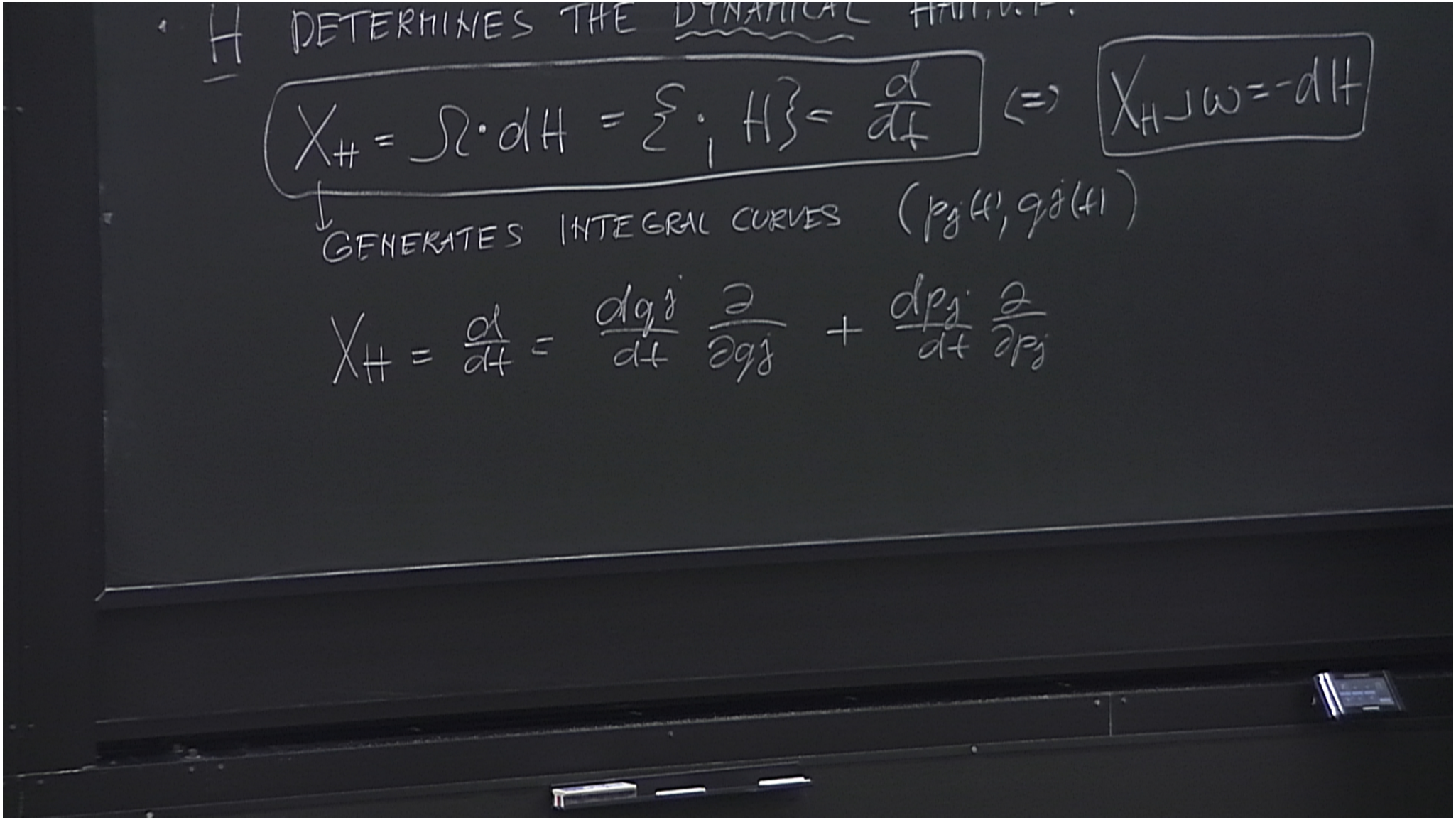
$q^i, \dot{q}^i$ $\qquad V \in T(TC) < \begin{matrix} P \in C & q^i \\ \delta \in & v = \dot{q}^i \frac{\partial}{\partial} \end{matrix}$

$\pi$

$C \ni Q = (q^i)$


· H DETERMINES THE DYNAMICAL HAM. V. F.


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$X_H = \Omega \cdot dH = \{\cdot, H\} = \frac{d}{dt}$ $\Leftrightarrow$ $X_H \lrcorner \omega = -dH$


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$X_H = \Omega \cdot dH = \{\xi, H\} = \frac{d}{dt}$ $\Leftrightarrow$ $X_H \lrcorner \omega = -dH$
GENERATES INTEGRAL CURVES $(p_j(t), q_j(t))$
$X_H = \frac{d}{dt} = \frac{dq_j}{dt}\frac{\partial}{\partial q_j} + \frac{dp_j}{dt}\frac{\partial}{\partial p_j}$

H DETERMINES THE DYNAMICAL

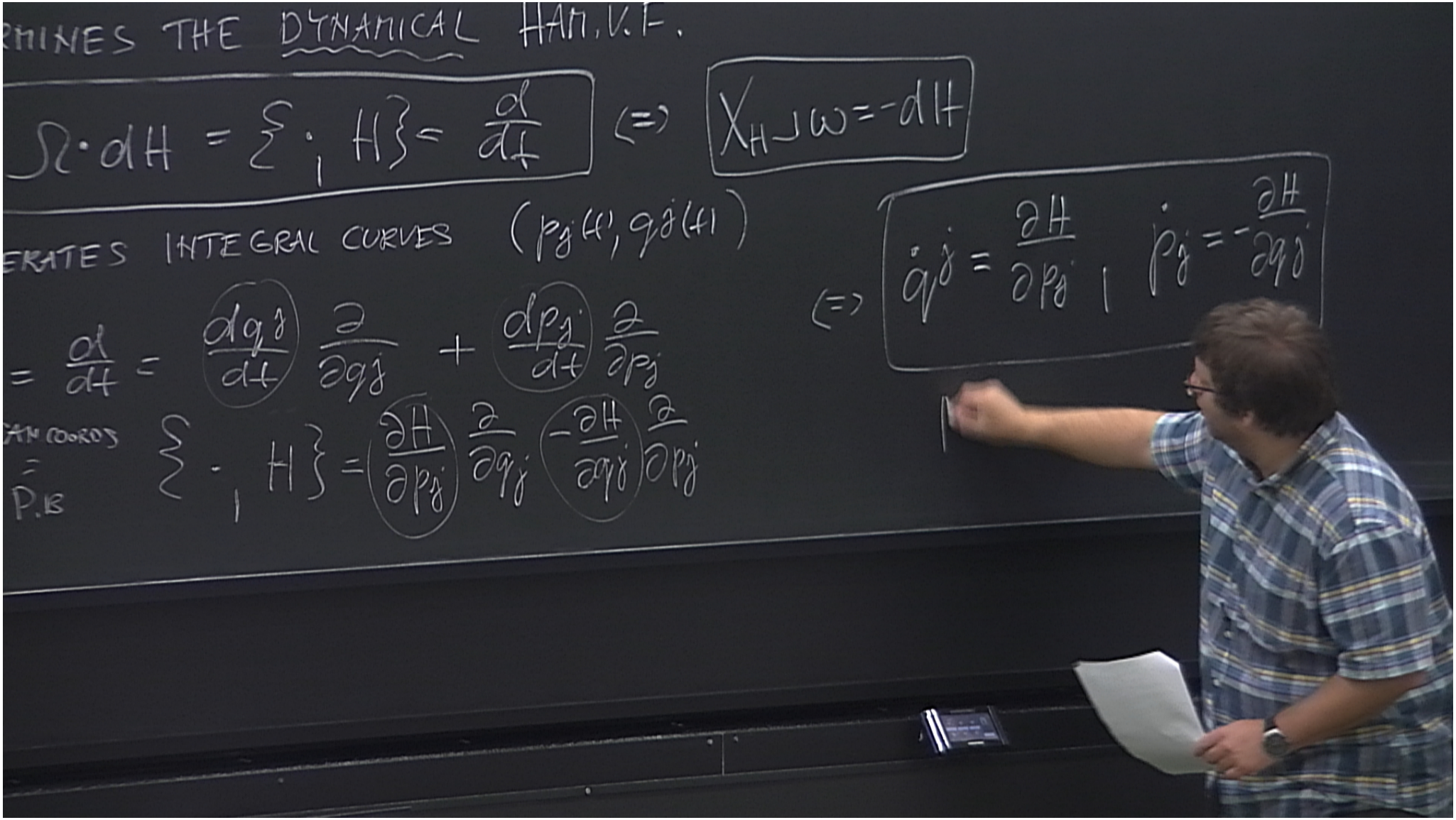
$$X_H = \Omega \cdot dH = \{\cdot, H\} = \frac{d}{dt} \iff X_H \lrcorner \omega = -dH$$

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$$X_H = \frac{d}{dt} = \frac{dq_j}{dt}\frac{\partial}{\partial q_j} + \frac{dp_j}{dt}\frac{\partial}{\partial p_j}$$

CAN COORDS $\{\cdot, H\} = \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j}$

P.B


RMINES THE DYNAMICAL HAM. V.F.
$\Omega \cdot dH = \{\cdot, H\} = \frac{d}{dt}$ $\Leftrightarrow$ $X_H \lrcorner \omega = -dH$
ERATES INTEGRAL CURVES $(p_j(t), q_j(t))$
$\Leftrightarrow$ $\dot{q}_j = \frac{\partial H}{\partial p_j}$ , $\dot{p}_j = -\frac{\partial H}{\partial q_j}$
$= \frac{d}{dt} = \frac{dq_j}{dt}\frac{\partial}{\partial q_j} + \frac{dp_j}{dt}\frac{\partial}{\partial p_j}$
AN COORDS
P.B
$\{\cdot, H\} = \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} \frac{-\partial H}{\partial q_j}\frac{\partial}{\partial p_j}$

MINES THE DYNAMICAL HAM. V.F.

$$\Omega \cdot dH = \{\cdot, H\} = \frac{d}{dt} \iff X_H \lrcorner \omega = -dH$$

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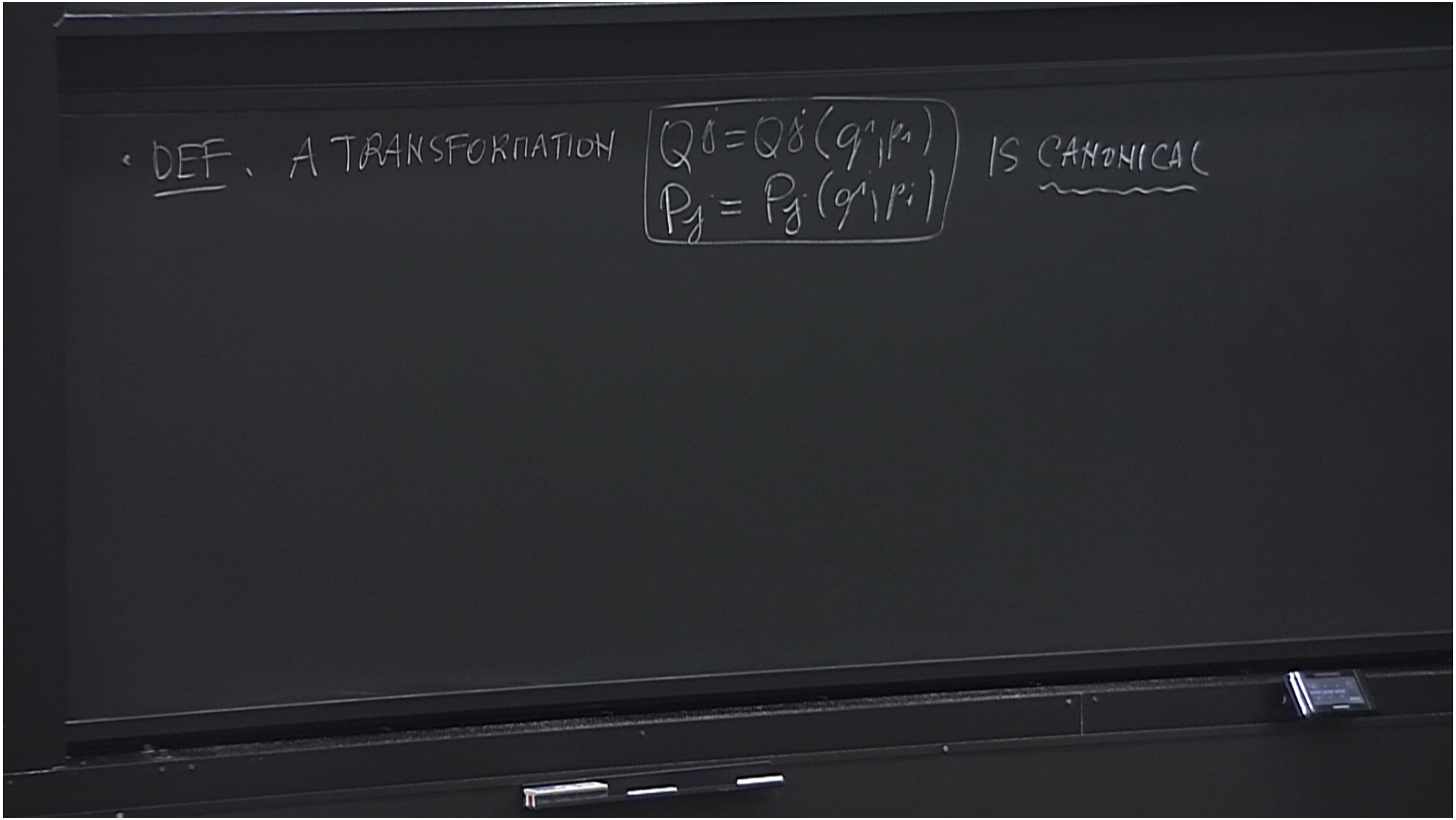
$$= \frac{d}{dt} = \frac{dq_j}{dt}\frac{\partial}{\partial q_j} + \frac{dp_j}{dt}\frac{\partial}{\partial p_j} \iff \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

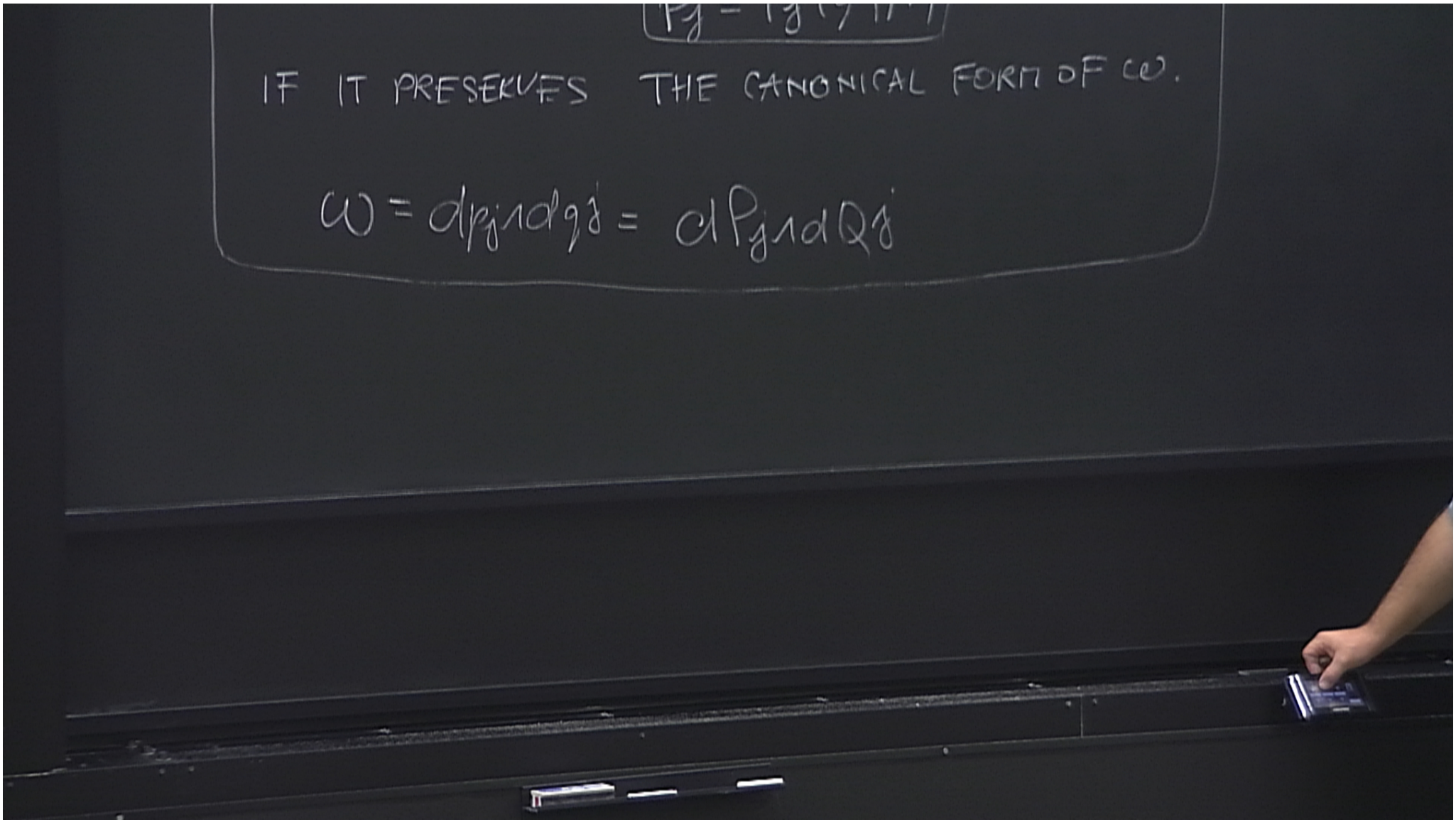
HAMILTON'S EQS.

AN COORDS

P.B

$$\{\cdot, H\} = \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j}$$


• DEF. A TRANSFORMATION $Q^i = Q^i(q^j, p_j)$ $P_j = P_j(q^j, p_i)$ IS CANONICAL


IF IT PRESERVES THE CANONICAL FORM OF ω.
ω = dp_j ∧ dq_j = dP_j ∧ dQ_j

$P_j = P_j(q_i, p_i)$ IS CANONICAL

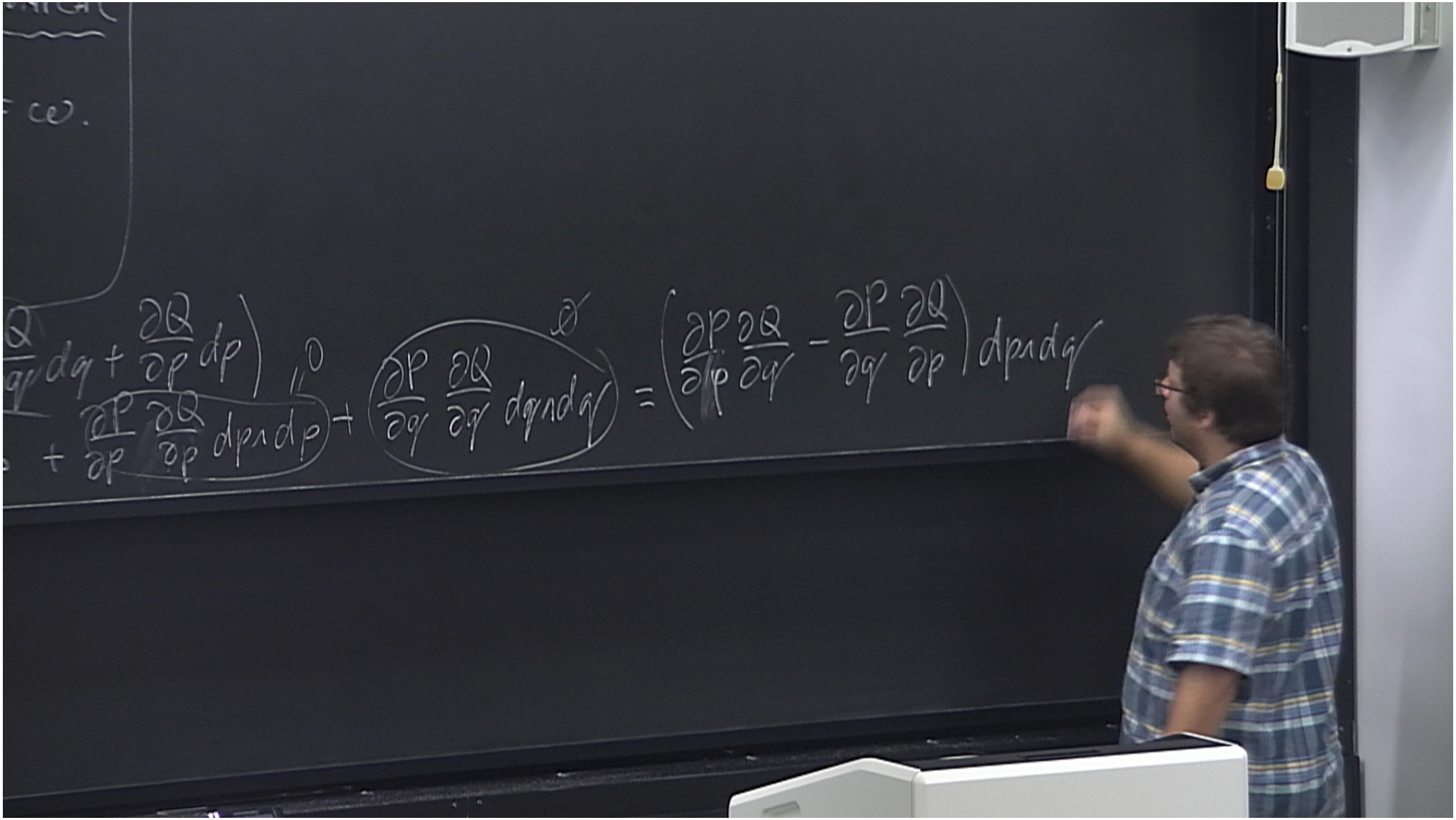
IF IT PRESERVES THE CANONICAL FORM OF $\omega$.

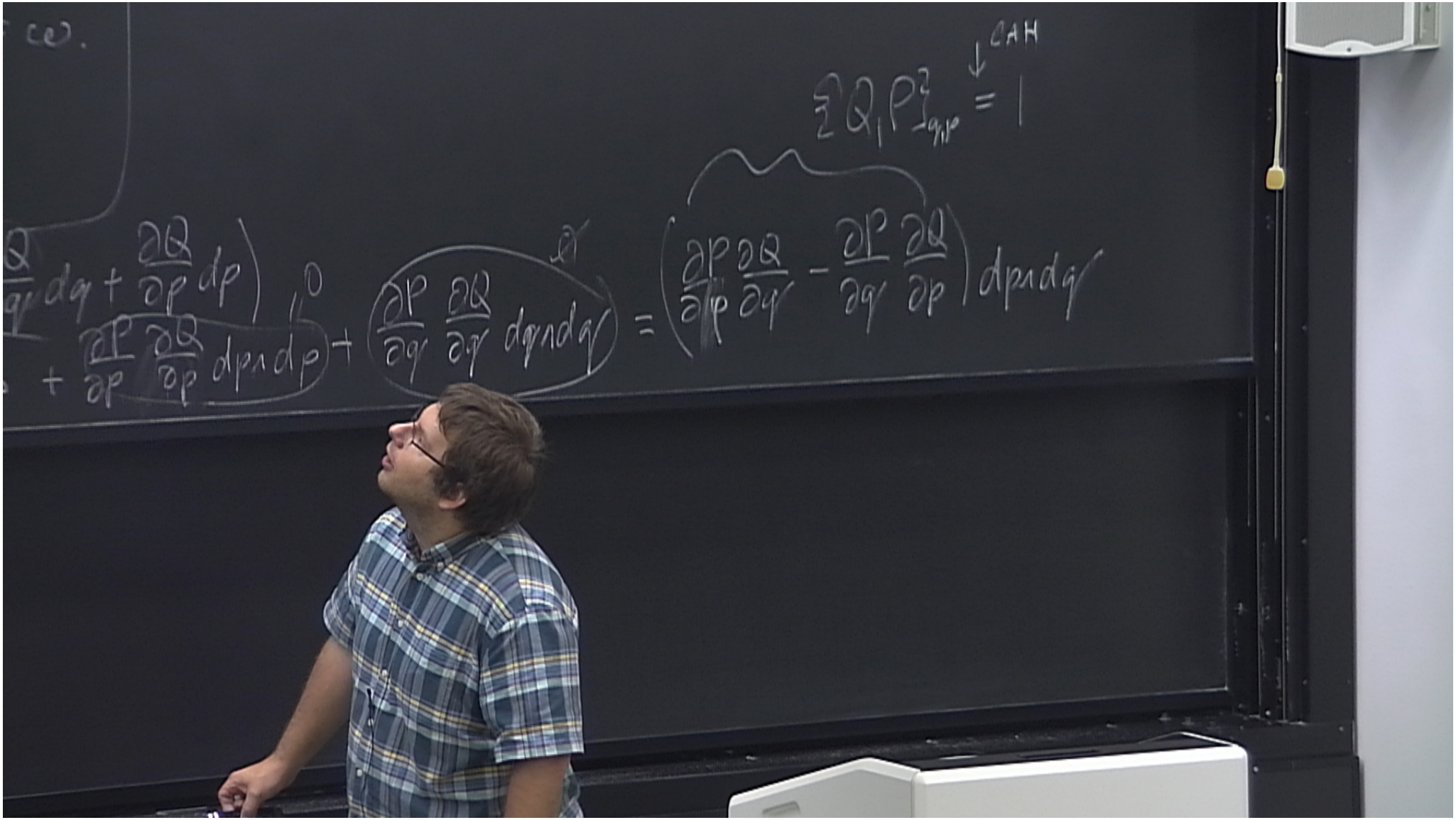
$$\omega = dp_j \wedge dq_j = dP_j \wedge dQ_j$$

* $n=1$:

$$\omega = dP \wedge dQ = \left(\frac{\partial P}{\partial p}dp + \frac{\partial P}{\partial q}dq\right) \wedge \left(\frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp\right)$$

$$= \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q}dp \wedge dq + \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}dq \wedge dp$$

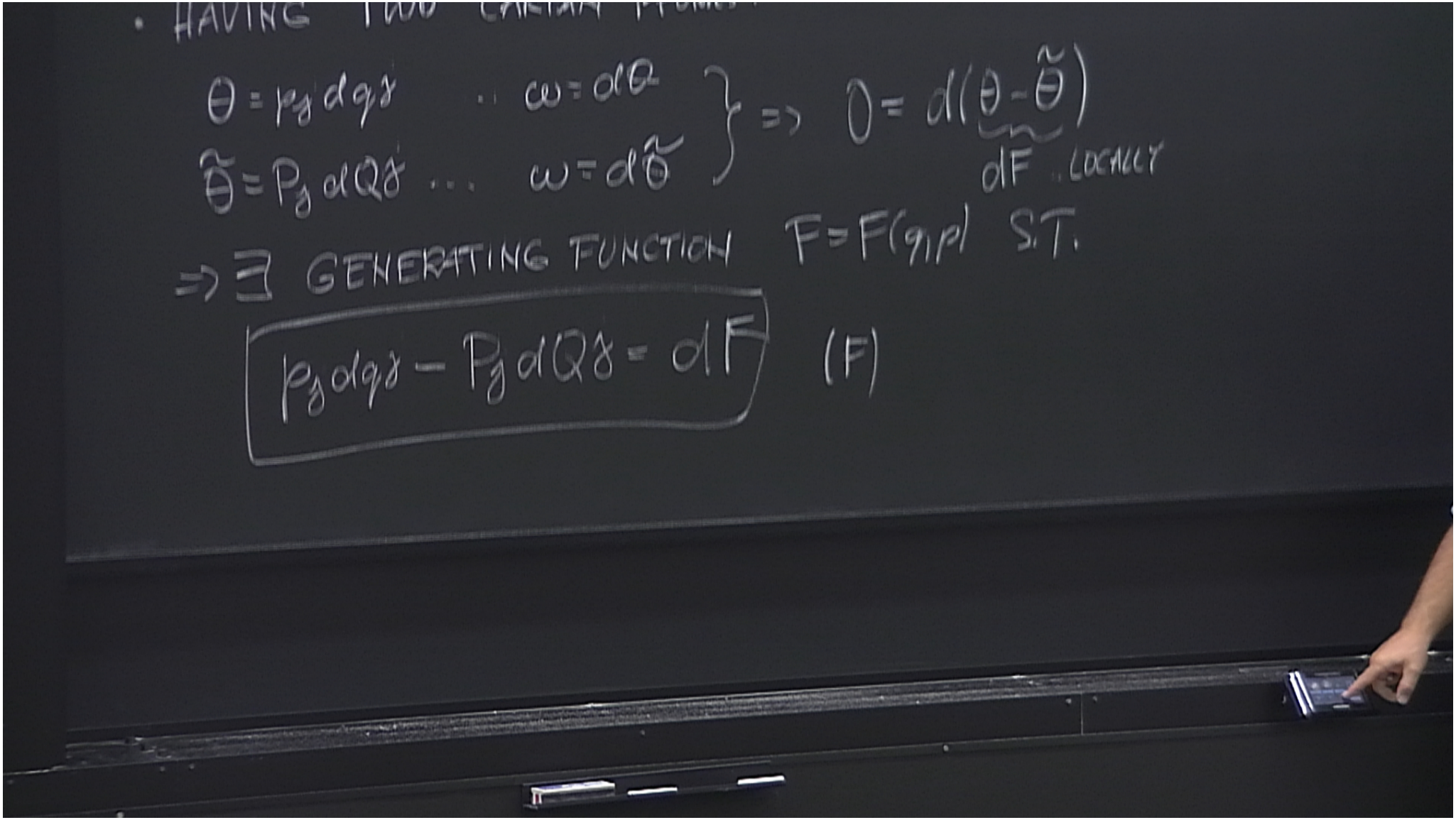

$\frac{\partial Q}{\partial q} dq + \frac{\partial Q}{\partial p} dp$
$+ \frac{\partial P}{\partial p}\frac{\partial Q}{\partial p} dp \wedge dp + \frac{\partial P}{\partial q}\frac{\partial Q}{\partial q} dq \wedge dq = \left(\frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}\right) dp \wedge dq$

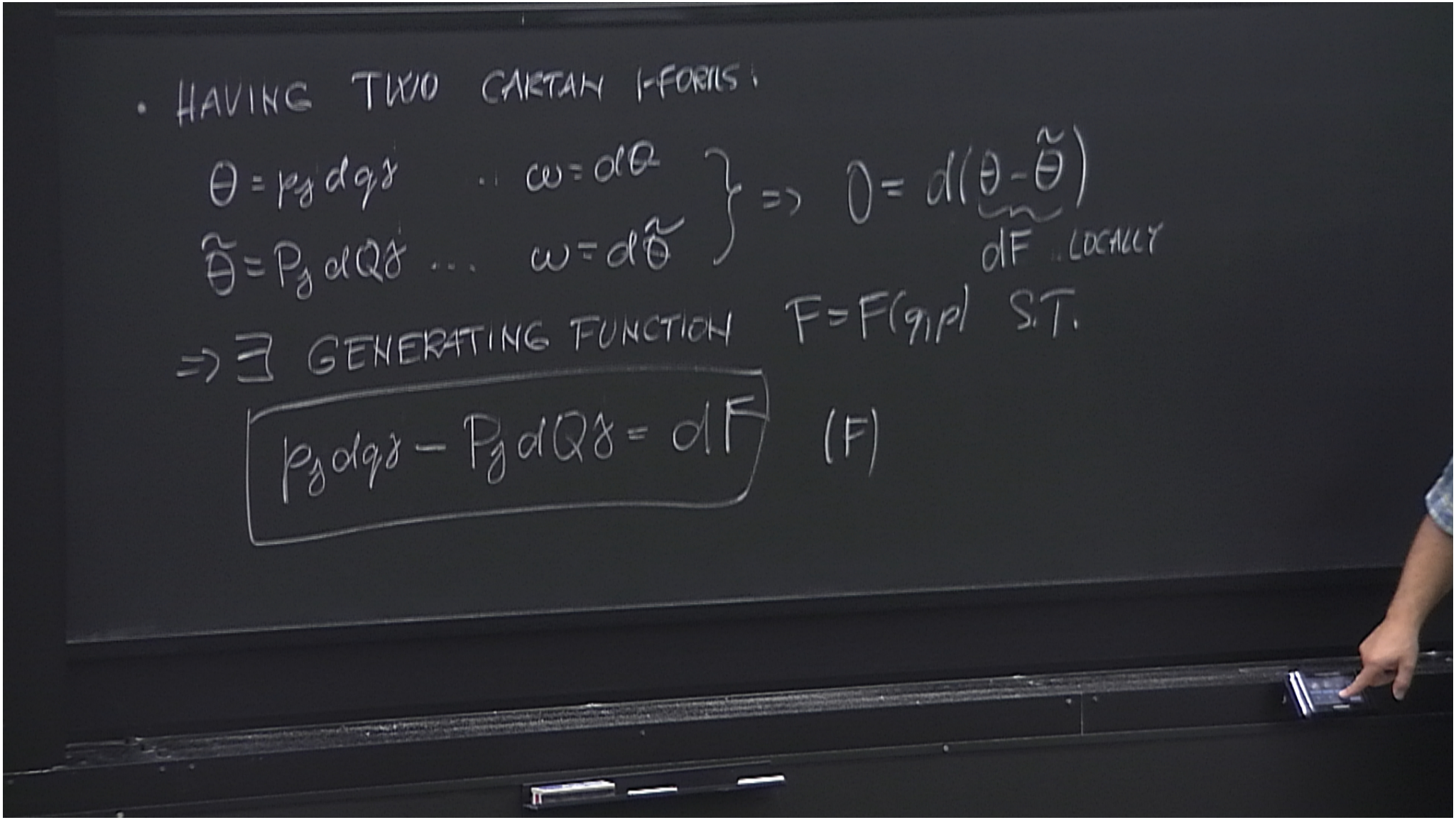

CAH
$\sum Q_i P^i_{q,p} = 1$
$dp \wedge dq$
$dp \wedge dq = \left(\frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}\right) dp \wedge dq$

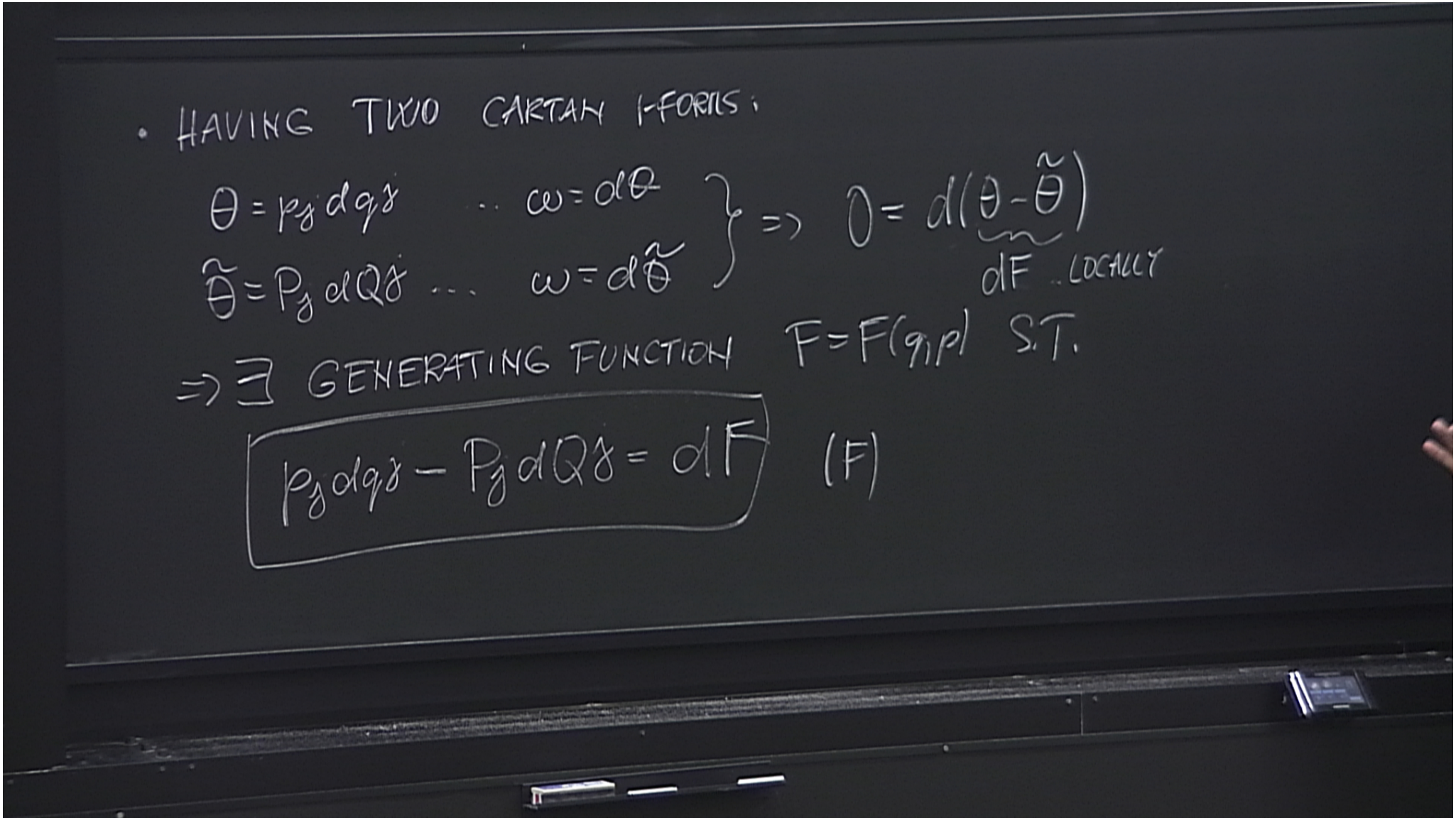
$$= \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q}\, dp \wedge dq + \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}\, dq \wedge dp + \frac{\partial P}{\partial p}\frac{\partial Q}{\partial p}\, dp \wedge dp$$

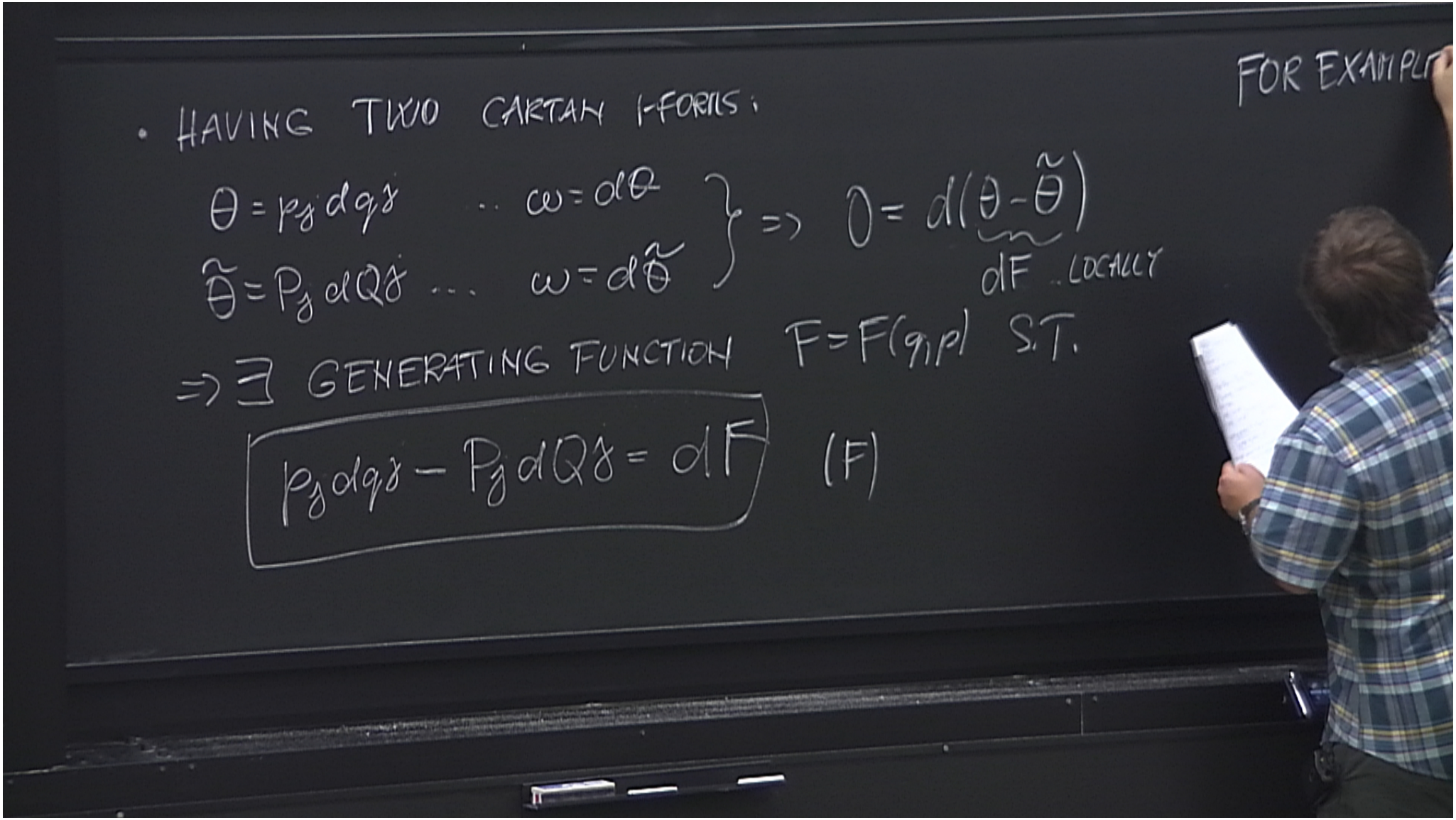
- HAVING TWO CARTAN 1-FORMS:

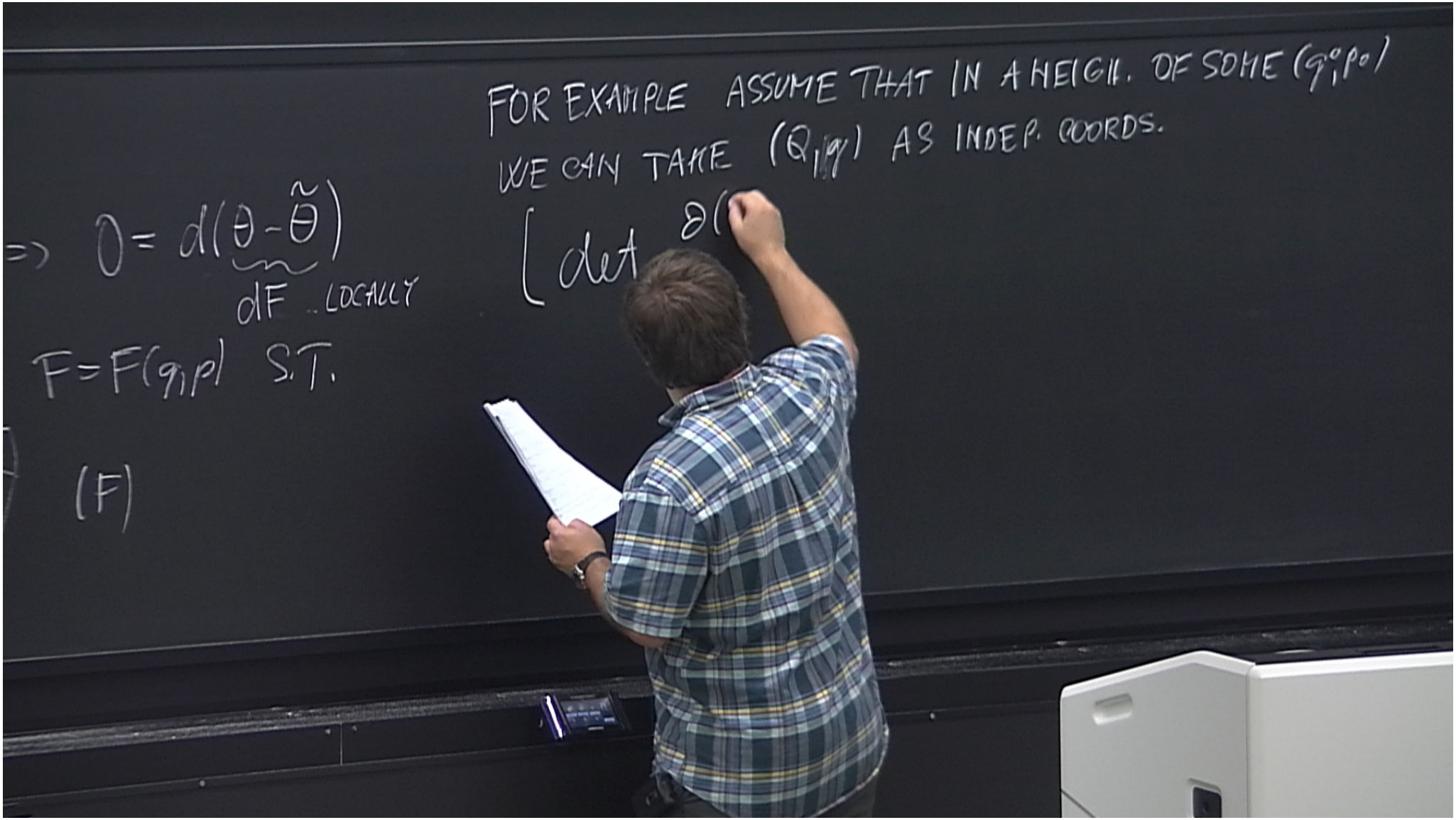
$$\left.\begin{array}{ll} \Theta = p_\gamma\, dq_\gamma & \dots \quad \omega = d\Theta \\ \tilde{\Theta} = P_\gamma\, dQ_\gamma & \dots \quad \omega = d\tilde{\Theta} \end{array}\right\} \Rightarrow 0 = d(\Theta - \tilde{\Theta})$$

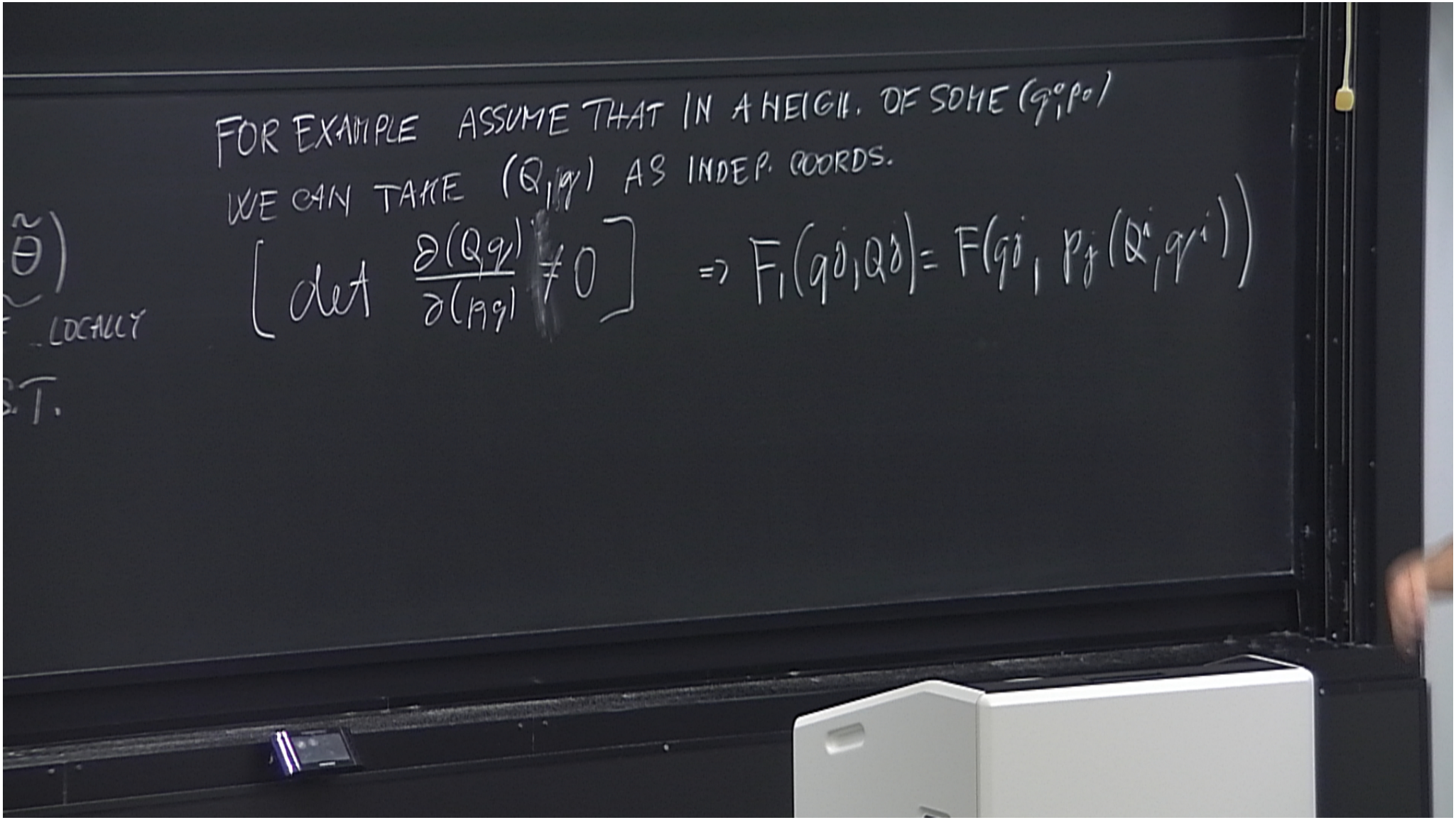

HAVING
$\Theta = p_j dq^j$ .. $\omega = d\Theta$
$\tilde{\Theta} = P_j dQ^j$ ... $\omega = d\tilde{\Theta}$
$\Rightarrow 0 = d(\Theta - \tilde{\Theta})$
$dF$ . LOCALLY
$\Rightarrow \exists$ GENERATING FUNCTION $F = F(q,p)$ S.T.
$p_j dq^j - P_j dQ^j = dF$ (F)

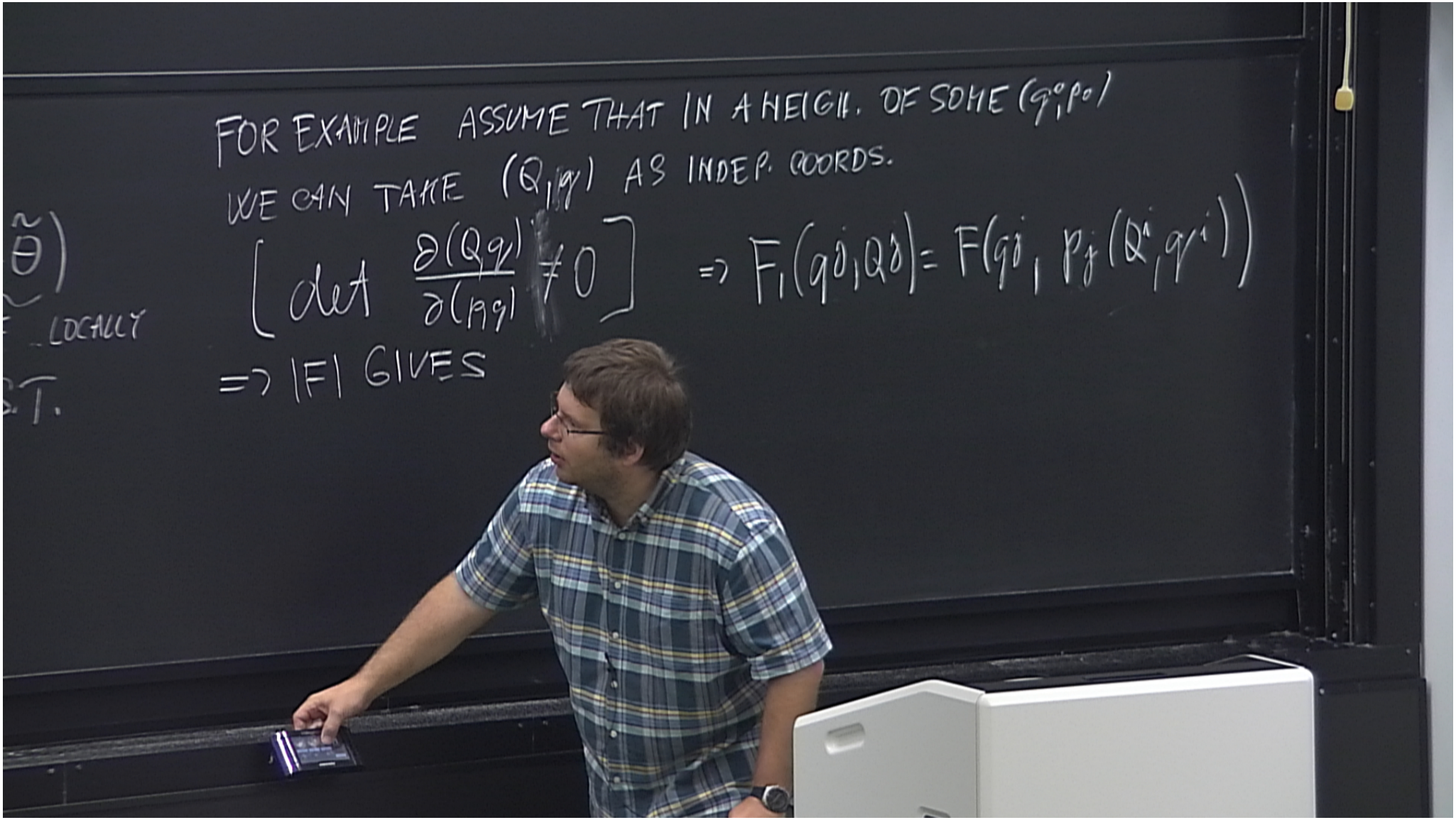

• HAVING TWO CARTAN 1-FORMS:
$\theta = p_j\, dq^j$ .. $\omega = d\theta$
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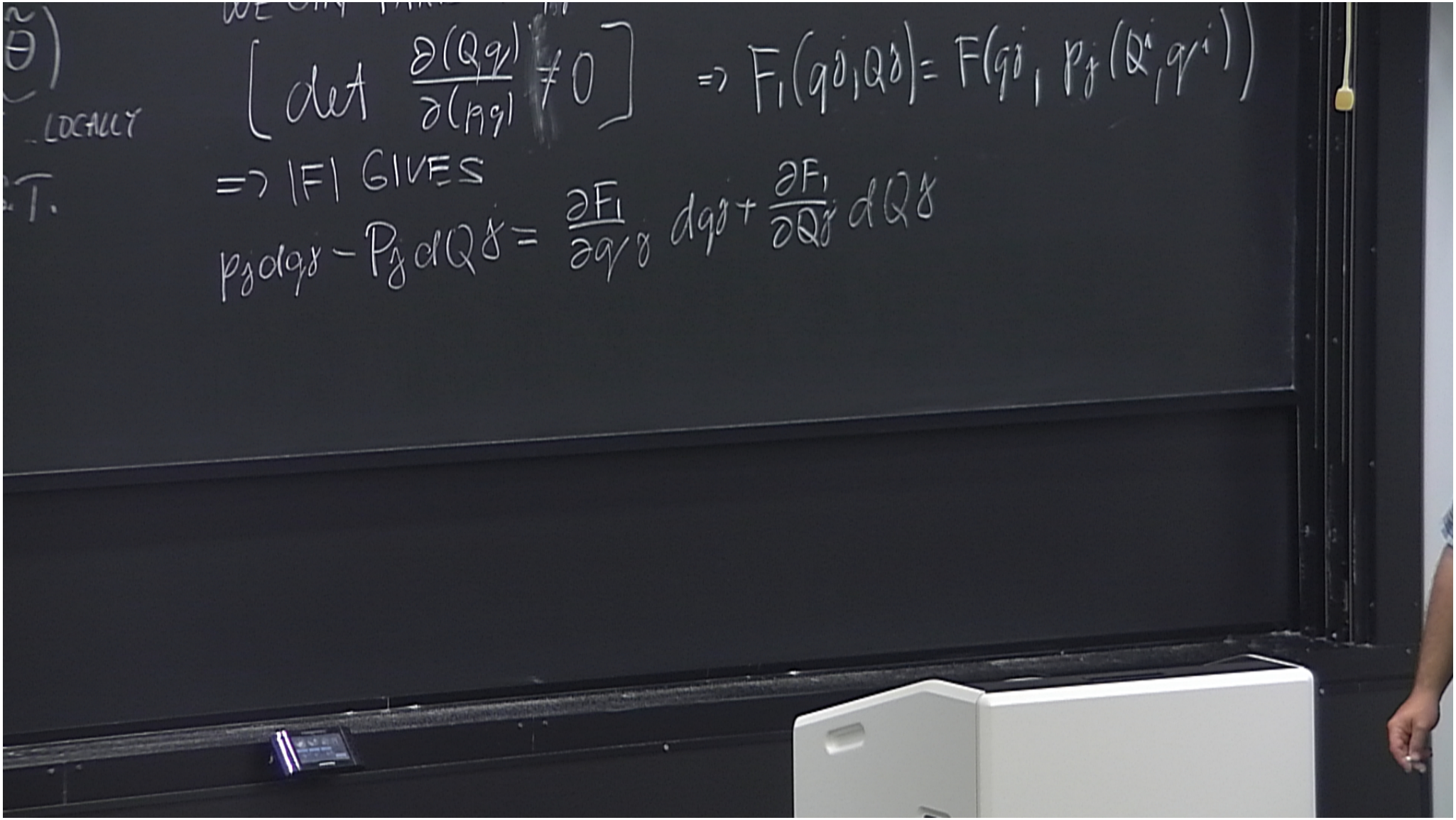

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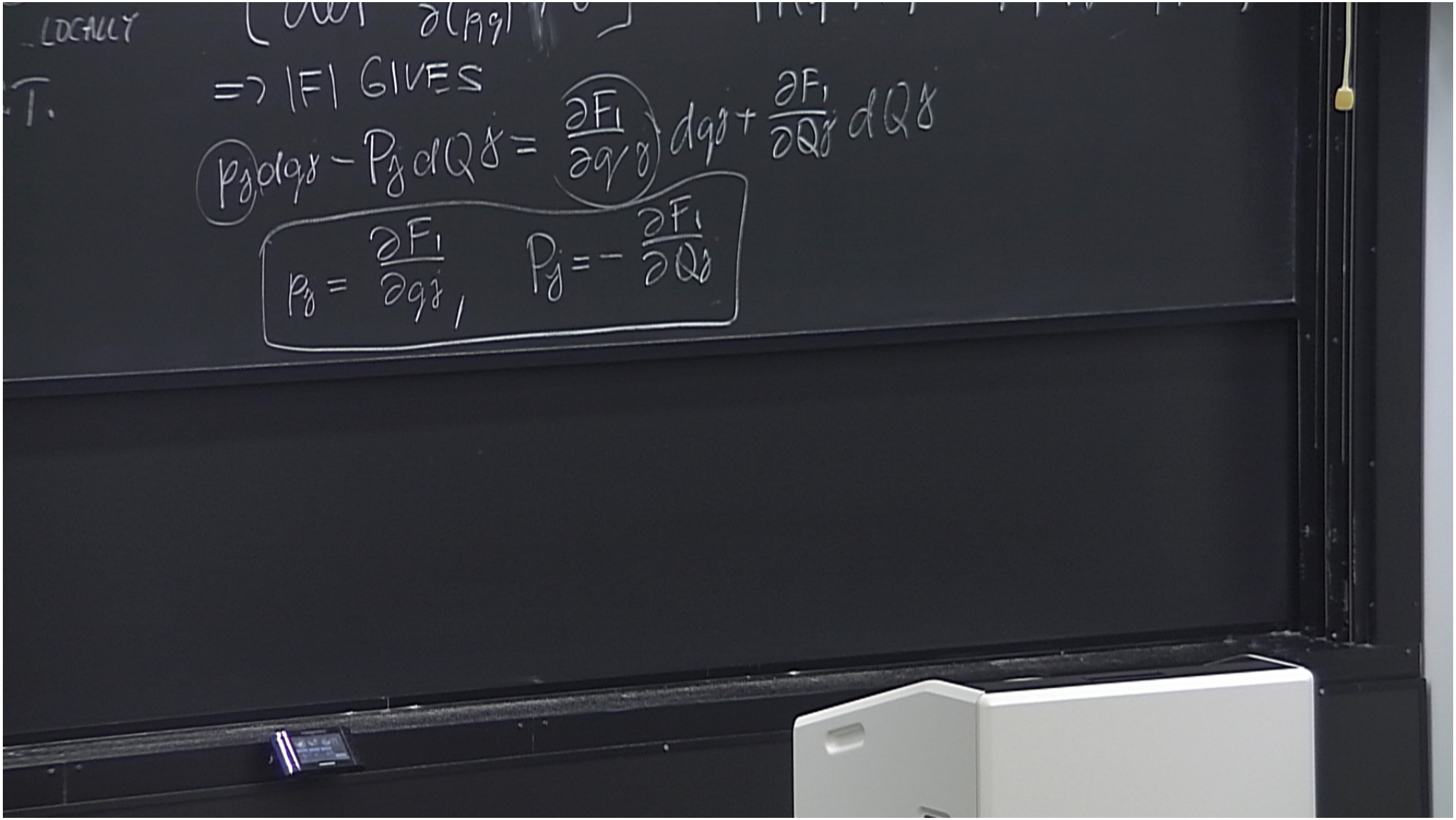

• HAVING TWO CARTAN 1-FORMS:
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$\tilde{\theta} = P_j\, dQ^j$ ... $\omega = d\tilde{\theta}$
$\Rightarrow 0 = d(\theta - \tilde{\theta})$
$d$F .. LOCALLY
$\Rightarrow \exists$ GENERATING FUNCTION $F = F(q,P)$ S.T.
$p_j dq^j - P_j dQ^j = dF$ (F)
FOR EXAMPL

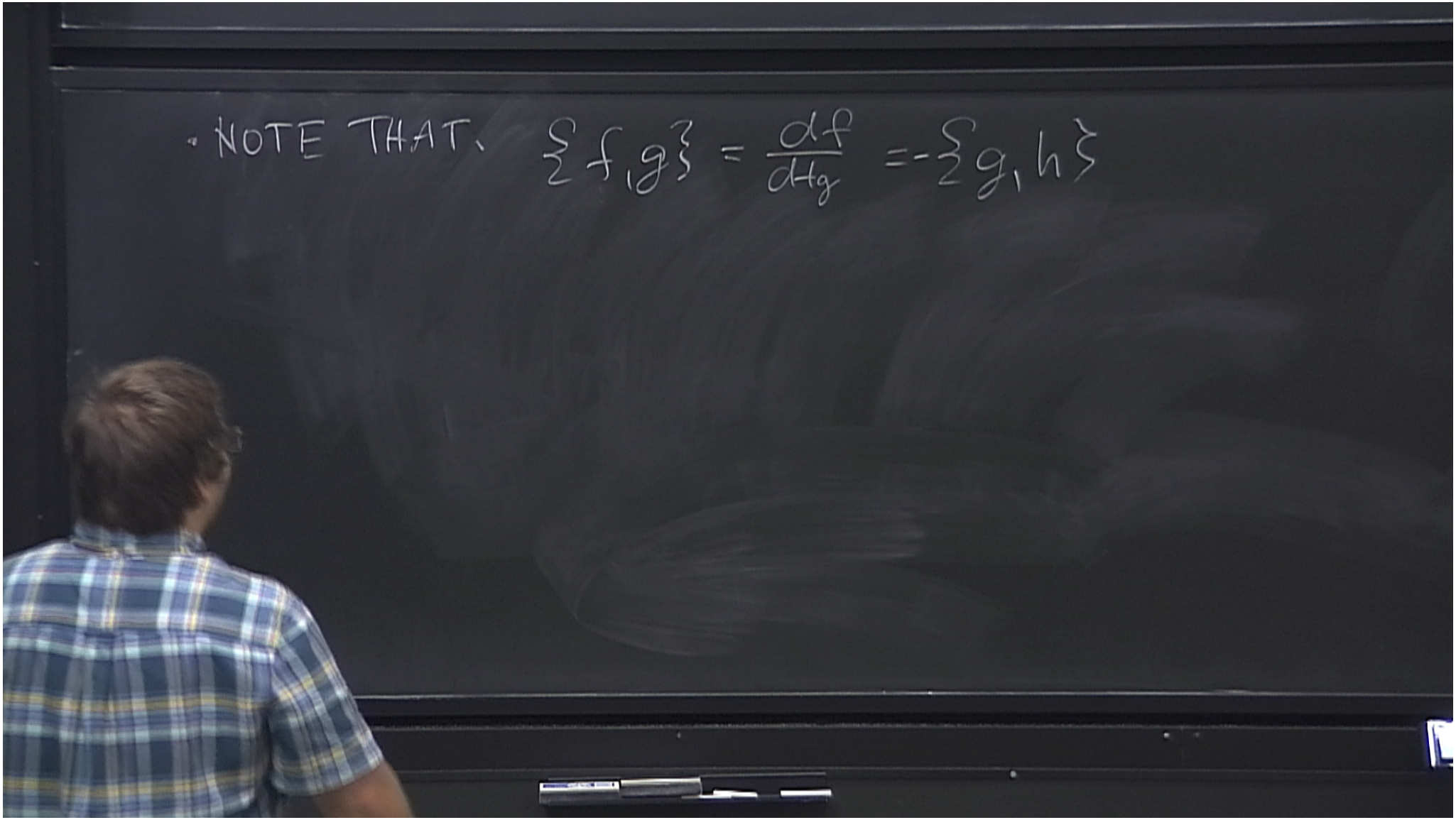

FOR EXAMPLE ASSUME THAT IN A NEIGH. OF SOME $(q_0, p_0)$
WE CAN TAKE $(Q_1, p_1)$ AS INDEP. COORDS.
$[\det \partial($
$\Rightarrow 0 = d(\theta - \tilde{\theta})$
$dF$ LOCALLY
$F = F(q, p)$ S.T.
(F)

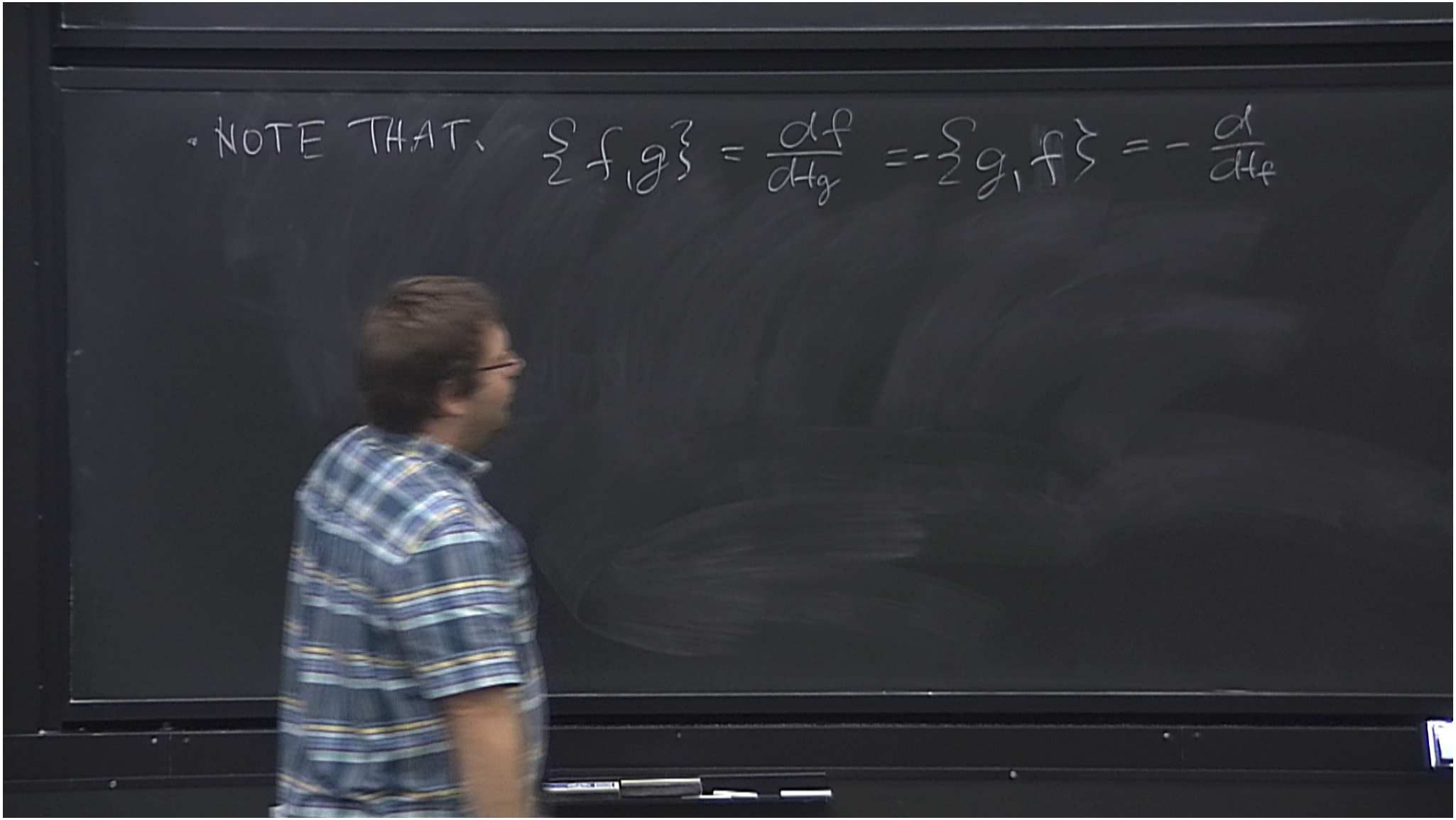

FOR EXAMPLE ASSUME THAT IN A NEIGH. OF SOME $(q^0, p^0)$
WE CAN TAKE $(Q, q)$ AS INDEP. COORDS.
$\left[\det \frac{\partial(Q,q)}{\partial(p,q)} \neq 0\right] \Rightarrow F_1(q^0, Q^0) = F(q^0, p_j(Q^i, q^i))$
LOCALLY


FOR EXAMPLE ASSUME THAT IN A NEIGH. OF SOME $(q^0, p^0)$
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$\Rightarrow |F|$ GIVES


LOCALLY
$\left[\det \frac{\partial(Q,q)}{\partial(p,q)} \neq 0\right] \Rightarrow F_1(q^i, Q^i) = F(q^i, p_j(Q^i, q^i))$
$\Rightarrow |F_1|$ GIVES
$p_i dq^i - P_i dQ^i = \frac{\partial F_1}{\partial q^i} dq^i + \frac{\partial F_1}{\partial Q^i} dQ^i$


LOCALLY
T.
$\Rightarrow |F|$ GIVES
$p_j dq_j - P_j dQ_j = \frac{\partial F_1}{\partial q_j} dq_j + \frac{\partial F_1}{\partial Q_j} dQ_j$
$p_j = \frac{\partial F_1}{\partial q_j}, \quad P_j = -\frac{\partial F_1}{\partial Q_j}$


• NOTE THAT. $\{f,g\} = \frac{df}{dt_g} = -\{g,h\}$

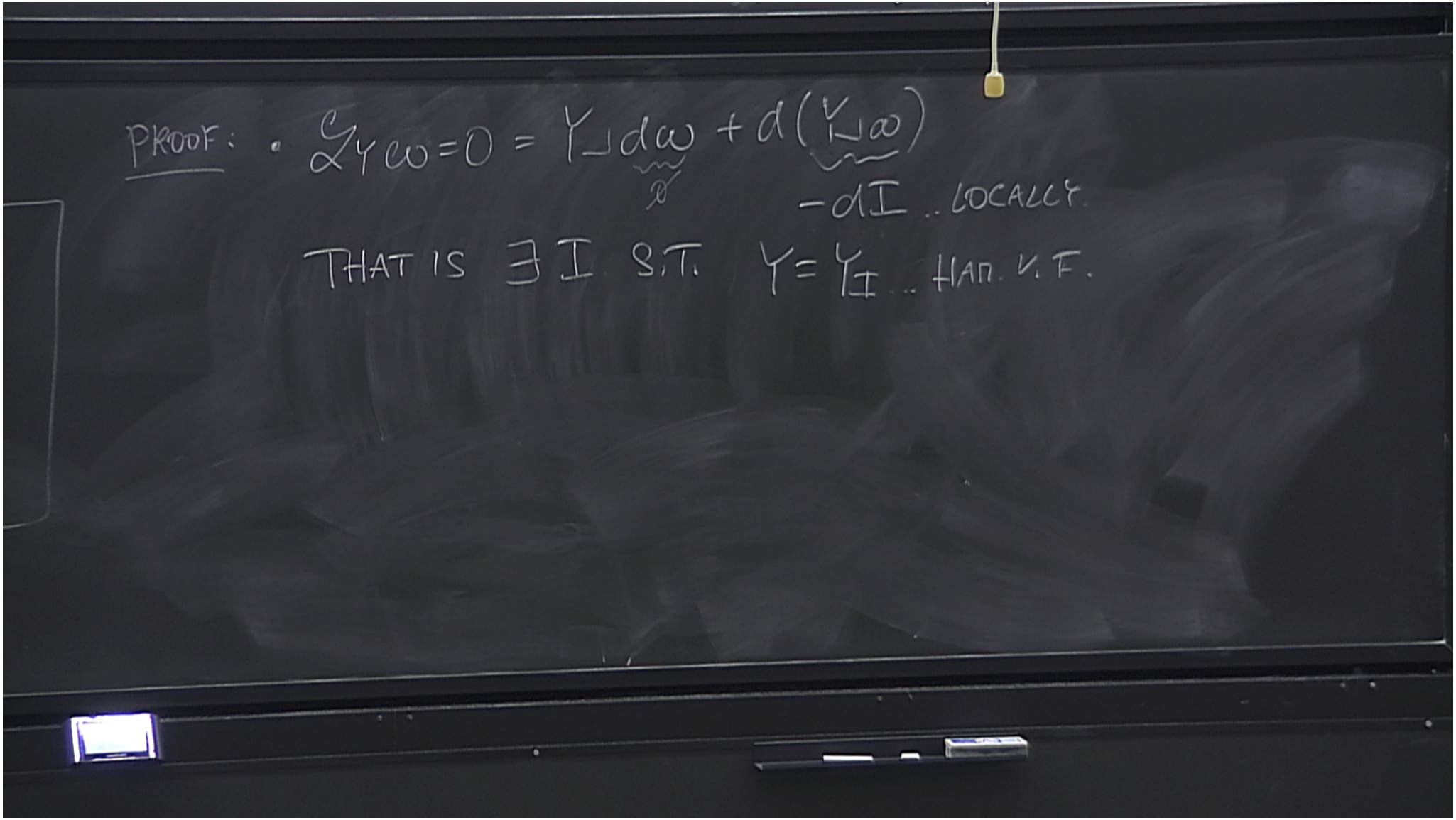

• NOTE THAT. $\{f,g\} = \frac{df}{dt_g} = -\{g,f\} = -\frac{d}{dt_f}$

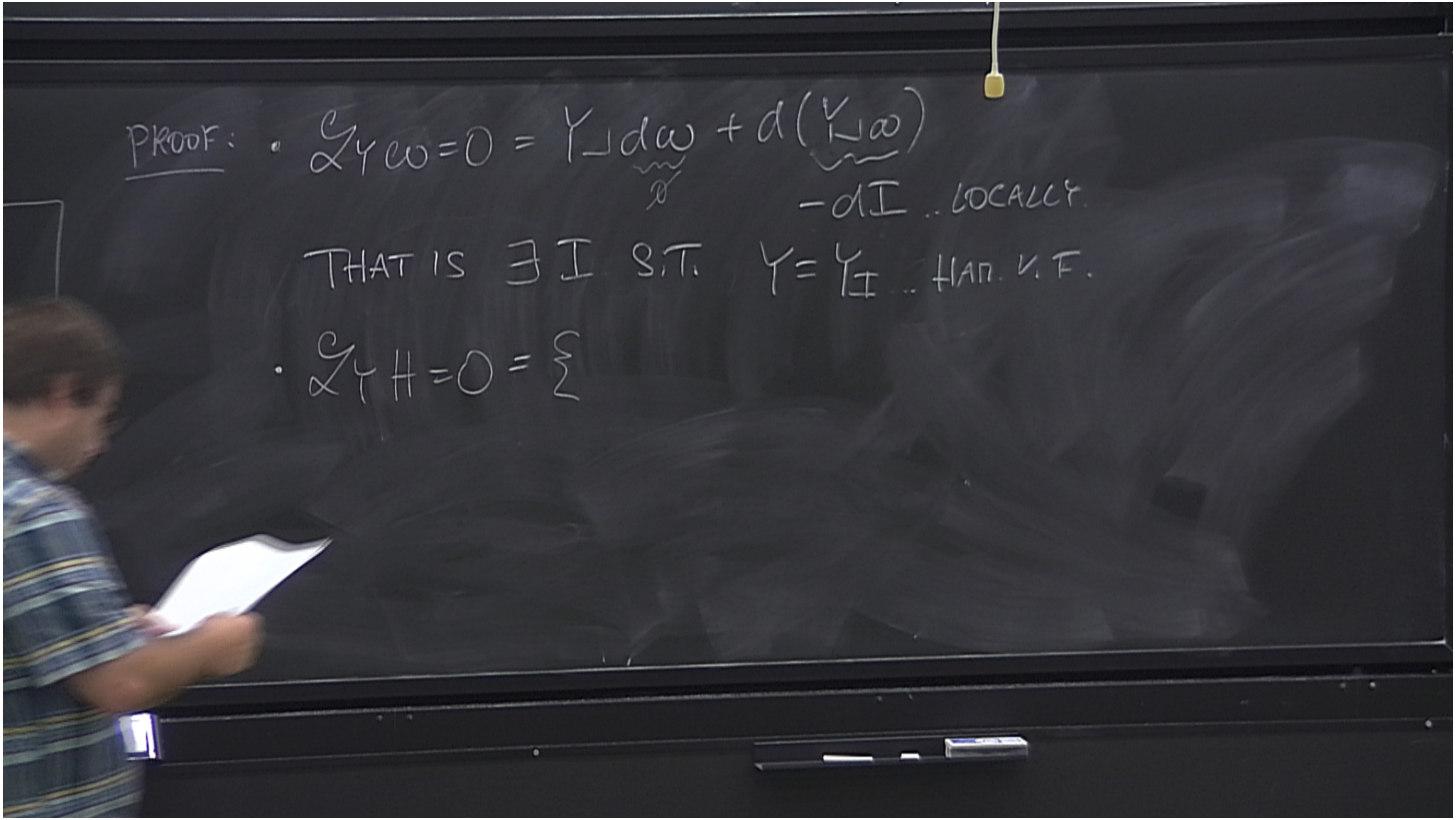
- NOTE THAT. $\{f,g\} = \frac{df}{dt_g} = -\{g,f\} = -\frac{dg}{dt_f}$

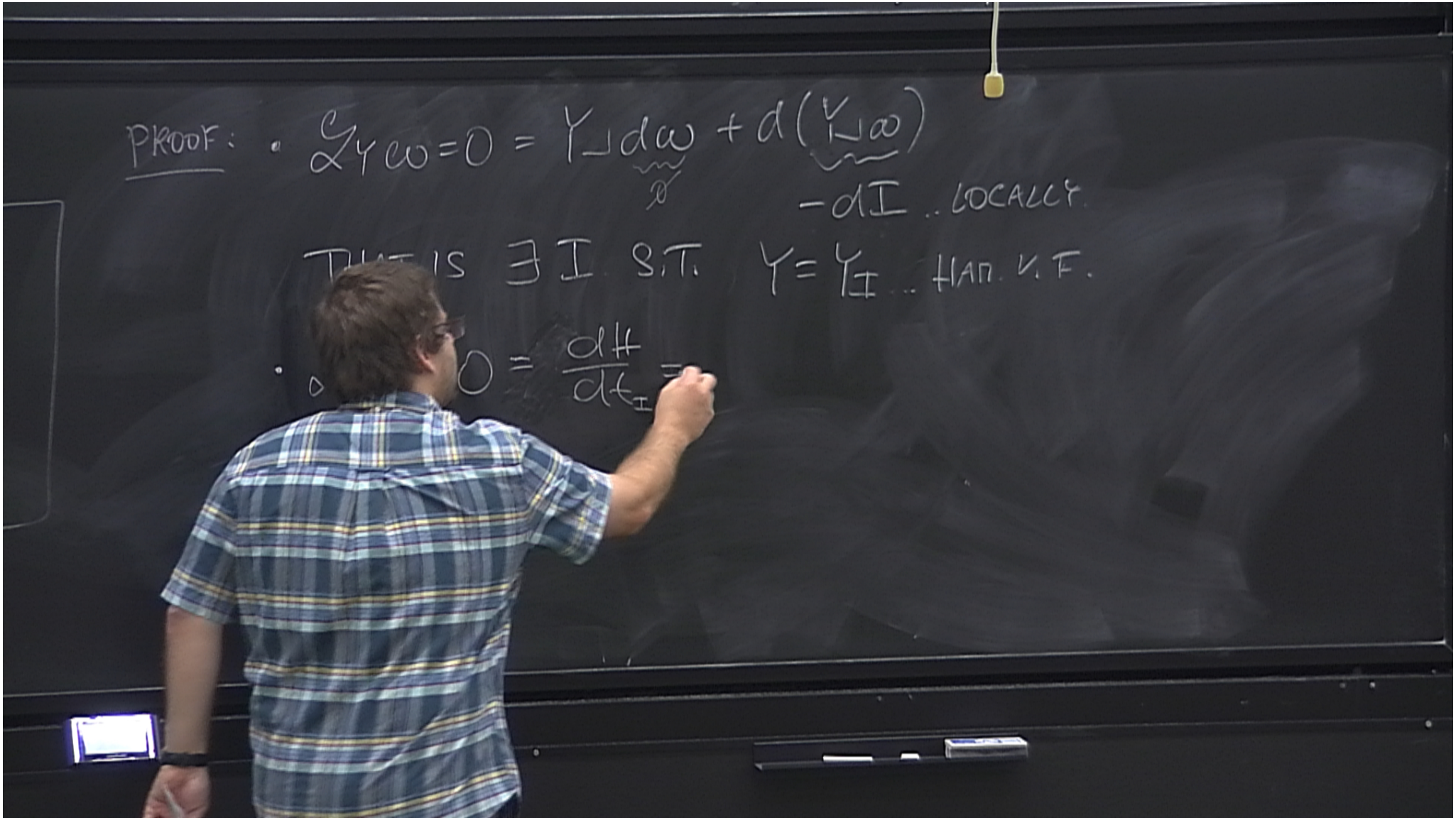
- VERSION 3. NOETHER'S THEOREM (HAMILTONIAN VERSION)

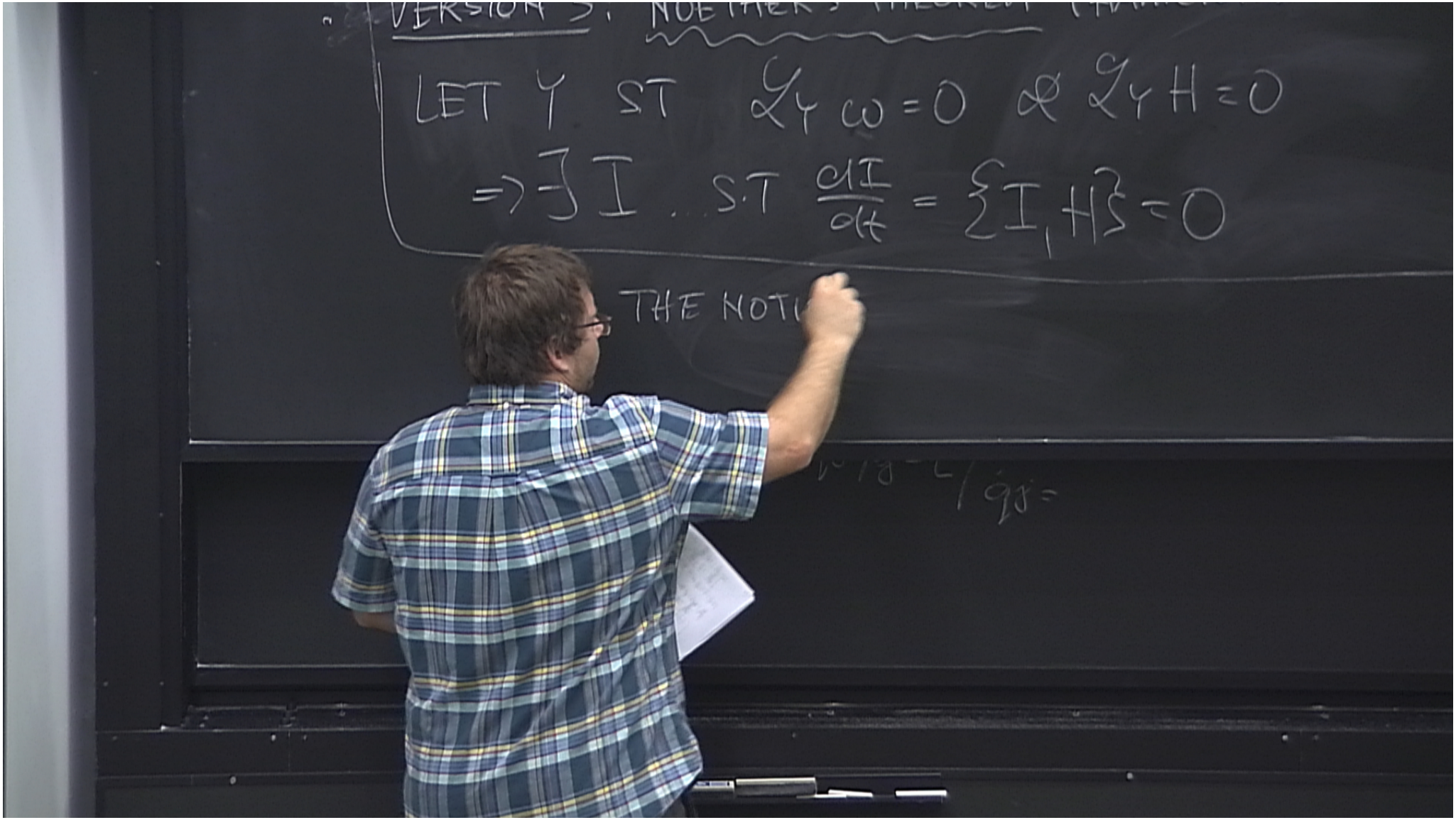
LET $Y$ S.T $\mathcal{L}_Y \omega = 0$ & $\mathcal{L}_Y H = 0$

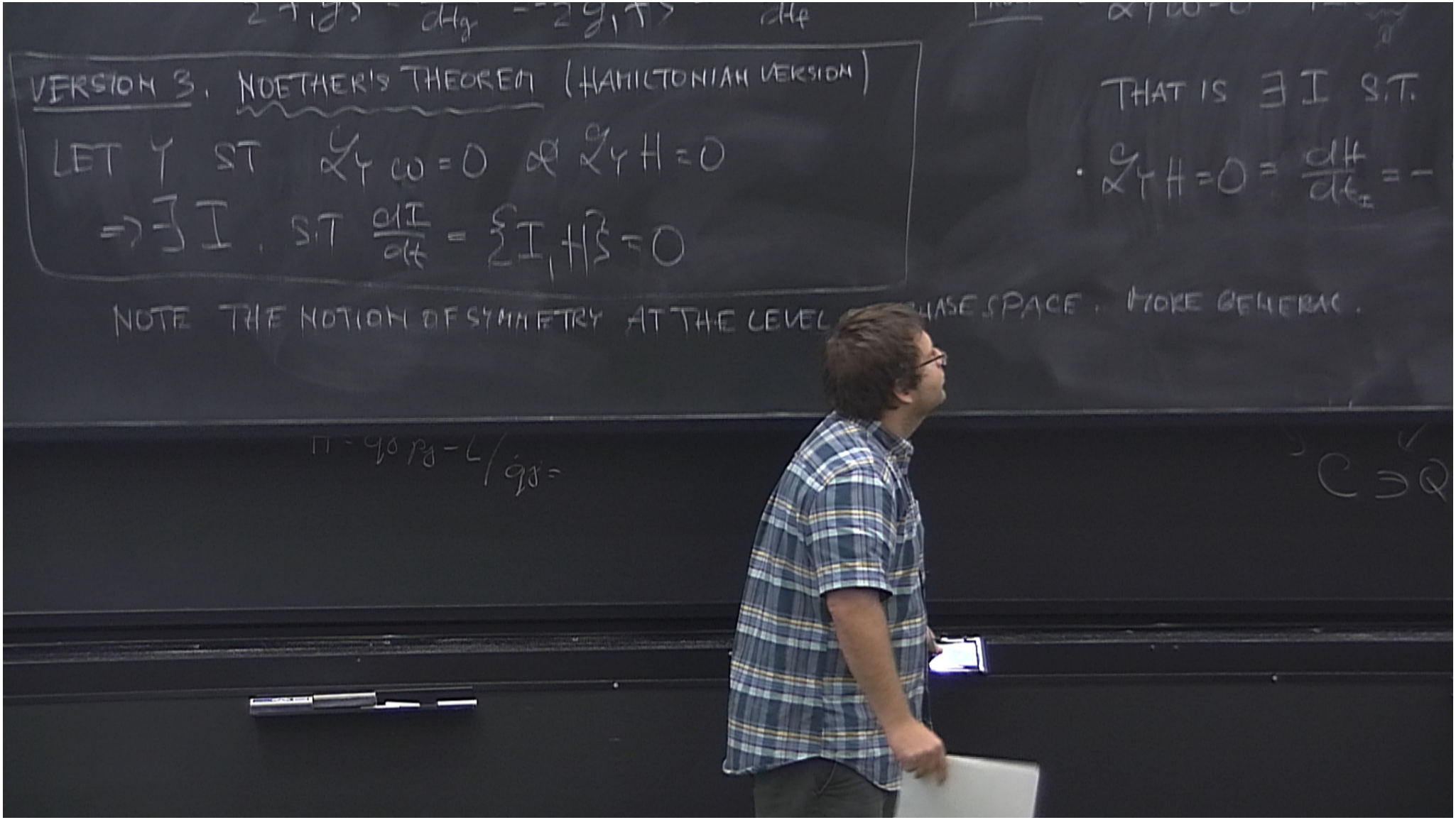
$\Rightarrow \exists I$ S.T $\frac{dI}{dt} = \{I, H\} = 0$

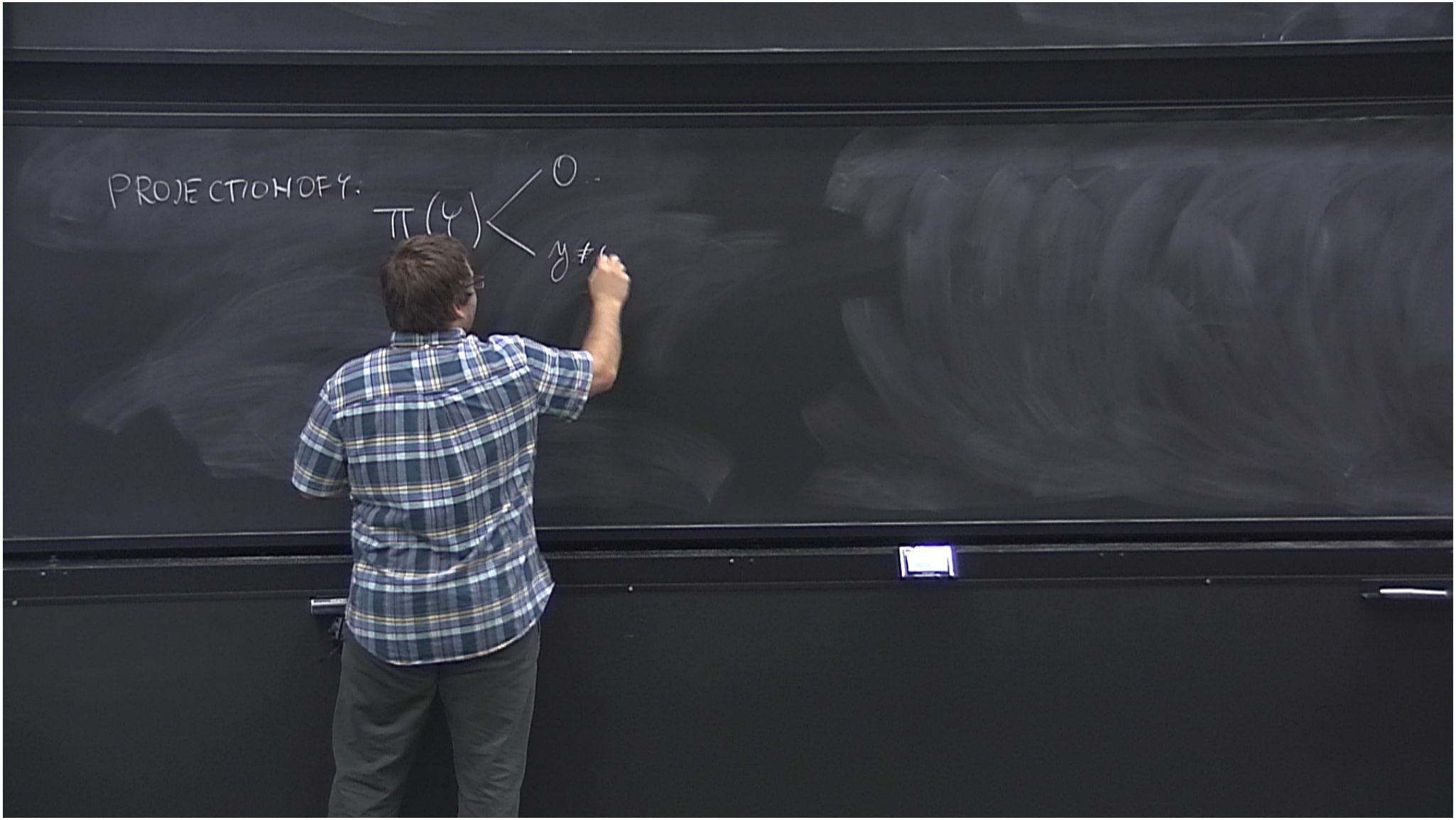

PROOF: • $\mathcal{L}_Y \omega = 0 = Y \lrcorner d\omega + d(Y \lrcorner \omega)$
$-dI$ .. LOCALLY
THAT IS $\exists I$ S.T. $Y = Y_I$ .. HAM. V. F.

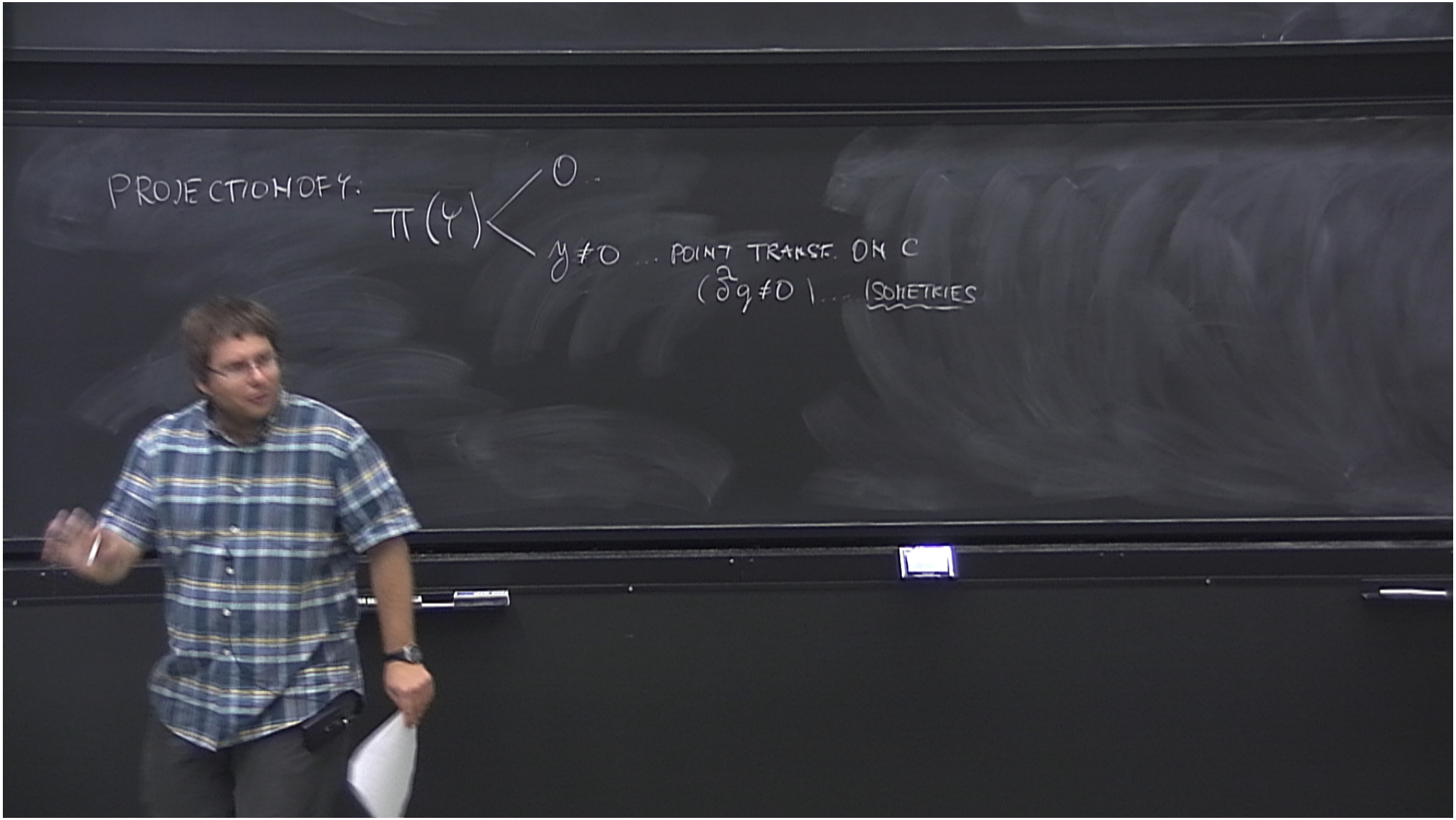

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THAT IS $\exists I$ S.T. $Y = Y_I$ .. HAM. V. F.
$\mathcal{L}_Y H = 0 = \{$

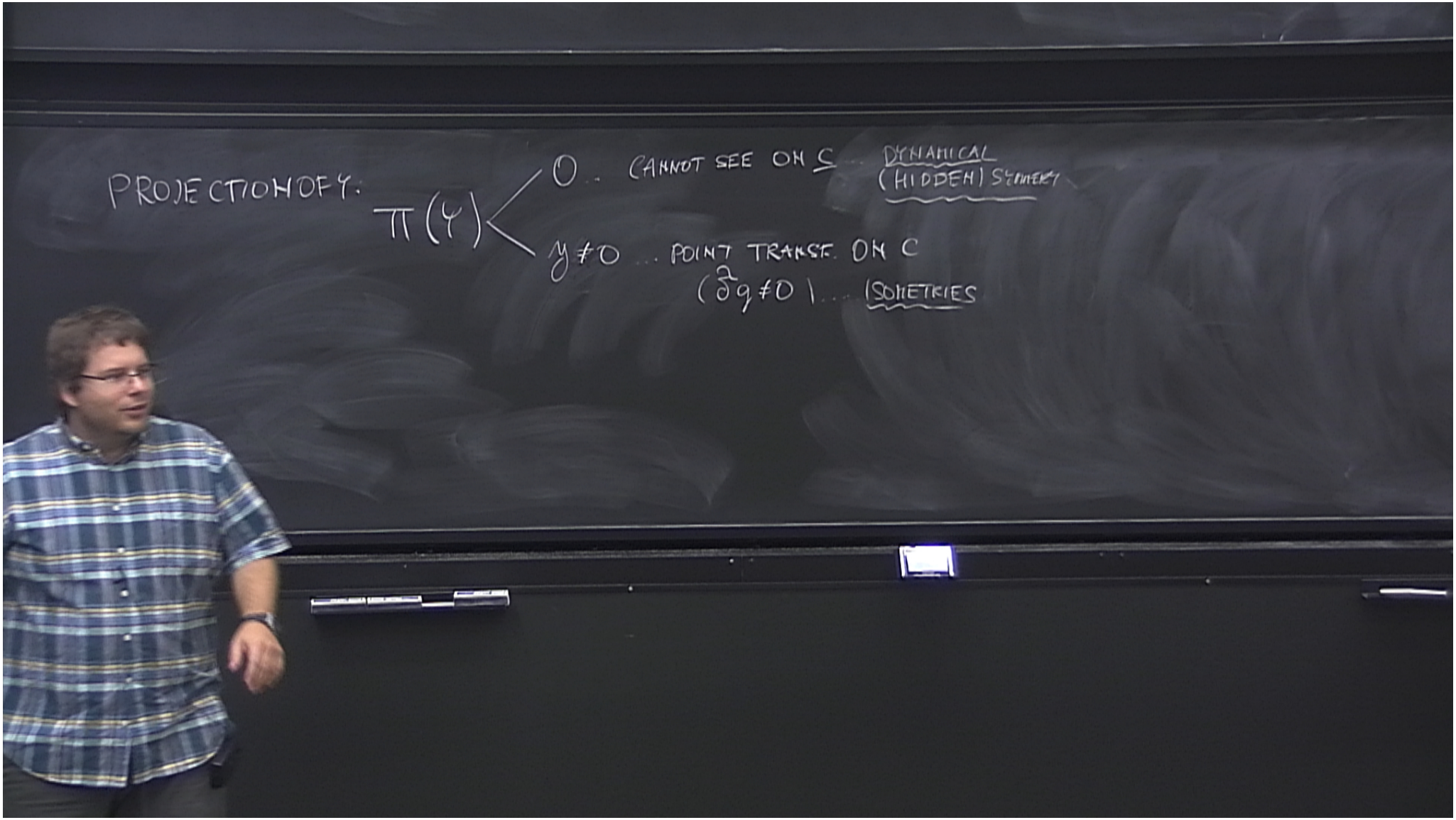

PROOF:
$\mathcal{L}_Y \omega = 0 = Y \lrcorner d\omega + d(Y \lrcorner \omega)$
$-dI$ .. LOCALLY
$Y = Y_I$ .. HAM. V. F.
$0 = \frac{dH}{dt_I}$

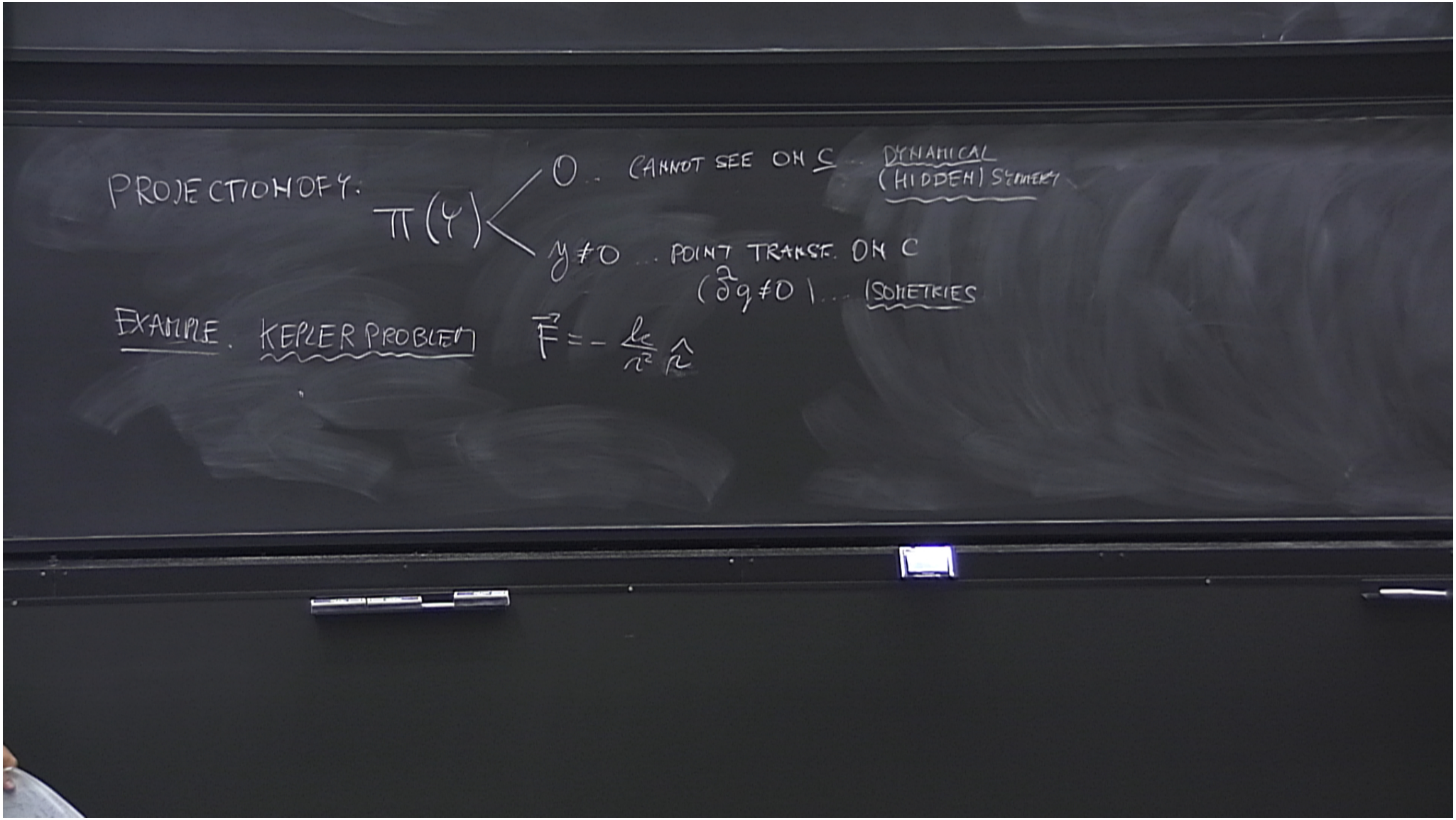

VERSION 3:
LET Y ST $\mathcal{L}_Y \omega = 0$ & $\mathcal{L}_Y H = 0$
$\Rightarrow \exists I$ S.T $\frac{dI}{dt} = \{I, H\} = 0$
THE NOTI

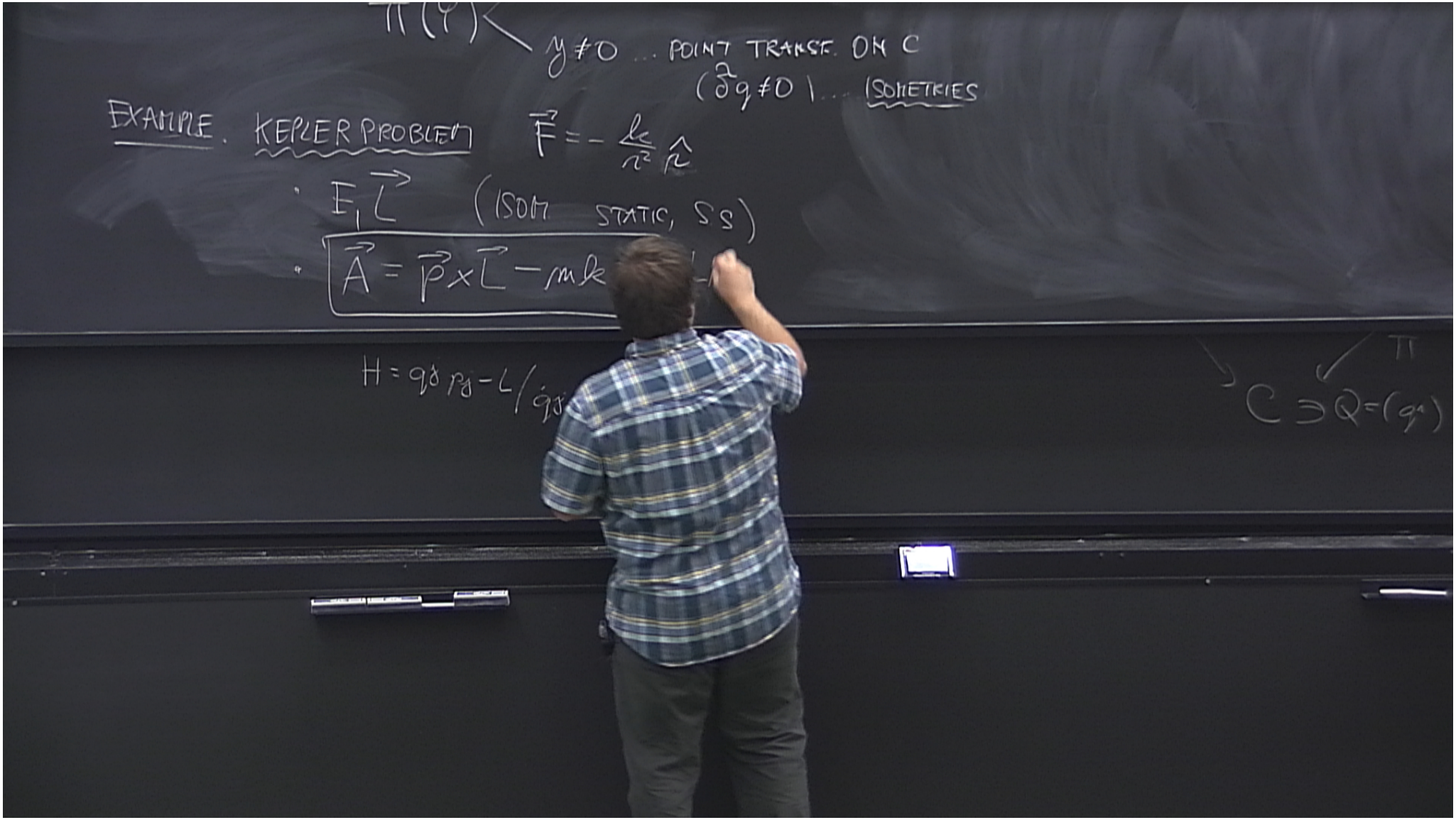

VERSION 3. NOETHER'S THEOREM (HAMILTONIAN VERSION)
LET Y ST $\mathcal{L}_Y \omega = 0$ & $\mathcal{L}_Y H = 0$
$\Rightarrow \exists I$ S.T $\frac{dI}{dt} = \{I, H\} = 0$
THAT IS $\exists I$ S.T.
$\mathcal{L}_Y H = 0 = \frac{dH}{dt_I} = -$
NOTE THE NOTION OF SYMMETRY AT THE LEVEL
PHASE SPACE. MORE GENERAL.
$\pi = q_0 p_0 - L / q_0 =$
$C \supset Q$


PROJECTION OF γ:
π(γ)
0


PROJECTION OF Y:
π(Y)
0
y≠0
POINT TRANSF. ON C
(∃q≠0)
SYMETRIES


PROJECTION OF Y:
π(Y)
0 .. CANNOT SEE ON C .. DYNAMICAL (HIDDEN) SYMMETRY
y ≠ 0 .. POINT TRANSF. ON C
(∂y ≠ 0) .. ISOMETRIES


PROJECTION OF Y:
π(Y)
0 .. CANNOT SEE ON C .. DYNAMICAL (HIDDEN) SYMMETRY
y ≠ 0 .. POINT TRANSF. ON C
(∂q ≠ 0) .. ISOMETRIES
EXAMPLE. KEPLER PROBLEM
F = - k/r² r̂

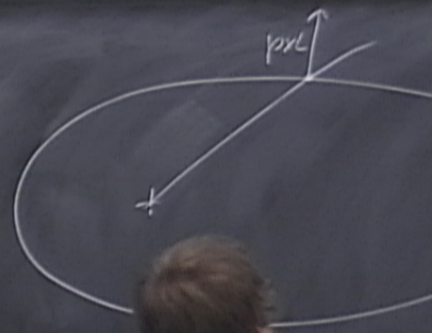

$\pi(q)$
$y \neq 0$ ... POINT TRANSF. ON C
$(\partial_q \neq 0)$ ... ISOMETRIES
EXAMPLE. KEPLER PROBLEM $\vec{F} = -\frac{k}{r^2}\hat{r}$
$E, \vec{L}$ (ISOM STATIC, SS)
$\vec{A} = \vec{p} \times \vec{L} - mk$
$H = q\dot{\partial}_q - L(q,$
$C \ni Q = (q^1)$
$\pi$

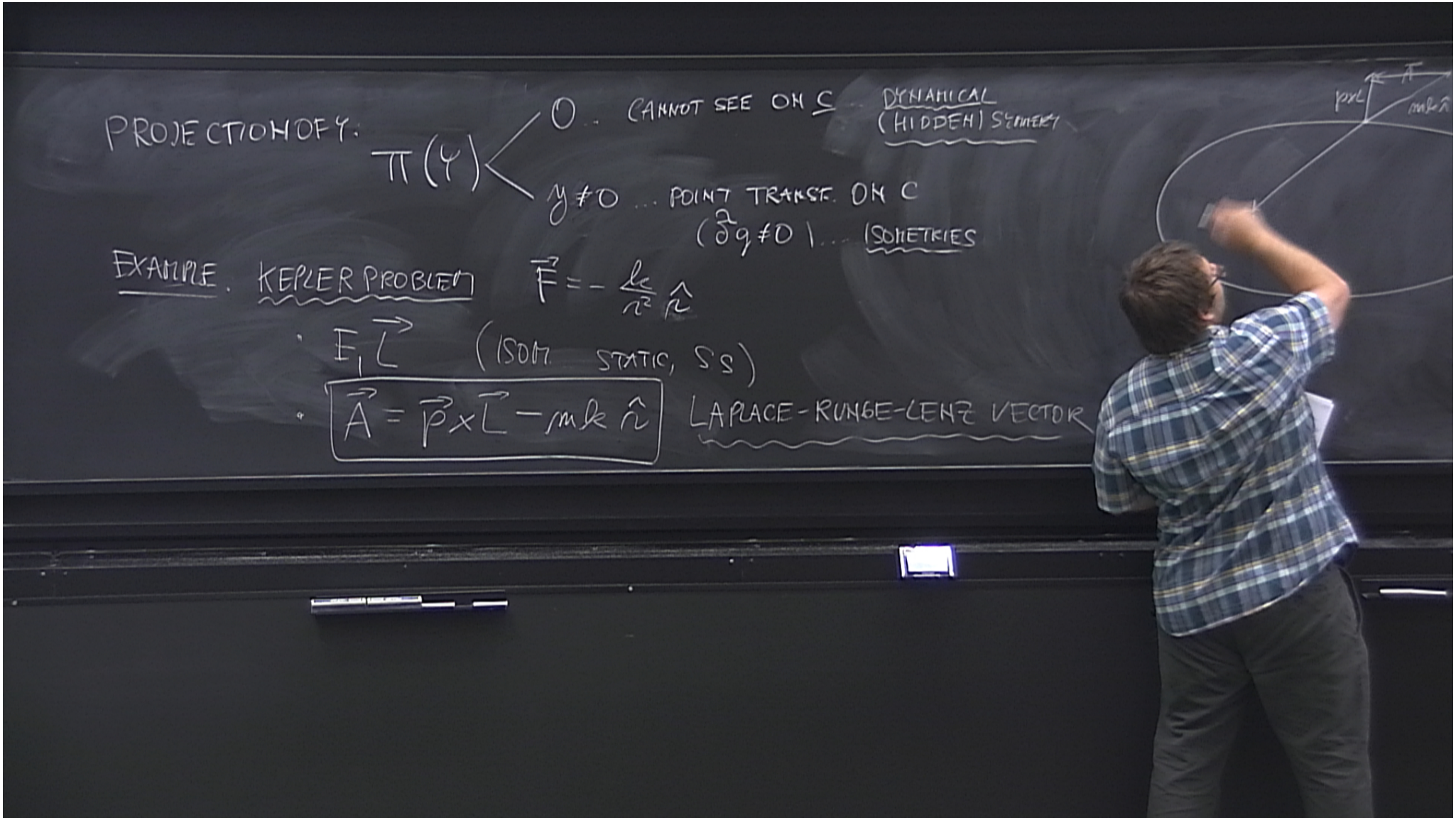
PROJECTION OF $\gamma$: $\pi(\gamma)$ < $0$ .. CANNOT SEE ON $C$ .. DYNAMICAL (HIDDEN) SYMMETRY

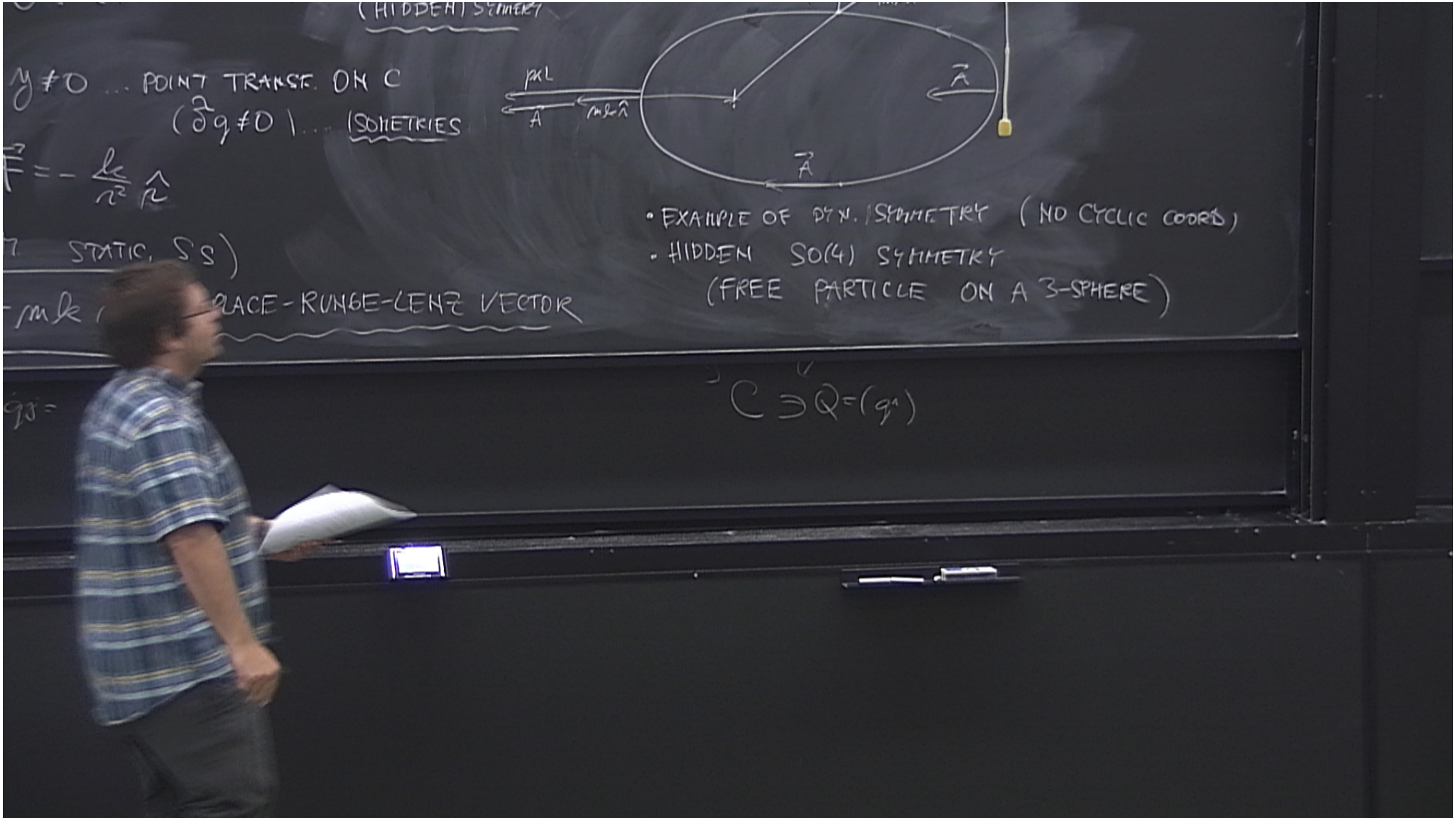
$y \neq 0$ ... POINT TRANSF. ON $C$ ($\partial q \neq 0$) ... SYMETRIES

EXAMPLE. KEPLER PROBLEM $\vec{F} = -\frac{k}{r^2}\hat{r}$

- $E, \vec{L}$ (ISOM STATIC, SS)
- $\boxed{\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}}$ LAPLACE-RUNGE-LENZ VECTOR




PROJECTION OF Y:
π(Y)
0 .. CANNOT SEE ON C .. DYNAMICAL (HIDDEN) SYMMETRY
y≠0 .. POINT TRANSF. ON C (∂q≠0) .. ISOMETRIES
EXAMPLE. KEPLER PROBLEM
F = -k/r² r̂
E, L (ISOM STATIC, SS)
A = p×L - mk r̂
LAPLACE-RUNGE-LENZ VECTOR
p×L
A
mkr̂


(HIDDEN) SYMMETRY
$\gamma \neq 0$ ... POINT TRANSF. ON C
($\partial_q \neq 0$) ... (SO)METRIES
$\vec{F} = -\frac{k}{r^2}\hat{r}$
STATIC, SS)
PLACE-RUNGE-LENZ VECTOR
$\vec{p} \times \vec{L}$
$\vec{A}$
$m\dot{\vec{r}} \times \hat{r}$
$\vec{A}$
$\vec{A}$
• EXAMPLE OF DYN. SYMMETRY (NO CYCLIC COORD.)
• HIDDEN SO(4) SYMMETRY
(FREE PARTICLE ON A 3-SPHERE)
$C \ni Q = (q^1)$

$\tilde{\Delta} = P_i dQ^i \quad \omega = d\tilde{\Delta}$

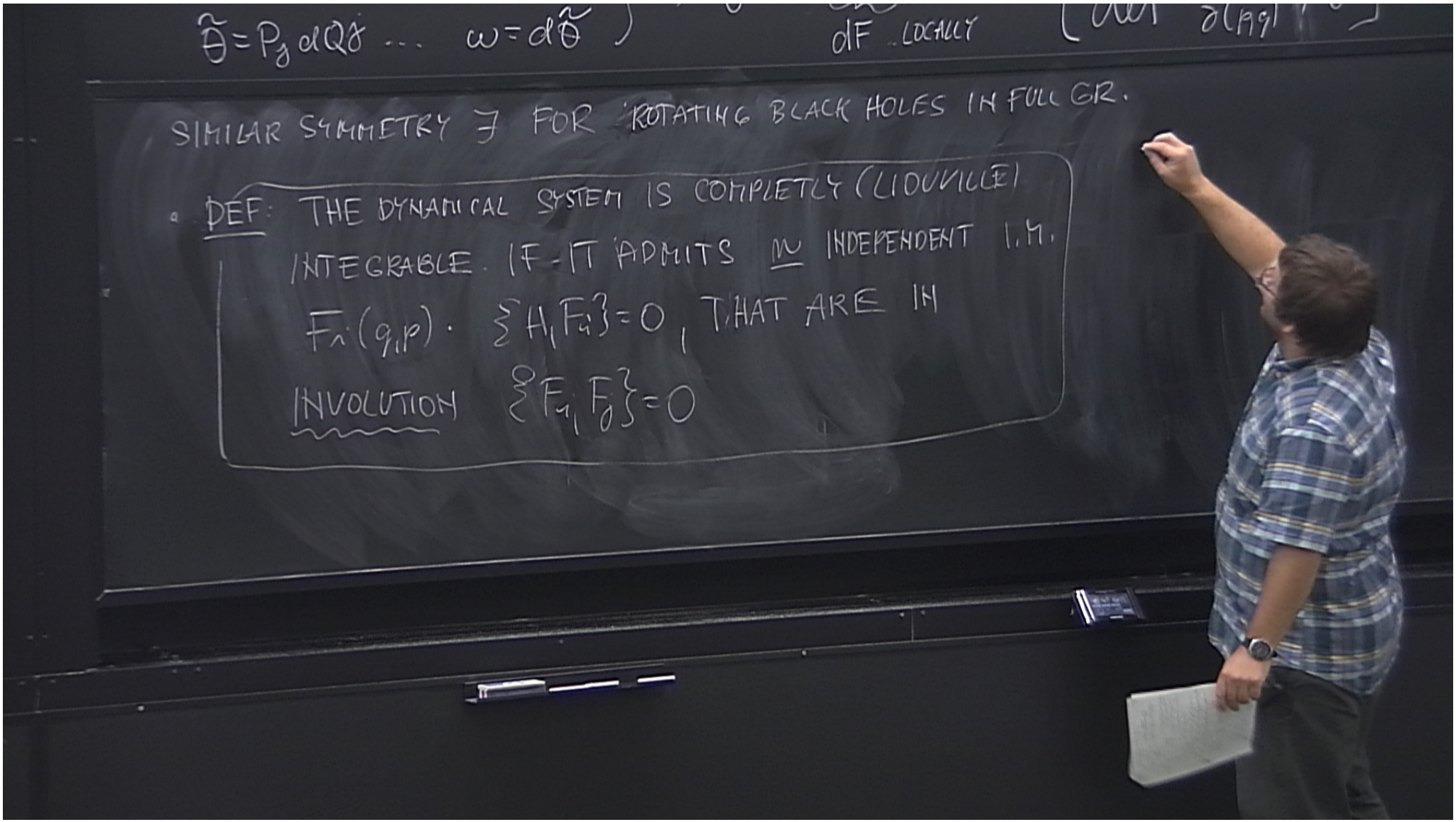
SIMILAR SYMMETRY $\exists$ FOR ROTATING BLACK HOLES IN FULL GR.

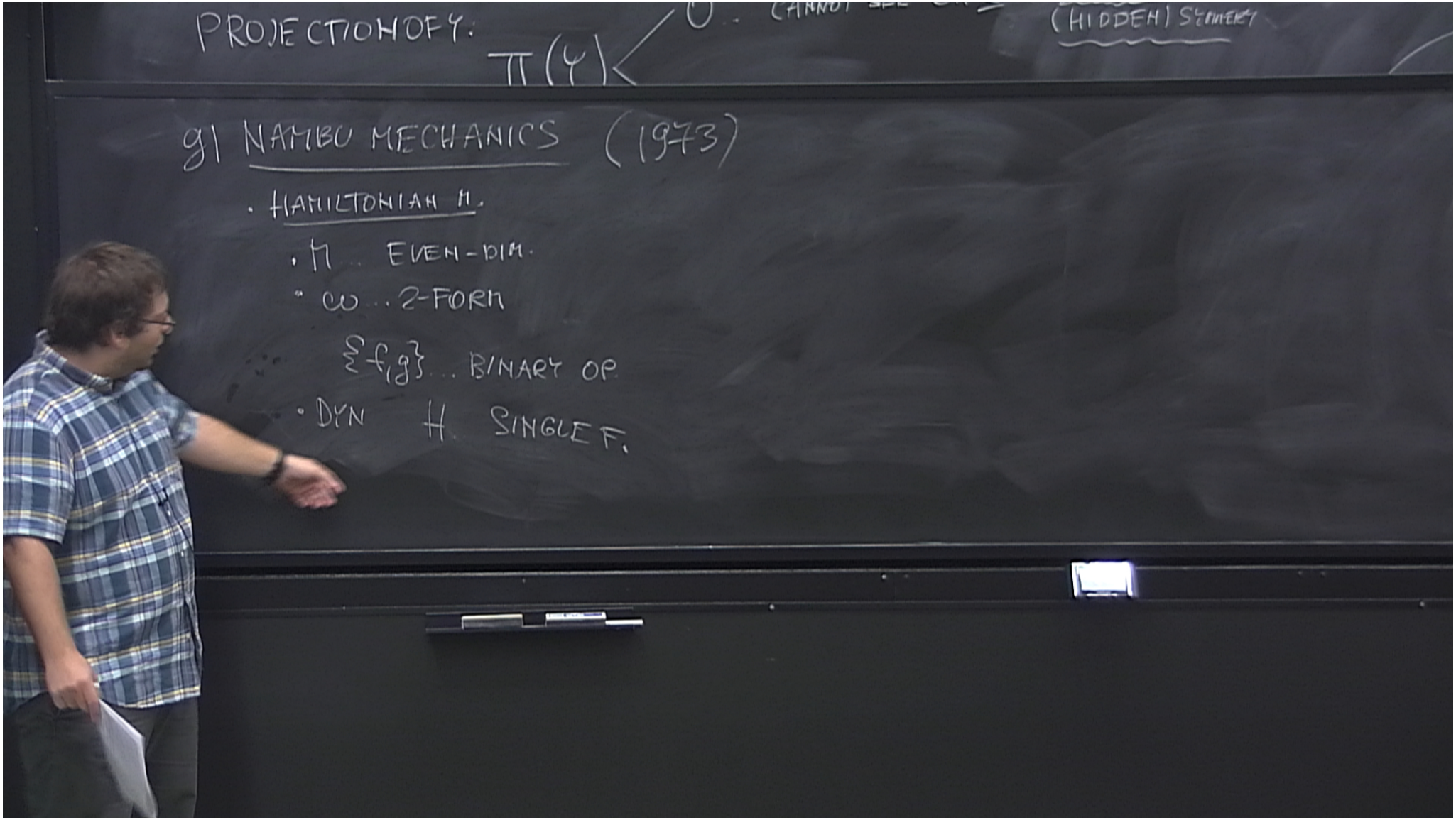
- DEF: THE DYNAMICAL SYSTEM IS COMPLETELY (LIOUVILLE) INTEGRABLE IF IT ADMITS $N$ INDEPENDENT I.M. $F_i(q,p)$ · $\{H, F_i\} = 0$, THAT ARE IN INVOLUTION $\{F_i, F_j\} = 0$

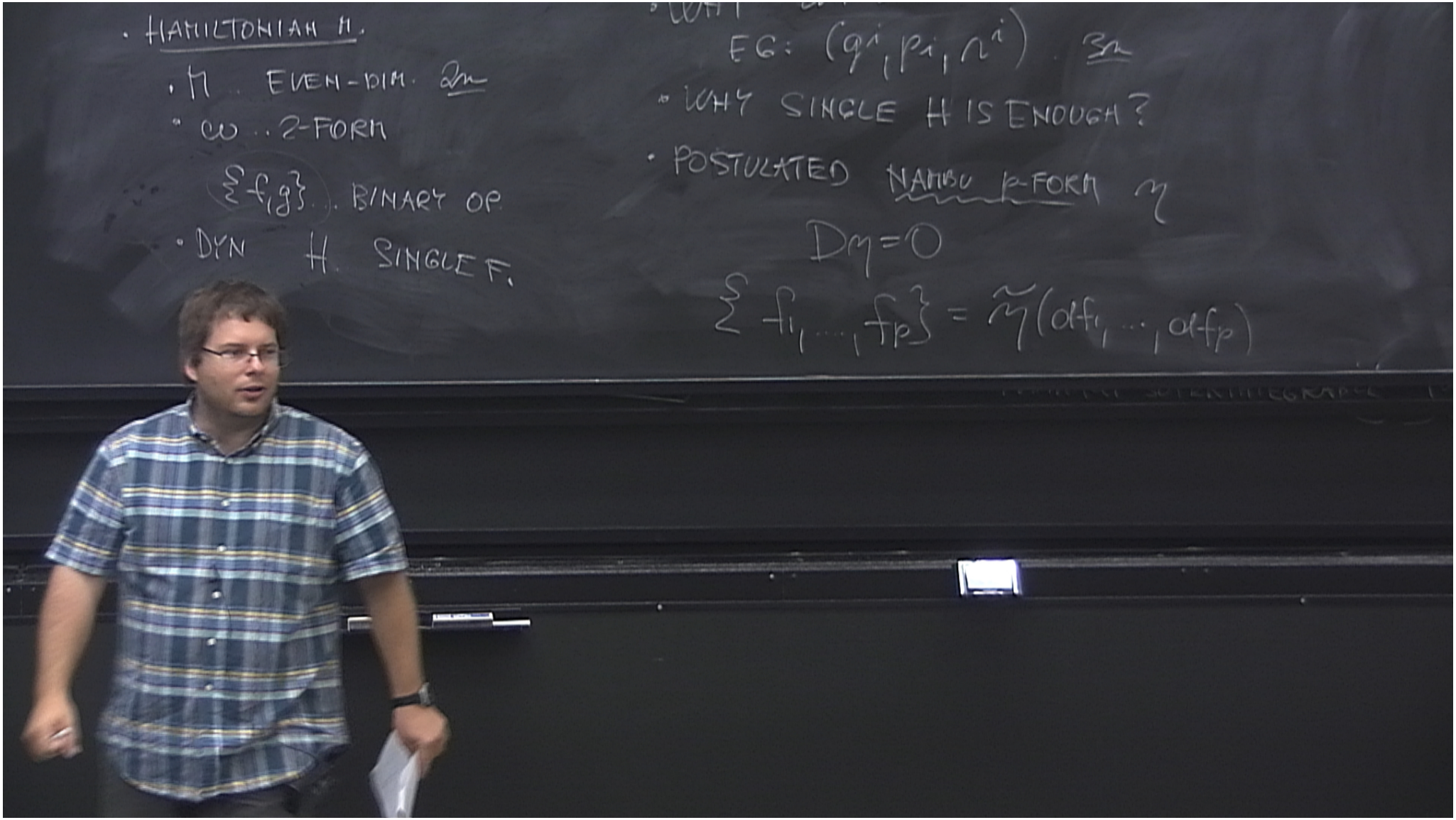
$\tilde{\Theta} = P_j dQ^j \ldots \quad \omega = d\tilde{\Theta}$ )  $dF$ LOCALLY

SIMILAR SYMMETRY $\exists$ FOR ROTATING BLACK HOLES IN FULL GR.

- DEF: THE DYNAMICAL SYSTEM IS COMPLETELY (LIOUVILLE) INTEGRABLE. IF IT ADMITS $N$ INDEPENDENT I.M. $F_i(q,p)$. $\{H, F_i\} = 0$, THAT ARE IN INVOLUTION $\{F_i, F_j\} = 0$




PROJECTION OF Y:
π(Y)
(HIDDEN) SYMMETRY
9| NAMBU MECHANICS (1973)
· HAMILTONIAN H.
· M .. EVEN-DIM.
· ω .. 2-FORM
{f,g} .. BINARY OP
· DYN H SINGLE F.


· HAMILTONIAN H.
· H EVEN-DIM. 2n
· ω 2-FORM
{f,g} BINARY OP
· DYN H SINGLE F.
EG: (q^i, p_i, ?^i) 3n
· WHY SINGLE H IS ENOUGH?
· POSTULATED NAMBU k-FORM η
Dη = 0
{f_1, ..., f_p} = η̃(df_1, ..., df_p)