Title: Classical Mechanics - David Kubiznak

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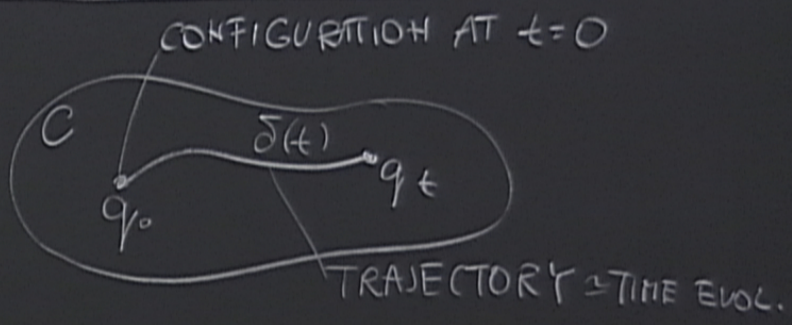
Abstract:

f) GEOMETRIC FORMULATION OF LAGRANGIAN MECHANICS

- CONFIGURATION SPACE = C

  "PHOTO OF THE SYSTEM"

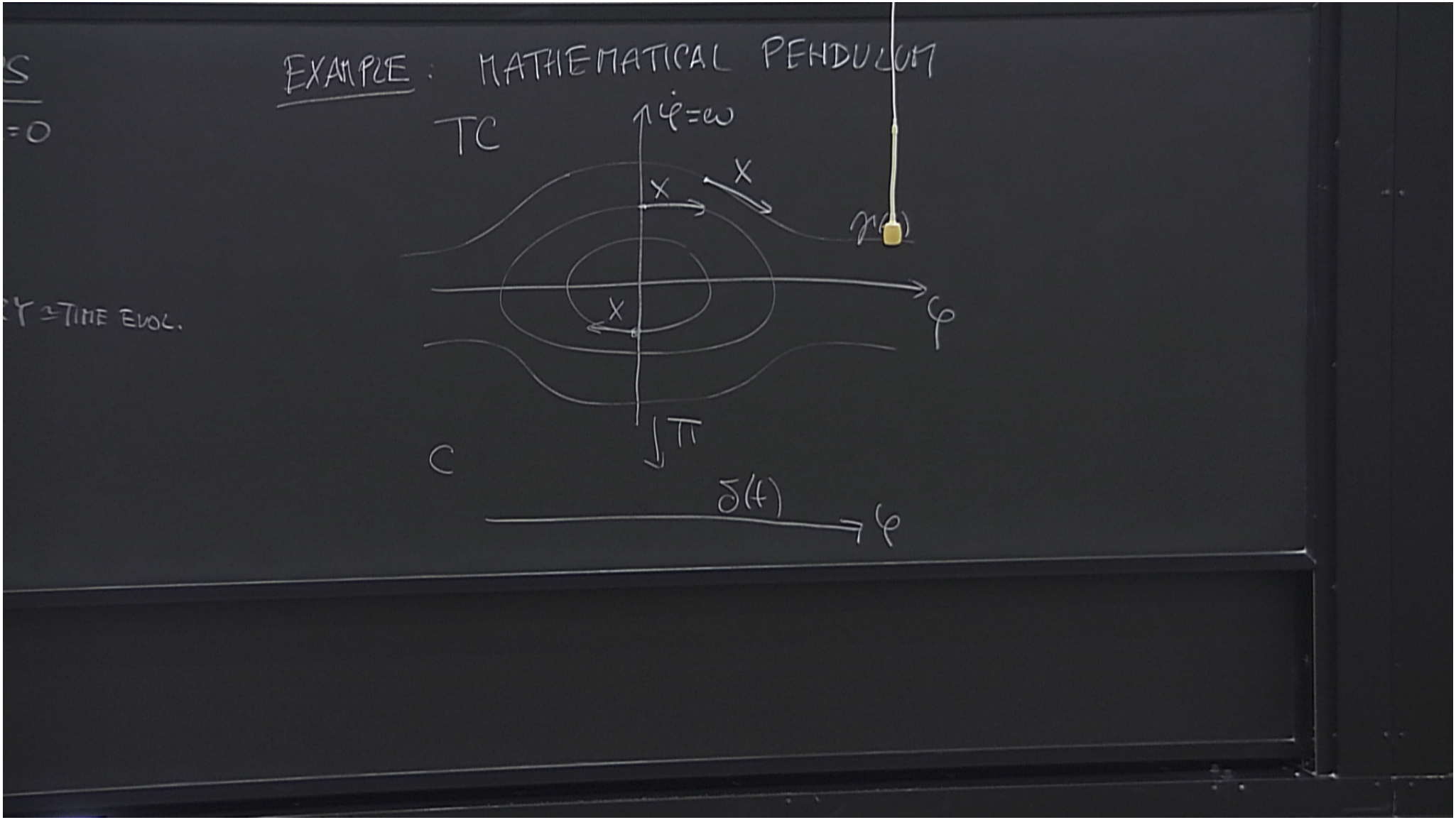
  n DOF $(q^1, \dots, q^n)$



- VELOCITY PHASE SPACE = TC

  "PHYSICAL STATE OF THE SYSTEM" ... $\gamma(t)$

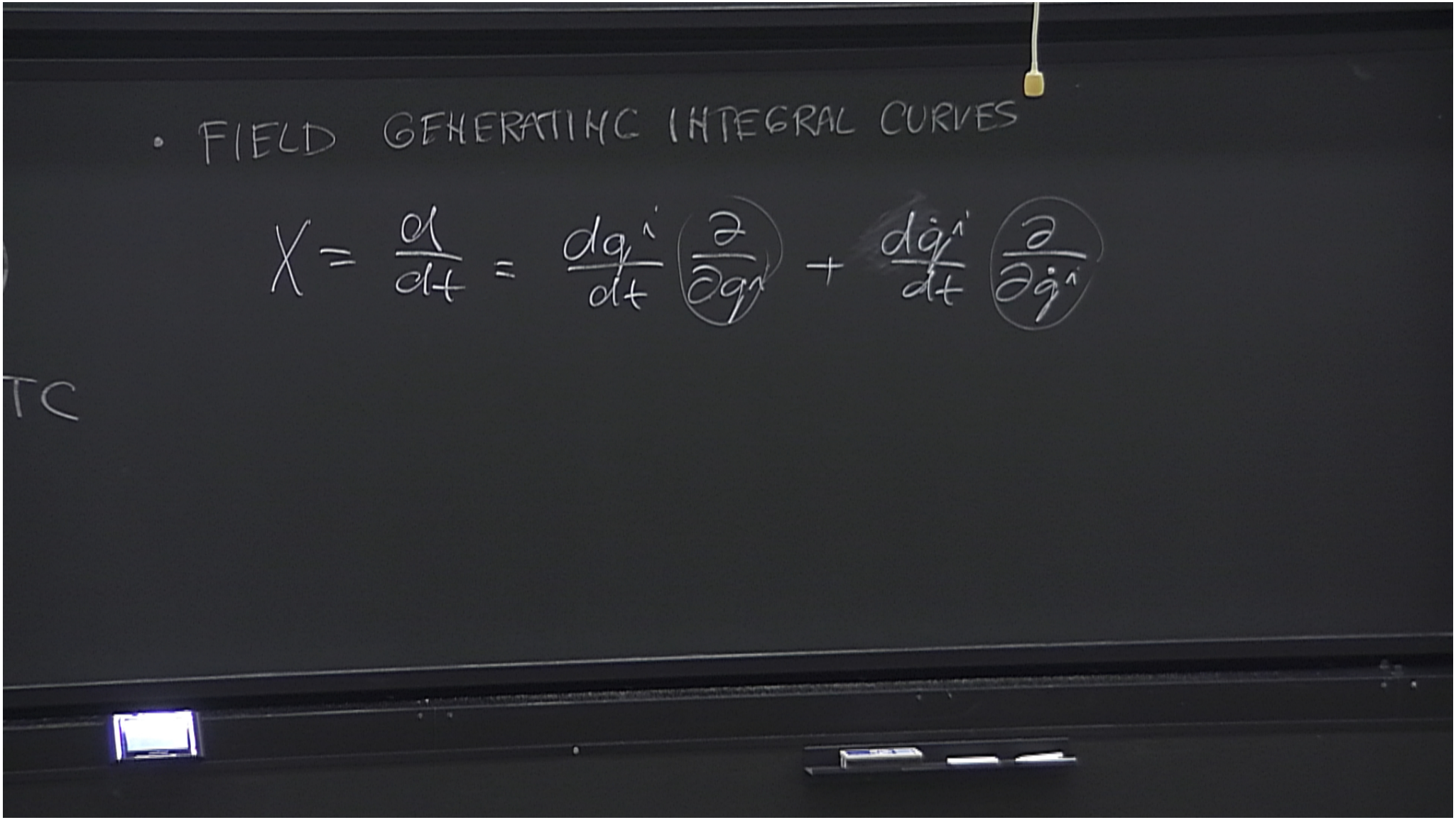
  $2n$ DOF $(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$

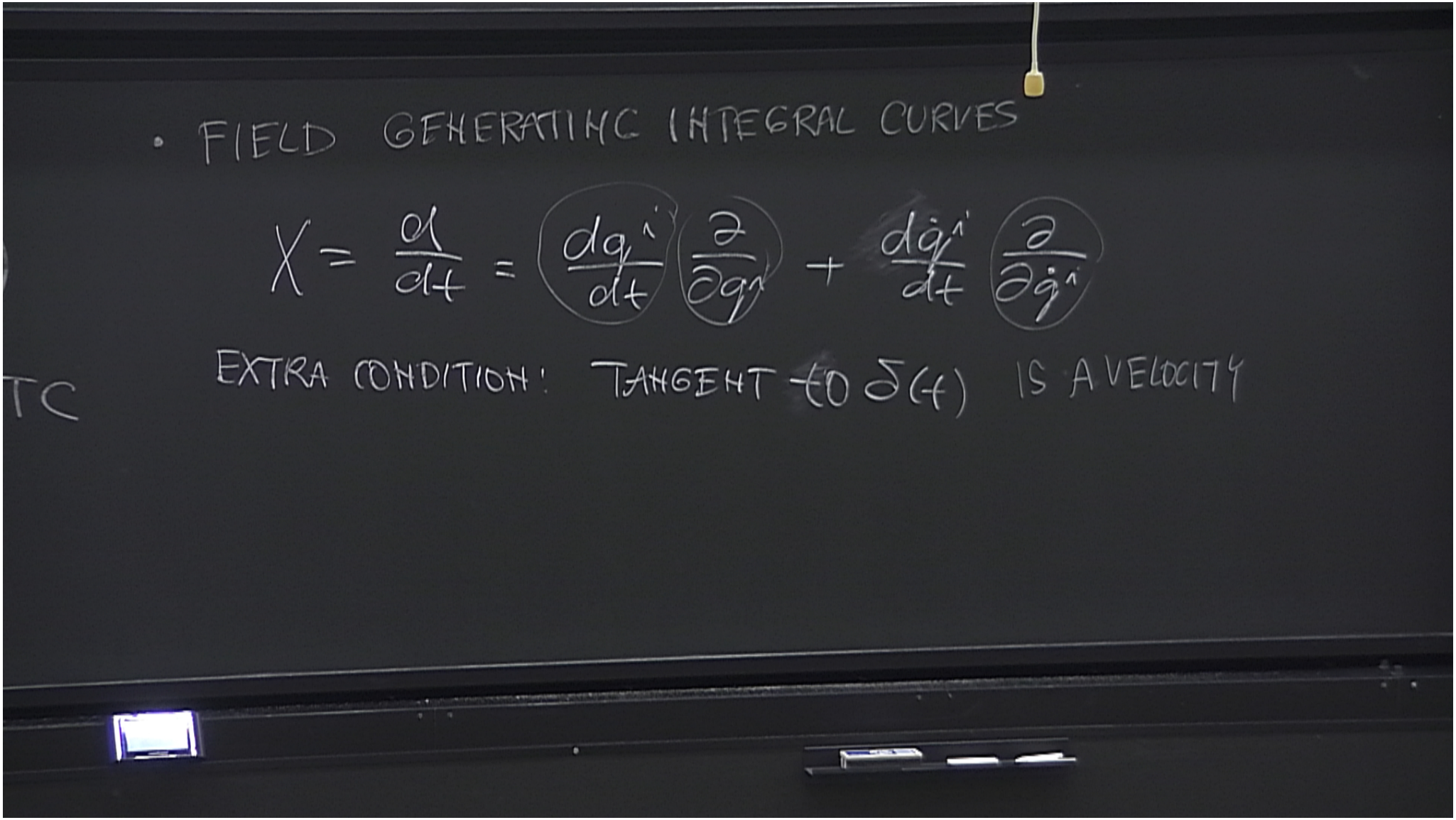

EXAMPLE: MATHEMATICAL PENDULUM
TC
$\dot{\varphi}=\omega$
X
X
X
φ
π
C
$\delta(t)$
φ

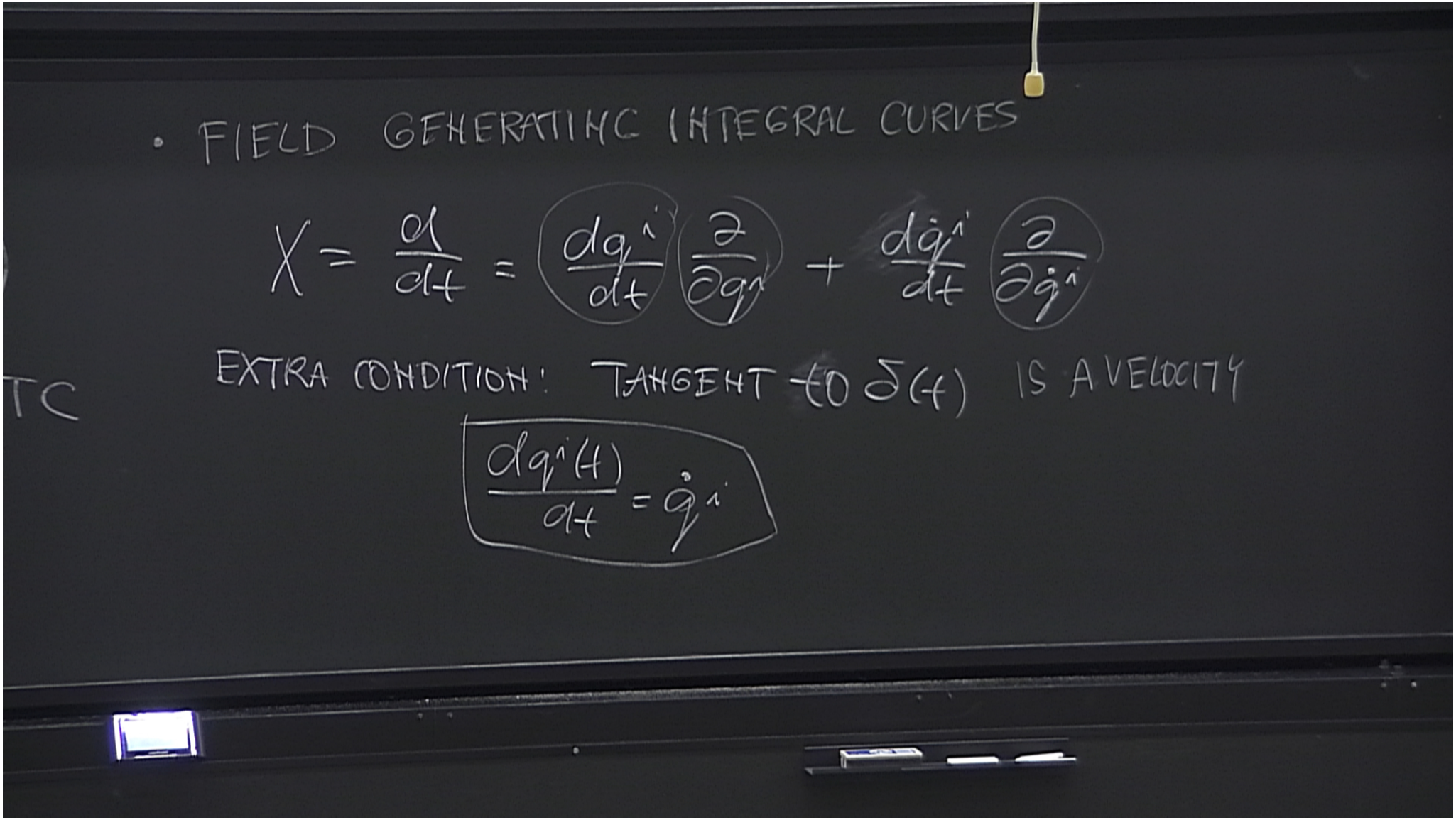
- LAGRANGIAN $L : TC \to \mathbb{R}$
- DYNAMICAL VECTOR FIELD $X \in T(TC)$

  DETERMINES INTEGRAL CURVES $\gamma(t)$ ON TC

  TRAJECT. $\delta(t) = \pi(\gamma(t))$


• FIELD GENERATING INTEGRAL CURVES
$X = \frac{d}{dt} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{d\dot{q}^i}{dt}\frac{\partial}{\partial \dot{q}^i}$
TC


• FIELD GENERATING INTEGRAL CURVES
$X = \frac{d}{dt} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{d\dot{q}^i}{dt}\frac{\partial}{\partial \dot{q}^i}$
EXTRA CONDITION: TANGENT TO $\delta(t)$ IS A VELOCITY
TC


• FIELD GENERATING INTEGRAL CURVES
$X = \frac{d}{dt} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{d\dot{q}^i}{dt}\frac{\partial}{\partial \dot{q}^i}$
EXTRA CONDITION: TANGENT TO $\delta(t)$ IS A VELOCITY
$\frac{dq^i(t)}{dt} = \dot{q}^i$

$2m$ DOF $(q^1, \dots, q^m, \dot{q}^1, \dots, \dot{q}^m)$

$\delta(t)$ →

LAGRANGIAN $L : TC \to \mathbb{R}$

DYNAMICAL VECTOR FIELD $X \in T(TC)$

DETERMINES INTEGRAL CURVES $\gamma(t)$ ON TC

TRAJECT. $\delta(t) = \tau(\gamma(t))$

$$X = \dot{q}^i \frac{\partial}{\partial q^i} + W^i(q_s, \dot{q}_s) \frac{\partial}{\partial \dot{q}^i}$$

• FIELD GENERATING INTEGRAL CURVES

$$X = \frac{d}{dt} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{d\dot{q}^i}{dt}\frac{\partial}{\partial \dot{q}^i}$$

EXTRA CONDITION: TANGENT TO $\delta(t)$ IS A

$$\frac{dq^i(t)}{dt} = \dot{q}^i$$

GIAN $L : TC \to \mathbb{R}$

AL VECTOR FIELD $X \in \mathfrak{X}(TC)$

KMINES INTEGRAL CURVES $\gamma(t)$ ON TC

$\delta(t) = \tau(\gamma(t))$

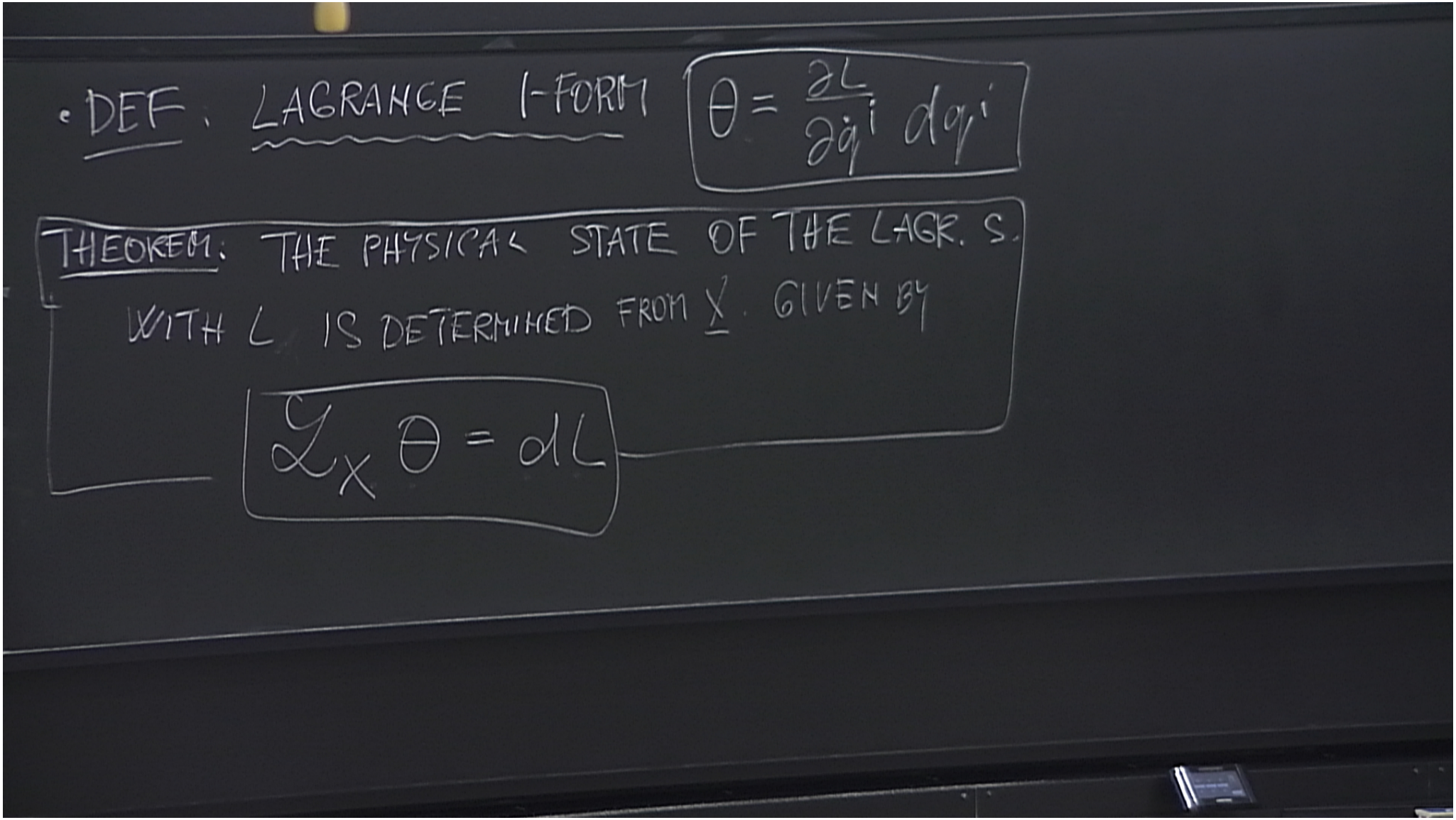
• FIELD GENERATING INTEGR

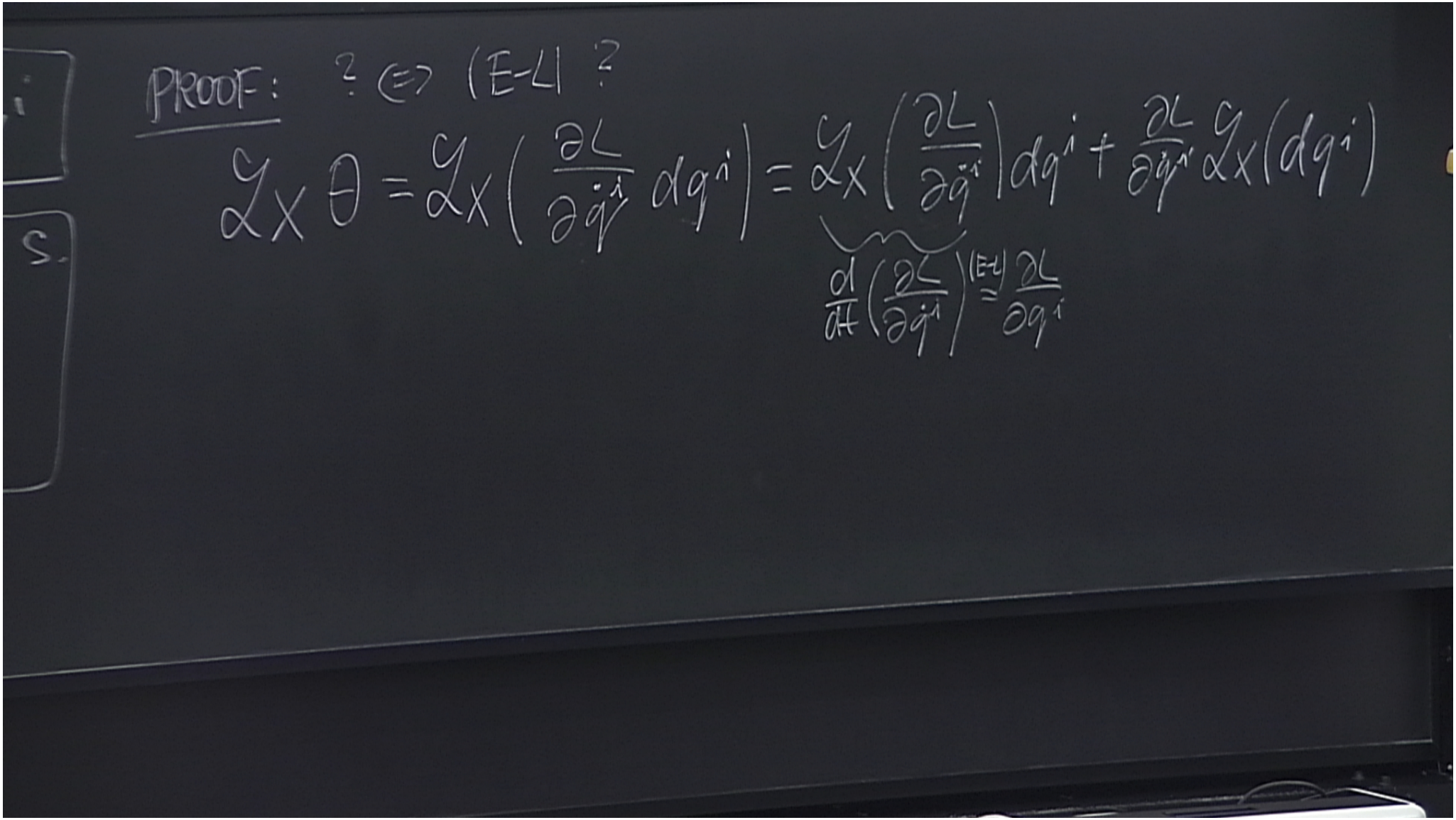
$X = \frac{d}{dt} = \left(\frac{dq^i}{dt}\right)\left(\frac{\partial}{\partial q^i}\right)$

EXTRA CONDITION: TANGENT

$\frac{dq^i(t)}{dt} = \dot{q}^i$

$X = \dot{q}^i \frac{\partial}{\partial q^i} + W^i(q^\beta, \dot{q}^\beta) \frac{\partial}{\partial \dot{q}^i}$ SPECIAL


• DEF. LAGRANGE 1-FORM $\theta = \frac{\partial L}{\partial \dot{q}^i} dq^i$
THEOREM: THE PHYSICAL STATE OF THE LAGR. S. WITH L IS DETERMINED FROM X GIVEN BY
$\mathcal{L}_X \theta = dL$


PROOF: ? ⇔ (E-L) ?
$\mathcal{L}_X\theta = \mathcal{L}_X\left(\frac{\partial L}{\partial \dot{q}^i} dq^i\right) = \mathcal{L}_X\left(\frac{\partial L}{\partial \dot{q}^i}\right) dq^i + \frac{\partial L}{\partial \dot{q}^i} \mathcal{L}_X(dq^i)$
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) \overset{(E-L)}{=} \frac{\partial L}{\partial q^i}$

$$\Theta = \frac{\partial L}{\partial \dot{q}^i} dq^i$$

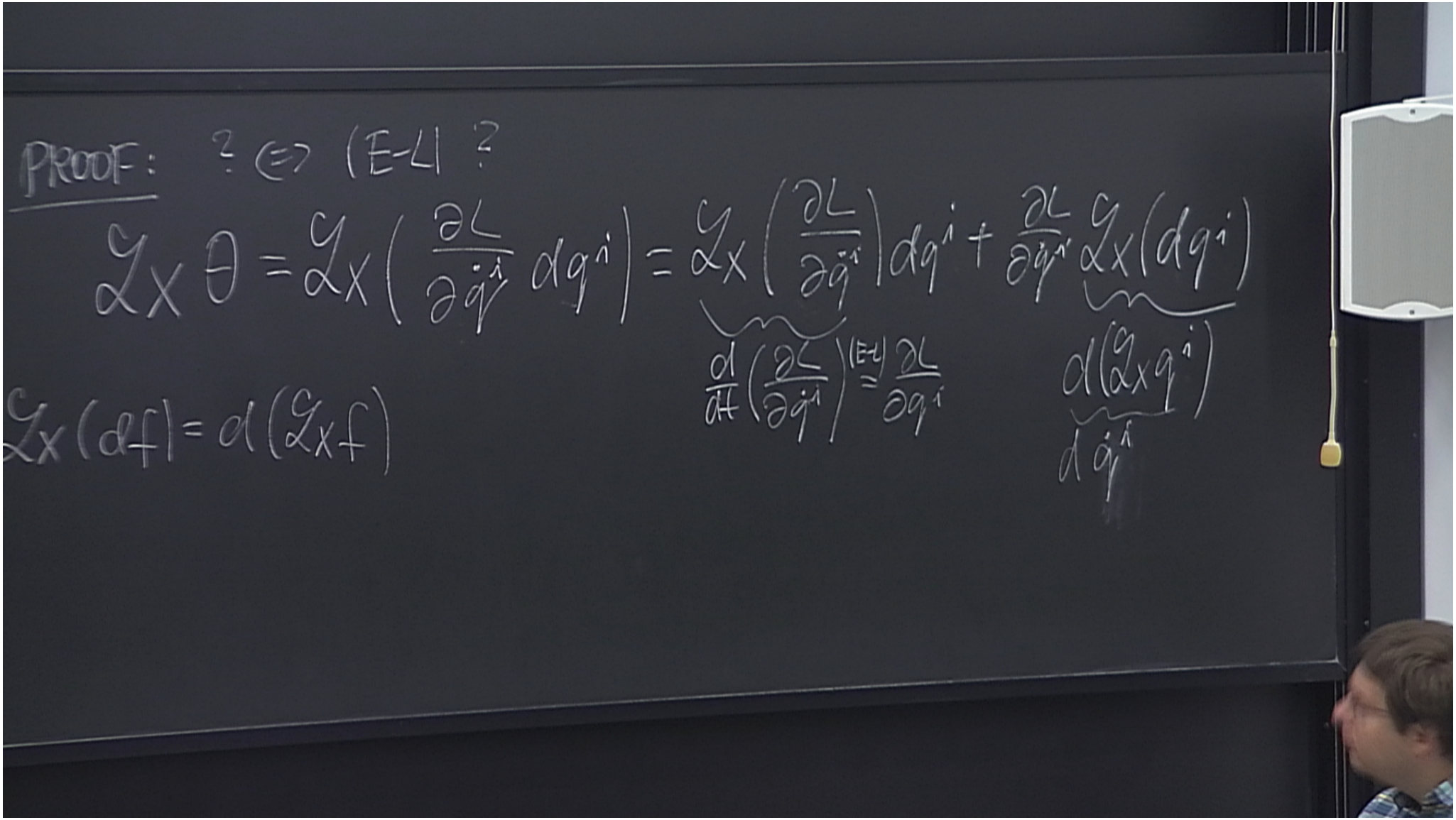
E OF THE LAGR. S.
ON $\underline{X}$. GIVEN BY

PROOF: ? $\Leftrightarrow$ (E-L) ?

$$\mathcal{L}_X \Theta = \mathcal{L}_X \left(\frac{\partial L}{\partial \dot{q}^i} dq^i \right) = \mathcal{L}_X \left(\frac{\partial L}{\partial \dot{q}^i} \right) dq^i$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) \overset{(E-L)}{=} \frac{\partial L}{\partial q^i}$$

$$\mathcal{L}_X (df) = d(\mathcal{L}_X f)$$

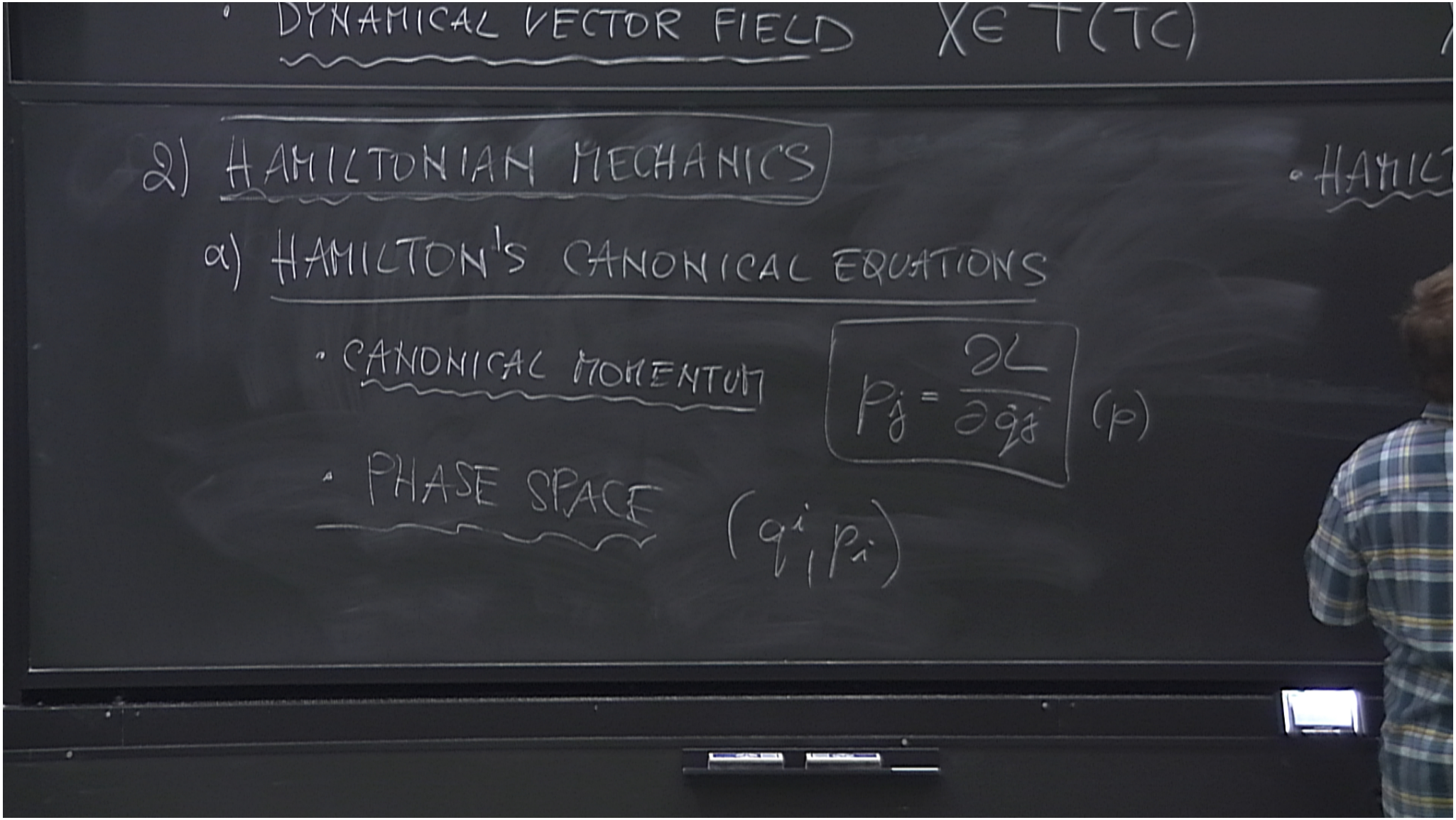

PROOF: ? ⇔ (E-L) ?
$\mathcal{L}_X \Theta = \mathcal{L}_X\left(\frac{\partial L}{\partial \dot{q}^i} dq^i\right) = \mathcal{L}_X\left(\frac{\partial L}{\partial \dot{q}^i}\right) dq^i + \frac{\partial L}{\partial \dot{q}^i} \mathcal{L}_X(dq^i)$
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) \overset{(E-L)}{=} \frac{\partial L}{\partial q^i}$
$d\left(\mathcal{L}_X q^i\right)$
$d\dot{q}^i$
$\mathcal{L}_X(df) = d\left(\mathcal{L}_X f\right)$

PROOF: $\stackrel{?}{\Leftrightarrow}$ (E-L) ?

$$\mathcal{L}_X \Theta = \mathcal{L}_X\left(\frac{\partial L}{\partial \dot{q}^i} dq^i\right) = \underbrace{\mathcal{L}_X\left(\frac{\partial L}{\partial q^i}\right)}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) \overset{(E-L)}{=} \frac{\partial L}{\partial q^i}} dq^i + \frac{\partial L}{\partial \dot{q}^i} \underbrace{\mathcal{L}_X(dq^i)}_{d\left(\frac{\mathcal{L}_X q^i}{d\dot{q}^i}\right)}$$

$$\mathcal{L}_X(df) = d(\mathcal{L}_X f)$$

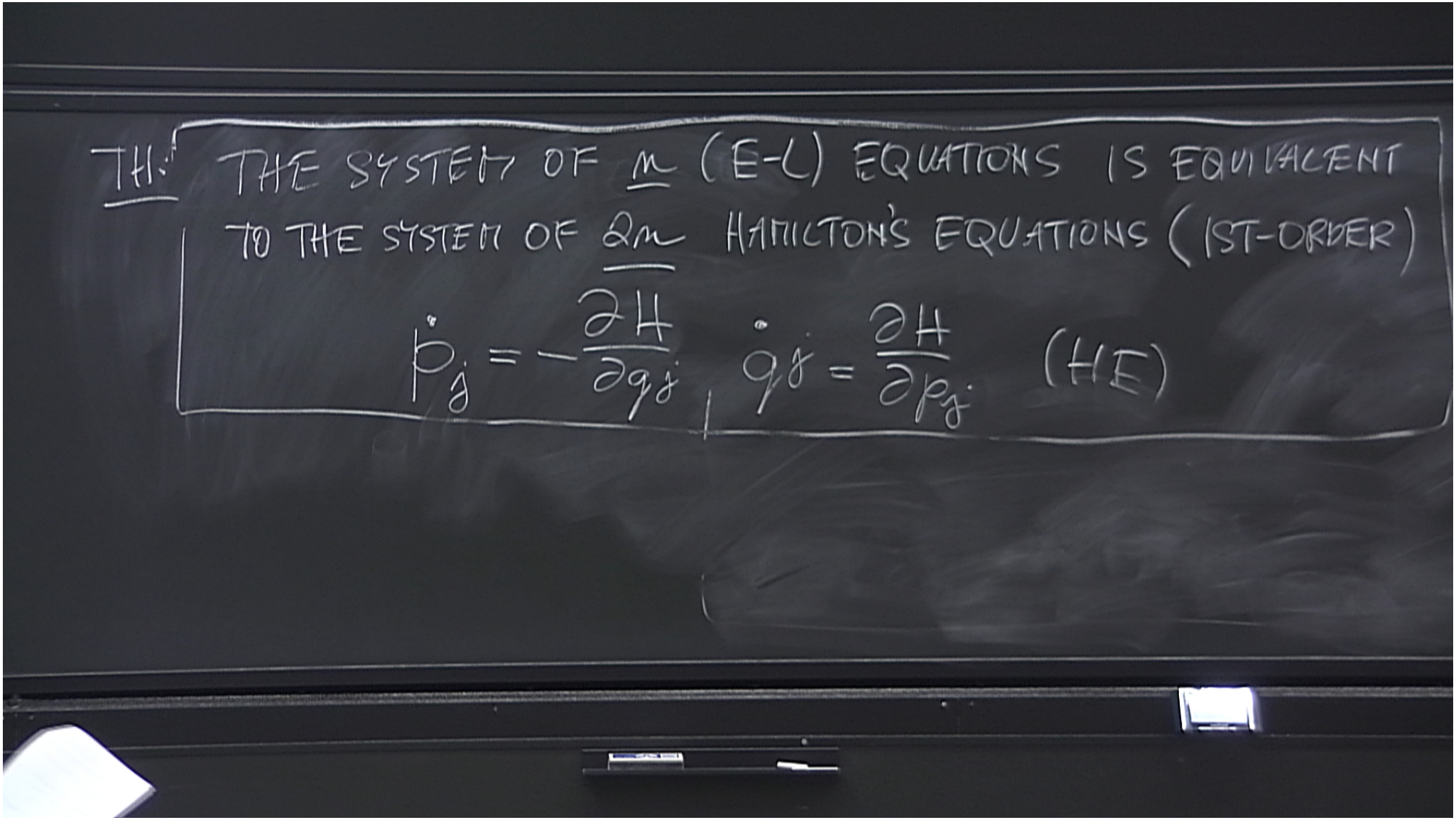
$$= \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i = dL \quad \square.$$


• DYNAMICAL VECTOR FIELD $X \in \Gamma(TC)$
2) HAMILTONIAN MECHANICS
a) HAMILTON'S CANONICAL EQUATIONS
• CANONICAL MOMENTUM $p_j = \frac{\partial L}{\partial \dot{q}^j}$ (P)
• PHASE SPACE (q^i, p_i)

$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right) \frac{\partial}{\partial q^i} + \left(\frac{\partial}{\partial t}\right) \frac{\partial}{\partial \dot{q}^i}$

- HAMILTONIAN ≡ LEGENDRE TRANSF OF LAGRANGIAN.

$$H(q^i, p_i, t) = p_j \dot{q}^j - L \Big/ \dot{q}^j = \dot{q}^j(q^i, p_i, t)$$


TH: THE SYSTEM OF n (E-L) EQUATIONS IS EQUIVALENT
TO THE SYSTEM OF 2n HAMILTON'S EQUATIONS (1ST-ORDER)
$\dot{p}_j = -\frac{\partial H}{\partial q_j}$, $\dot{q}_j = \frac{\partial H}{\partial p_j}$ (HE)

TH: THE SYSTEM OF n (E-L) EQUATIONS IS EQUIVALENT TO THE SYSTEM OF $2n$ HAMILTON'S EQUATIONS (1ST-ORDER)

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j} \qquad (HE)$$

PROOF: VARIAT. P.

$$0 = \delta S = \delta \int_1^2 (p_i \dot{q}^i - H)\, dt = \int_1^2 \left(p_i \frac{d}{dt}\delta q^i + \dot{q}^i \delta p_i \right.$$

TH: THE SYSTEM OF n (E-L) EQUATIONS IS EQUIVALENT TO THE SYSTEM OF $2n$ HAMILTON'S EQUATIONS (1ST-ORDER)

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PROOF: VARIAT. P.

$$0 = \delta S = \delta \int_1^2 (p_i \dot{q}^i - H)\, dt = \int_1^2 \left(p_i \frac{d}{dt}\delta q^i + \dot{q}^i \delta p_i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i \right.$$

SYSTEM OF n (E-L) EQUATIONS IS EQUIVALENT
SYSTEM OF $2n$ HAMILTON'S EQUATIONS (1ST-ORDER)

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j} \qquad (HE)$$

RIAT. P.

$$S = \delta \int_1^2 (p_i \dot{q}^i - H)\, dt = \int_1^2 \left(p_i \frac{d}{dt}\delta q^i + \dot{q}^i \delta p_i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i \right) dt$$

$$= \left[p_i \delta q^i \right]_1^2 + \int_1^2 \left\{ \delta p_i \left(\dot{q}^i - \frac{\partial H}{\partial p_i} \right) - \delta q^i \left(\dot{p}_i + \frac{\partial H}{\partial q^i} \right) \right\} dt$$

$= [p_i \partial q^i]_{+}$

$\frac{dH}{dt} = \frac{\partial H}{\partial t}$ SO IF $H \neq H(t) \Rightarrow$ $H(q,p) = \text{CONST}$

b) POISSON BRACKETS

DEF: LET f, g BE TWO PHASE SPACE FUNCTIONS.
THEIR CANONICAL POISSON BRACKET $\{f,g\}$ NEW F.

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}$$

$\{\ \} = \text{CONST}$

PROPERTIES.

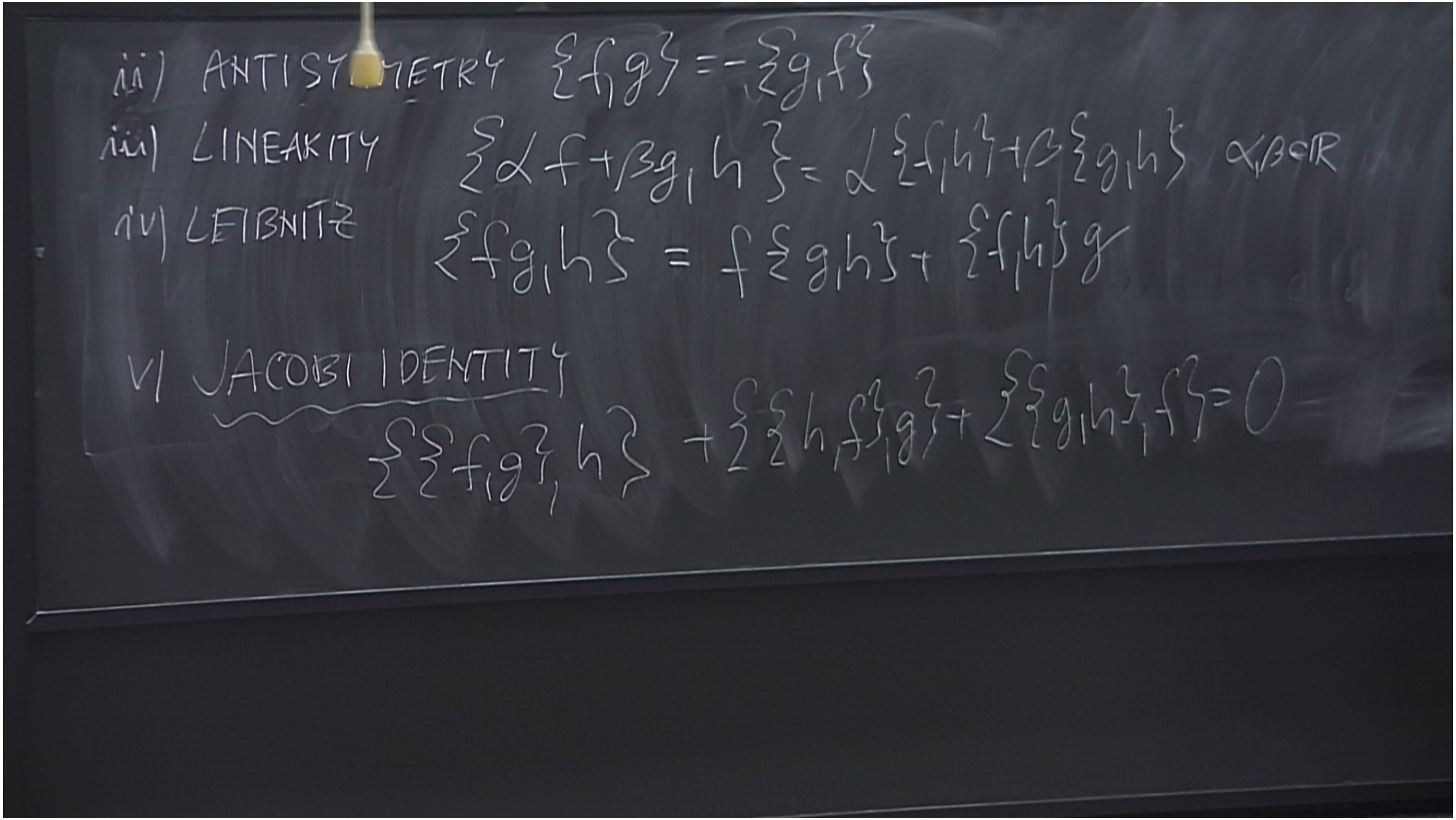
i) FUNDAMENTAL BRACKETS

$$\{q^i, p_j\} = \delta^i_j, \quad \{q^i, q^j\} = 0 = \{p_i, p_j\}$$

IN QM: $(q^i, p_i) \to (\hat{q}^i, \hat{p}_i)$ ... OPER.

REPLACE $\{\ ,\ \} \to$ COMMUTATOR $[\ ,\ ]$

$$[\hat{q}^i, \hat{p}_j] = i\hbar\,\delta^i_j$$

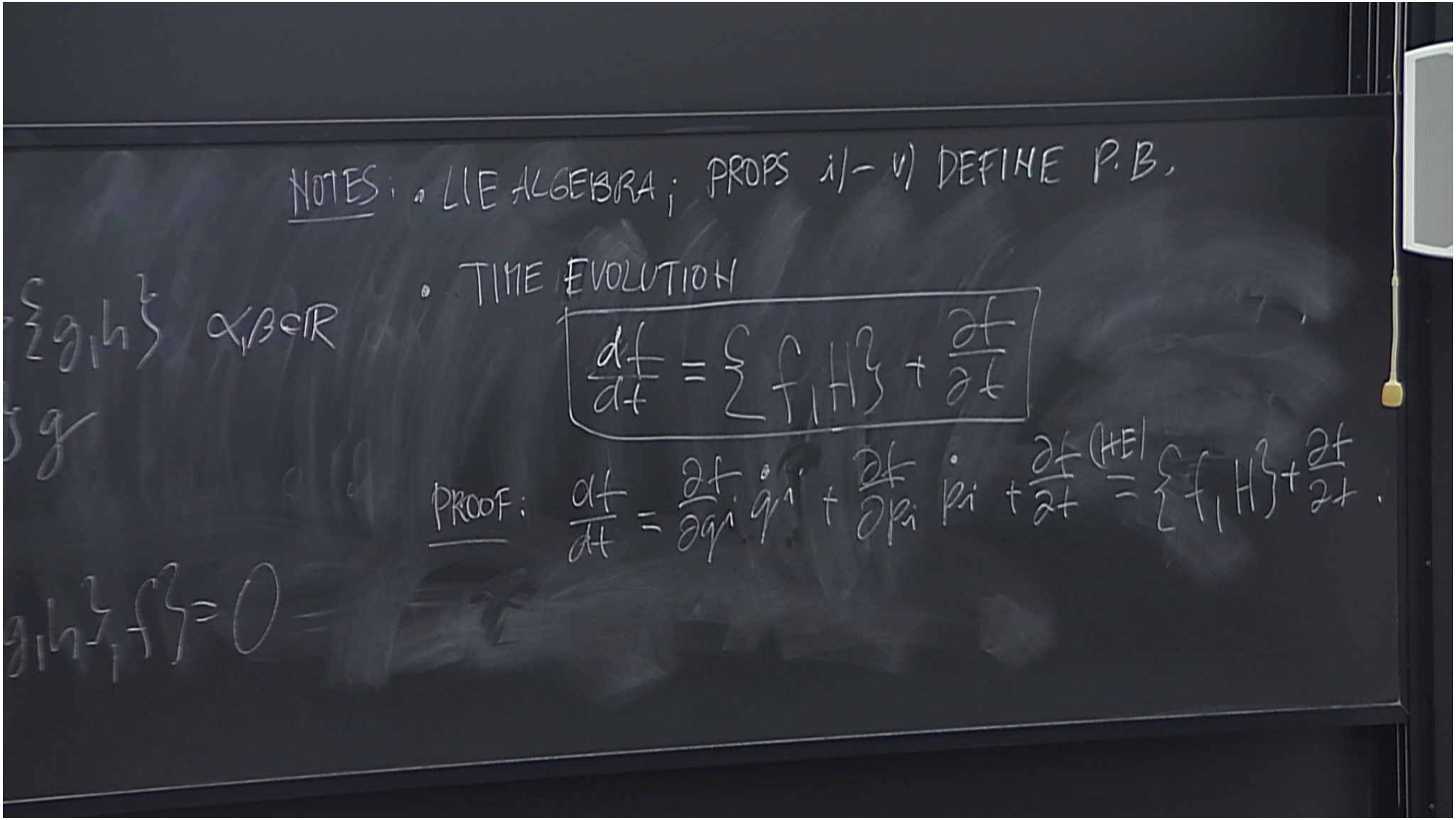

ii) ANTISYMMETRY $\{f,g\} = -\{g,f\}$
iii) LINEARITY $\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$ $\alpha,\beta \in \mathbb{R}$
iv) LEIBNITZ $\{fg,h\} = f\{g,h\} + \{f,h\}g$
v) JACOBI IDENTITY
$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$

NOTES: • LIE ALGEBRA; PROPS i/- v) DEFINE P.B.

• TIME EVOLUTION

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

PROOF: $\frac{df}{dt} = \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t} =$


NOTES: • LIE ALGEBRA; PROPS i/- v) DEFINE P.B.
• TIME EVOLUTION
$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$
PROOF: $\frac{df}{dt} = \frac{\partial f}{\partial q^i}\dot{q}^i + \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t} \overset{(HE)}{=} \{f, H\} + \frac{\partial f}{\partial t}$.

NOTES: • LIE ALGEBRA; PROPS 1/-1/ DEFINE P.B.

$\{\alpha g + \beta h, f\} = \alpha\{g,f\} + \beta\{h,f\}$ $\alpha,\beta \in \mathbb{R}$

• TIME EVOLUTION

$$\boxed{\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}}$$

PROOF: $\frac{df}{dt} = \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t} \overset{(HE)}{=} \{f, H\} + \frac{\partial f}{\partial t}$

• INTEGRAL OF MOTION $\boxed{\{I, H\} = 0}$ $\{I_1, I_2\} = I_3$

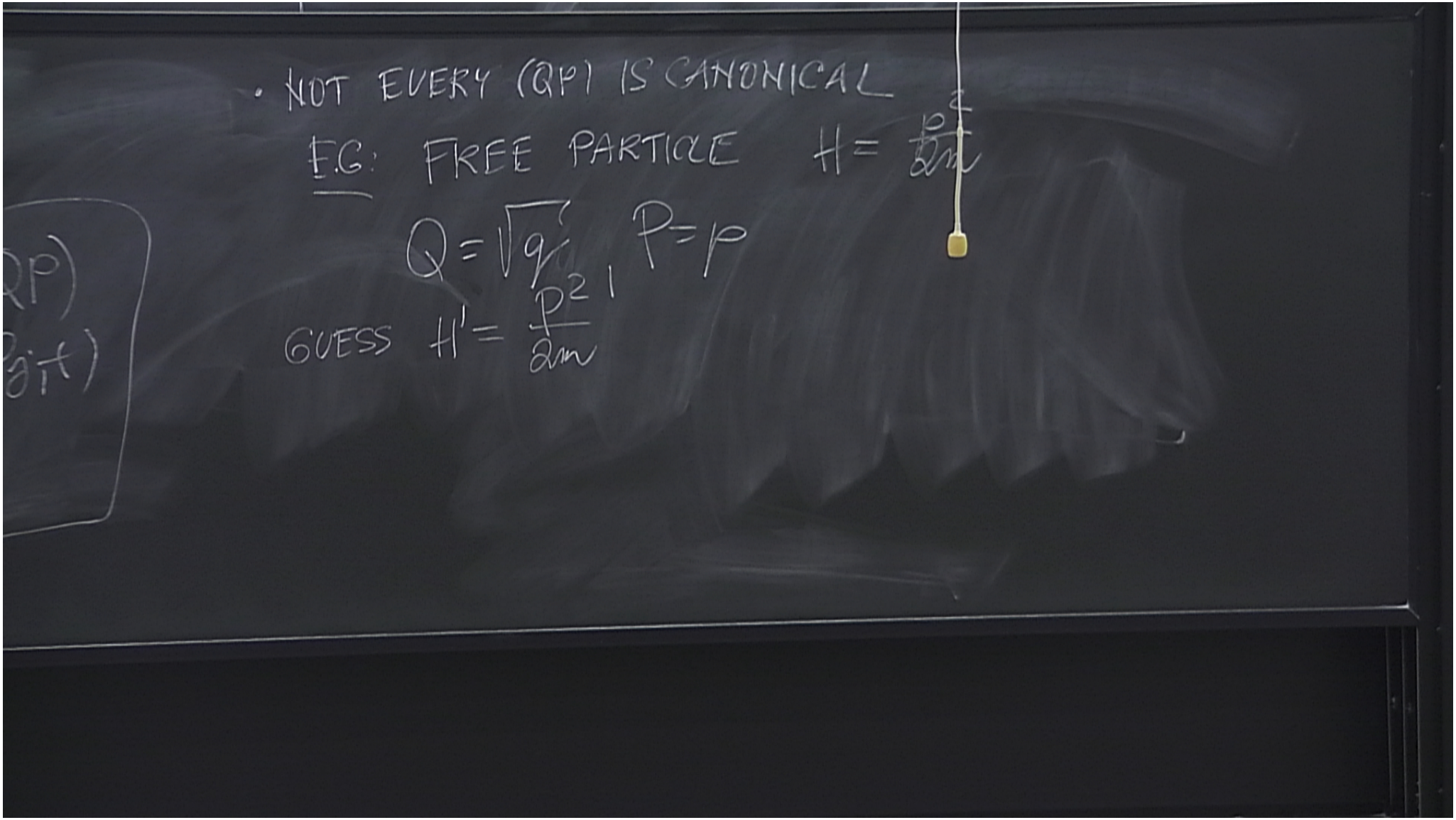
c) CANONICAL TRANSFORMATIONS

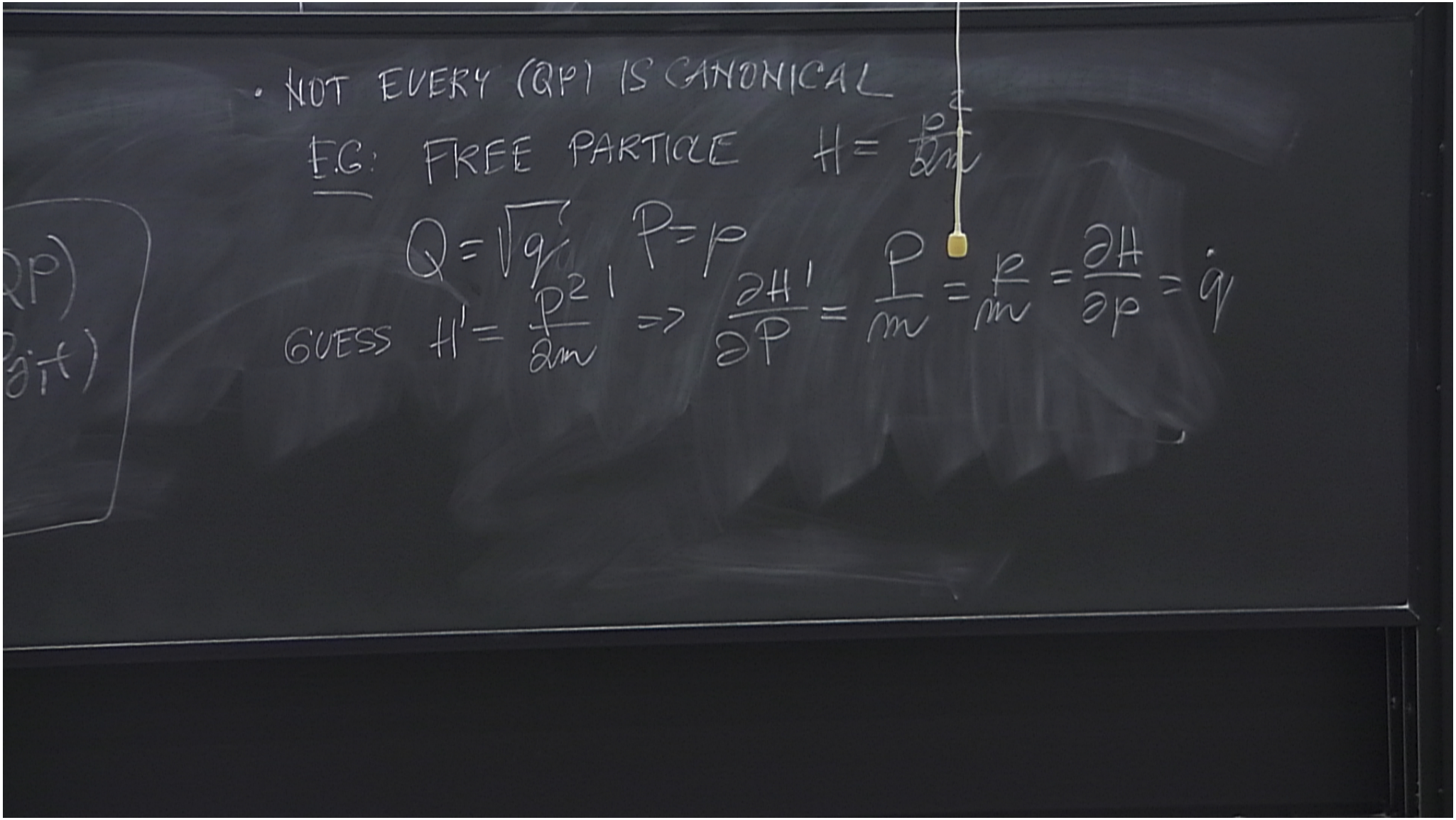
DEF: CANONICAL TRANSF

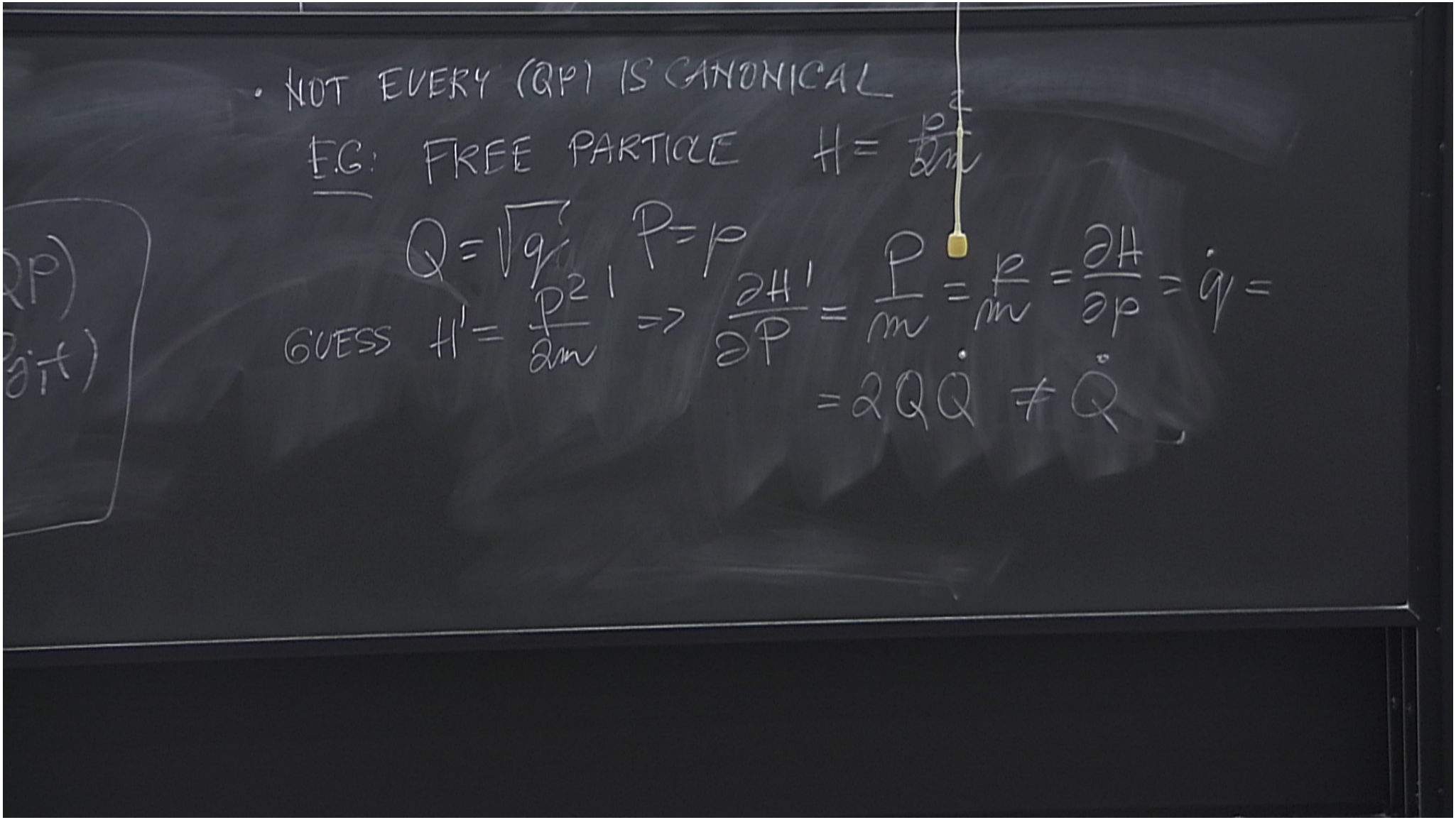
$$Q_j = Q_j(q_i, p_i, t)$$
$$P_j = P_j(q_i, p_i, t) \quad (QP)$$

PRESERVES THE FORM OF (HE). $\exists\ H' = H'(Q_j, P_j, t)$

S.T.: $\dfrac{\partial H'}{\partial P_j} = \dot{Q}_j$; $\dfrac{\partial H'}{\partial Q_j} = -\dot{P}_j$


• NOT EVERY (QP) IS CANONICAL
E.G: FREE PARTICLE $H = \frac{p^2}{2m}$
$Q = \sqrt{q}$, $P = p$
GUESS $H' = \frac{P^2}{2m}$


• NOT EVERY (QP) IS CANONICAL
E.G: FREE PARTICLE $H = \frac{p^2}{2m}$
$Q = \sqrt{q}$, $P = p$
GUESS $H' = \frac{P^2}{2m}$ => $\frac{\partial H'}{\partial P} = \frac{P}{m} = \frac{p}{m} = \frac{\partial H}{\partial p} = \dot{q}$


• NOT EVERY (QP) IS CANONICAL
E.G: FREE PARTICLE $H = \frac{p^2}{2m}$
$Q = \sqrt{q}$, $P = p$
GUESS $H' = \frac{P^2}{2m}$ => $\frac{\partial H'}{\partial P} = \frac{P}{m} = \frac{p}{m} = \frac{\partial H}{\partial p} = \dot{q} =$
$= 2Q\dot{Q} \neq \dot{Q}$

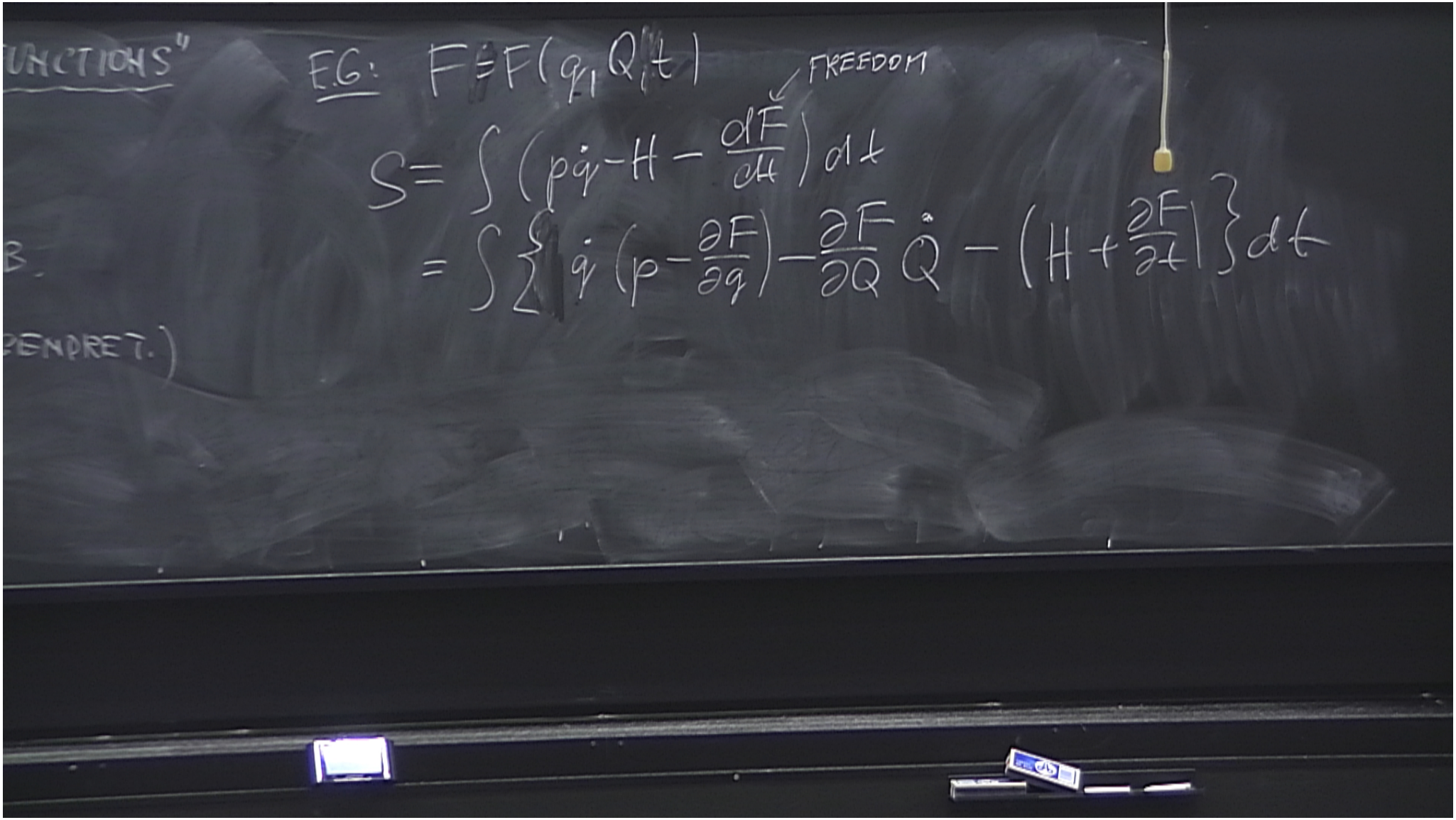
4) POISSON BRACKETS

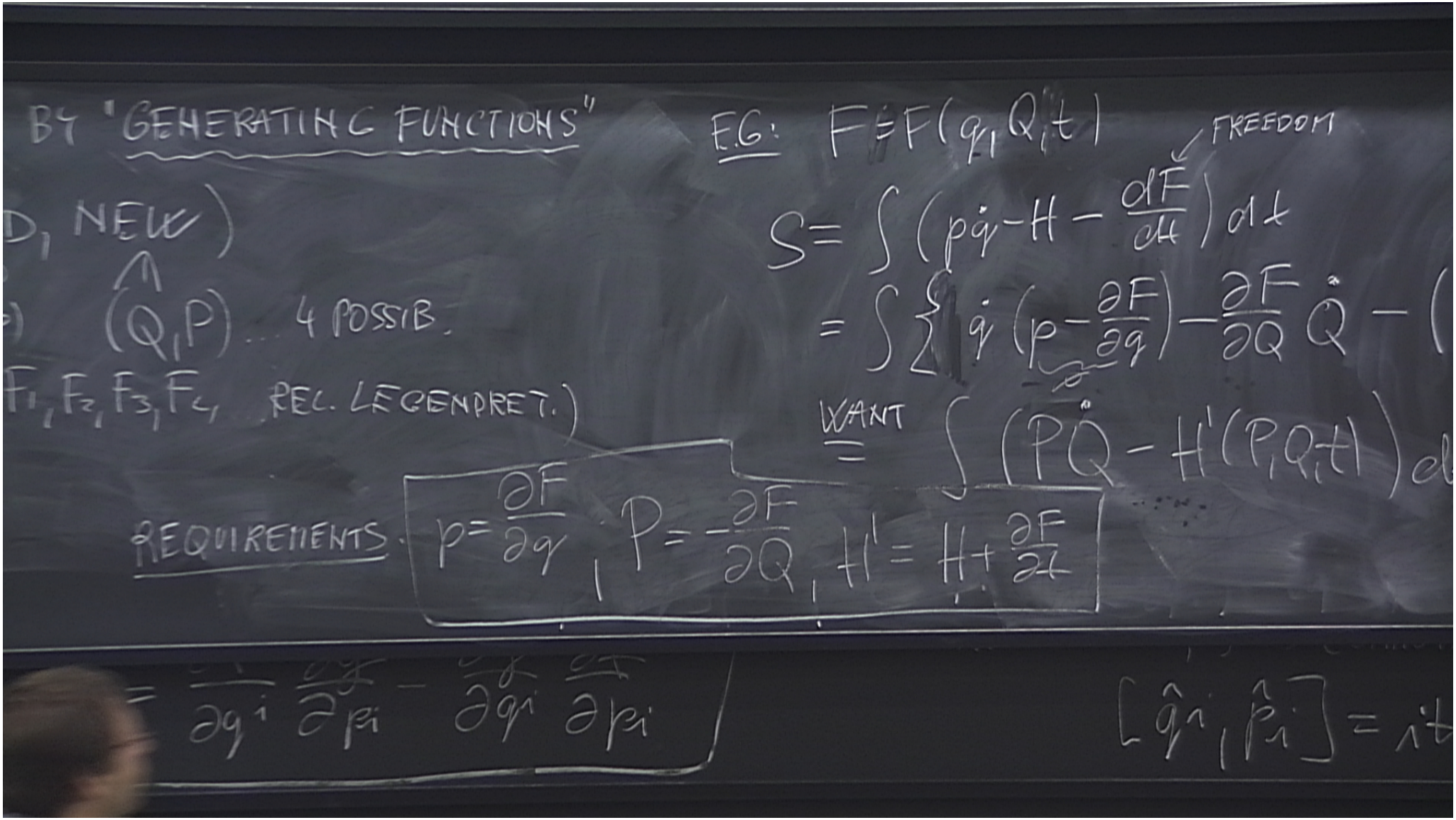
- C.T. GENERATED BY "GENERATING FUNCTIONS" E.G: F

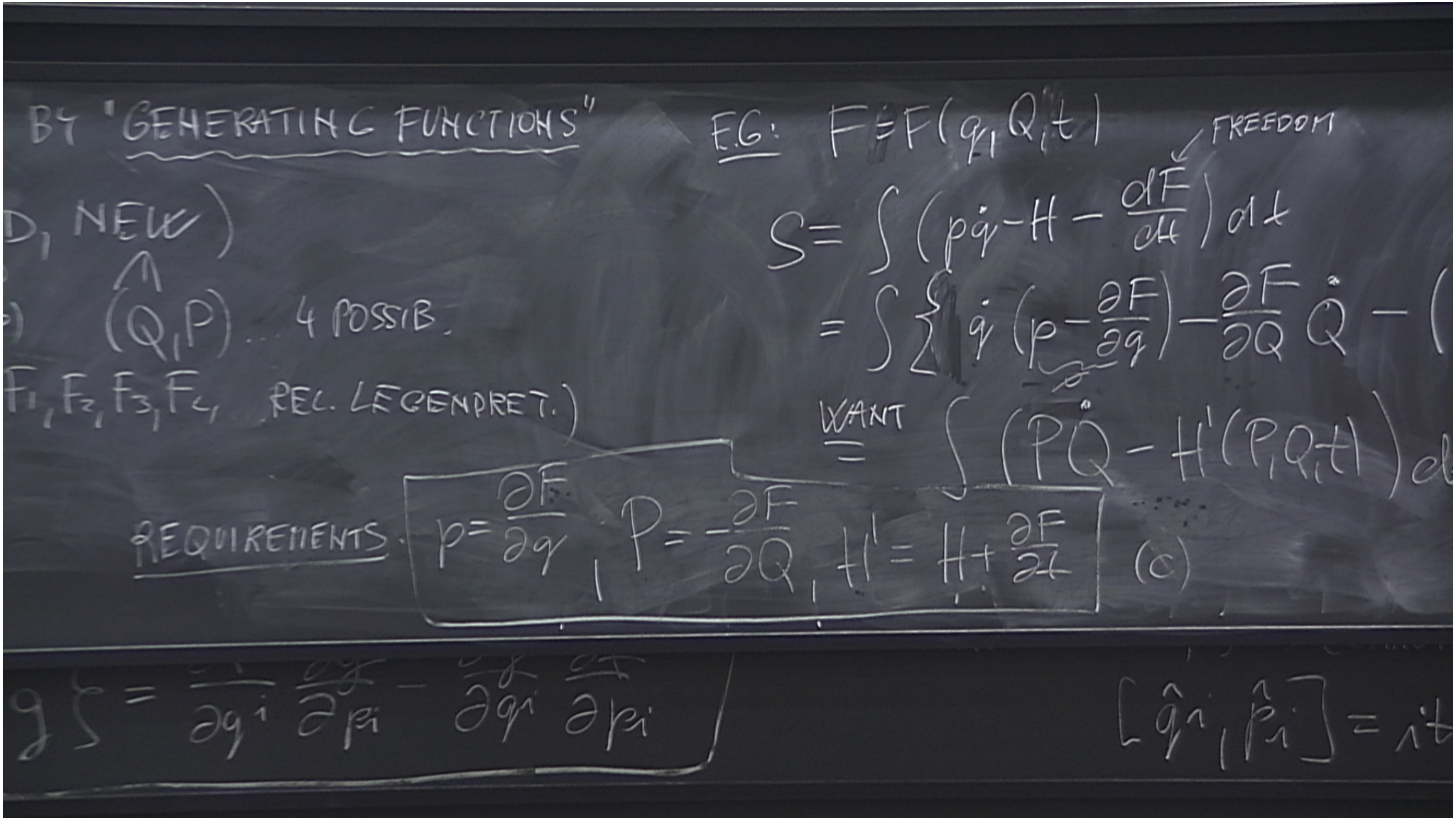
$$F = F(\text{OLD}, \text{NEW})$$

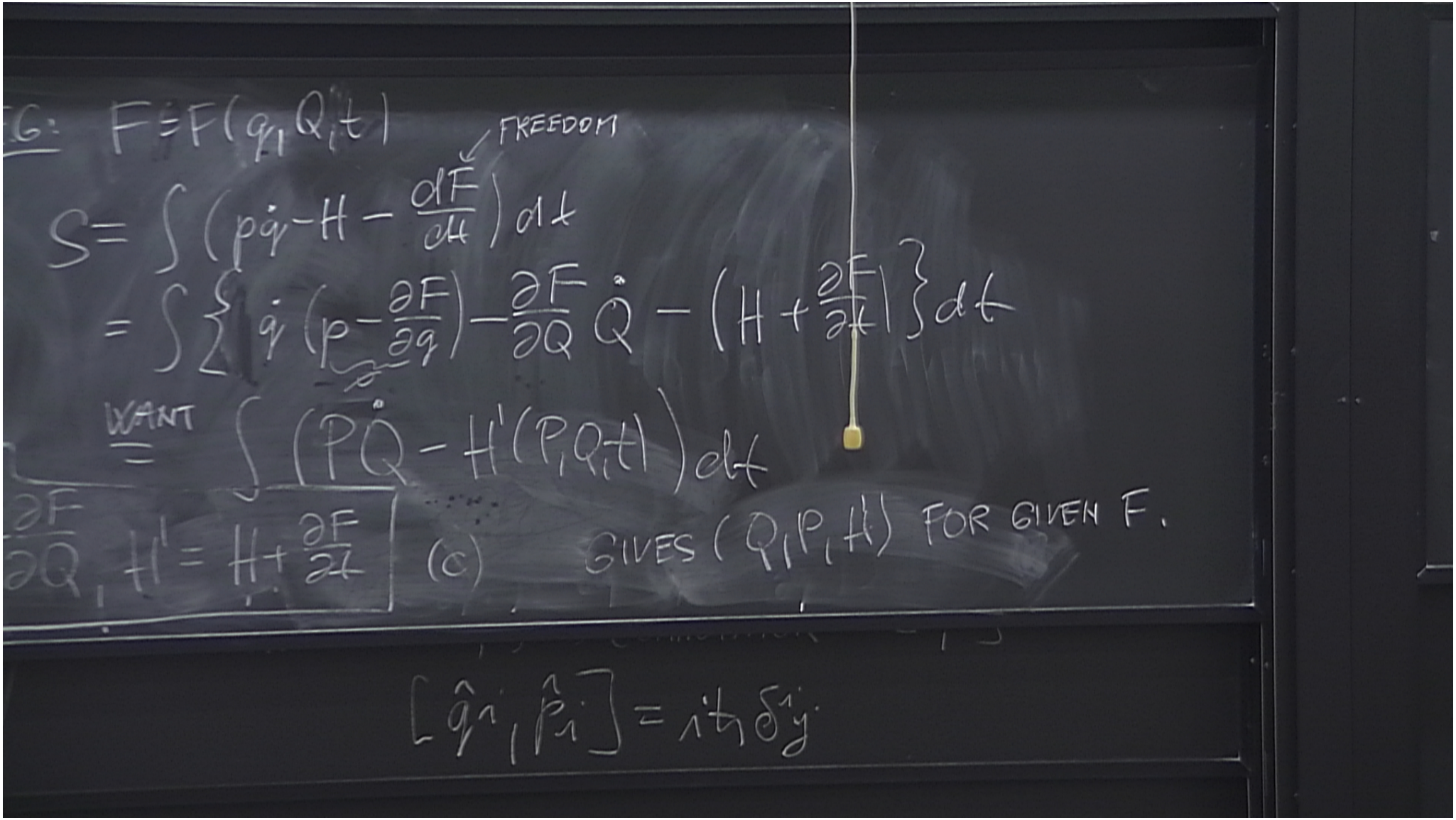
$$(q,p) \qquad (Q,P) \quad \ldots \text{4 POSSIB.}$$

$(F_1, F_2, F_3, F_4$ REL. LEGENDRE.$)$


UNCTIONS"
EG: F = F(q, Q, t)
FREEDOM
S = ∫ (pq̇ − H − dF/dt) dt
= ∫ { q̇ (p − ∂F/∂q) − ∂F/∂Q Q̇ − (H + ∂F/∂t) } dt
B.
ENDRET.)


BY "GENERATING FUNCTIONS"
D, NEW)
(Q, P) ... 4 POSSIB.
F1, F2, F3, F4 REL. LEGENDRE.)
E.G: F = F(q, Q, t)
FREEDOM
S = ∫ (pq̇ − H − ∂F/∂t) dt
= ∫ Σ q̇ (p − ∂F/∂q) − ∂F/∂Q Q̇ − (
WANT = ∫ (PQ̇ − H'(P,Q,t)) dt
REQUIREMENTS
p = ∂F/∂q , P = −∂F/∂Q , H' = H + ∂F/∂t
= ∂q_i ∂p_i − ∂q_i ∂p_i
[q̂_i, p̂_i] = iℏ


BY "GENERATING FUNCTIONS"
D, NEW)
(Q, P) ... 4 POSSIB.
F1, F2, F3, F4 REL. LEGENDRET.)
E.G: F = F(q, Q, t)
FREEDOM
S = ∫ (pq̇ − H − ∂F/∂t) dt
= ∫ Σ q̇ (p − ∂F/∂q) − ∂F/∂Q Q̇ − (
WANT = ∫ (PQ̇ − H'(P,Q,t)) dt
REQUIREMENTS
p = ∂F/∂q , P = −∂F/∂Q , H' = H + ∂F/∂t (c)
[q̂ᵢ, p̂ⱼ] = iℏ


EG: $F = F(q, Q, t)$
FREEDOM
$S = \int \left(p\dot{q} - H - \frac{dF}{dt} \right) dt$
$= \int \left\{ \dot{q}\left(p - \frac{\partial F}{\partial q}\right) - \frac{\partial F}{\partial Q}\dot{Q} - \left(H + \frac{\partial F}{\partial t}\right) \right\} dt$
WANT $\int \left(P\dot{Q} - H'(P, Q, t) \right) dt$
$\frac{\partial F}{\partial Q}, H' = H + \frac{\partial F}{\partial t}$ (c) GIVES (Q, P, H') FOR GIVEN F.
$[\hat{q}^i, \hat{p}_j] = i\hbar\delta^i_j$

REMARKS. a) P.B ARE INVARIANT UNDER C.T.

$$\{f,g\}_{q,p} = \{f,g\}_{Q,P}$$

REMARKS. a) P.B ARE INVARIANT UNDER C.T.

$$\{f,g\}_{q,p} = \{f,g\}_{Q,P}$$

Q,P NEW CAN. COORDS

$$\{Q^i, P_j\} = \delta^i_j \qquad \{Q^i, Q^j\} = 0 = \{P_i, P_j\}$$

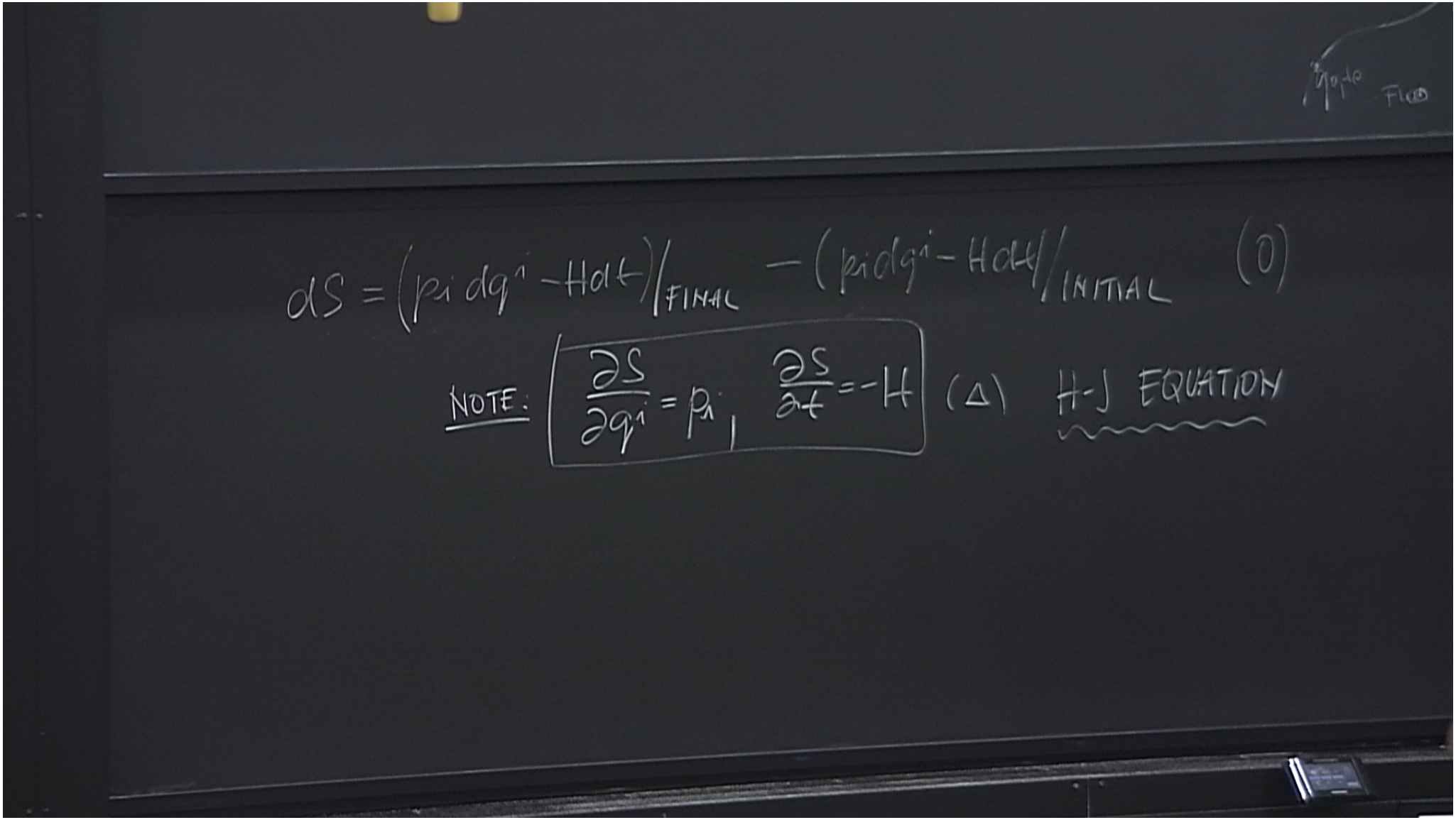
ii) "TIME EVOLUTION ≡ CHANGE OF COORDS"

TIME EVOLUTION = C.T GENERATED BY F=-S

- S AS A FUNCTION OF COORDS & TIME

$P_i, P_j\}$

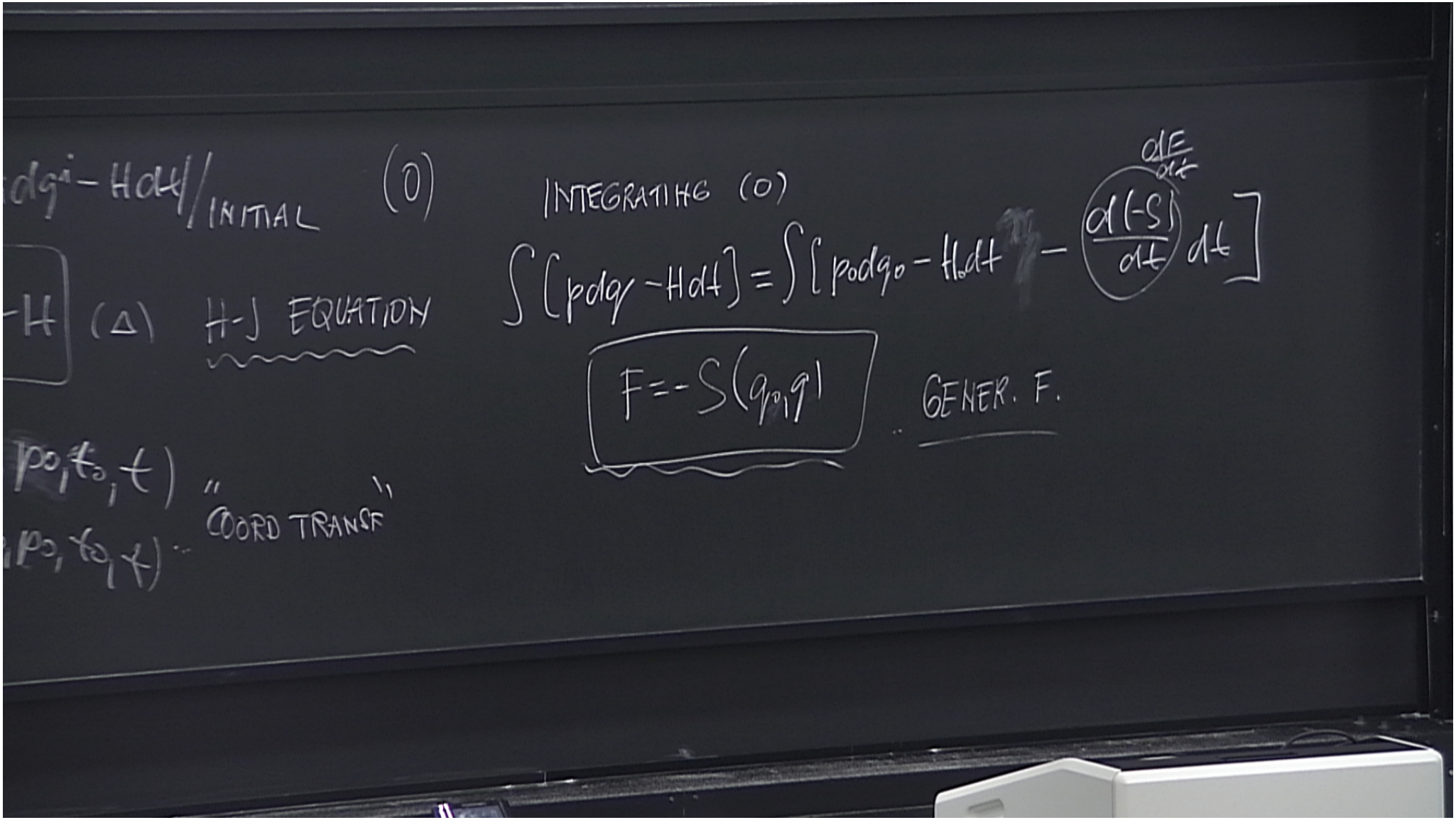
$$S = \int_{t_0, q_0}^{t, q} L\,dt \overset{\text{ON-SHELL}}{=} \overset{\text{FIXED TRAJ}}{=} S(q_0, t_0, q, t)$$


$dS = (p_i\,dq^i - H\,dt)\big|_{\text{FINAL}} - (p_i\,dq^i - H\,dt)\big|_{\text{INITIAL}}$ (0)
NOTE: $\frac{\partial S}{\partial q^i} = p_i, \quad \frac{\partial S}{\partial t} = -H$ (Δ) H-J EQUATION

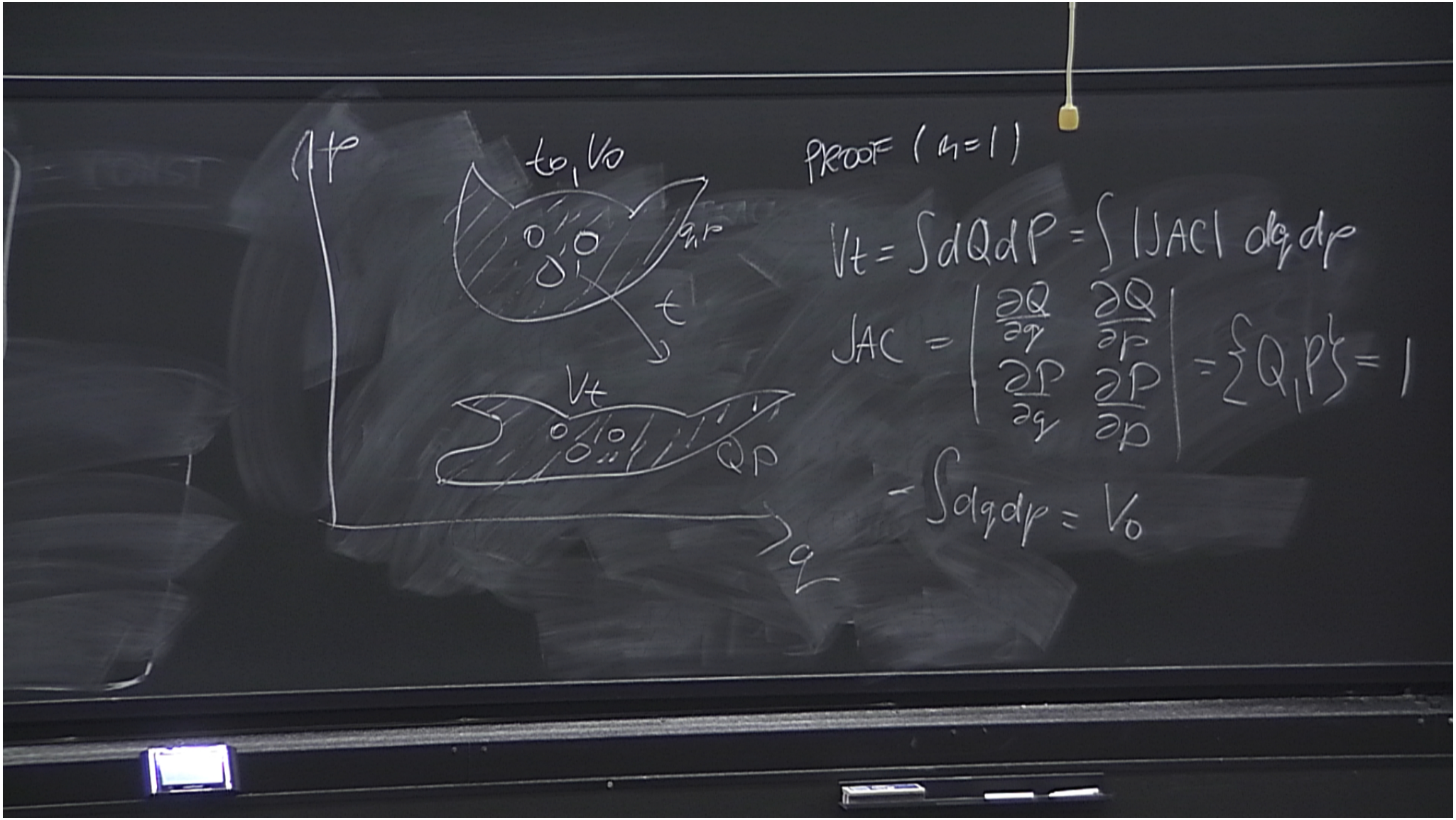
$$dS = (p_i\, dq^i - H dt)\big|_{FINAL} - (p_i dq^i - H dt)\big|_{INITIAL} \quad (0)$$

NOTE: $\boxed{\frac{\partial S}{\partial q^i} = p_i, \quad \frac{\partial S}{\partial t} = -H}$ (Δ) H-J EQUATION

- TIME EVOLUTION $\quad q(t) = q(q_0, p_0, t_0, t)$ "COORD TRANSF"
$\quad\quad\quad


dq^i − H dt INITIAL (0)
INTEGRATING (0)
∫[p dq − H dt] = ∫[p₀ dq₀ − H₀ dt − d(−S)/dt dt]
dE/dt
−H (Δ) H-J EQUATION
F = −S(q₀, q)
GENER. F.
p₀, t₀, t) "COORD TRANSF"
p₀, t₀, t)

iii) LIOUVILLE'S TH. VOLUME OF THE PHASE SPACE REMAINS INVARIANT UNDER CAN. T. PHASE FLOW TIME EV. PRESERVES THIS VOLUME



PROOF ($n=1$)

$$V_t = \int dQ\,dP = \int |JAC|\, dq\,dp$$

$$JAC = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \{Q, P\} = 1$$

$$= \int dq\,dp = V_0$$