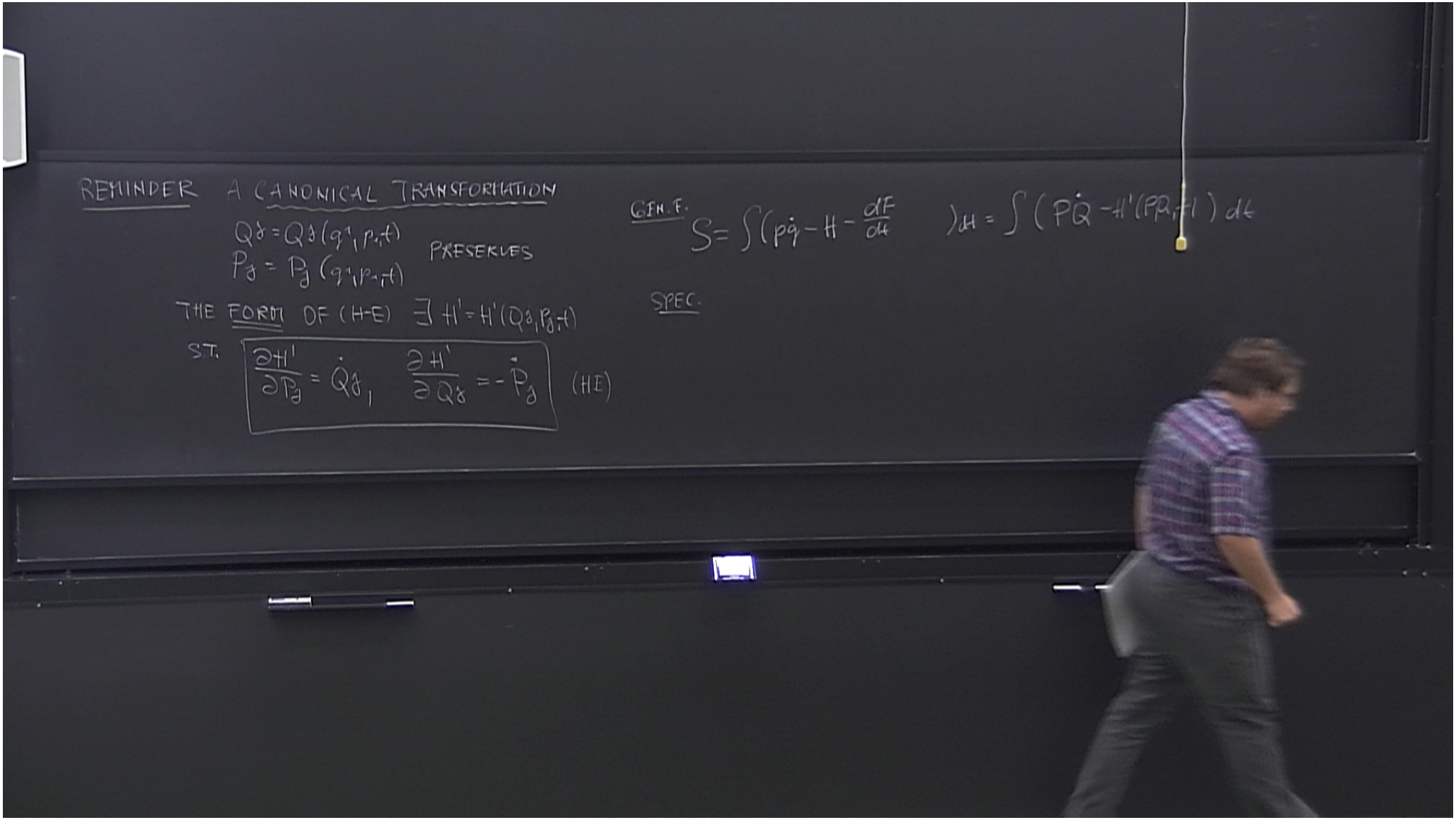
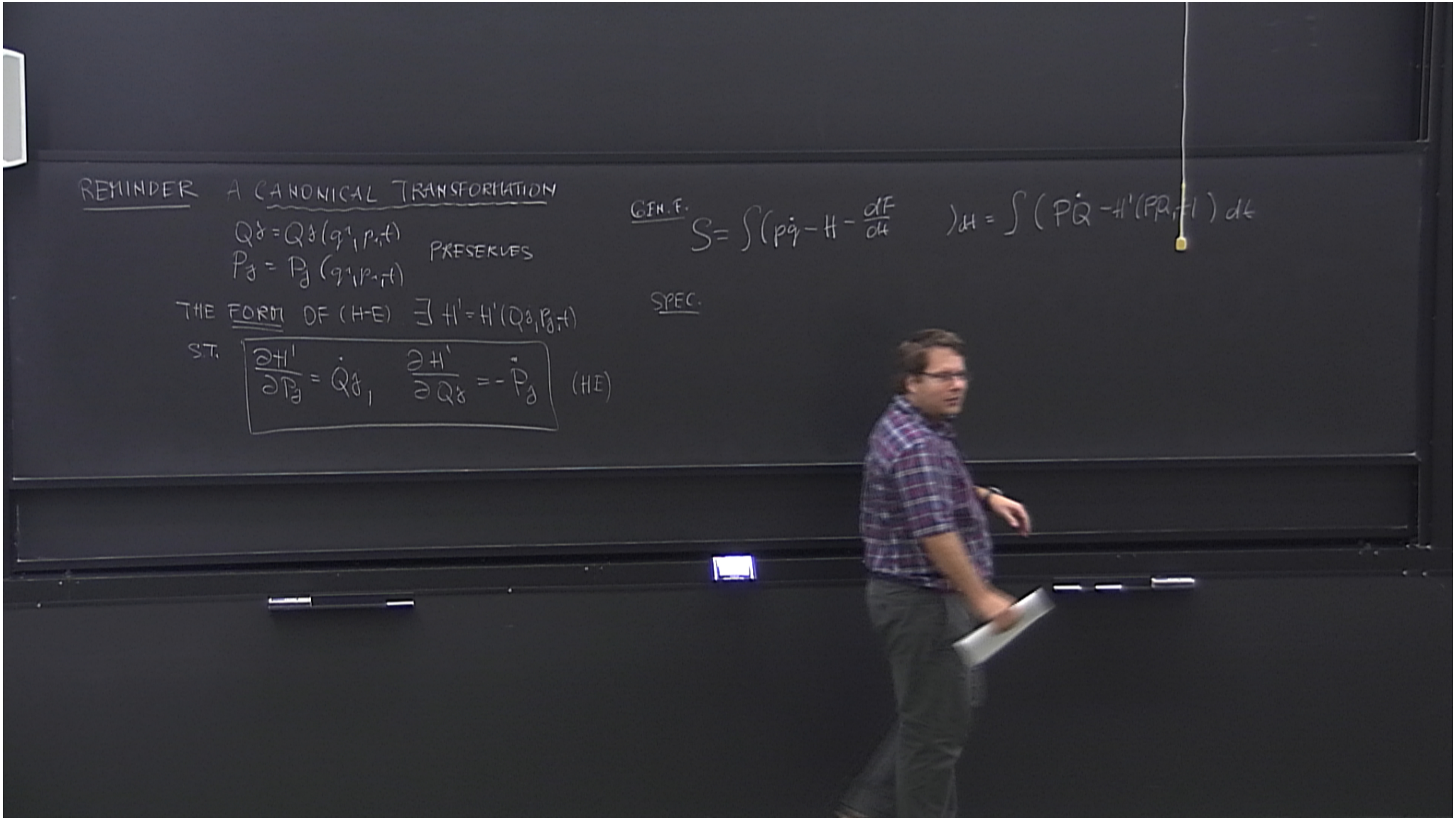
Title: Classical Mechanics - David Kubiznak

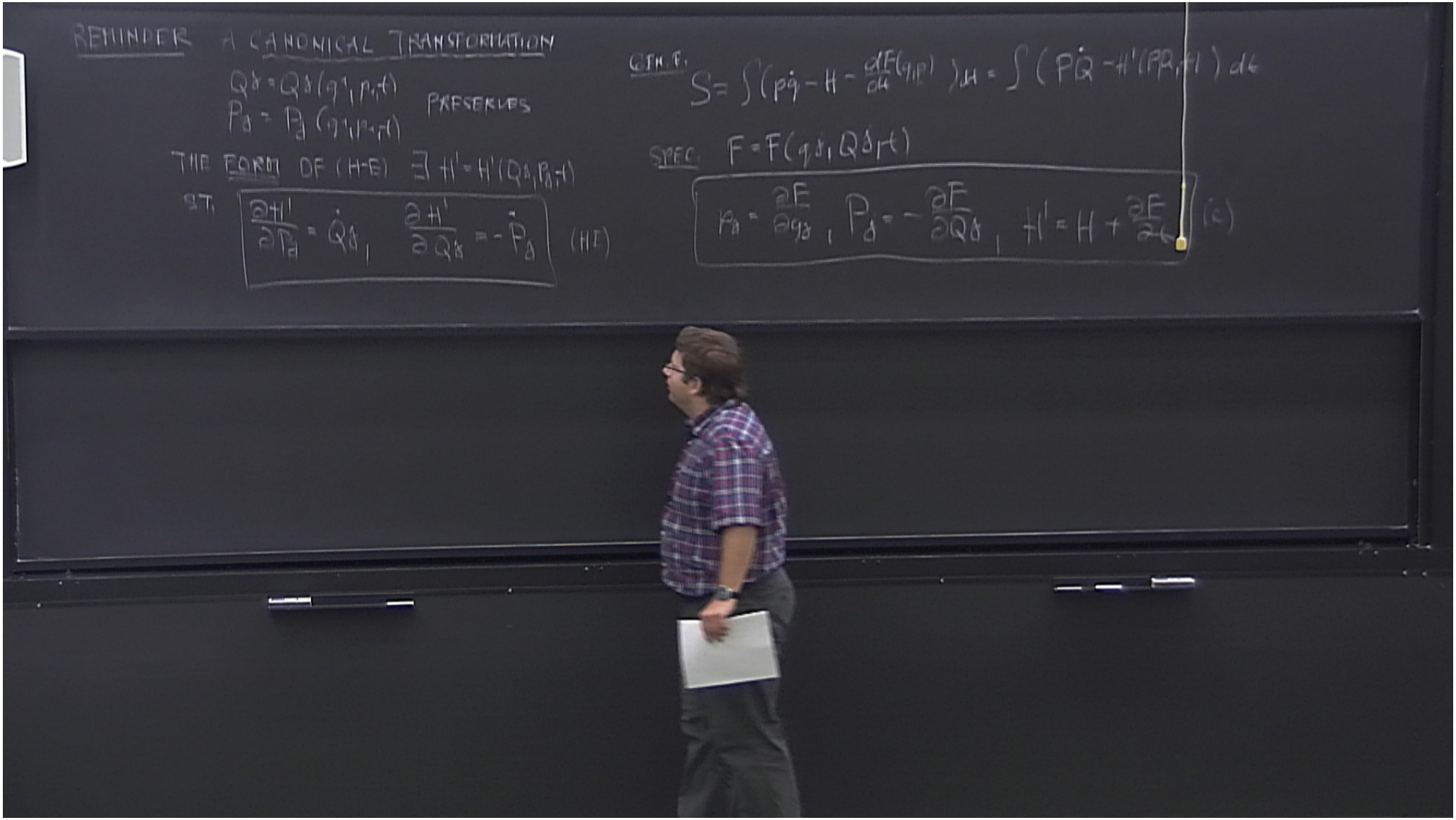
Date: Aug 25, 2015 09:00 AM

URL: http://pirsa.org/15080101

Abstract:


REMINDER A CANONICAL TRANSFORMATION
$Q_\gamma = Q_\gamma(q^1, P_i, t)$
$P_\gamma = P_\gamma(q^1, P_i, t)$
PRESERVES
GEN. F.
$S = \int (p\dot{q} - H - \frac{dF}{dt}) dt = \int (P\dot{Q} - H'(P,Q,t)) dt$
THE FORM OF (H-E) $\exists H' = H'(Q_\gamma, P_\gamma, t)$
SPEC.
ST. $\frac{\partial H'}{\partial P_\gamma} = \dot{Q}_\gamma, \quad \frac{\partial H'}{\partial Q_\gamma} = -\dot{P}_\gamma$ (HE)


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SPEC.


REMINDER A CANONICAL TRANSFORMATION
$Q^\alpha = Q^\alpha(q^\beta, p_\beta, t)$
$P_\alpha = P_\alpha(q^\beta, p_\beta, t)$
PRESERVES
THE FORM OF (H-E) $\exists\ H' = H'(Q^\alpha, P_\alpha, t)$
S.T. $\frac{\partial H'}{\partial P_\alpha} = \dot{Q}^\alpha, \quad \frac{\partial H'}{\partial Q^\alpha} = -\dot{P}_\alpha$ (HE)
GEN. F. $S = \int (p\dot{q} - H - \frac{dF(q,P)}{dt})\,dt = \int (P\dot{Q} - H'(P,Q,t))\,dt$
SPEC. $F = F(q^\alpha, Q^\alpha, t)$
$p_\alpha = \frac{\partial F}{\partial q^\alpha}, \quad P_\alpha = -\frac{\partial F}{\partial Q^\alpha}, \quad H' = H + \frac{\partial F}{\partial t}$ (C)

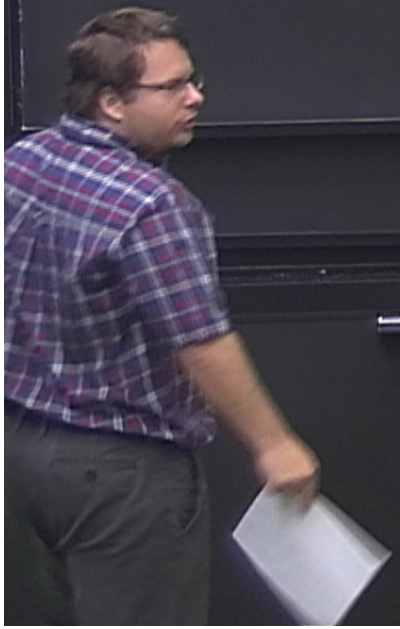
$\frac{\partial P_\beta}{\partial t} = Q_\beta, \quad \frac{\partial Q_\beta}{\partial t} = -P_\beta$ (HE)

$P_\beta = \frac{\partial S}{\partial q_\beta}, \; Q_\beta = \frac{\partial S}{\partial Q_\beta}, \; H' = H + \frac{\partial S}{\partial t}$

d) HAMILTON-JACOBI THEORY

INGENIOUS IDEA. LET'S FIND A SPECIAL GEN. F.
S S.T. THE NEW HAMILTONIAN $H' = 0$

THEN. • (HE)



$\dot{P}_\beta = Q_\beta$, $\dot{Q}_\beta = -P_\beta$ (HE)

$P_\beta = \frac{\partial g_\beta}{\partial Q_\beta}$, $H' = H + \frac{\partial}{\partial t}$

d) HAMILTON-JACOBI THEORY

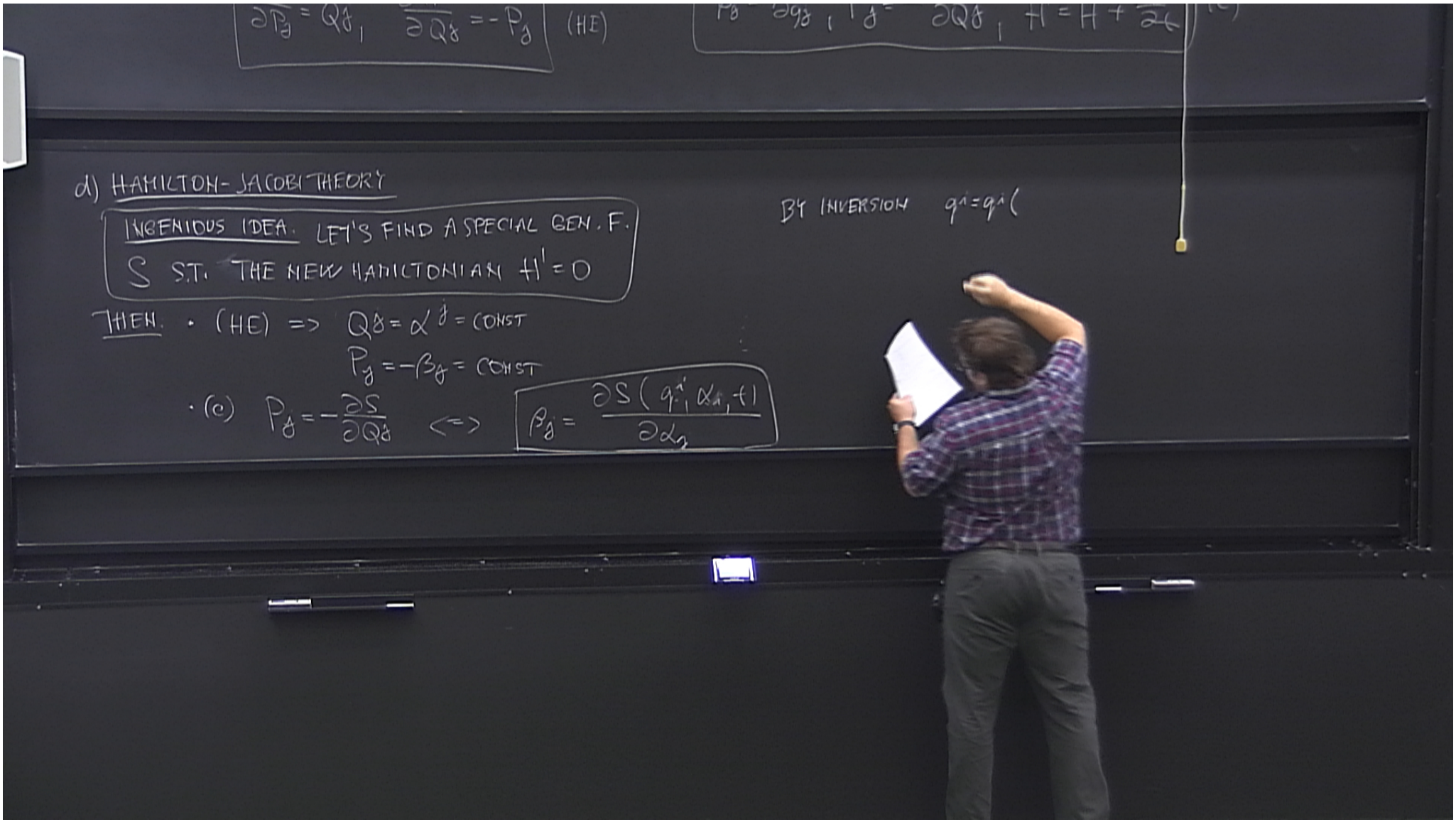
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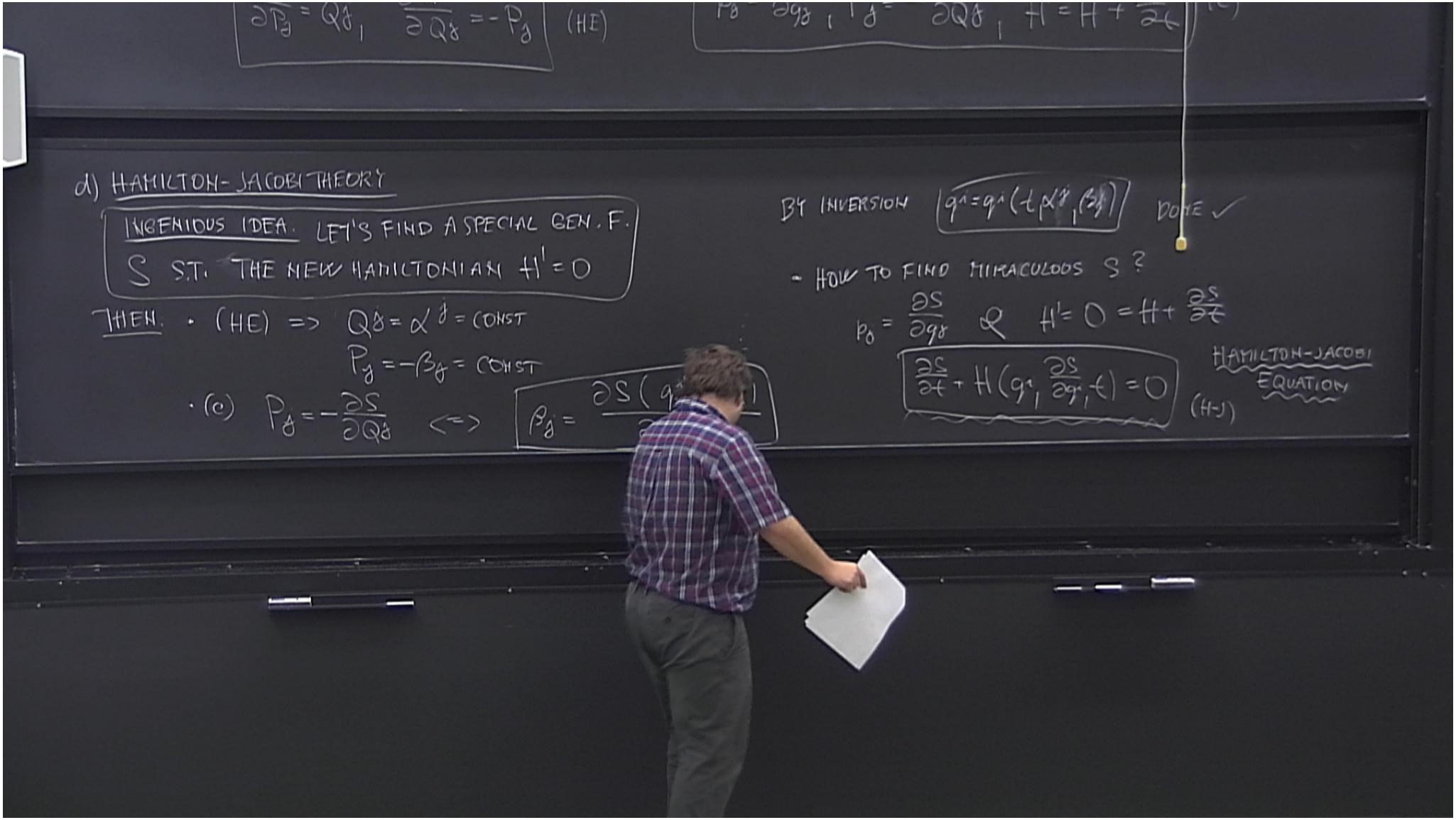
BY INVERSION $q^i = q^i($

THEN. • (HE) $\Rightarrow$ $Q_\beta = \alpha_\beta = \text{CONST}$

$P_\beta = -\beta_\beta = \text{CONST}$

• (c) $P_\beta = -\frac{\partial S}{\partial Q_\beta}$ $\Leftrightarrow$ $\beta_\beta = \frac{\partial S(q^i, \alpha_\beta, t)}{\partial \alpha_\beta}$




$\frac{\partial P_\gamma}{\partial t} = \dot{Q}_\gamma$, $\frac{\partial H}{\partial Q_\gamma} = -\dot{P}_\gamma$ (HE)
d) HAMILTON-JACOBI THEORY
INGENIOUS IDEA. LET'S FIND A SPECIAL GEN. F.
S S.T. THE NEW HAMILTONIAN $H' = 0$
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$P_\gamma = -\beta_\gamma = \text{CONST}$
• (c) $P_\gamma = -\frac{\partial S}{\partial Q_\gamma}$ <=> $\beta_\gamma = \frac{\partial S(q^i, \alpha_\gamma, t)}{\partial \alpha_\gamma}$
BY INVERSION $q^i = q^i(t, \alpha_\gamma, \beta_\gamma)$ DONE ✓
- HOW TO FIND MIRACULOUS S ?
$p_\gamma = \frac{\partial S}{\partial q_\gamma}$ & $H' = 0 = H + \frac{\partial S}{\partial t}$
$\frac{\partial S}{\partial t} + H(q_i, \frac{\partial S}{\partial q_i}, t) = 0$ (H-J)
HAMILTON-JACOBI EQUATION

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HAMILTON-JACOBI EQUATION

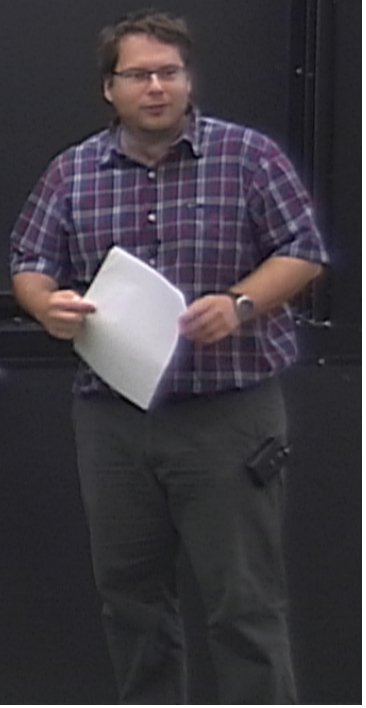
• 1 PDE FOR $S = S(q_i, t)$

↑ ACTION AS A FUNCTION OF "COORDINATES".

• INTERESTED IN "COMPLETE INTEGRAL" $S(q_i, \alpha_i, t)$

RATHER THAN "GENERAL INTEGRAL" (ARBITRARY F.)

• USE SEPARATION OF VARIABLES



S S.T. THE NEW HAMILTONIAN $H' = 0$

THEN: • (HE) $\Rightarrow$ $Q_\beta = \alpha_\beta = \text{CONST}$

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• (c) $P_\beta = -\dfrac{\partial S}{\partial Q_\beta}$ $\Leftrightarrow$ $\boxed{\beta_\beta = \dfrac{\partial S(q_i, \alpha_i, t)}{\partial \alpha_\beta}}$

- HOW TO FIND MIRACULOUS S ?

$p_\beta = \dfrac{\partial S}{\partial q_\beta}$ & $H' = 0 = H + \dfrac{\partial S}{\partial t}$

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HAMILTON-JACOBI EQUATION

• 1 PDE FOR $S = S(q_\beta, t)$

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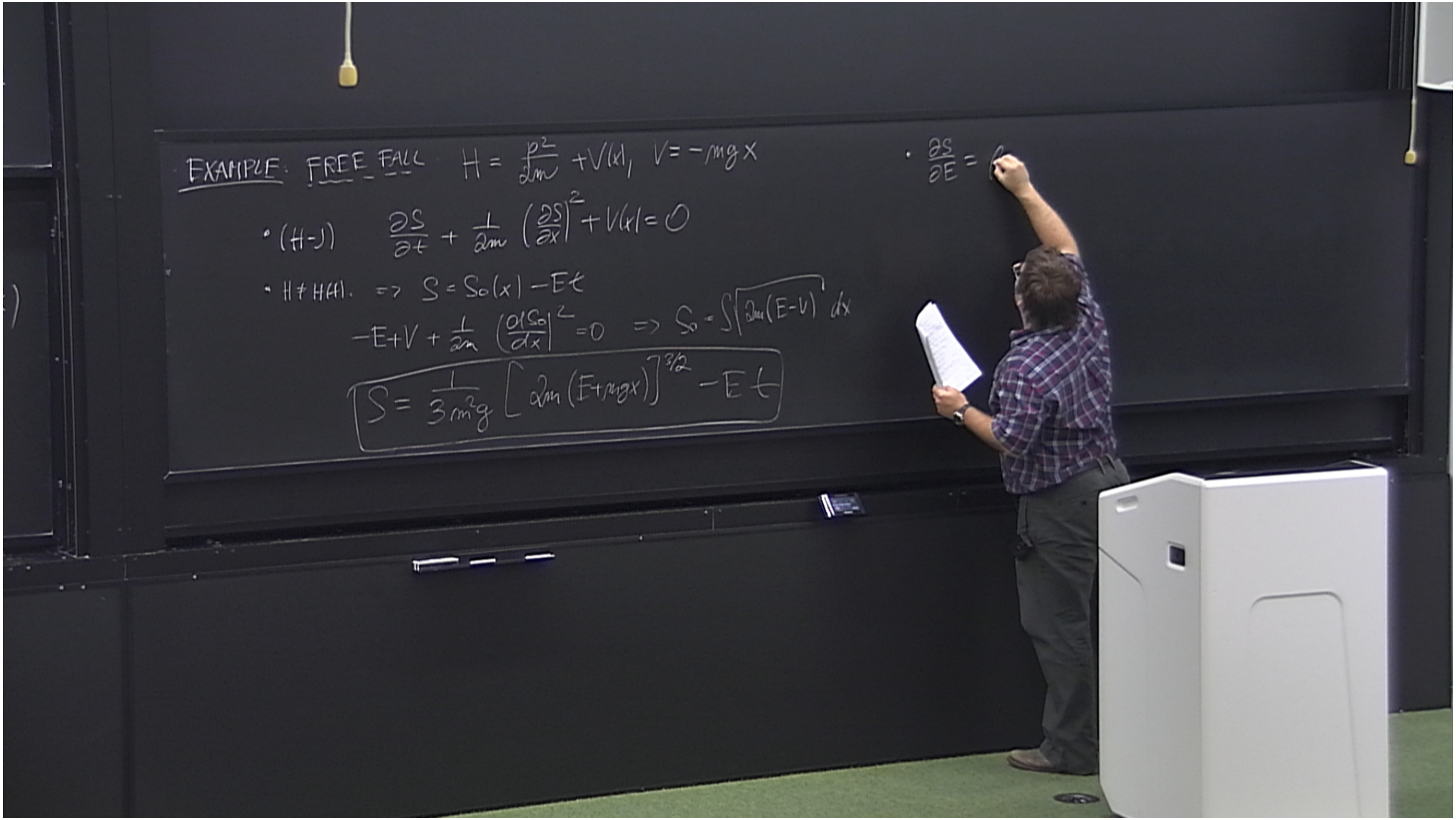
$\dfrac{\partial^2 f}{\partial t^2} - v^2 \dfrac{\partial^2 f}{\partial x^2} = 0$ $f = f(x - vt)$,

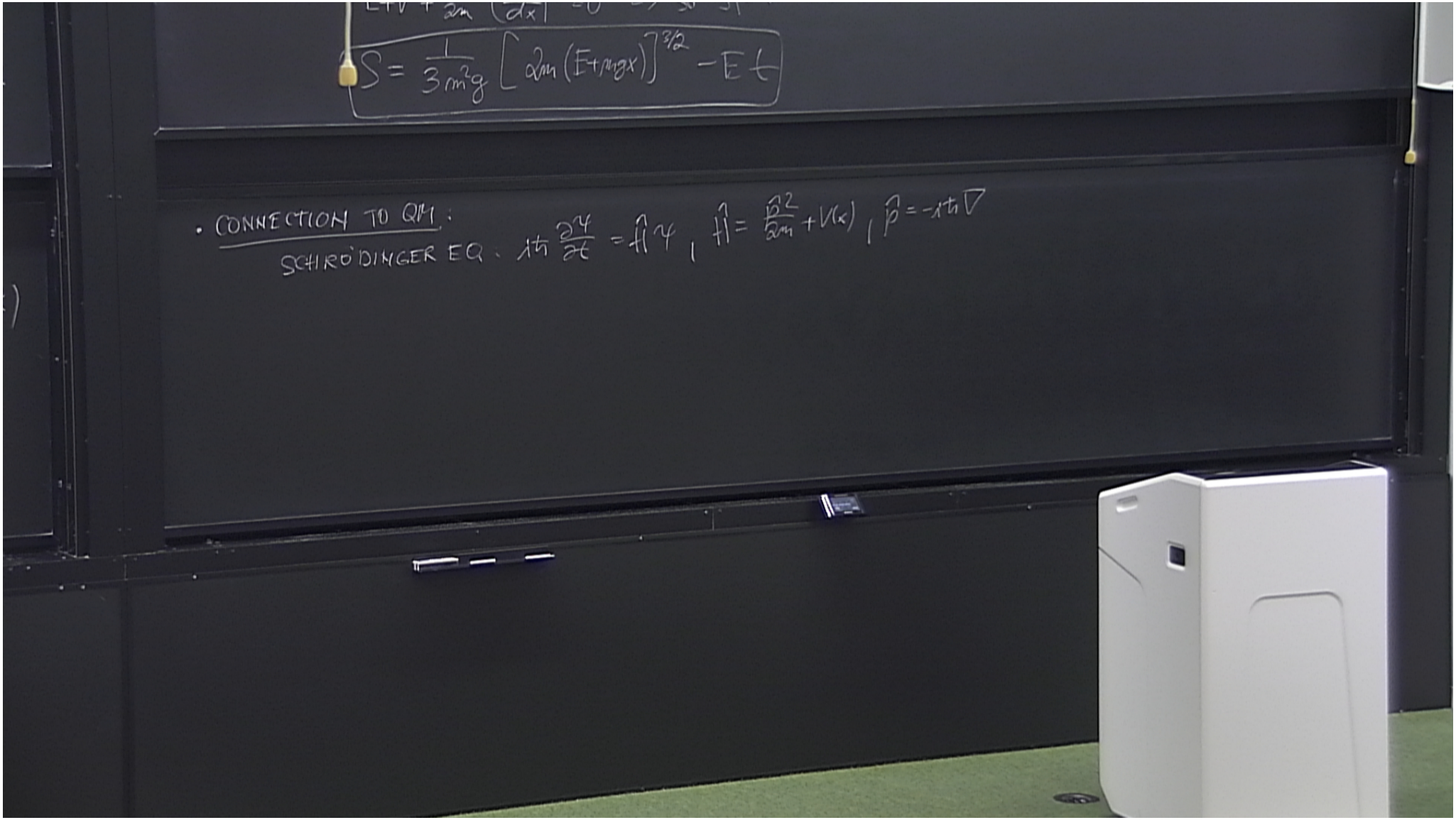
$H \neq H(t) \Rightarrow S(q_\beta, t) = S_0(q_\beta) - Et$

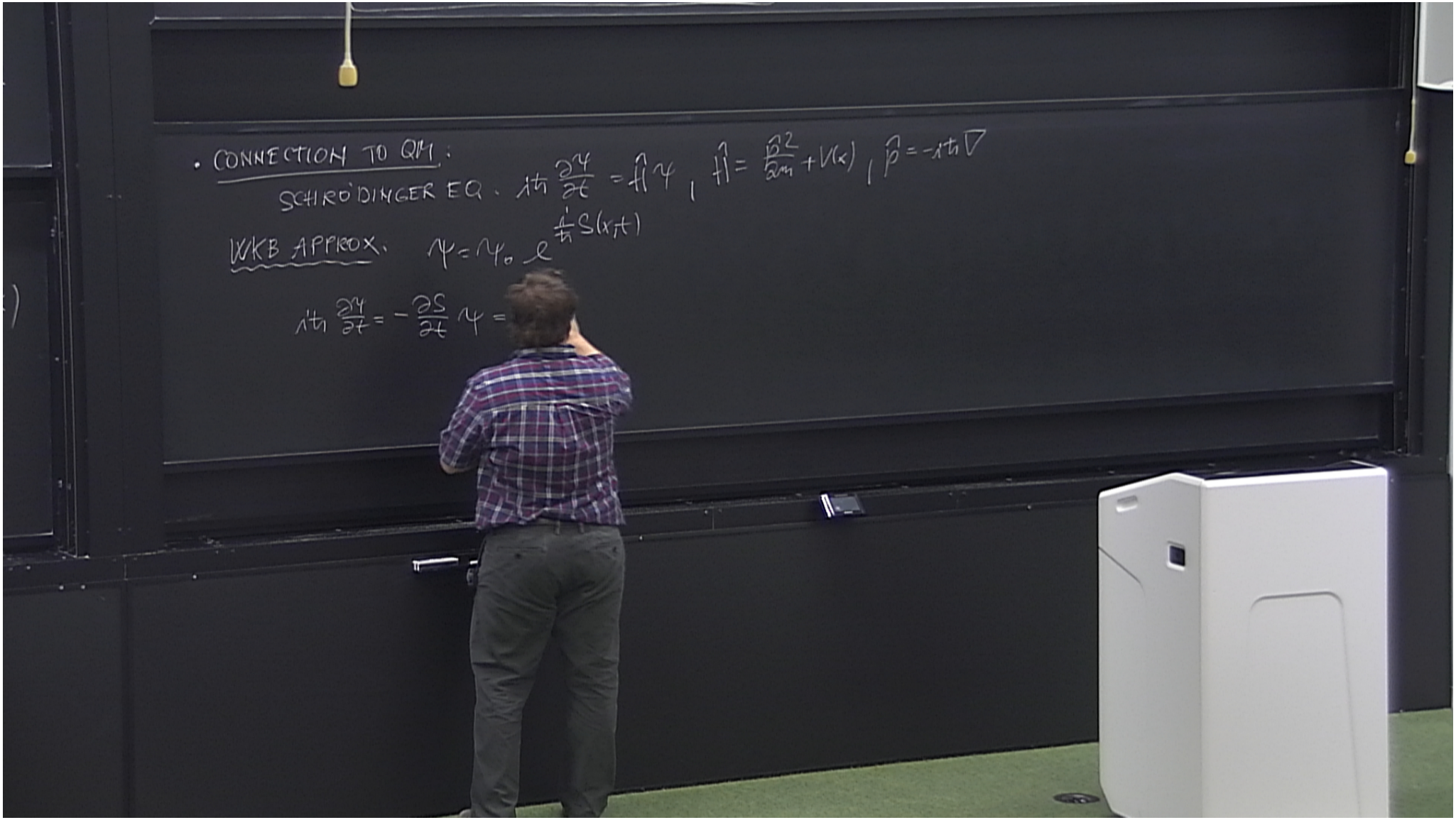
$H \neq H(q_c) \Rightarrow S(q_\beta, t) = \alpha_c q_c + S_c(q_1, \ldots \hat{q_c}, \ldots q_n, t)$

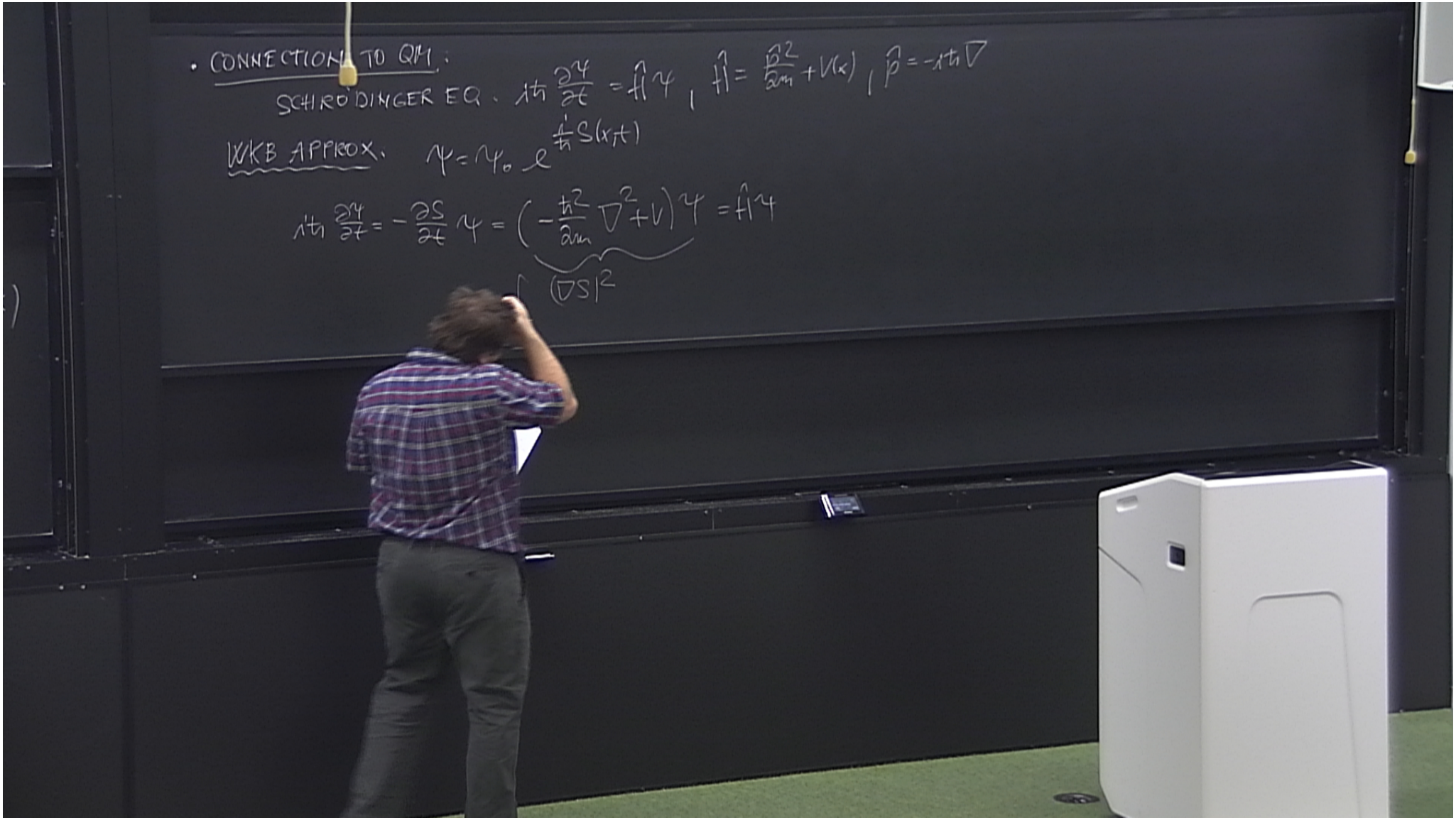
COMPLETE SEP.

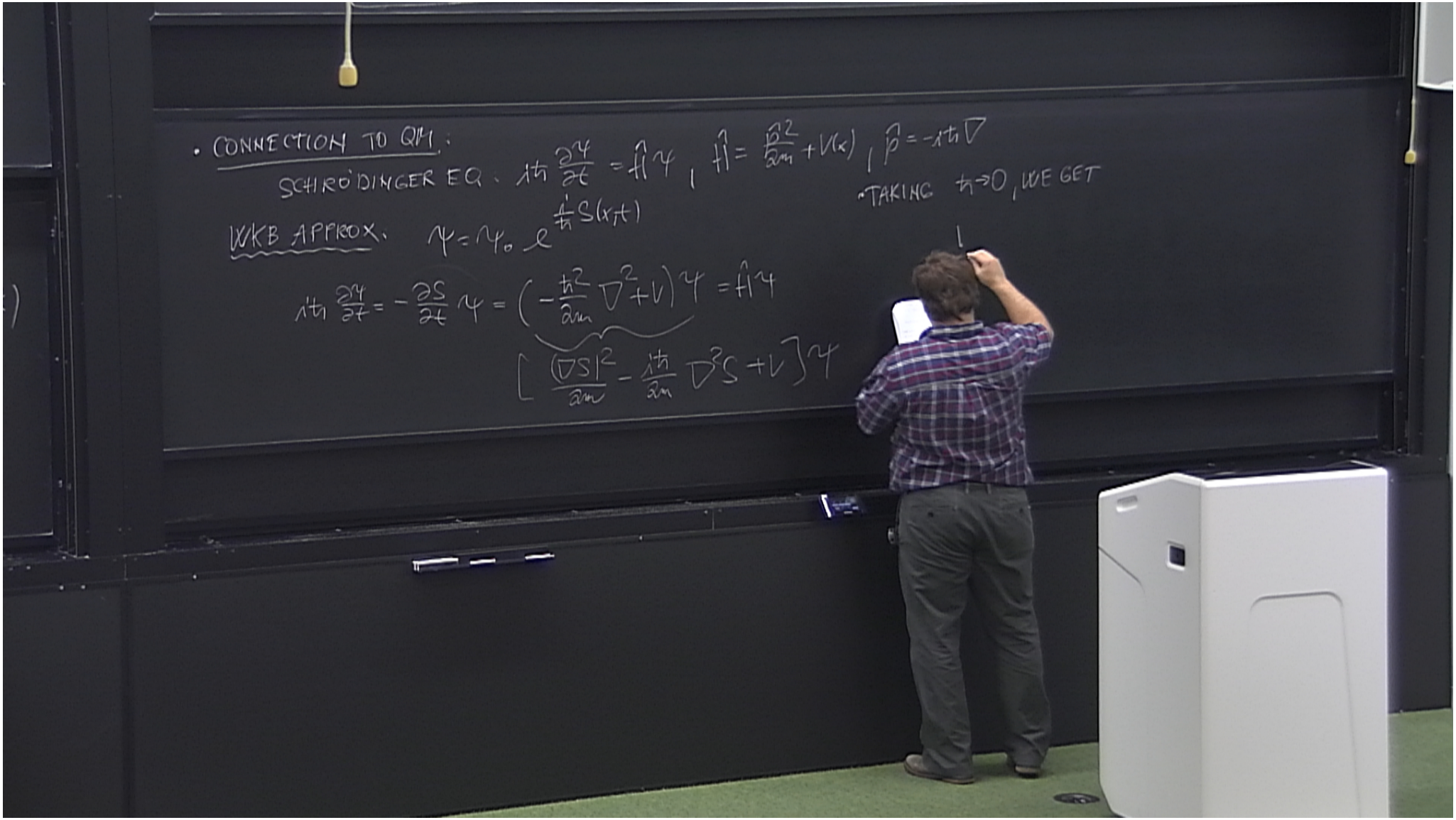
$S(q_\beta, t) = S_1(q_1) + \ldots + S_n(q_n)$ ADDITIVE SEPAR.


EXAMPLE: FREE FALL $H = \frac{p^2}{2m} + V(x)$, $V = -mgx$
• $\frac{\partial S}{\partial E} =$
• (H-J) $\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x) = 0$
• H ≠ H(t). $\Rightarrow S = S_0(x) - Et$
$-E + V + \frac{1}{2m}\left(\frac{\partial S_0}{\partial x}\right)^2 = 0 \Rightarrow S_0 = \int \sqrt{2m(E-V)}\,dx$
$S = \frac{1}{3m^2g}\left[2m(E+mgx)\right]^{3/2} - Et$


$S = \frac{1}{3m^2g}\left[2m(E+mgx)\right]^{3/2} - Et$
• CONNECTION TO QM:
SCHRÖDINGER EQ. $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$, $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$, $\hat{p} = -i\hbar\nabla$


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WKB APPROX: $\Psi = \Psi_0 e^{\frac{i}{\hbar}S(x,t)}$
$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\partial S}{\partial t}\Psi =$


• CONNECTION TO QM:
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WKB APPROX. $\psi = \psi_0 e^{\frac{i}{\hbar}S(x,t)}$
$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = \hat{H}\psi$
$|\nabla S|^2$


• CONNECTION TO QM:
SCHRÖDINGER EQ.: $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$, $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$, $\hat{p} = -i\hbar\nabla$
WKB APPROX.: $\psi = \psi_0 e^{\frac{i}{\hbar}S(x,t)}$
• TAKING $\hbar \to 0$, WE GET
$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = \hat{H}\psi$
$\left[\frac{(\nabla S)^2}{2m} - \frac{i\hbar}{2m}\nabla^2 S + V\right]\psi$

S S.T. THE NEW HAMILTONIAN $H' = 0$

THEN. • (HE) $\Rightarrow$ $Q^\delta = \alpha^\delta = \text{CONST}$

$P_\gamma = -\beta_\gamma = \text{CONST}$

• (c) $P_\delta = -\frac{\partial S}{\partial Q^\delta} \iff \beta_\gamma = \frac{\partial S(q^i, \alpha_i, t)}{\partial \alpha_\gamma}$

- HOW TO FIND MIRACULOUS S ?

$p_\delta = \frac{\partial S}{\partial q^\delta}$ & $H' = 0 = H + \frac{\partial S}{\partial t}$

$\frac{\partial S}{\partial t} + H(q^i, \frac{\partial S}{\partial q^i}, t) = 0$ (H-J)

HAMILTON-JACOBI EQUATION

2) SYMPLECTIC GEOMETRY

DEF: LET M^{2n} BE EVEN-DIM. MANIFOLD.

A SYMPLECTIC STRUCTURE ON M IS A

CLOSED NON-DEGENERATE 2-FORM ω

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HAMILTON-JACOBI EQUATION

e) SYMPLECTIC GEOMETRY

DEF: LET M^{2n} BE EVEN-DIM MANIFOLD.

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THAT IS. i) $d\omega = 0$

ii) $\forall X \neq 0 \; \exists Y$ S.T $\omega(X, Y) \neq 0$

$X, Y \in TM$

THE PAIR (M^{2n}, ω) IS CALLED

S S.T. THE NEW HAMILTONIAN $H' = 0$

THEN: • (HE) $\Rightarrow$ $Q^\delta = \alpha^\delta = \text{CONST}$

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NOTE: •

S S.T. THE NEW HAMILTONIAN $H' = 0$

THEN. • (HE) => $Q\dot{q} = \alpha^{\dot{q}} = \text{CONST}$

- HOW TO FIND MIRACULOUS S ?

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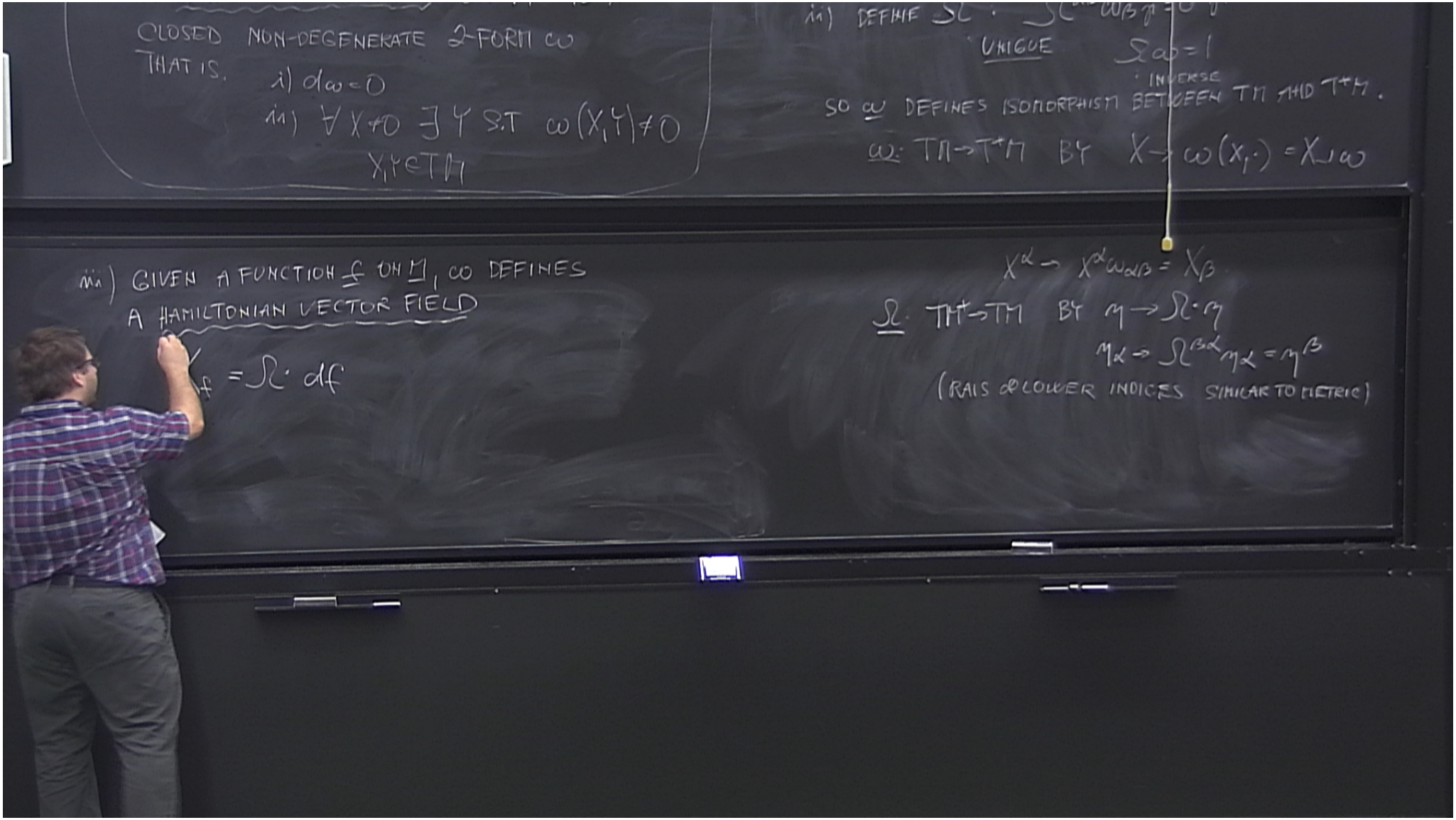
NOTE: i) NON-DEGENERACY OF ω => $\dim M = 2n$

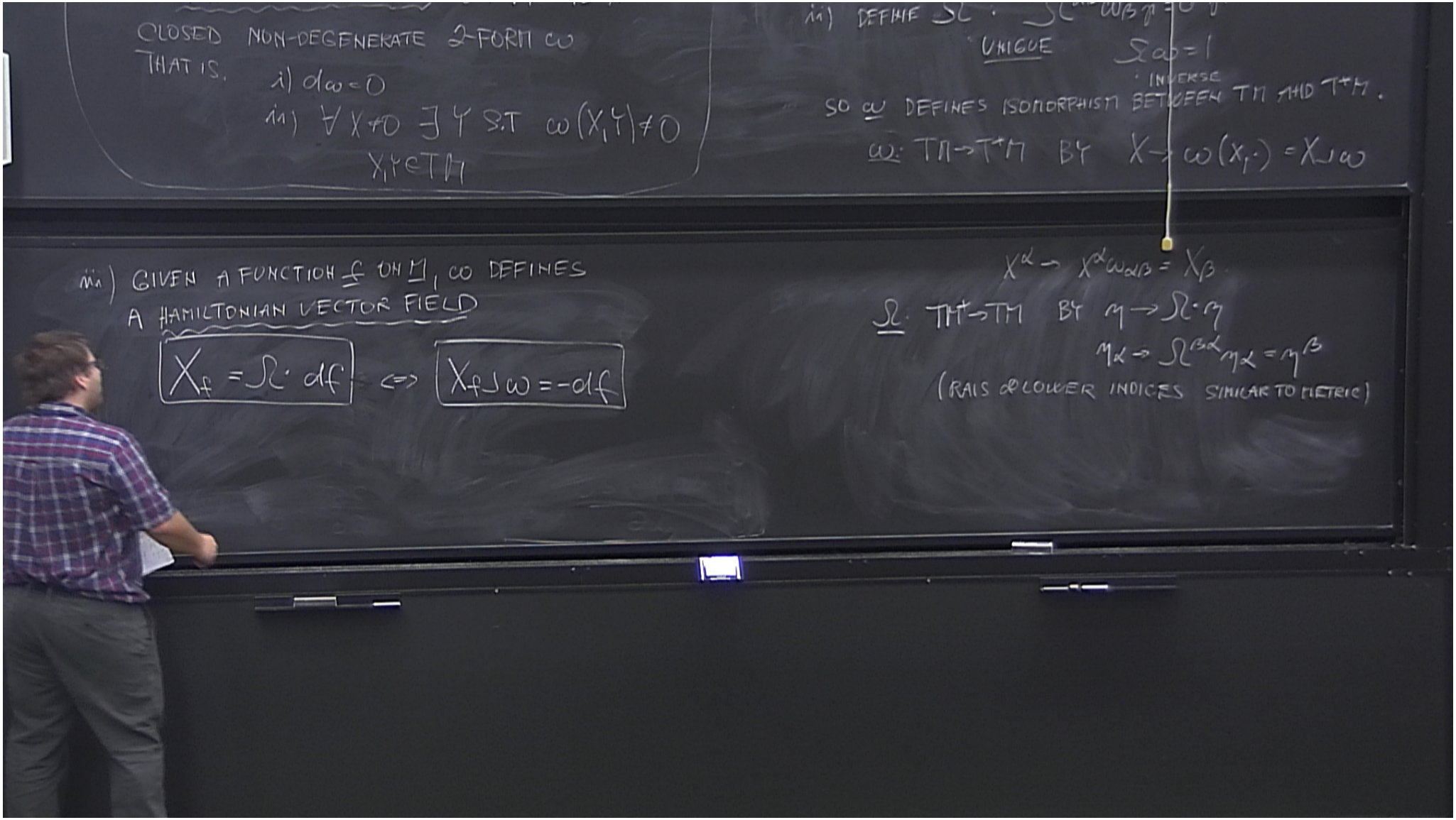
ii) DEFINE $\mathcal{L}$: $\mathcal{L}^{\alpha\beta}\omega_{\beta\gamma} = \delta^\alpha_\gamma$

UNIQUE $\mathcal{L}\omega = 1$

INVERSE

SEPARATION OF VARIABLE $\frac{\partial^2 f}{\partial t^2} - v^2 \frac{\partial^2 f}{\partial x^2} = 0$ $f = f(x - vt)$


CLOSED NON-DEGENERATE 2-FORM ω
THAT IS.
i) dω = 0
ii) ∀ X ≠ 0 ∃ Y S.T ω(X,Y) ≠ 0
X,Y ∈ TM
ii) DEFINE
UNIQUE
Ωω = 1
INVERSE
SO ω DEFINES ISOMORPHISM BETWEEN TM AND T*M.
ω: TM → T*M BY X → ω(X,·) = X⌋ω
iii) GIVEN A FUNCTION f ON M, ω DEFINES
A HAMILTONIAN VECTOR FIELD
X_f = Ω · df
X^α → X^α ω_αβ = X_β
Ω: TM* → TM BY η → Ω · η
η_α → Ω^βα η_α = η^β
(RAIS & LOWER INDICES SIMILAR TO METRIC)


CLOSED NON-DEGENERATE 2-FORM ω
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X_f = Σ · df ⟺ X_f ⌟ ω = -df
X^α → X^α ω_αβ = X_β
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(RAIS & LOWER INDICES SIMILAR TO METRIC)

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i) $d\omega = 0$

ii) $\forall X \neq 0 \; \exists Y \; \text{S.T} \; \omega(X,Y) \neq 0$

$X, Y \in TM$

ii) DEFINE Ω : $\Omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^{\alpha}_{\gamma}$

UNIQUE $\Omega\omega = 1$

INVERSE

SO ω DEFINES ISOMORPHISM BETWEEN TM AND T^*M.

$\omega : TM \to T^*M$ BY $X \to \omega(X, \cdot) = X \lrcorner \omega$

$X^\alpha \to X^\alpha \omega_{\alpha\beta} = X_\beta$

$\Omega : TM^* \to TM$ BY $\eta \to \Omega \cdot \eta$

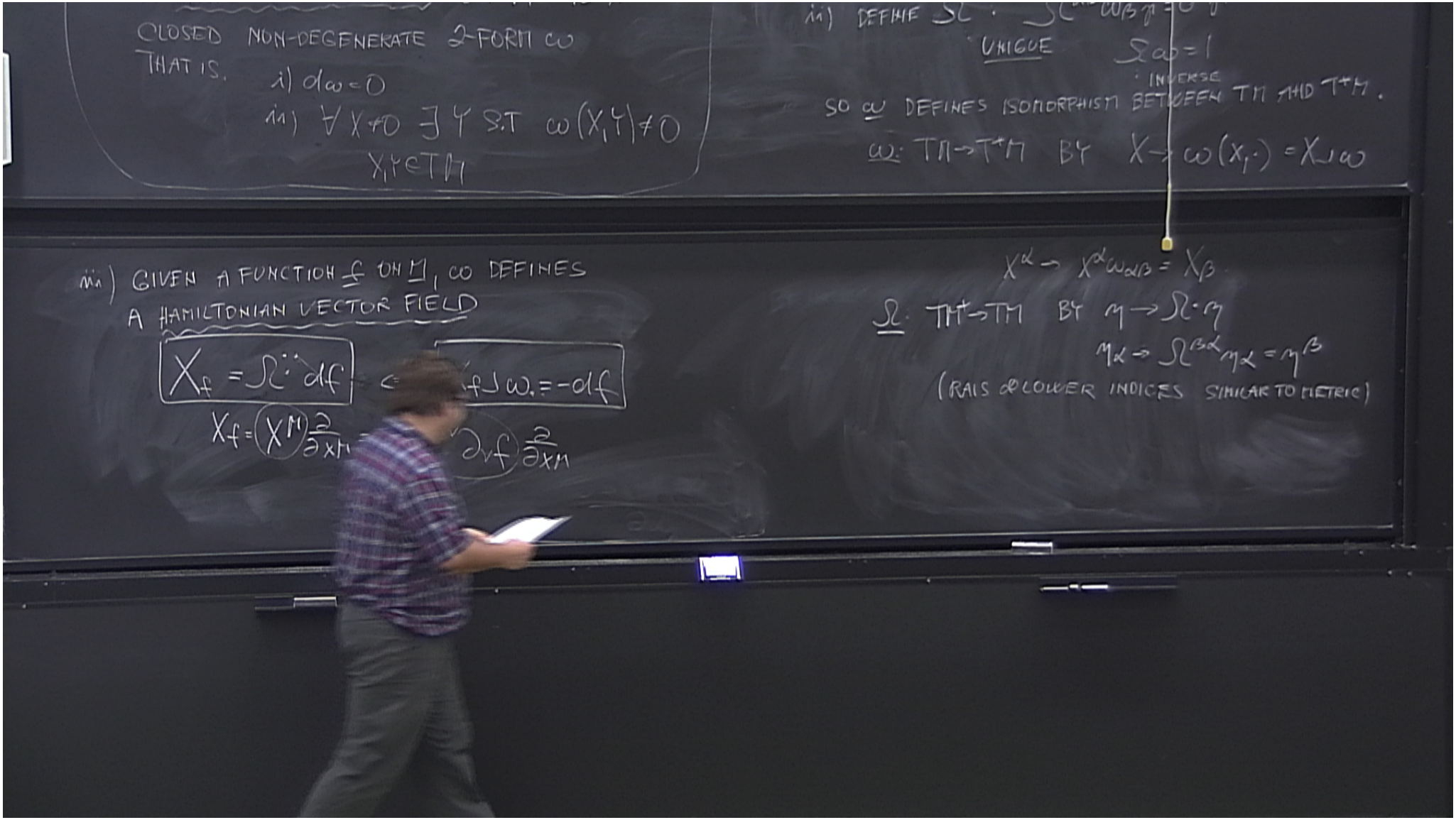
$\eta_\alpha \to \Omega^{\beta\alpha}\eta_\alpha = \eta^\beta$

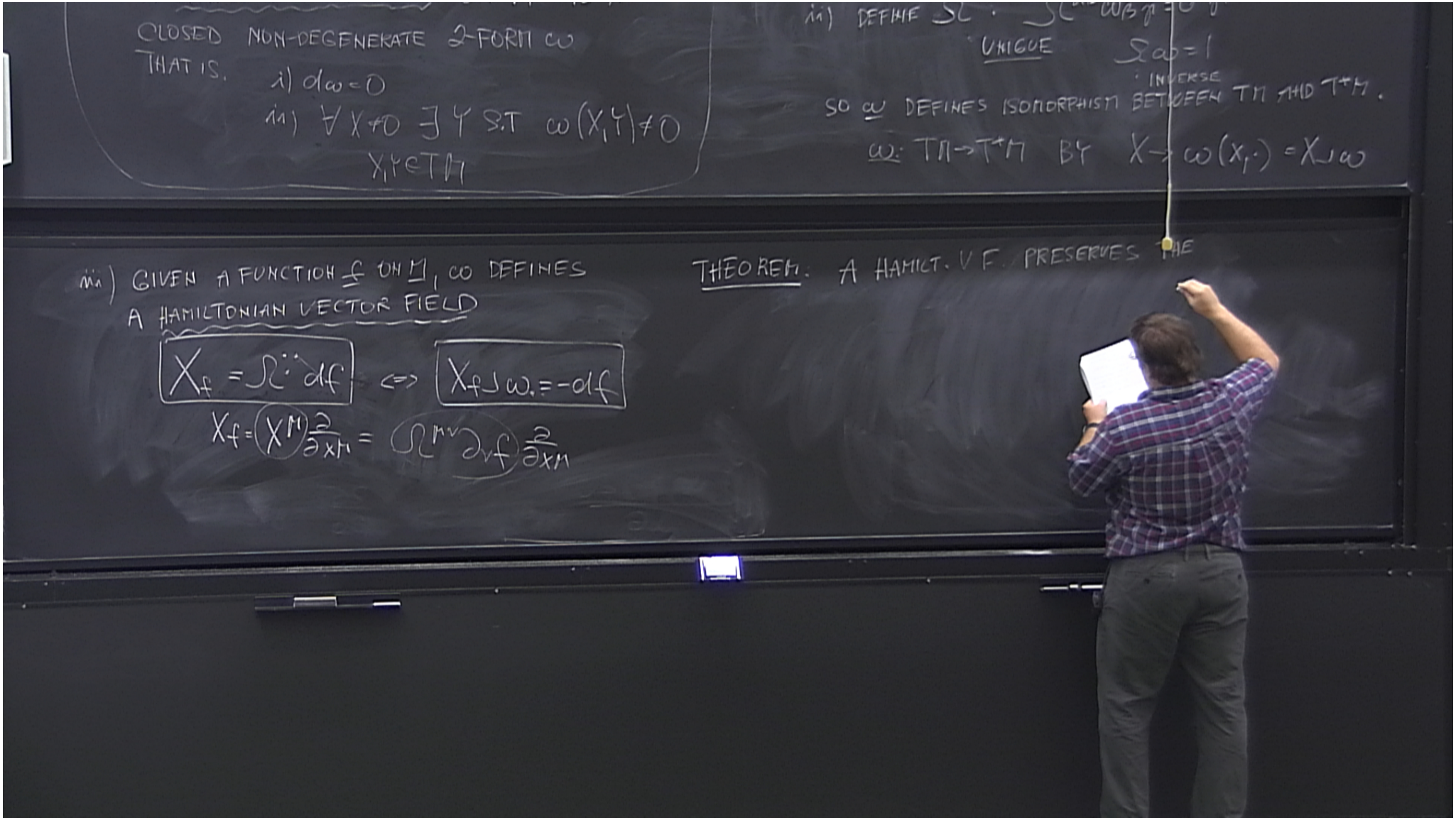
(RAIS & LOWER INDICES SIMILAR TO METRIC)

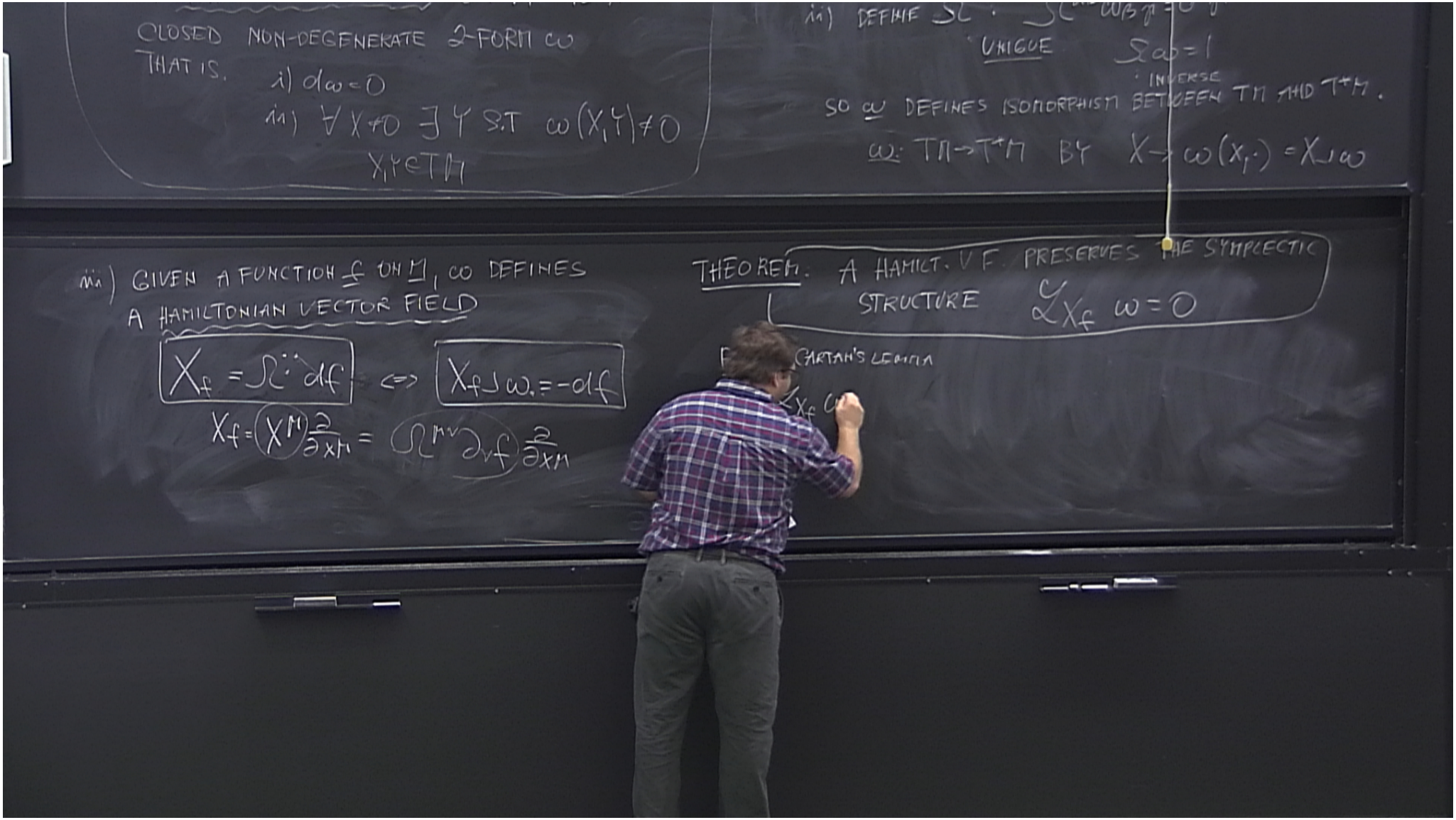
iii) GIVEN A FUNCTION f ON M, ω DEFINES A HAMILTONIAN VECTOR FIELD

$X_f = \Omega \cdot df \iff X_f \lrcorner \omega = -df$

$X_f = X^\mu \frac{\partial}{\partial x^\mu} = \Omega^{\mu\nu}\partial_\nu f \frac{\partial}{\partial x^\mu}$


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X_f = Σ : df
X_f = X^M ∂/∂x^M
∂_M f ∂/∂x^M
X^α → X^α ω_αβ = X_β
Σ: TM → TM BY η → Σ · η
η_α → Σ^βα η_α = η^β
(RAIS & LOWER INDICES SIMILAR TO METRIC)


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INVERSE
ω: TM → T*M BY X → ω(X,·) = X ⌟ ω
iii) GIVEN A FUNCTION f ON M, ω DEFINES
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THEOREM: A HAMILT. V.F. PRESERVES THE


CLOSED NON-DEGENERATE 2-FORM ω
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ii) DEFINE
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INVERSE
SO ω DEFINES ISOMORPHISM BETWEEN TM AND T*M.
ω: TM → T*M BY X → ω(X, ·) = X ⌟ ω
iii) GIVEN A FUNCTION f ON M, ω DEFINES
A HAMILTONIAN VECTOR FIELD
X_f = Ω · df ⟺ X_f ⌟ ω = −df
X_f = X^μ ∂/∂x^μ = Ω^μν ∂_ν f ∂/∂x^μ
THEOREM. A HAMILT. V.F. PRESERVES THE SYMPLECTIC STRUCTURE L_{X_f} ω = 0
CARTAN'S LEMMA
L_{X_f}

$X_1 \in TM$

$\omega: TM \to T^*M$ BY $X \to \omega(X, \cdot) = X \lrcorner \omega$

iii) GIVEN A FUNCTION f ON M, ω DEFINES A HAMILTONIAN VECTOR FIELD

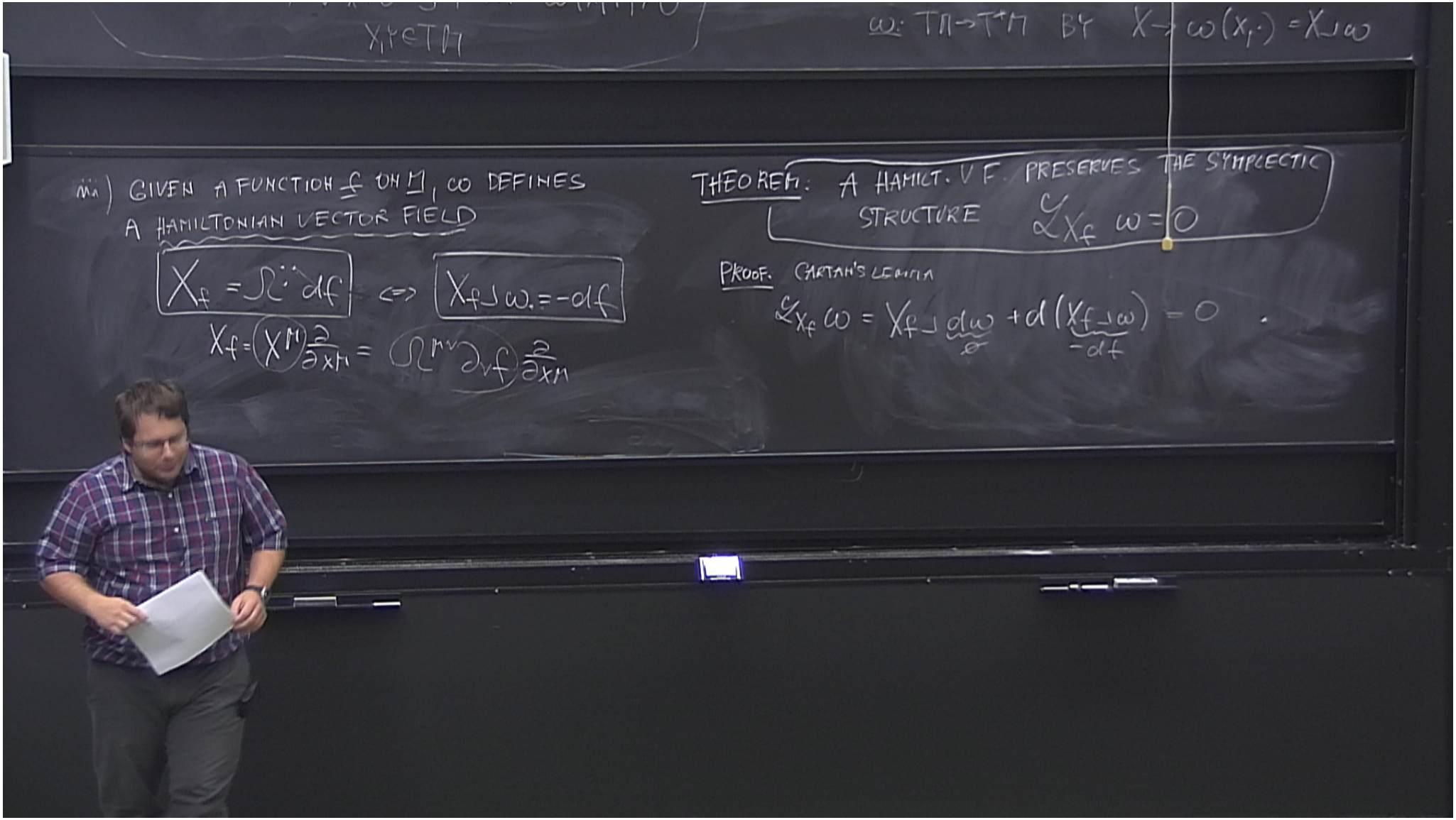
$$X_f = \Omega \cdot df \iff X_f \lrcorner \omega = -df$$

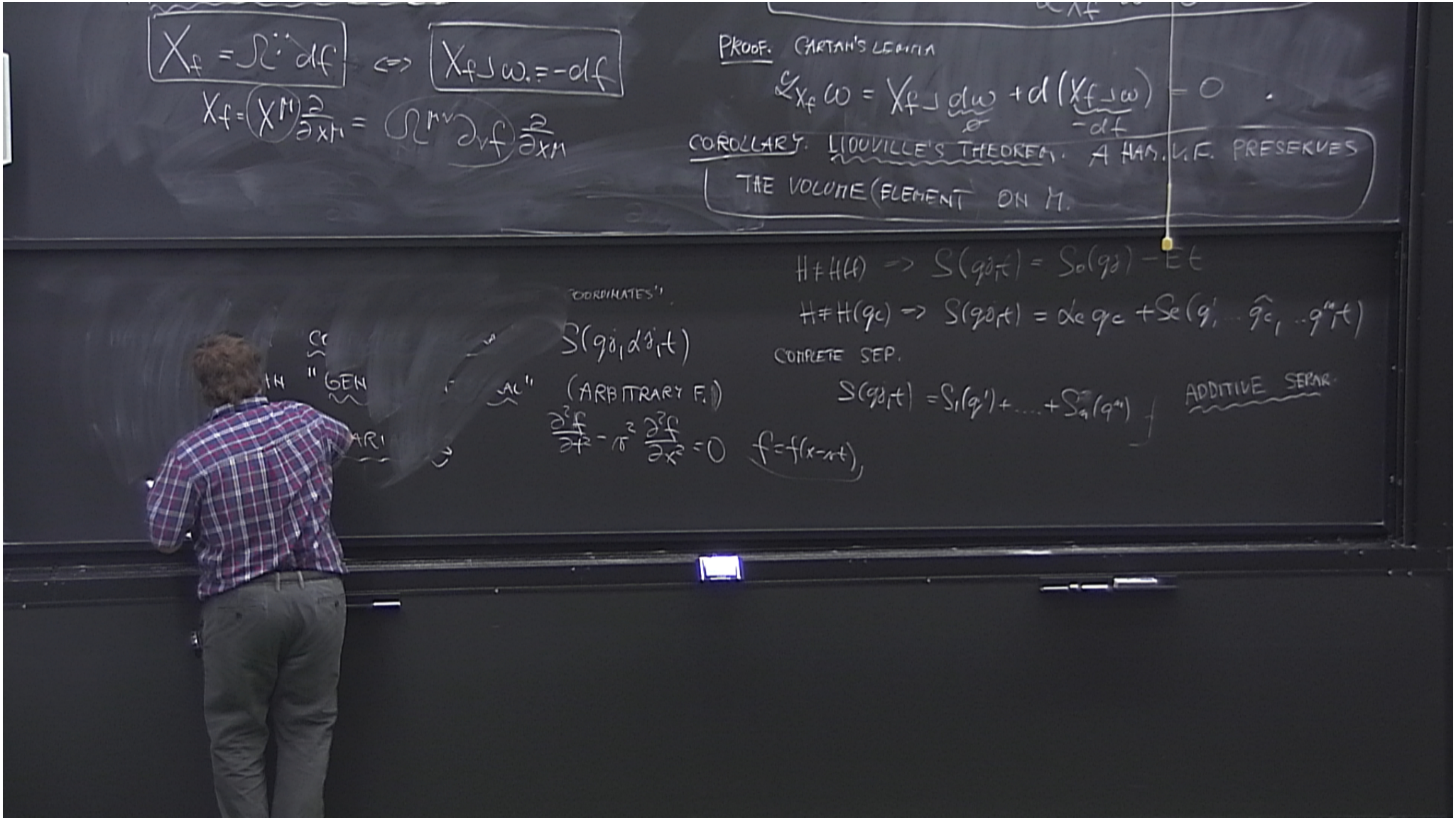
$$X_f = X^\mu \frac{\partial}{\partial x^\mu} = \Omega^{\mu\nu} \partial_\nu f \frac{\partial}{\partial x^\mu}$$

THEOREM: A HAMILT. V.F. PRESERVES THE SYMPLECTIC STRUCTURE $\mathcal{L}_{X_f} \omega = 0$

PROOF. CARTAN'S LEMMA

$$\mathcal{L}_{X_f} \omega = X_f \lrcorner \underbrace{d\omega}_{0} + d(\underbrace{X_f \lrcorner \omega}_{-df}) = 0 \,.$$




$X_f = \Omega^{\cdots} df$ <=> $X_f \lrcorner \omega_\cdot = -df$
$X_f = (X^\mu) \frac{\partial}{\partial x^\mu} = (\Omega^{\mu\nu} \partial_\nu f) \frac{\partial}{\partial x^\mu}$
PROOF. CARTAN'S LEMMA
$\mathcal{L}_{X_f} \omega = X_f \lrcorner d\omega + d(X_f \lrcorner \omega) = 0$
$d\omega$ $-df$
COROLLARY. LIOUVILLE'S THEOREM. A HAM. V.F. PRESERVES THE VOLUME (ELEMENT ON M.
"COORDINATES"
$S(q_i, \alpha_i, t)$
(ARBITRARY F.)
$\frac{\partial^2 f}{\partial t^2} - \pi^2 \frac{\partial^2 f}{\partial x^2} = 0$ $f = f(x - \pi t)$
$H \neq H(t) \Rightarrow S(q_i, t) = S_0(q_i) - Et$
$H \neq H(q_c) \Rightarrow S(q_i, t) = \alpha_c q_c + S_c(q_1, \ldots \hat{q}_c, \ldots q_n, t)$
COMPLETE SEP.
$S(q_i, t) = S_1(q_1) + \ldots + S_n(q_n)$ ADDITIVE SEPAR.

$X_f = \Omega \cdot df \iff X_f \lrcorner \omega = -df$

$X_f = X^\mu \frac{\partial}{\partial x^\mu} = \Omega^{\mu\nu} \partial_\nu f \frac{\partial}{\partial x^\mu}$

PROOF. CARTAN'S LEMMA

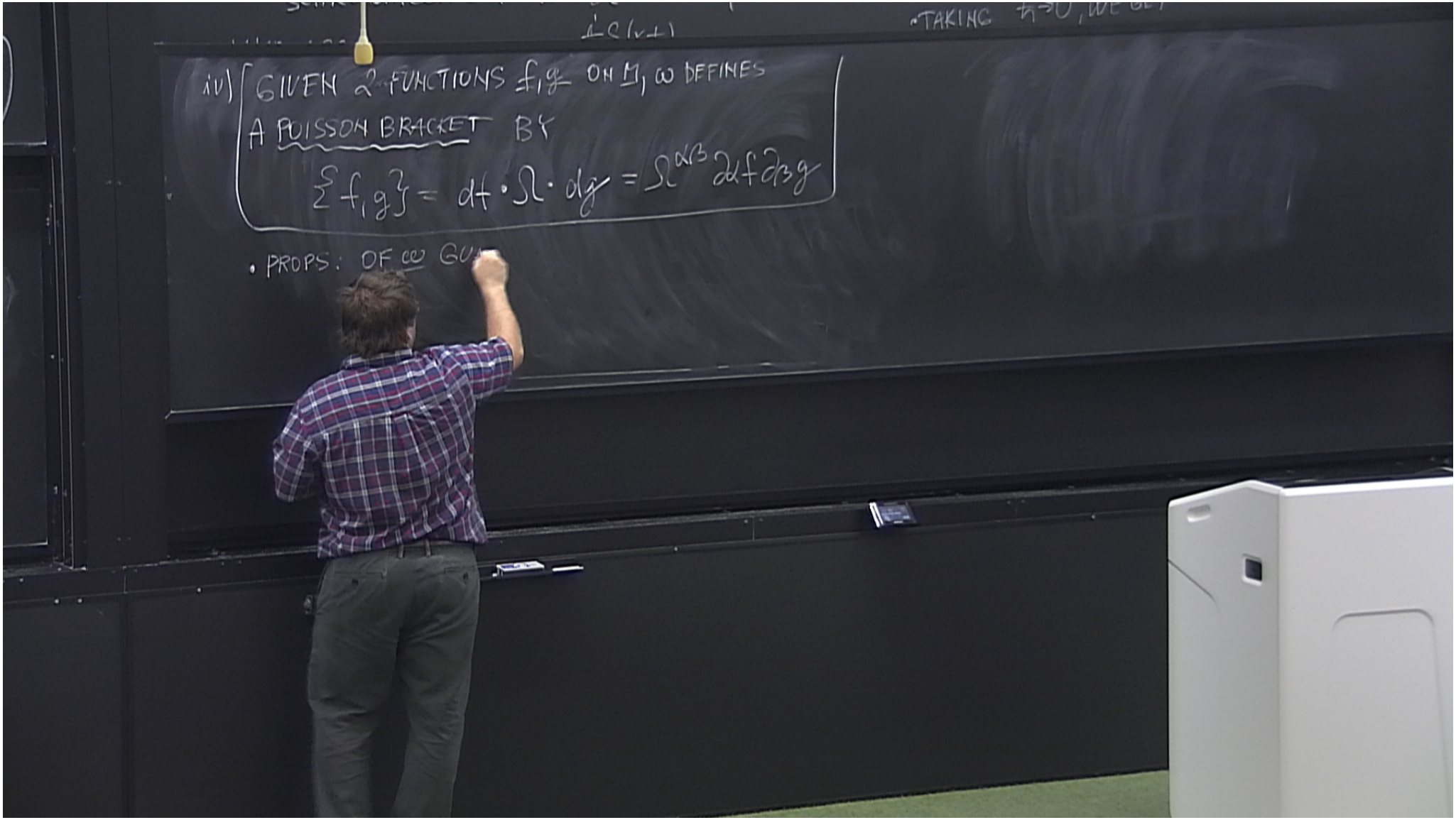
$\mathcal{L}_{X_f} \omega = X_f \lrcorner d\omega + d(X_f \lrcorner \omega) = 0$.

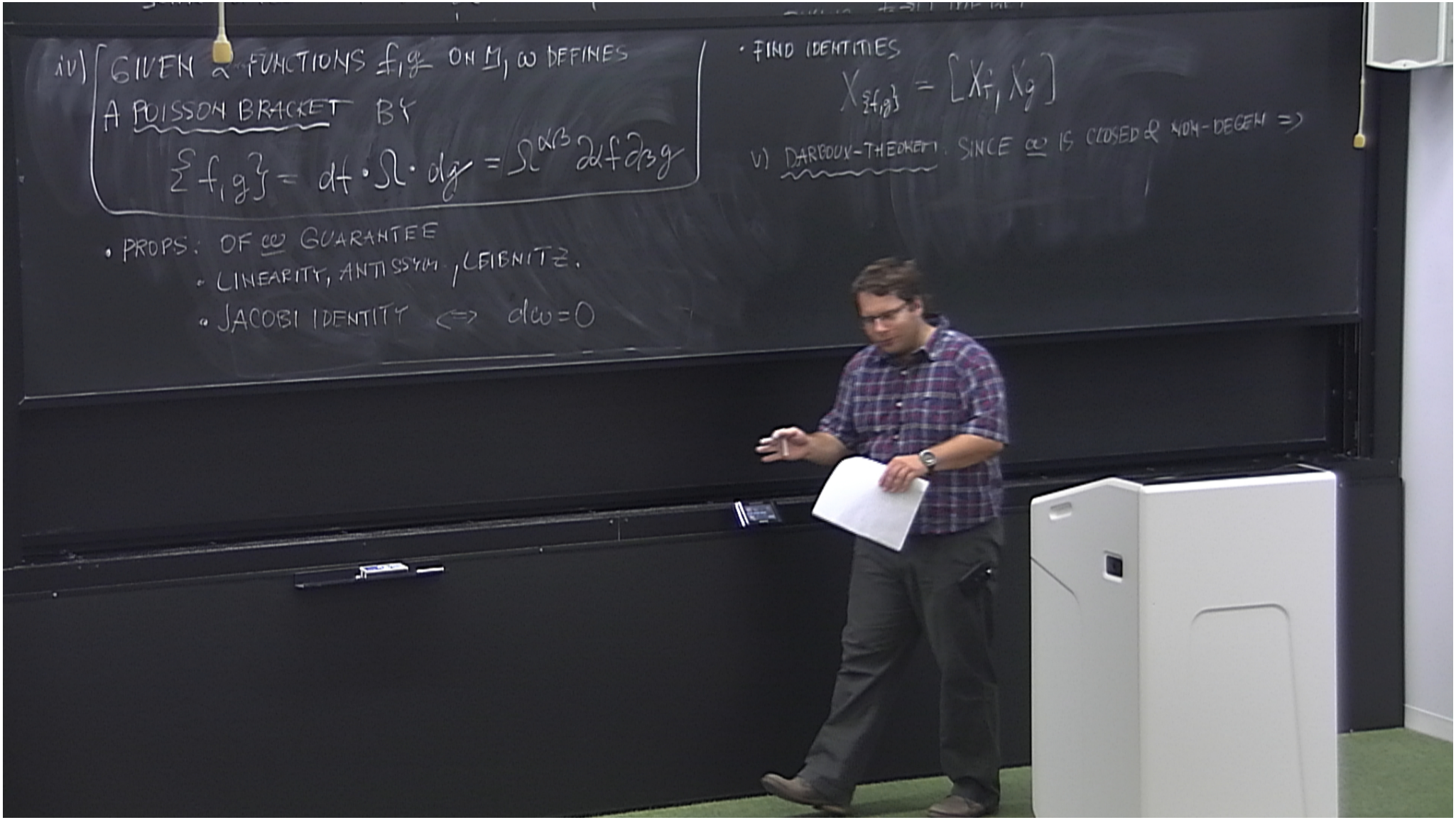
$d\omega = 0$, $X_f \lrcorner \omega = -df$

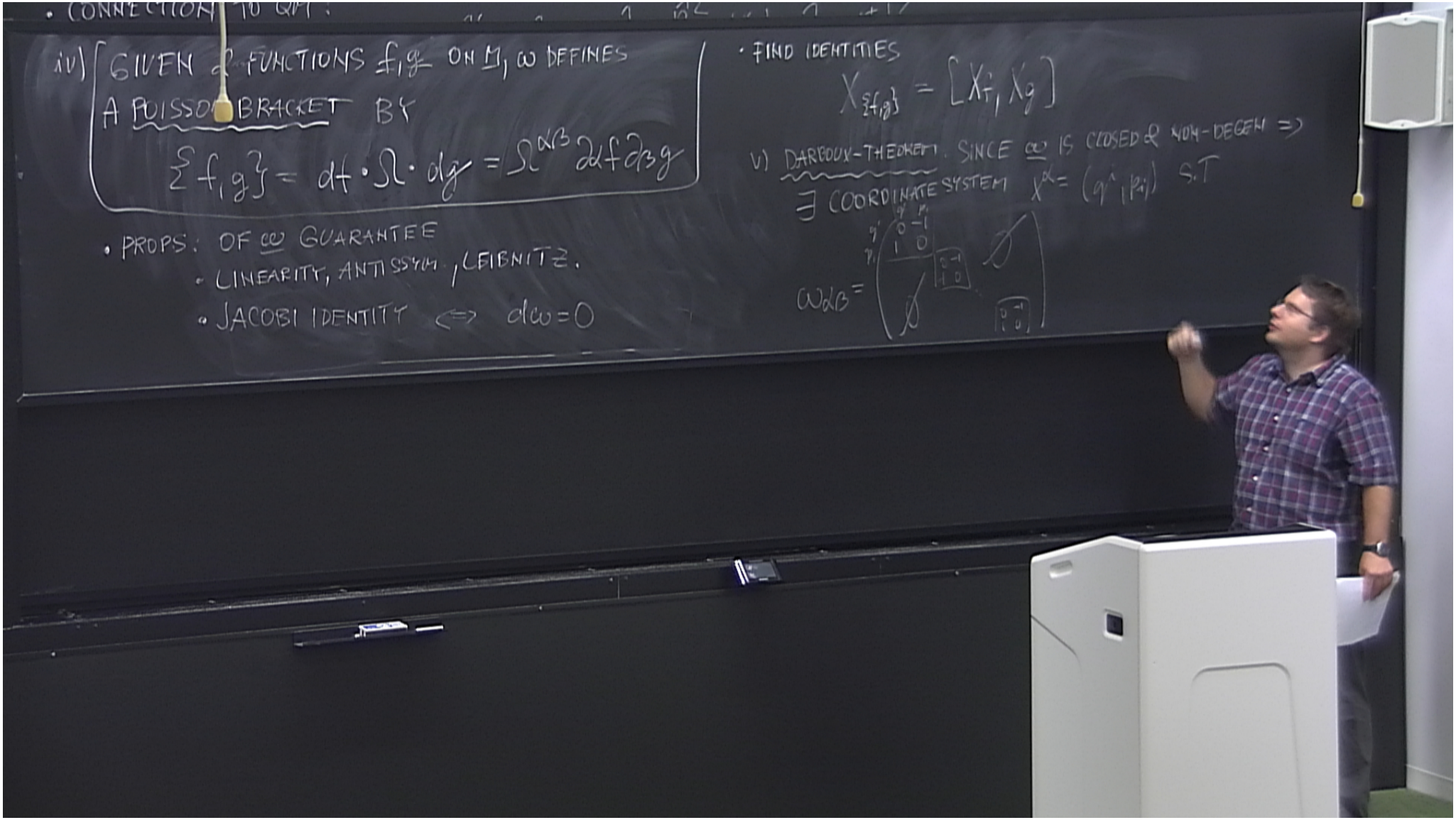
COROLLARY. LIOUVILLE'S THEOREM. A HAM. V.F. PRESERVES THE VOLUME (ELEMENT) ON M.

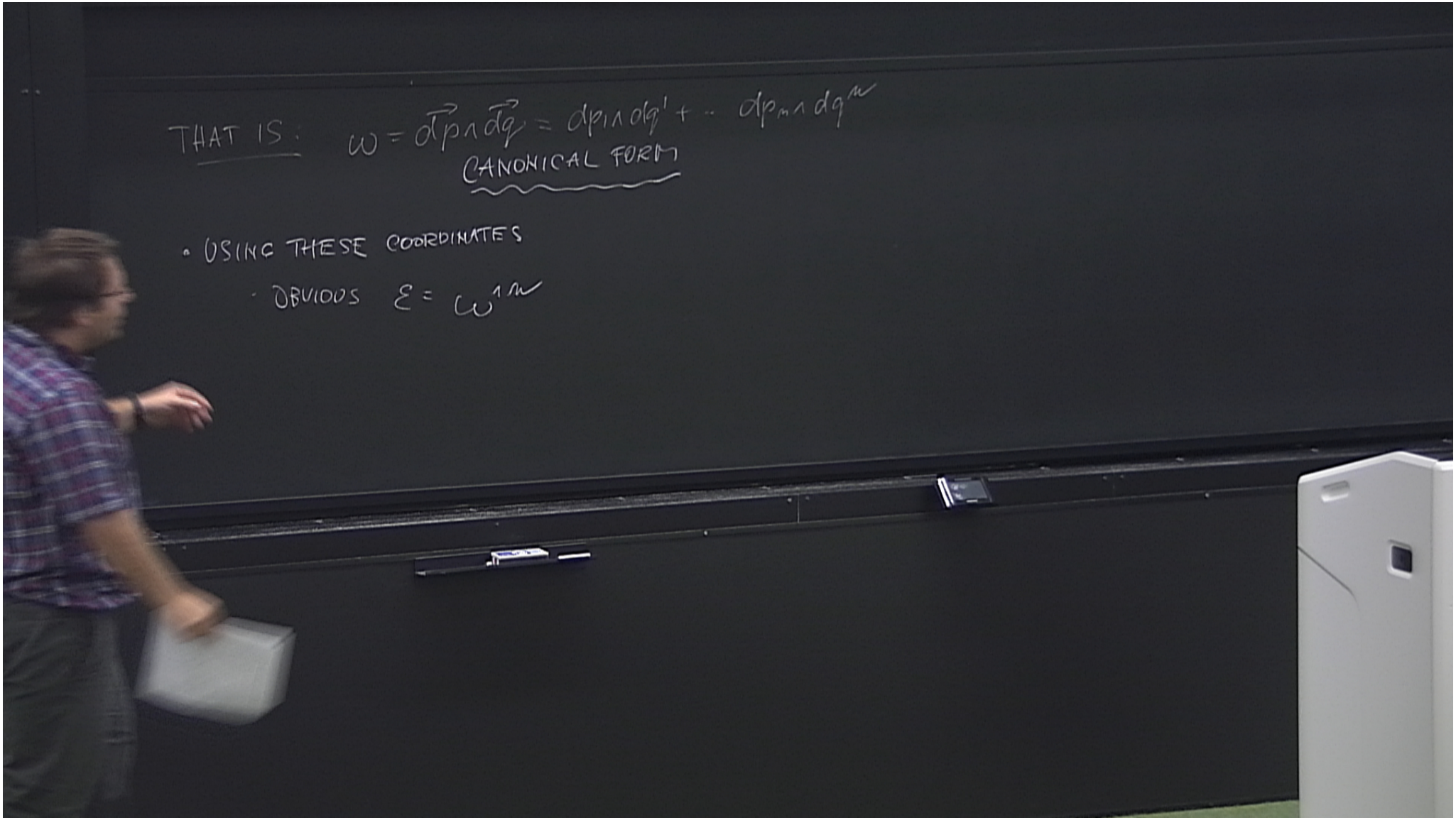
PROOF. · VOL. ELEMENT $\varepsilon = \omega^{\wedge m} = \frac{1}{m!} \omega \wedge \omega \wedge \ldots \wedge \omega$

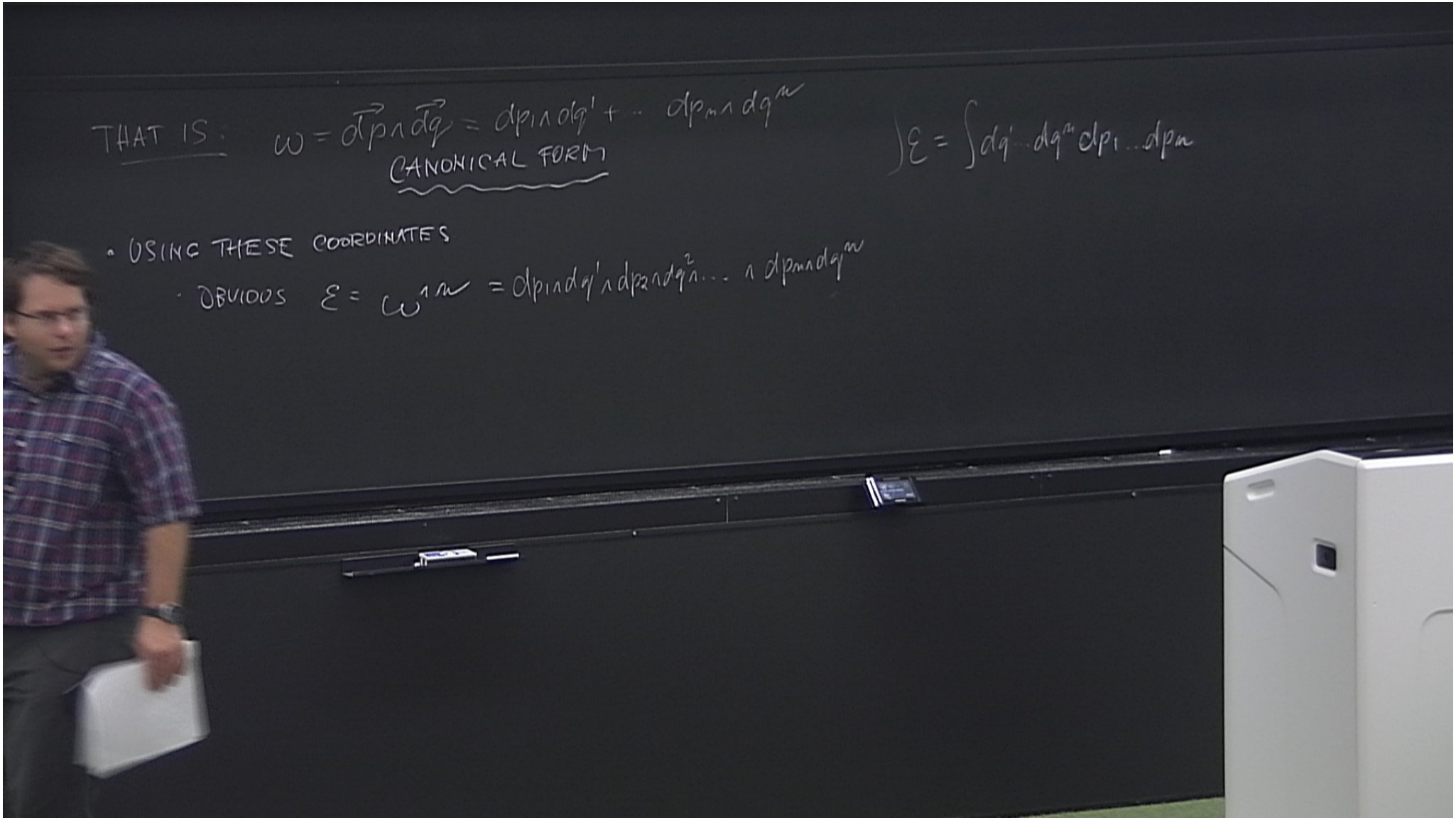
· $\mathcal{L}_{X_f} \varepsilon = \mathcal{L}_{X_f} (\omega \wedge \ldots \wedge \omega) \frac{1}{m!}$ $\overset{m \times}{\text{LEIBNIZ}}$ $\omega^{\wedge(m-1)} \wedge \mathcal{L}_{X_f} \omega = 0$.

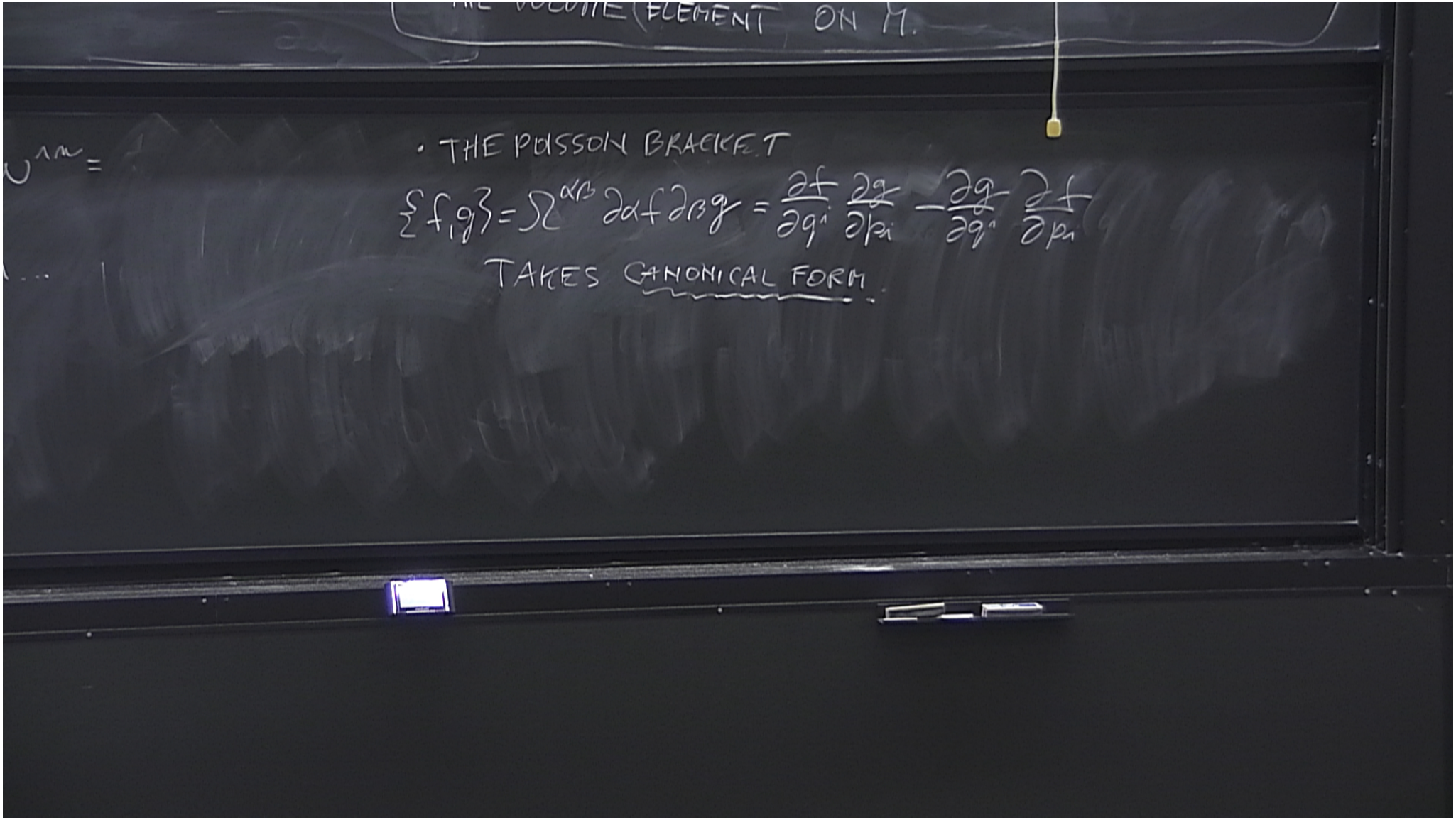

iv) GIVEN 2 FUNCTIONS f, g ON M, ω DEFINES
A POISSON BRACKET BY
$\{f,g\} = df \cdot \Omega \cdot dg = \Omega^{\alpha\beta}\partial_\alpha f \partial_\beta g$
• PROPS: OF ω GU
TAKING


iv) GIVEN 2 FUNCTIONS f, g ON M, ω DEFINES A POISSON BRACKET BY
$\{f,g\} = df \cdot \Omega \cdot dg = \Omega^{\alpha\beta}\partial_\alpha f \partial_\beta g$
• PROPS: OF ω GUARANTEE
• LINEARITY, ANTISSYM, LEIBNITZ.
• JACOBI IDENTITY ⇔ $d\omega = 0$
• FIND IDENTITIES
$X_{\{f,g\}} = [X_f, X_g]$
v) DARBOUX-THEOREM. SINCE ω IS CLOSED & NON-DEGEN ⇒


• CONNECTION TO QM:
iv) GIVEN 2 FUNCTIONS f, g ON M, ω DEFINES A POISSON BRACKET BY
$\{f,g\} = df \cdot \Omega \cdot dg = \Omega^{\alpha\beta}\partial_\alpha f \partial_\beta g$
• PROPS: OF ω GUARANTEE
• LINEARITY, ANTISYM, LEIBNITZ.
• JACOBI IDENTITY ⟺ $d\omega = 0$
• FIND IDENTITIES
$X_{\{f,g\}} = [X_f, X_g]$
v) DARBOUX-THEOREM: SINCE ω IS CLOSED & NON-DEGEN ⇒
∃ COORDINATE SYSTEM $X^\alpha = (q^i, p_i)$ S.T
$\omega_{\alpha\beta} =$


THAT IS: $\omega = d\vec{p} \wedge d\vec{q} = dp_1 \wedge dq^1 + \ldots dp_m \wedge dq^m$
CANONICAL FORM
• USING THESE COORDINATES
· OBVIOUS $\mathcal{E} = \omega^{\wedge m}$


THAT IS: $\omega = d\vec{p} \wedge d\vec{q} = dp_1 \wedge dq^1 + \dots dp_n \wedge dq^n$
CANONICAL FORM
$\mathcal{E} = \int dq^1 \dots dq^n dp_1 \dots dp_n$
• USING THESE COORDINATES
· OBVIOUS $\mathcal{E} = \omega^{1\dots n} = dp_1 dq^1 \wedge dp_2 dq^2 \wedge \dots \wedge dp_n dq^n$


ELEMENT ON M.
• THE POISSON BRACKET
$\{f,g\} = \Omega^{\alpha\beta}\partial_\alpha f \partial_\beta g = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}$
TAKES CANONICAL FORM.