Title: PSI 2015/2016 Classical Mechanics 2- David Kubiznak

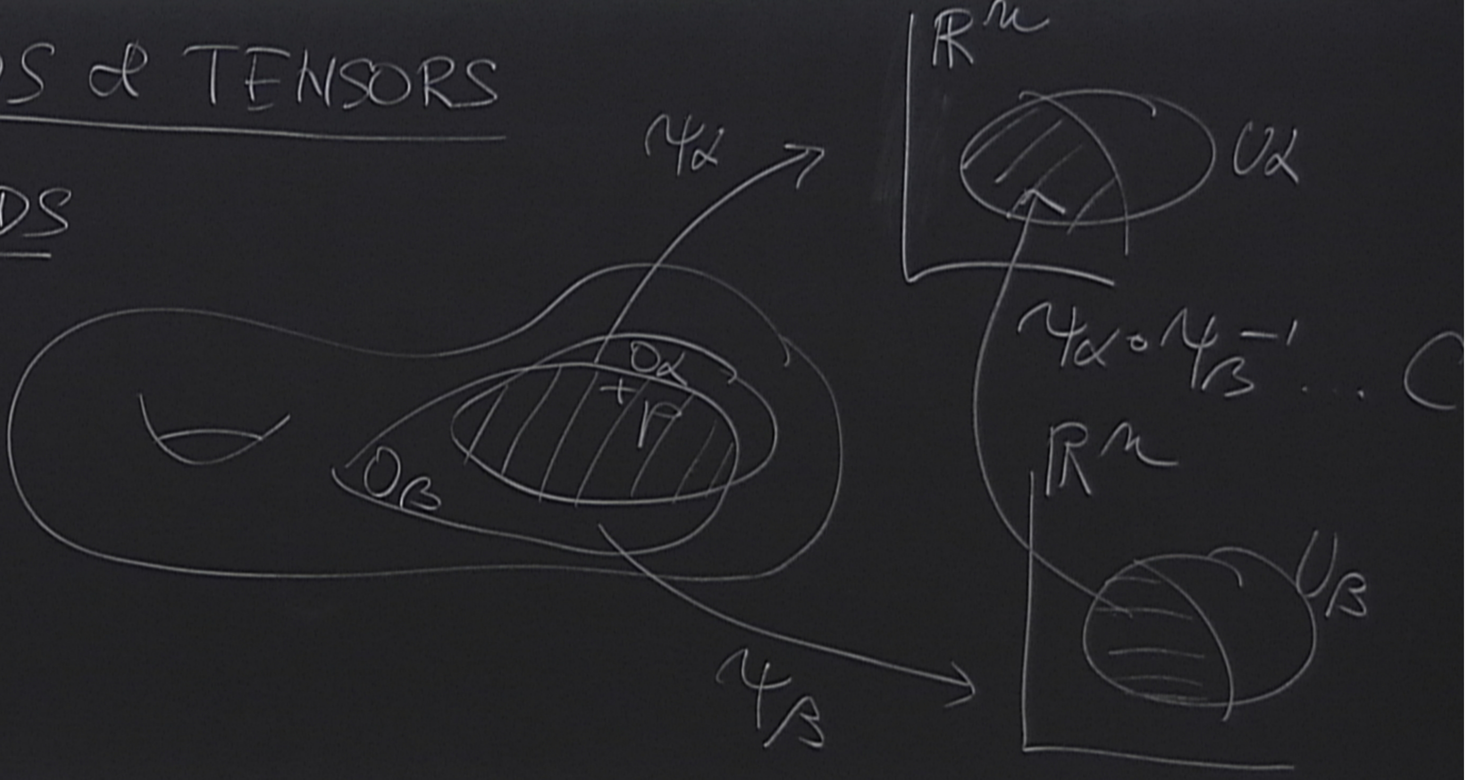
Date: Aug 19, 2015 09:00 AM

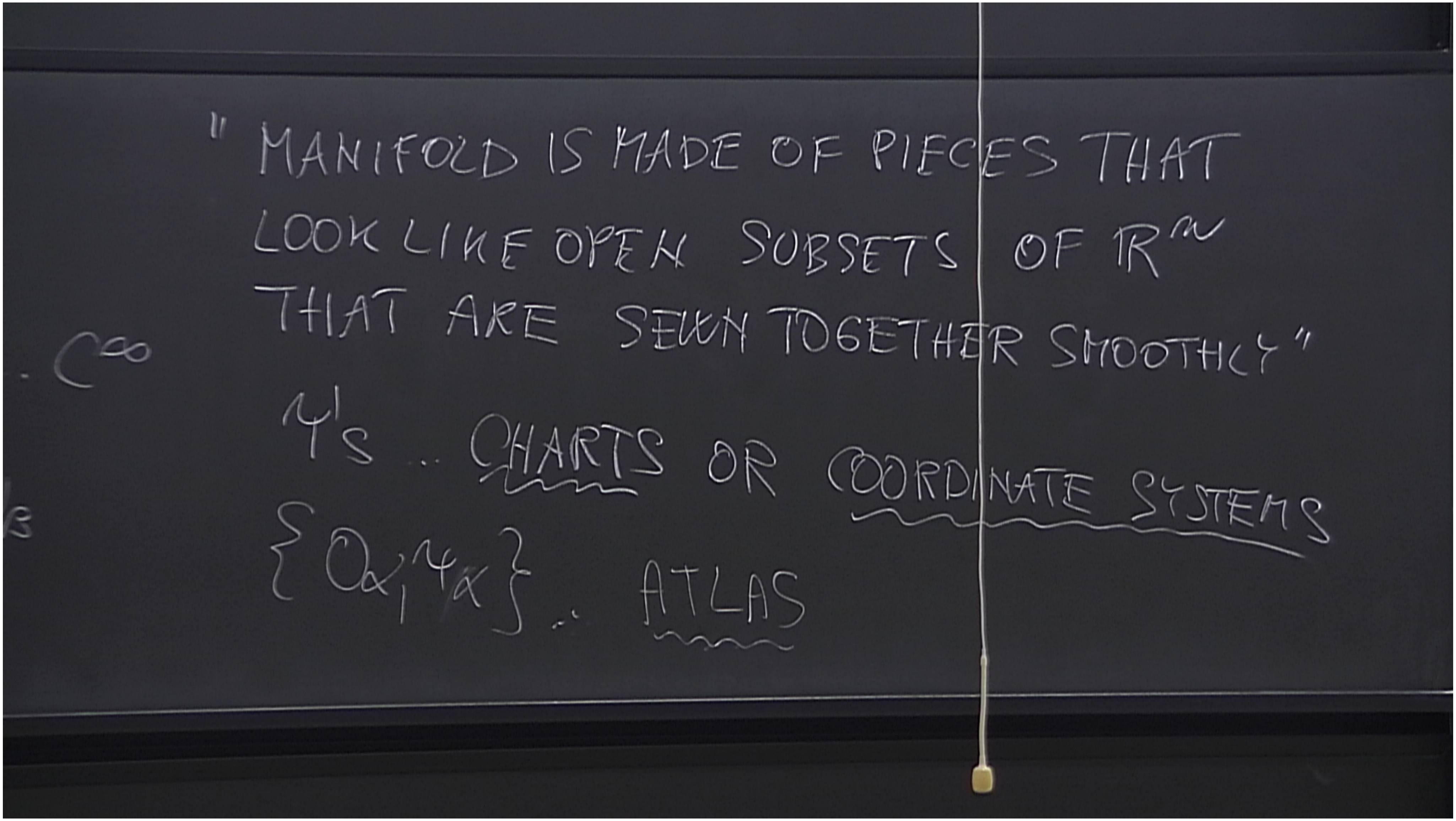
URL: http://pirsa.org/15080097

Abstract:

## c) MANIFOLDS & TENSORS

- MANIFOLDS



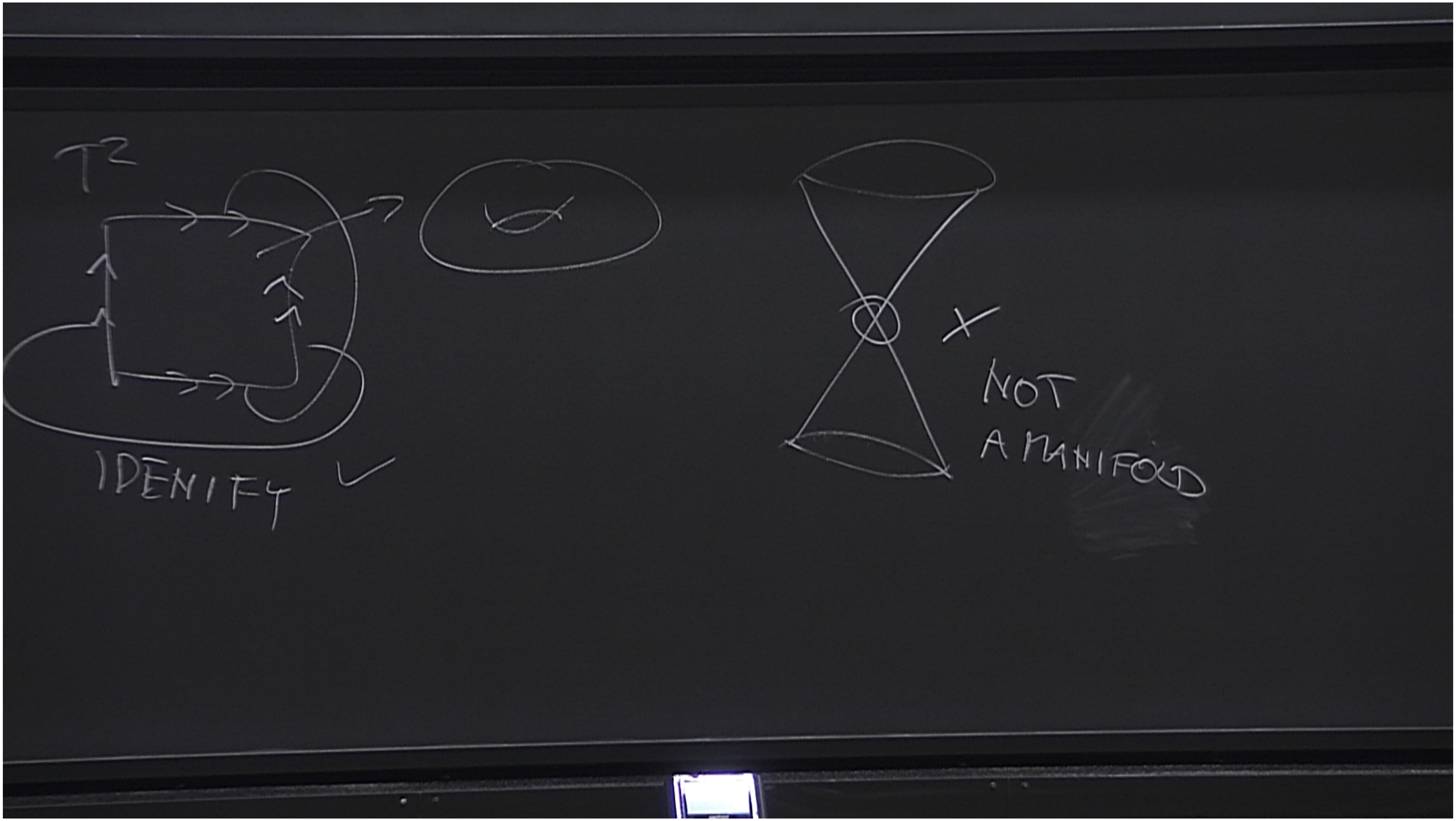


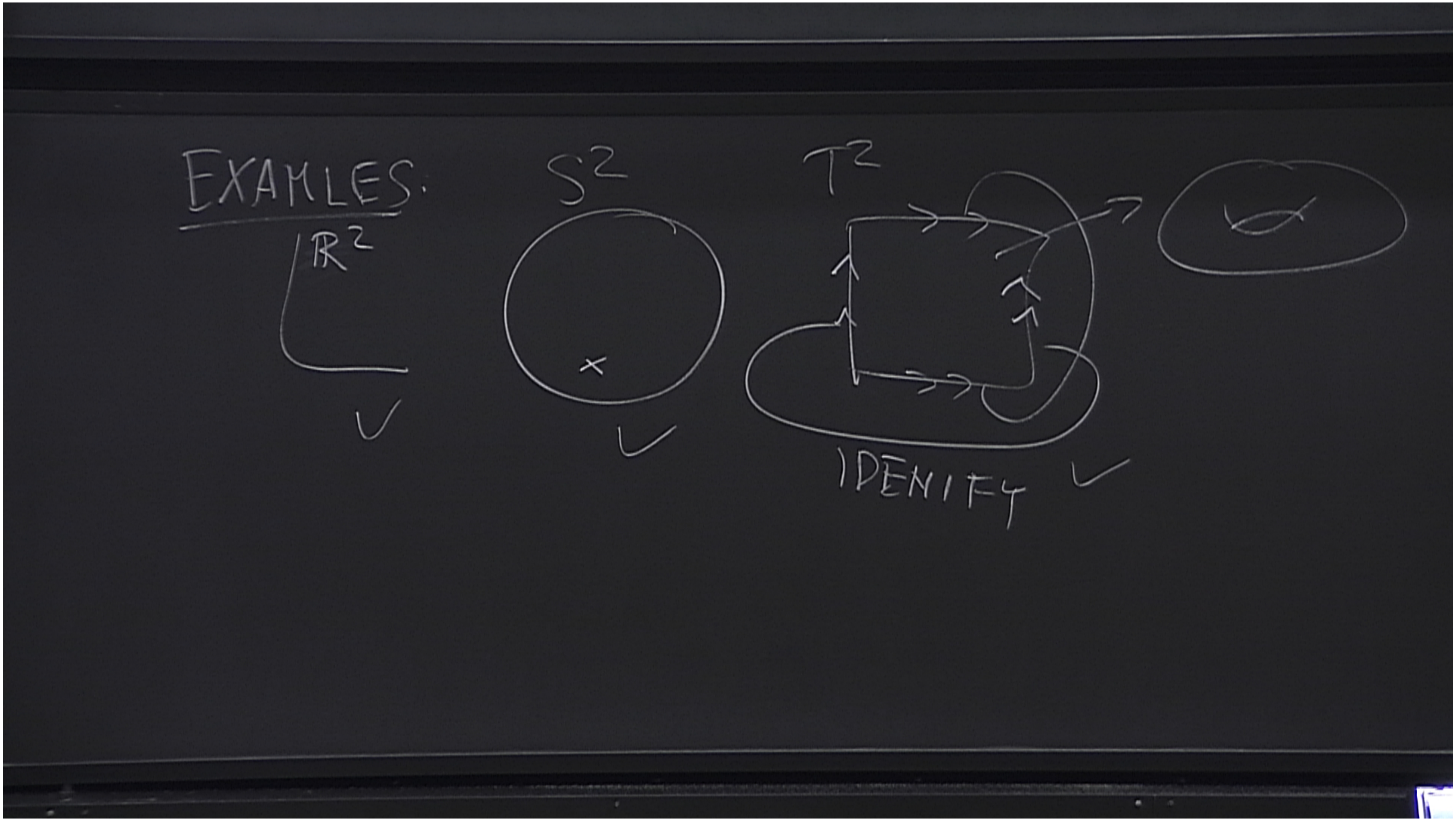
"MANIFOLD IS MADE OF PIECES THAT LOOK LIKE OPEN SUBSETS OF $\mathbb{R}^n$ THAT ARE SEWN TOGETHER SMOOTHLY"

. $C^\infty$

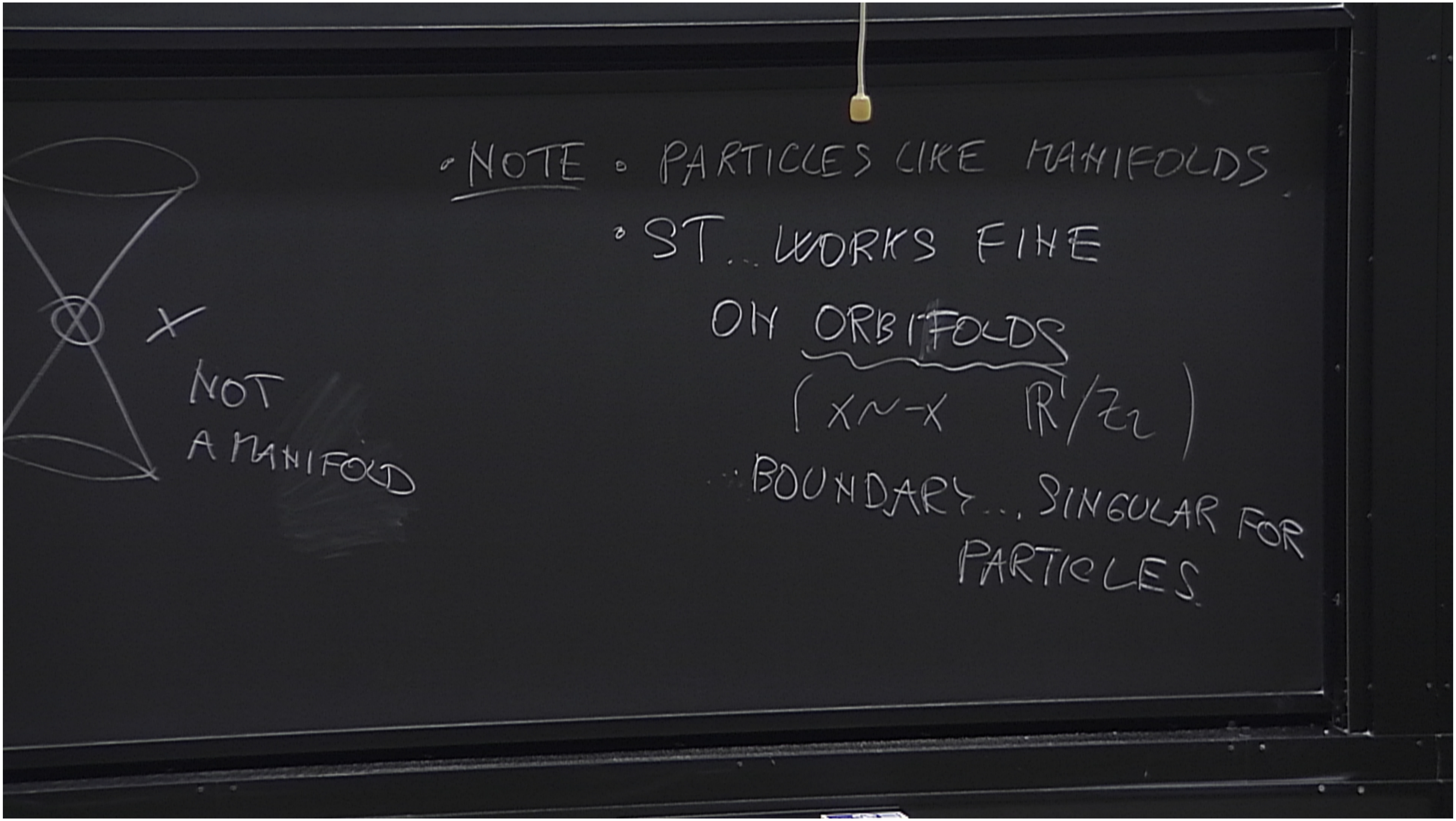
$\psi$'s ... CHARTS OR COORDINATE SYSTEMS

$\{O_\alpha, \psi_\alpha\}$ .. ATLAS


$T^2$
IDENIFY
✓
×
NOT
A MANIFOLD



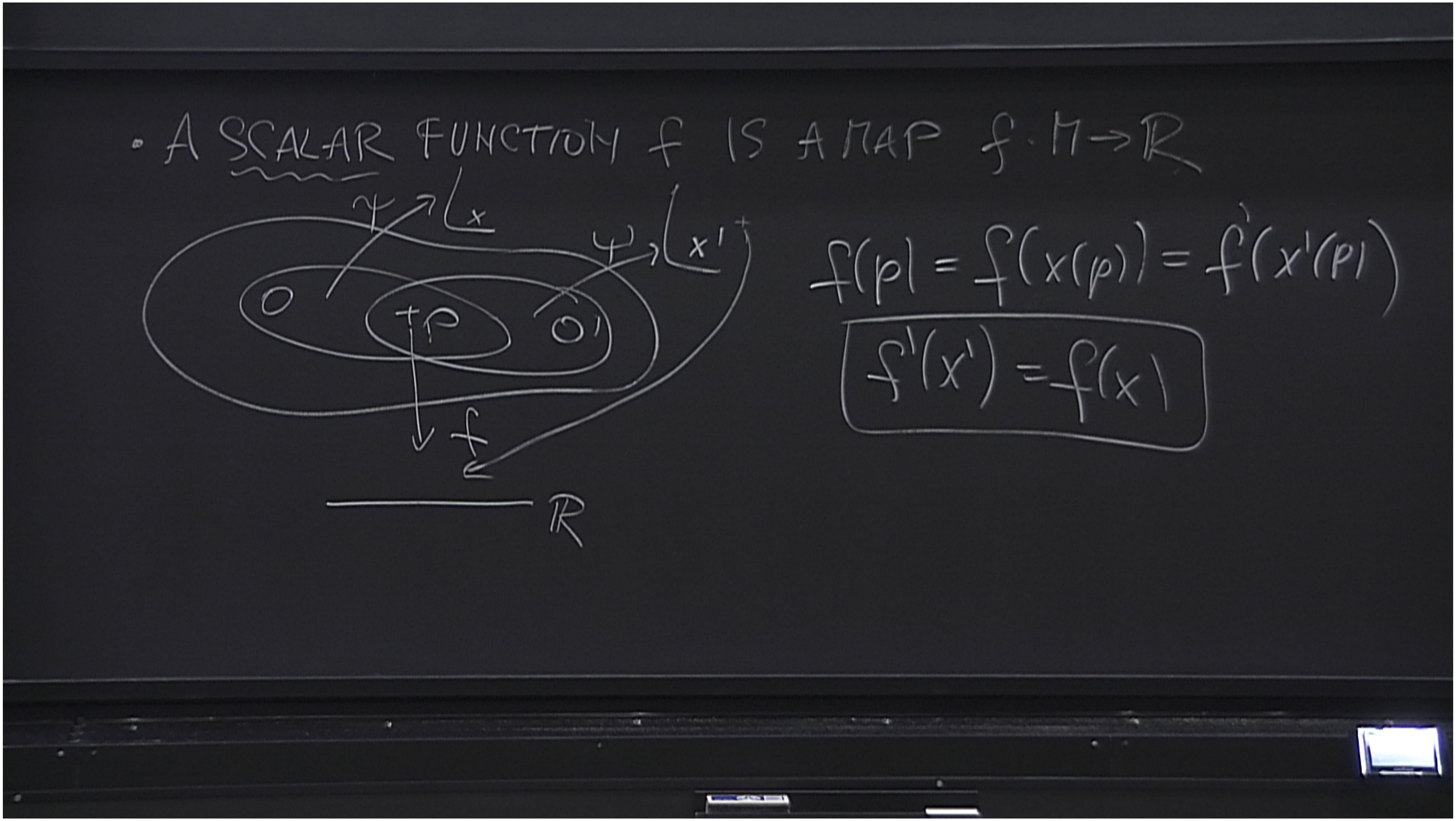
EXAMLES:
$\mathbb{R}^2$
✓
$S^2$
✓
$T^2$
IDENIFY
✓



NOT A MANIFOLD

- **NOTE** • PARTICLES LIKE MANIFOLDS
  - ST... WORKS FINE ON ORBIFOLDS ( $x \sim -x$   $\mathbb{R}/\mathbb{Z}_2$ )

  ... BOUNDARY... SINGULAR FOR PARTICLES



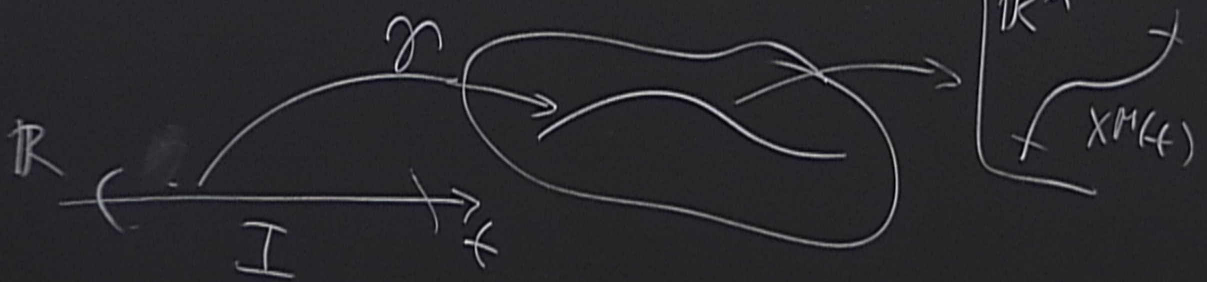
- A SCALAR FUNCTION $f$ IS A MAP $f: M \to \mathbb{R}$

$$f(p) = f(x(p)) = f'(x'(p))$$

$$\boxed{f'(x') = f(x)}$$

• A CURVE $\gamma$ ON $M$ IS A MAP $\gamma : I \subset \mathbb{R} \to M$ S.T.

$$(\psi_\alpha \circ \gamma)(t) = [x^1(t), x^2(t), \ldots, x^n(t)]$$ SMOOTH.

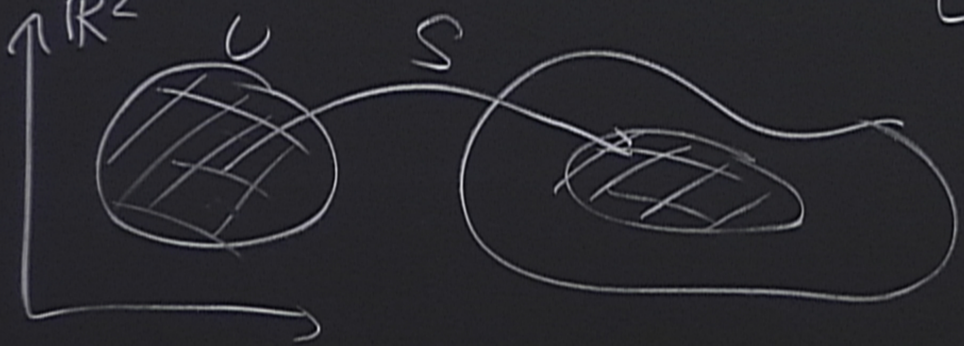


E.G. RIVER

TRAJECTORY (WORLDLINE)

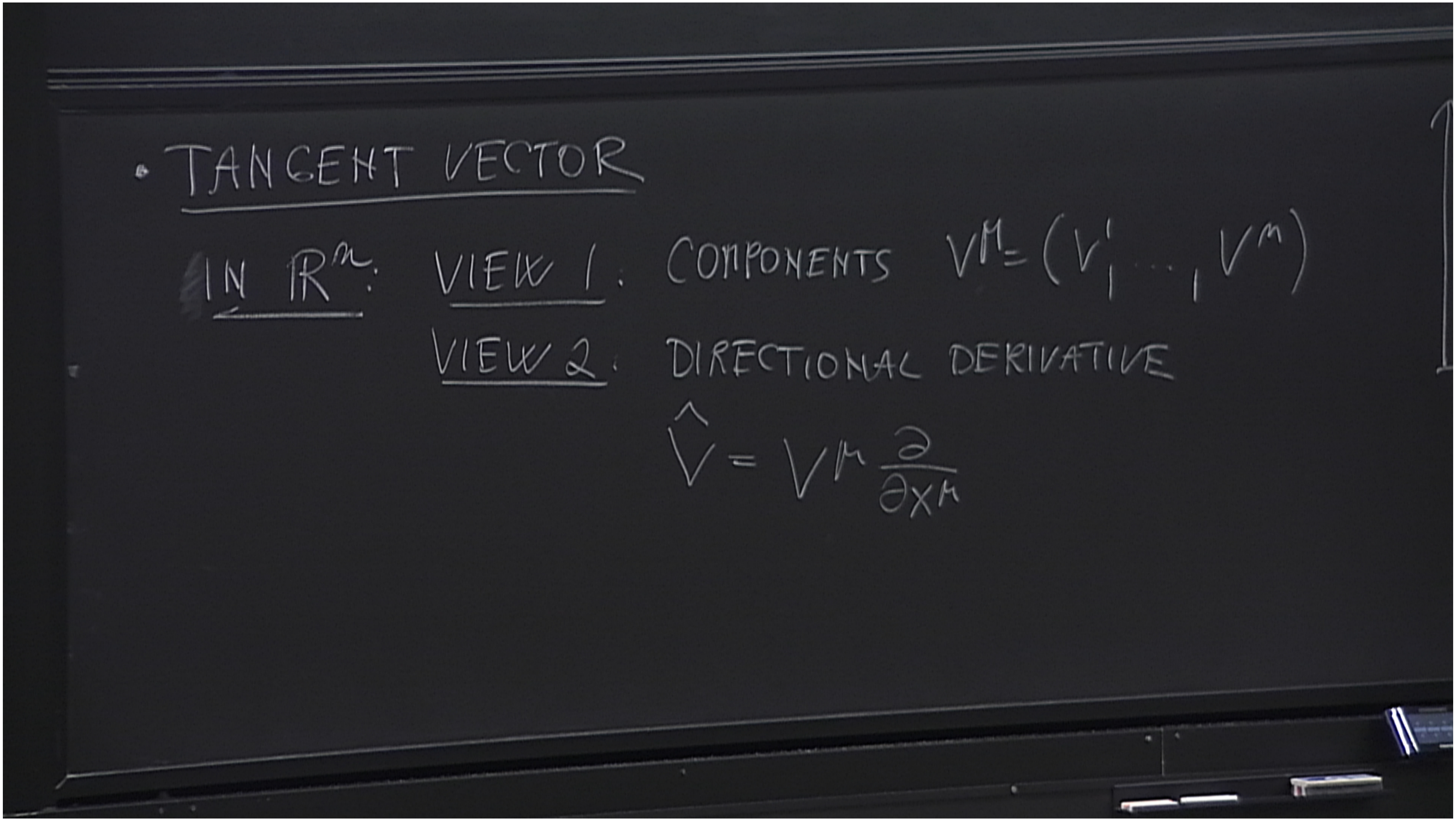
SIMILARLY: A SURFACE S.

$$[X^1(\tau,\sigma), \dots X^n(\tau,\sigma)]$$



E.G: LAKE

WORLDSHEET OF STRING

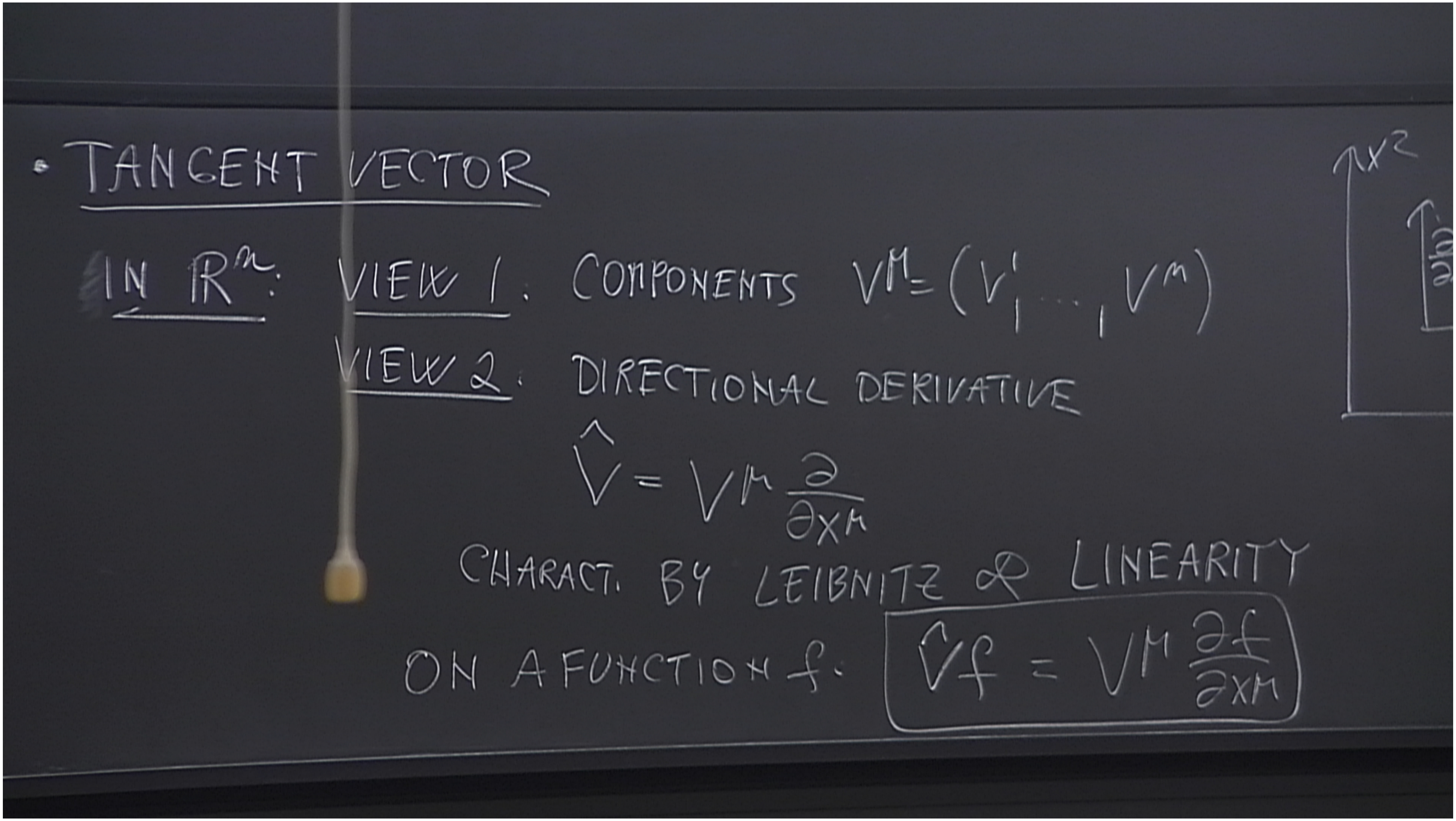


- TANGENT VECTOR

IN $\mathbb{R}^n$: VIEW 1: COMPONENTS $V^\mu = (V^1, \ldots, V^n)$

VIEW 2: DIRECTIONAL DERIVATIVE

$$\hat{V} = V^\mu \frac{\partial}{\partial x^\mu}$$



- TANGENT VECTOR

IN $\mathbb{R}^n$: VIEW 1: COMPONENTS $V^\mu = (V^1, \dots, V^n)$

VIEW 2: DIRECTIONAL DERIVATIVE

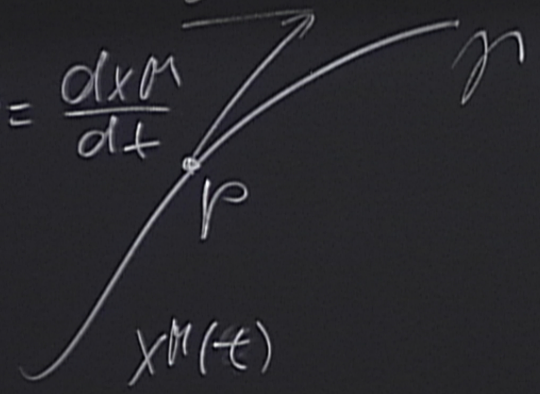
$$\hat{V} = V^\mu \frac{\partial}{\partial x^\mu}$$

CHARACT. BY LEIBNITZ & LINEARITY

ON A FUNCTION f: $\boxed{\hat{V} f = V^\mu \frac{\partial f}{\partial x^\mu}}$

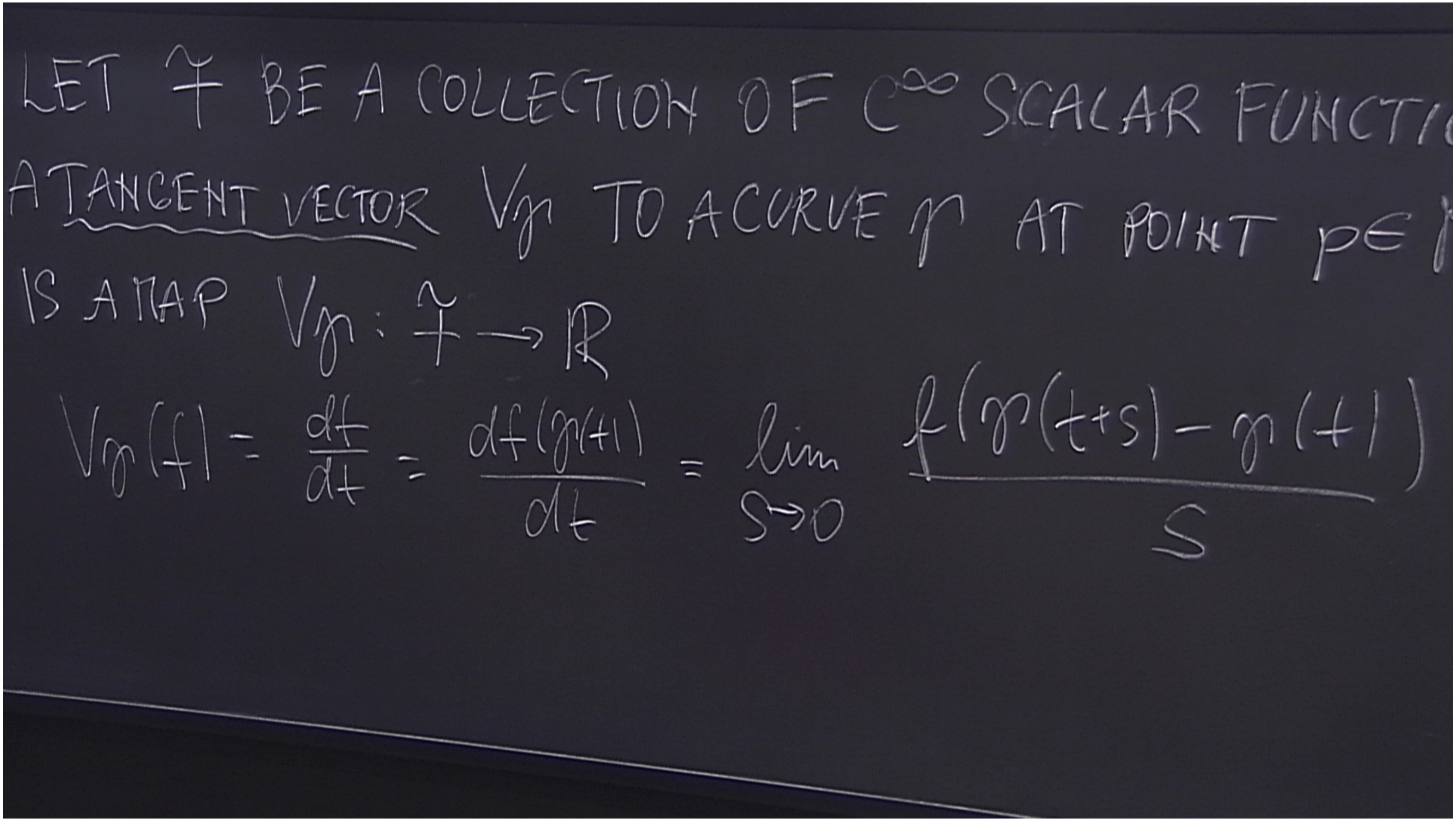
- <u>SPEC</u> TANGENT VECTOR TO A CURVE

$$V^\mu = \frac{dx^\mu}{dt}$$



$$\Rightarrow \quad \hat{V} f = \frac{dx^\mu}{dt}\frac{\partial f}{\partial x^\mu} = \frac{df}{dt}$$

"HOW f CHANGES ALONG γ AT POINT p."

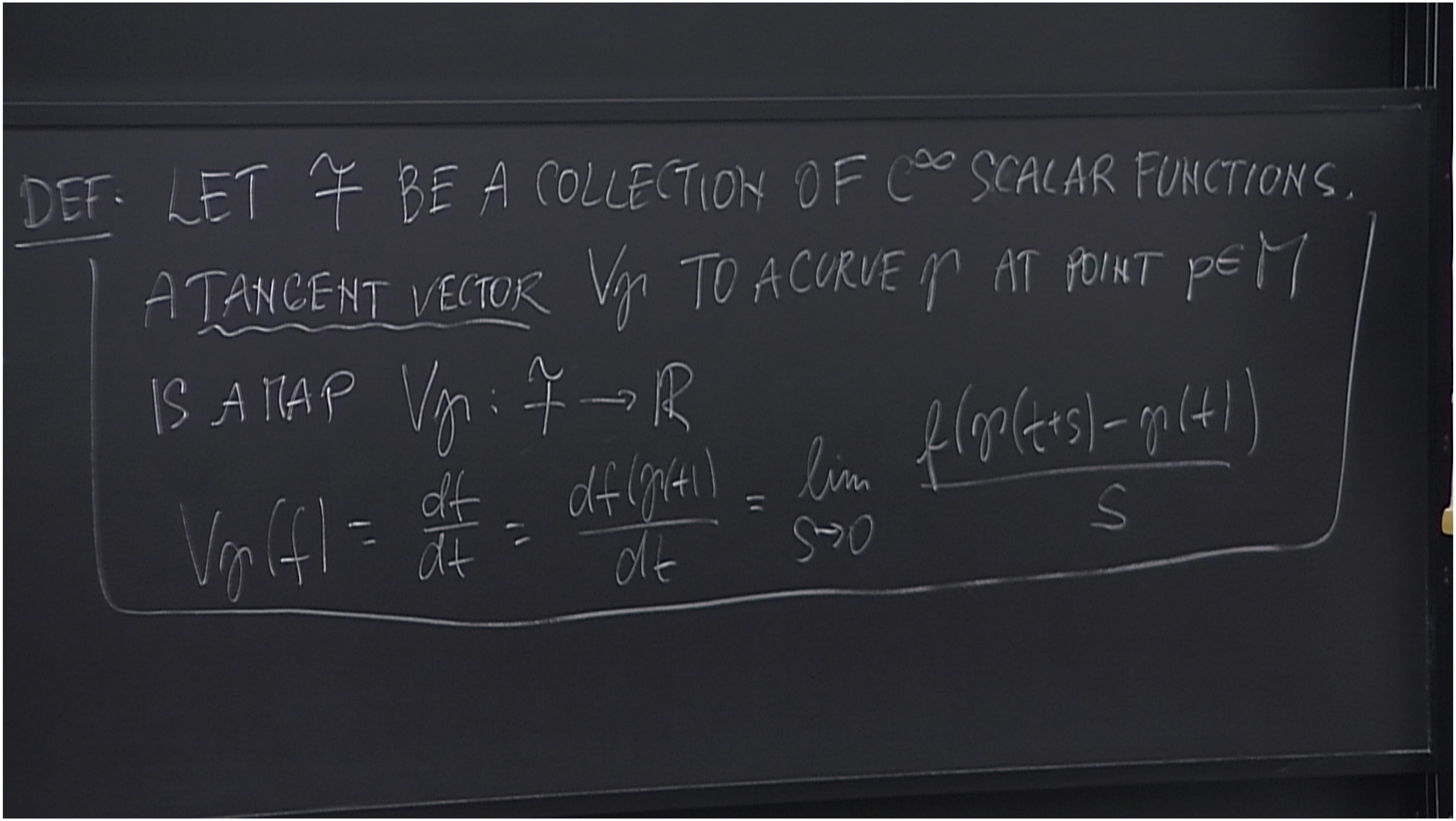


LET $\tilde{\mathcal{F}}$ BE A COLLECTION OF $C^\infty$ SCALAR FUNCTI

A TANGENT VECTOR $V_\gamma$ TO A CURVE $\gamma$ AT POINT $p \in$

IS A MAP $V_\gamma : \tilde{\mathcal{F}} \longrightarrow \mathbb{R}$

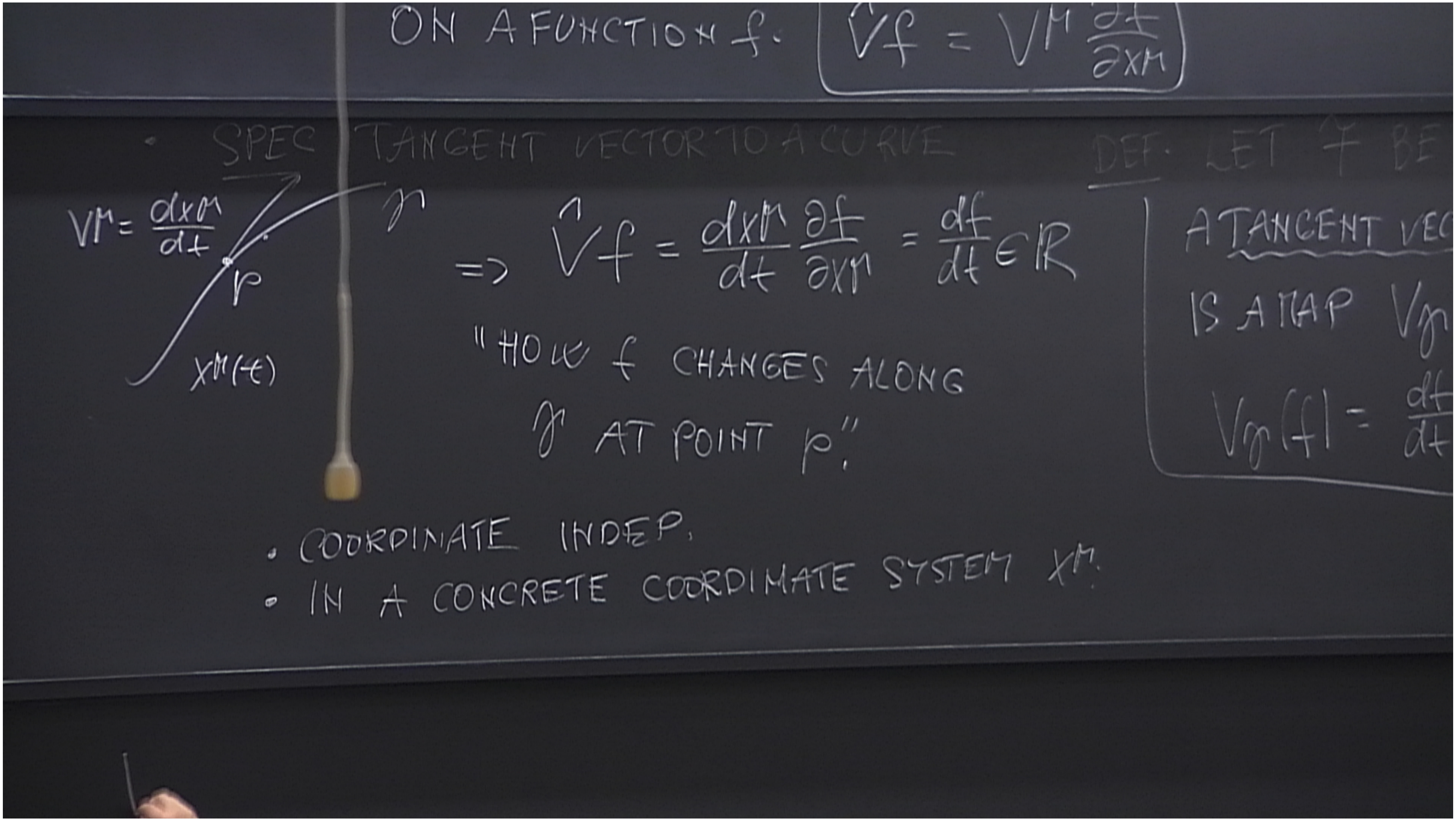
$$V_\gamma(f) = \frac{df}{dt} = \frac{df(\gamma(t))}{dt} = \lim_{s \to 0} \frac{f(\gamma(t+s) - \gamma(t))}{s}$$



**DEF:** LET $\tilde{\mathcal{F}}$ BE A COLLECTION OF $C^\infty$ SCALAR FUNCTIONS.

A TANGENT VECTOR $V_\gamma$ TO A CURVE $\gamma$ AT POINT $p \in M$ IS A MAP $V_\gamma : \tilde{\mathcal{F}} \to \mathbb{R}$

$$V_\gamma(f) = \frac{df}{dt} = \frac{df(\gamma(t))}{dt} = \lim_{s \to 0} \frac{f(\gamma(t+s) - \gamma(t))}{s}$$



ON A FUNCTION f. $\boxed{\hat{V} f = V^\mu \frac{\partial f}{\partial x^\mu}}$

• SPEC. TANGENT VECTOR TO A CURVE

$$\Rightarrow \hat{V} f = \frac{dx^\mu}{dt}\frac{\partial f}{\partial x^\mu} = \frac{df}{dt} \in \mathbb{R}$$

"HOW f CHANGES ALONG $\gamma$ AT POINT $p$."

DEF. LET F BE

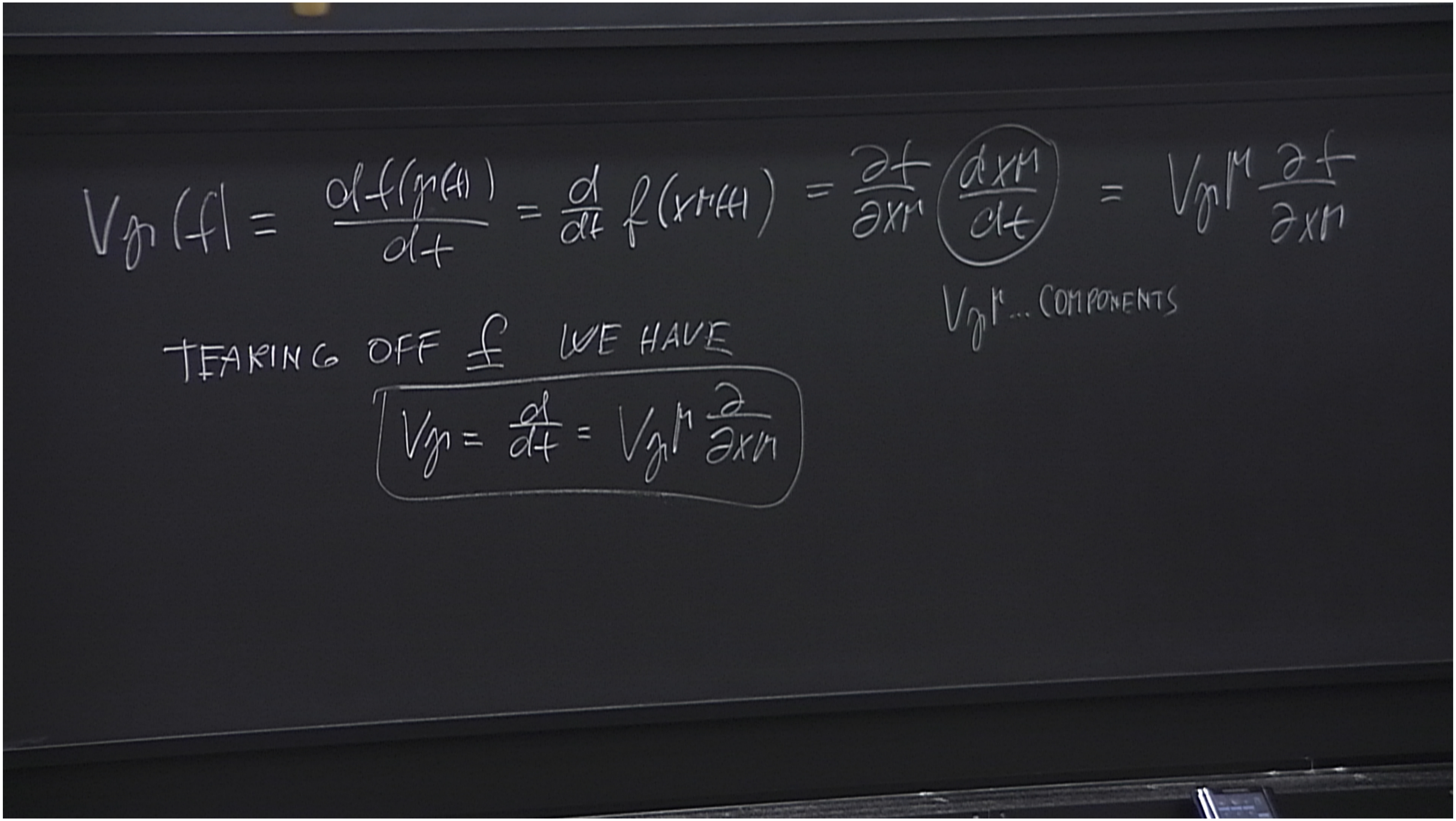
A TANGENT VEC

IS A MAP $V_\gamma$

$V_\gamma(f) = \frac{df}{dt}$

• COORDINATE INDEP.

• IN A CONCRETE COORDINATE SYSTEM $x^\mu$:



$$V_\gamma(f) = \frac{d f(\gamma(t))}{dt} = \frac{d}{dt} f(x^\mu(t)) = \frac{\partial f}{\partial x^\mu}\boxed{\frac{dx^\mu}{dt}} = V_\gamma^\mu \frac{\partial f}{\partial x^\mu}$$

$V_\gamma^\mu$... COMPONENTS

TEARING OFF $\underline{f}$ WE HAVE

$$\boxed{V_\gamma = \frac{d}{dt} = V_\gamma^\mu \frac{\partial}{\partial x^\mu}}$$

$\partial x^\mu$

TS

TOR

COMPONENTS

$$\boxed{V = \underbrace{V^\mu}_{\text{COMPONENTS}} \underbrace{\frac{\partial}{\partial x^\mu}}_{\text{COORDINATE BASIS}}}$$

$$V_\gamma(f) = \frac{d f(\gamma(t))}{dt} = \frac{d}{dt} f(x^\mu(t)) = \frac{\partial f}{\partial x^\mu} \boxed{\frac{dx^\mu}{dt}} = V_\gamma^\mu \frac{\partial f}{\partial x^\mu}$$

TEARING OFF $\underline{f}$ WE HAVE

$$\boxed{V_\gamma = \frac{d}{dt} = V_\gamma^\mu \frac{\partial}{\partial x^\mu}}$$

$V_\gamma^\mu$ ... COMPONENTS

TANGENT VECTOR

· CONSIDERING ALL POSSIBLE CURVES

COMPONENTS

$$\boxed{V = \boxed{V^\mu} \boxed{\frac{\partial}{\partial x^\mu}}}$$

• ONE CAN SHOW:

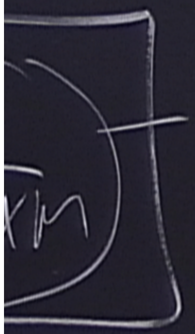
i) LINEARITY: $V(af+bg)=aV(f)+bV(g)$

$a,b\in\mathbb{R}$

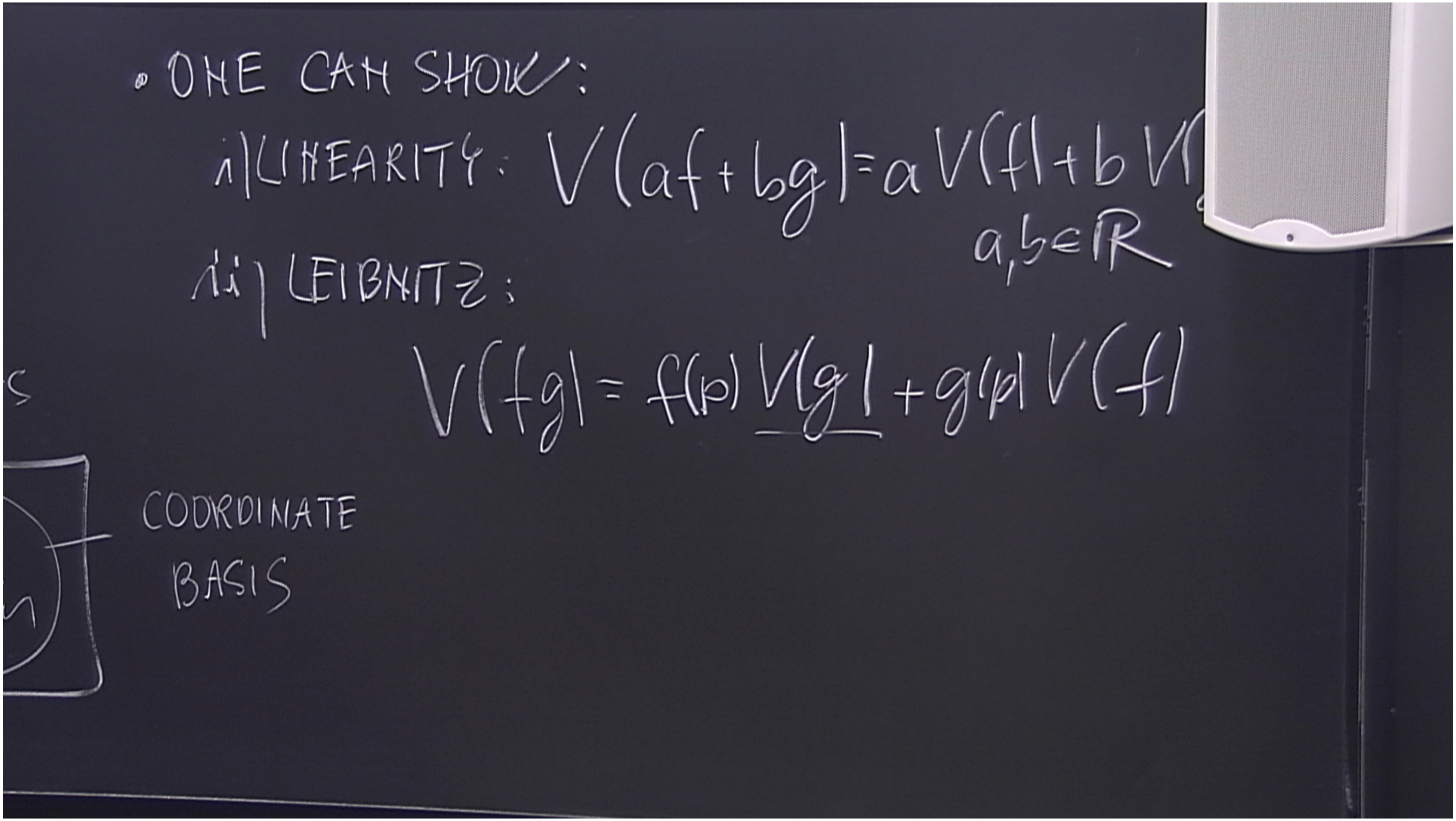
ii) LEIBNITZ:

$V(fg)=$

TS



COORDINATE BASIS



- ONE CAN SHOW:

i) LINEARITY: $V(af+bg)=aV(f)+bV($

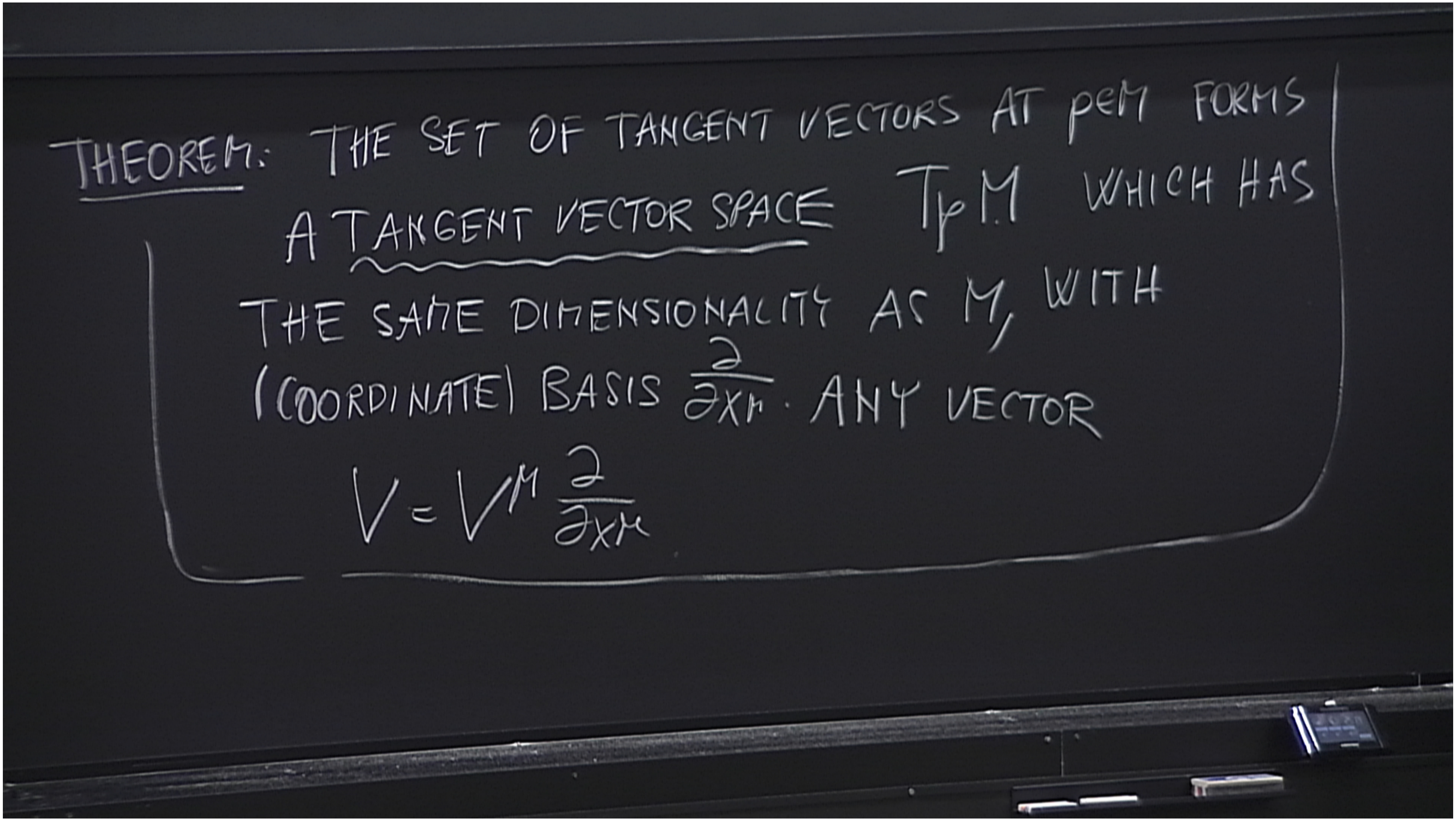
$a,b\in\mathbb{R}$

ii) LEIBNITZ:

$$V(fg)=f(p)\,\underline{V(g)}+g(p)\,V(f)$$

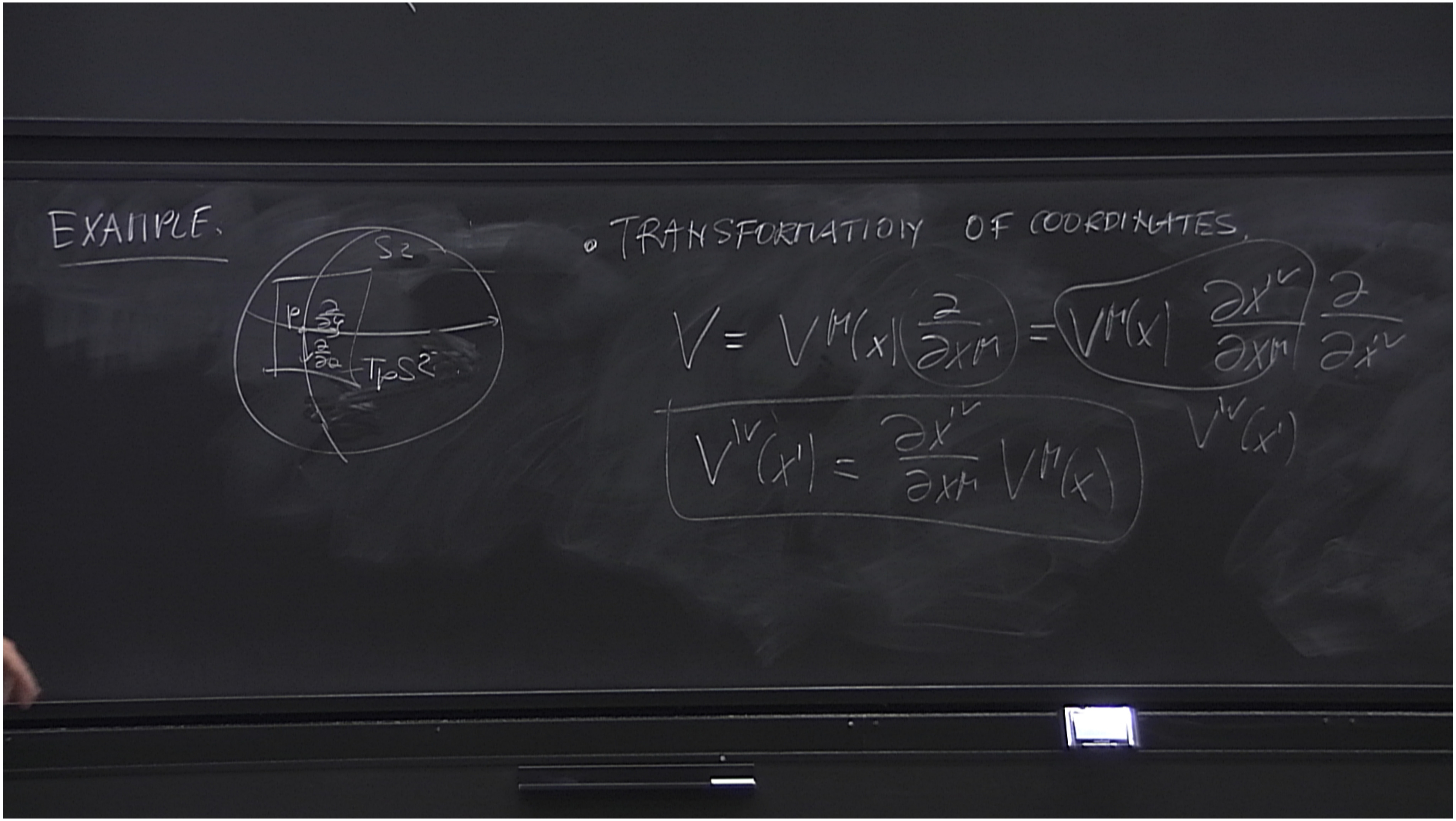
S

COORDINATE BASIS



**THEOREM:** THE SET OF TANGENT VECTORS AT $p \in M$ FORMS A TANGENT VECTOR SPACE $T_p M$ WHICH HAS THE SAME DIMENSIONALITY AS $M$, WITH (COORDINATE) BASIS $\frac{\partial}{\partial x^\mu}$. ANY VECTOR

$$V = V^\mu \frac{\partial}{\partial x^\mu}$$



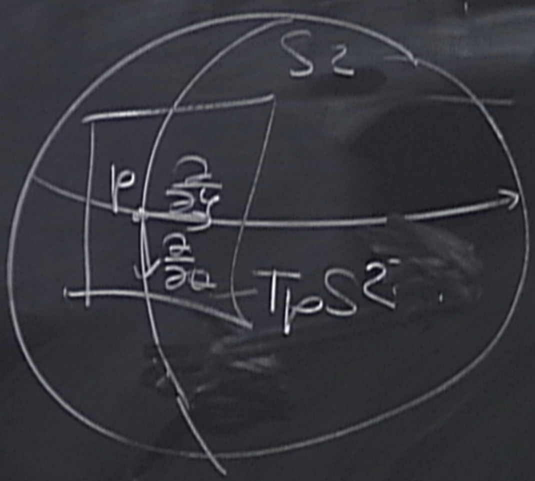
EXAMPLE.

• TRANSFORMATION OF COORDINATES.

$$V = V^\mu(x)\frac{\partial}{\partial x^\mu} = \underbrace{V^\mu(x)\frac{\partial x'^\nu}{\partial x^\mu}}_{V'^\nu(x')}\frac{\partial}{\partial x'^\nu}$$

$$\boxed{V'^\nu(x') = \frac{\partial x'^\nu}{\partial x^\mu}V^\mu(x)}$$

EXAMPLE.



- TRANSFORMATION OF CO

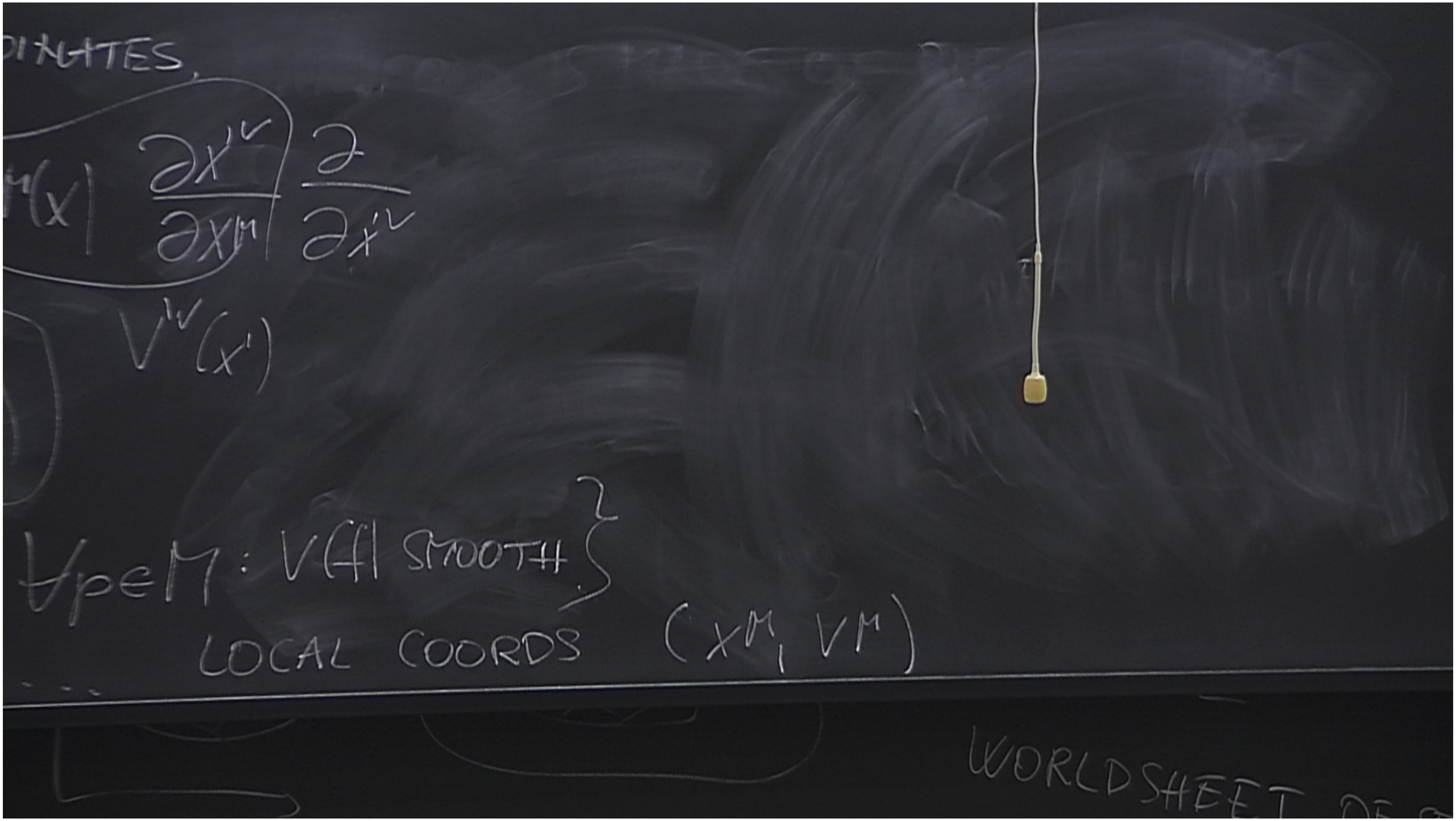
$$V = V^\mu(x) \frac{\partial}{\partial x^\mu} =$$

$$\boxed{V'^\nu(x') = \frac{\partial x'^\nu}{\partial x^\mu} V^\mu}$$

- EXTEND TO THE WHOLE MAN.

DEF: TANGENT VECTOR $V \equiv \{ V_p \in T_pM$

TANGENT BUNDLE $TM = \bigcup_p T_pM$



DINATES,

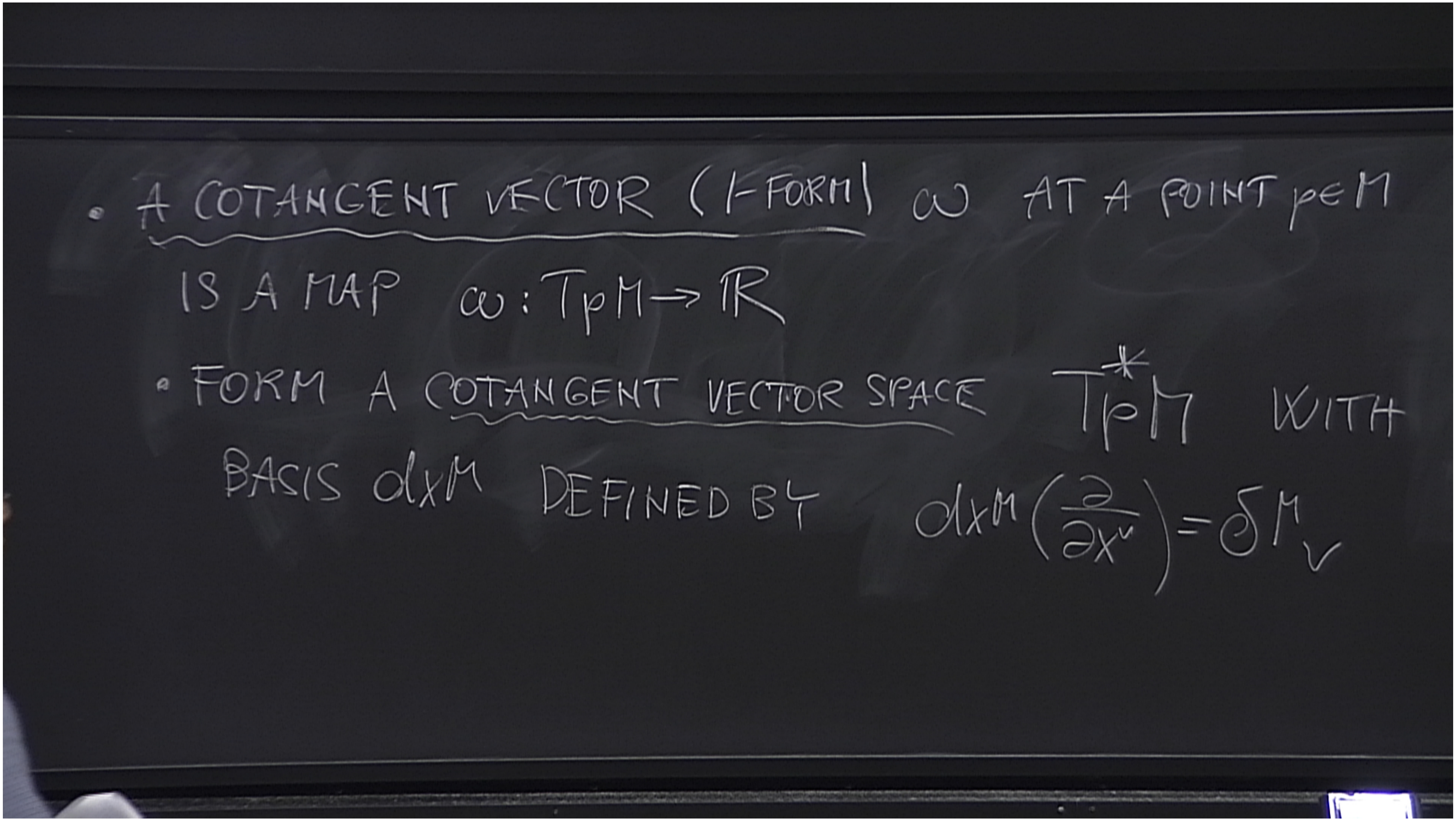
$$^\mu(x)\,\frac{\partial x'^\nu}{\partial x^\mu}\,\frac{\partial}{\partial x'^\nu}$$

$$V'^\nu(x')$$

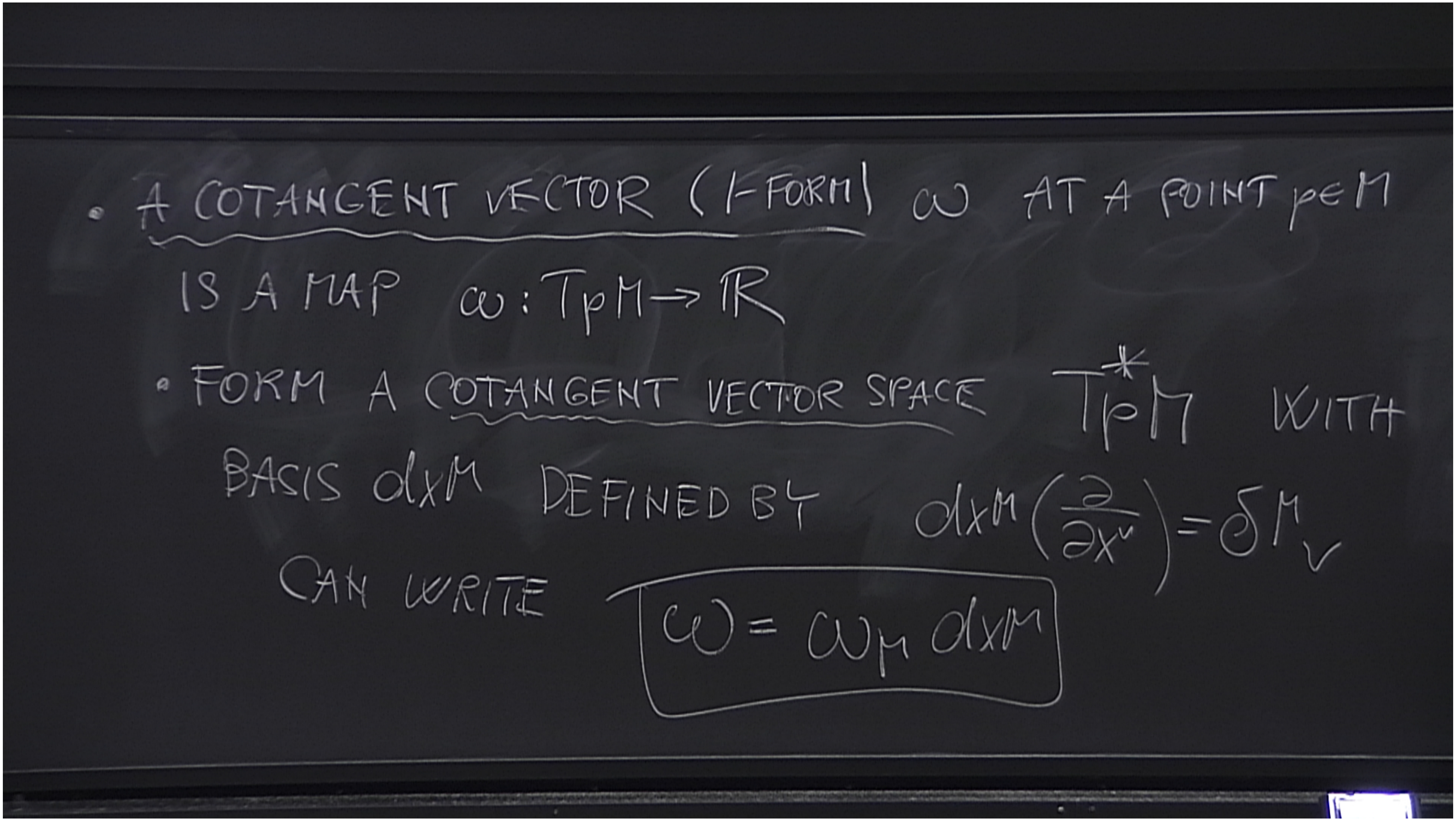
$\forall p \in M : V(p)$ SMOOTH $\}$

LOCAL COORDS $(x^\mu, V^\mu)$

WORLDSHEET OF



- A COTANGENT VECTOR (1-FORM) $\omega$ AT A POINT $p \in M$ IS A MAP $\omega : T_pM \to \mathbb{R}$
- FORM A COTANGENT VECTOR SPACE $T_p^*M$ WITH BASIS $dx^\mu$ DEFINED BY $dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu$



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- FORM A COTANGENT VECTOR SPACE $T_p^*M$ WITH BASIS $dx^\mu$ DEFINED BY $dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu$

CAN WRITE $\boxed{\omega = \omega_\mu \, dx^\mu}$

$T\ p \in M$

- CHANGIN COORDINATES

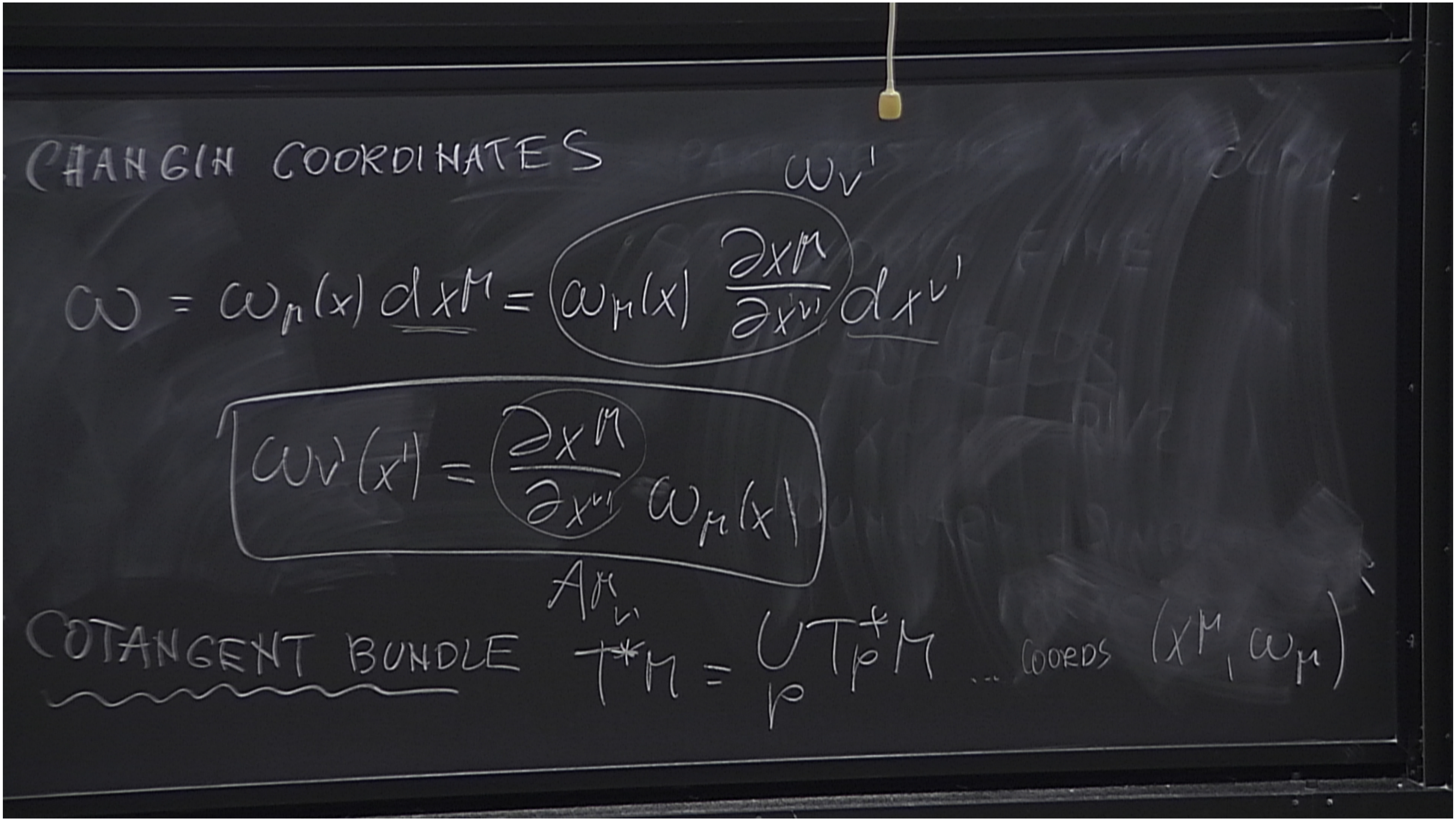
$$\omega = \omega_\mu(x)\,\underline{dx^\mu} = \overbrace{\omega_\mu(x)\,\frac{\partial x^\mu}{\partial x'^\nu}}^{\omega_\nu'}\,\underline{dx'^\nu}$$

WITH

$$\boxed{\omega_\nu'(x') = \underbrace{\frac{\partial x^\mu}{\partial x'^\nu}}_{A^\mu{}_\nu}\,\omega_\mu(x)}$$

$\delta^\mu{}_\nu$

- COTANGENT BUNDLE $T^*M = \bigcup_p T_p^*M$

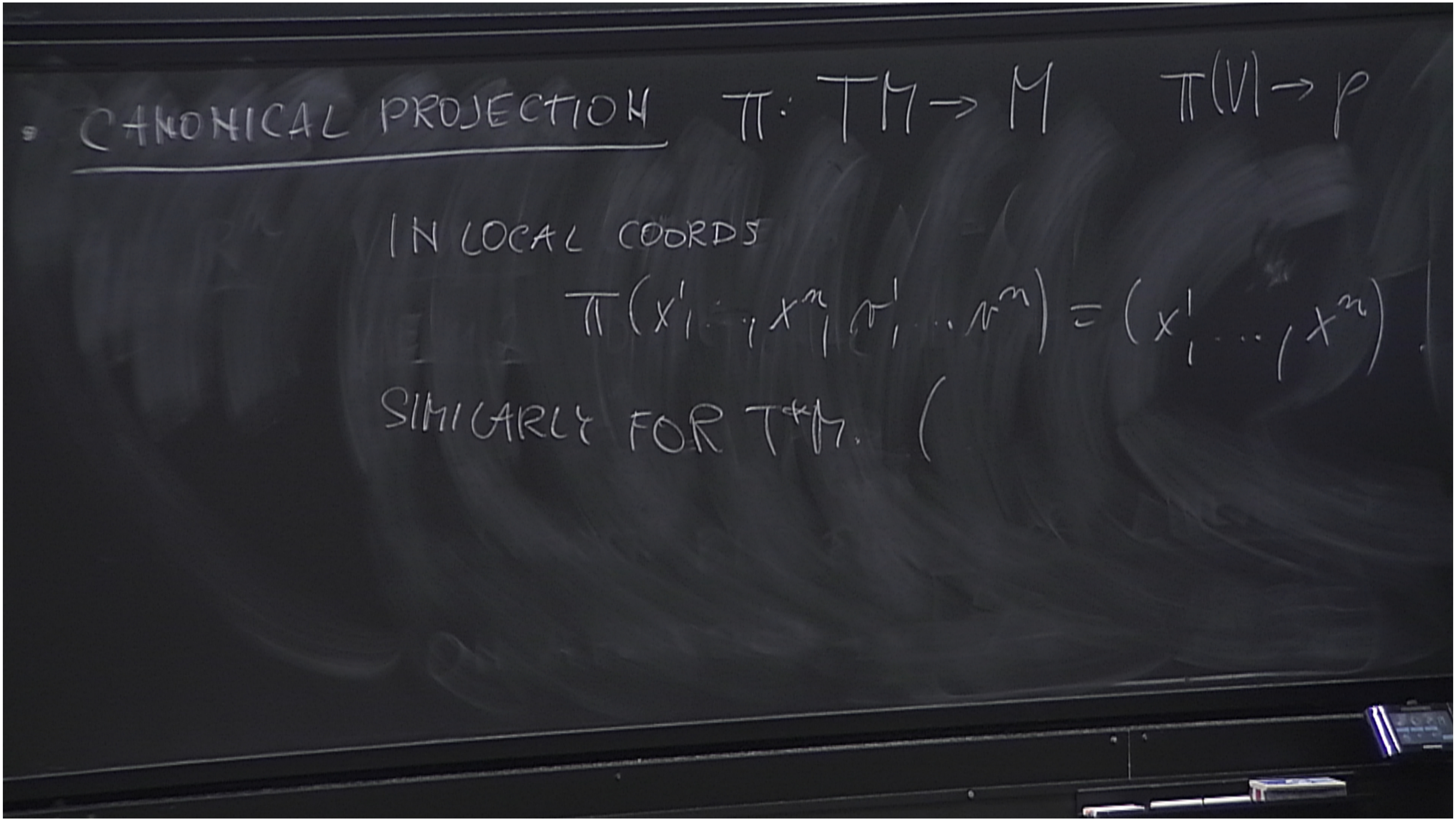


CHANGIN COORDINATES

$$\omega = \omega_\mu(x)\, \underline{dx^\mu} = \overset{\omega_{\nu'}}{\boxed{\omega_\mu(x) \frac{\partial x^\mu}{\partial x^{\nu'}}}}\, \underline{dx^{\nu'}}$$

$$\boxed{\omega_{\nu'}(x') = \underset{A^\mu{}_{\nu'}}{\boxed{\frac{\partial x^\mu}{\partial x^{\nu'}}}}\, \omega_\mu(x)}$$

COTANGENT BUNDLE $T^*M = \bigcup_p T_p^*M$ ... COORDS $(x^\mu, \omega_\mu)$

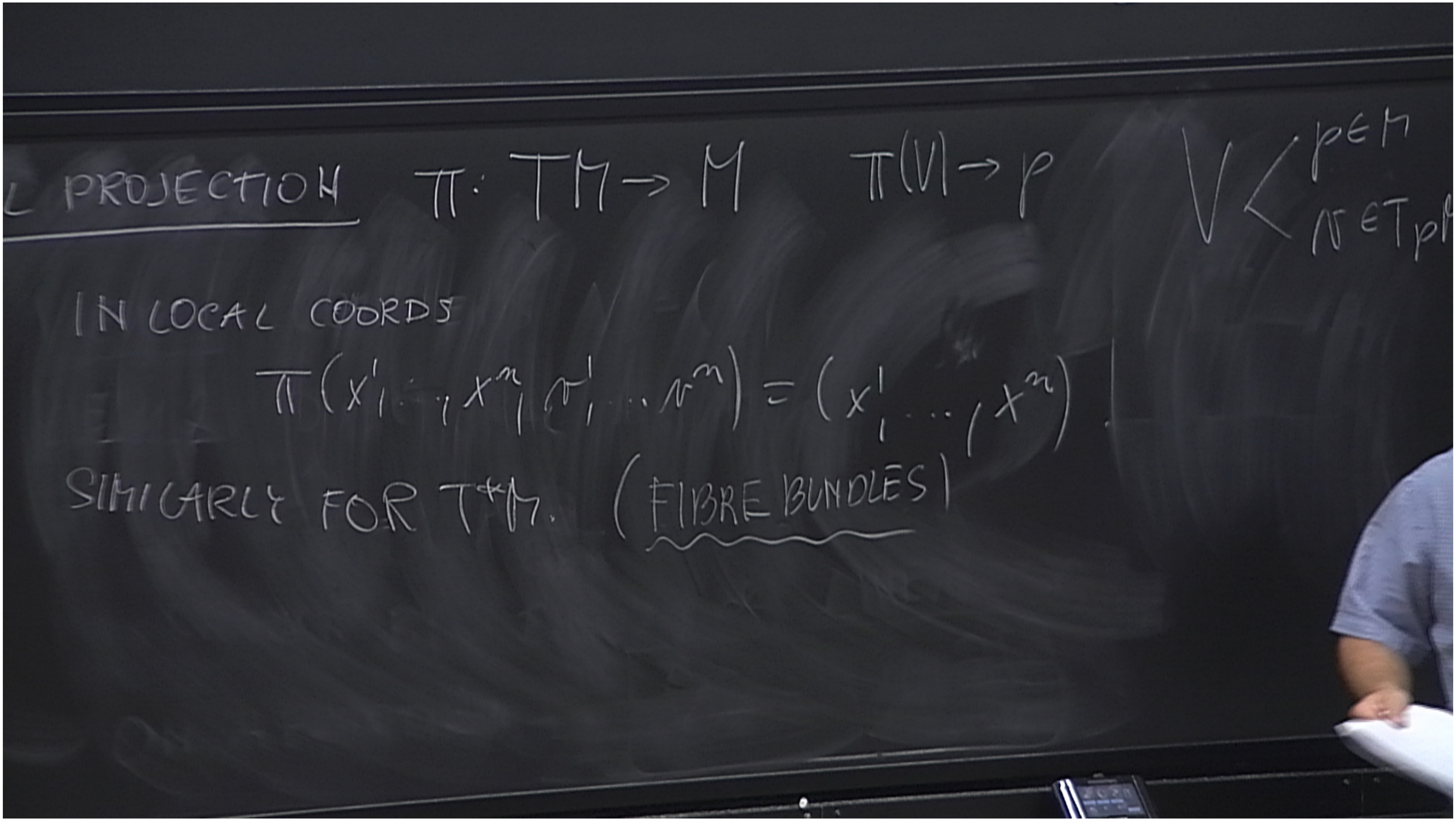


- CANONICAL PROJECTION $\pi : TM \to M$ $\quad \pi(V) \to p$

IN LOCAL COORDS

$$\pi(x^1,\dots,x^n, v^1,\dots,v^n) = (x^1,\dots,x^n).$$

SIMILARLY FOR $T^*M$. (

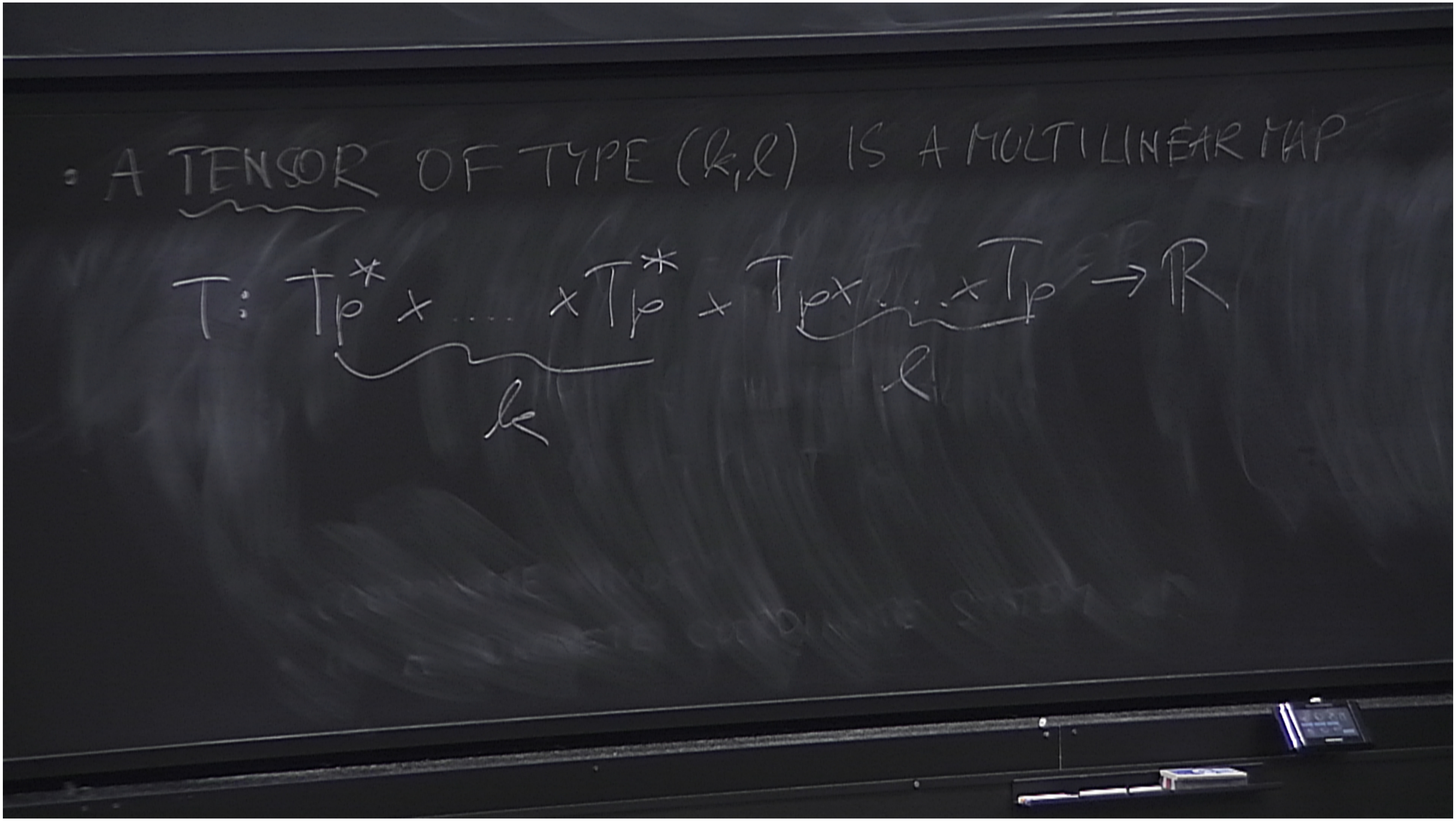


L PROJECTION $\pi : TM \to M$ $\quad \pi(v) \to p$ $\quad v < \begin{matrix} p \in M \\ v \in T_p M \end{matrix}$

IN LOCAL COORDS

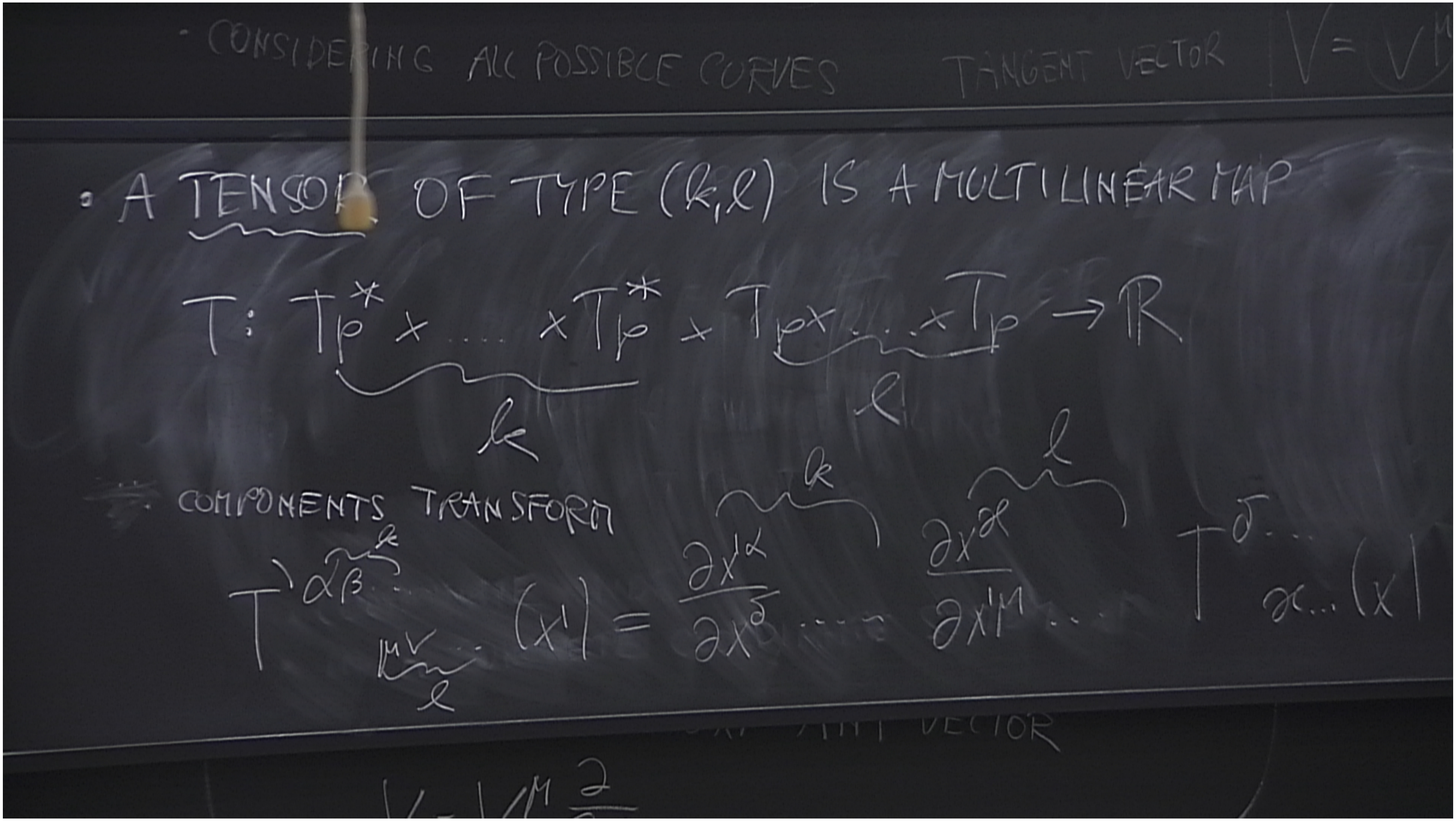
$$\pi(x^1, \ldots, x^n, v^1, \ldots v^n) = (x^1, \ldots, x^n).$$

SIMILARLY FOR $T^*M$. (FIBRE BUNDLES)



- A TENSOR OF TYPE $(k,l)$ IS A MULTILINEAR MAP

$$T: \underbrace{T_p^* \times \dots\dots \times T_p^*}_{k} \times \underbrace{T_p \times \dots \times T_p}_{l} \to \mathbb{R}$$



· CONSIDERING ALL POSSIBLE CURVES TANGENT VECTOR $V = V^\mu$

· A TENSOR OF TYPE $(k,l)$ IS A MULTILINEAR MAP

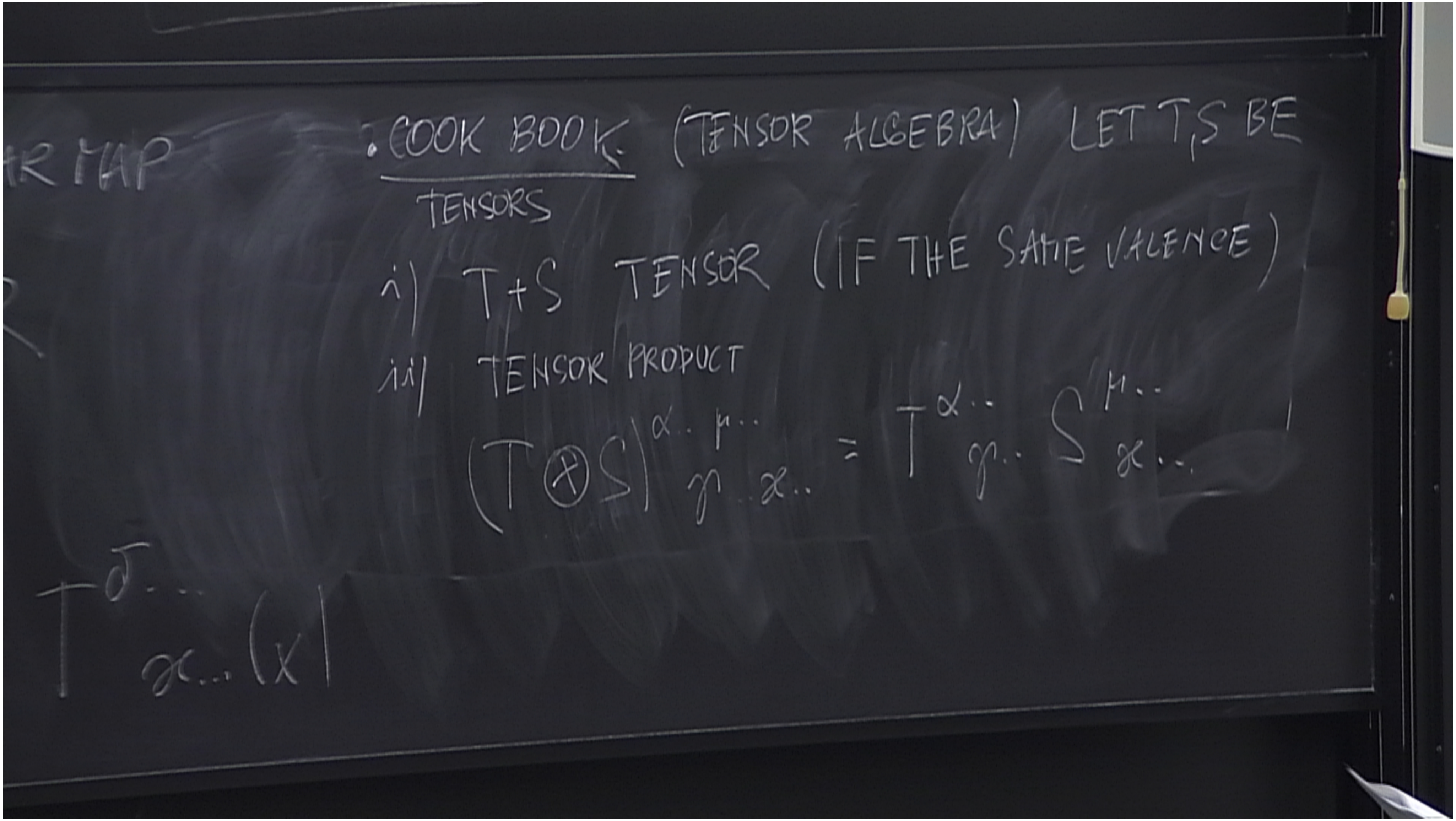
$$T: \underbrace{T_p^* \times \dots \times T_p^*}_{k} \times \underbrace{T_p \times \dots \times T_p}_{l} \rightarrow \mathbb{R}$$

· COMPONENTS TRANSFORM

$$T'^{\overbrace{\alpha\beta\dots}^{k}}{}_{\underbrace{\mu\nu\dots}_{l}}(x') = \overbrace{\frac{\partial x'^\alpha}{\partial x^\delta}\dots}^{k} \overbrace{\frac{\partial x^\sigma}{\partial x'^\mu}\dots}^{l} T^{\delta\dots}{}_{\sigma\dots}(x)$$

ANY VECTOR

$V = V^\mu \frac{\partial}{}$



AR MAP

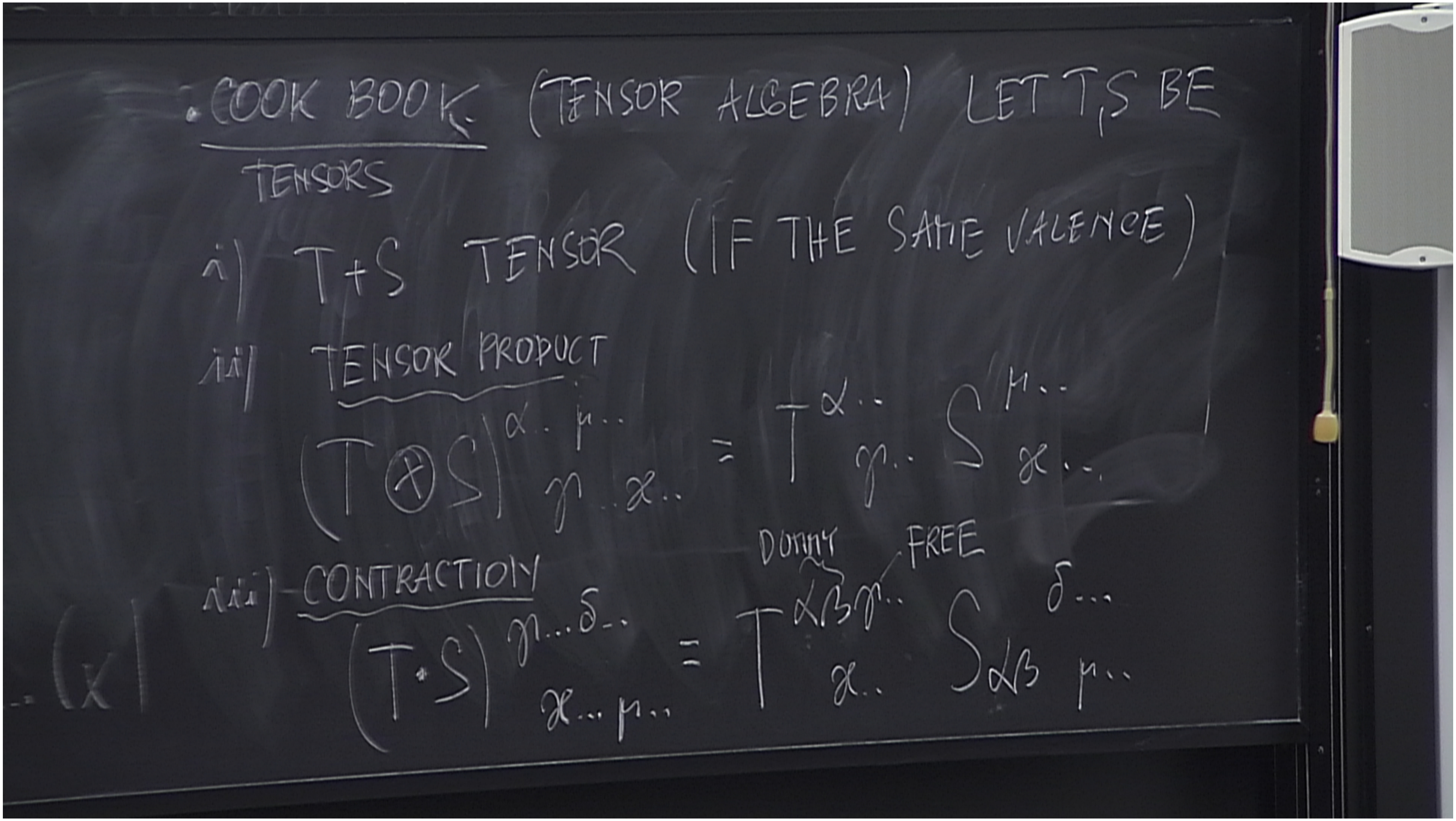
• COOK BOOK (TENSOR ALGEBRA) LET T, S BE TENSORS

i) T+S TENSOR (IF THE SAME VALENCE)

ii) TENSOR PRODUCT

$$(T\otimes S)^{\alpha..\,\mu..}{}_{\gamma..\,\chi..} = T^{\alpha..}{}_{\gamma..}\,S^{\mu..}{}_{\chi..}$$

$$T^{\delta\ldots}{}_{\chi\ldots}(x)$$



• COOK BOOK: (TENSOR ALGEBRA) LET T, S BE TENSORS

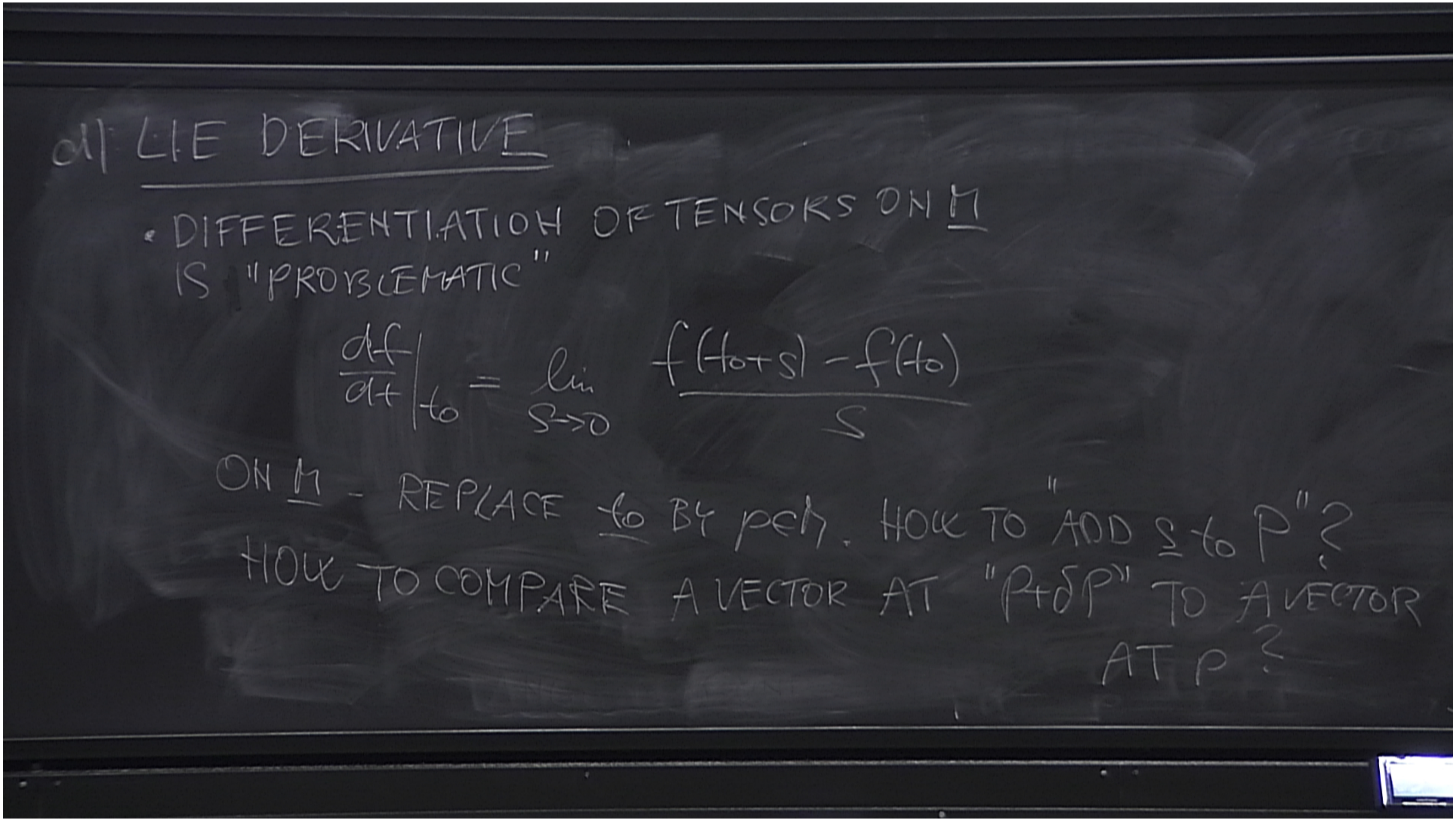
i) T+S TENSOR (IF THE SAME VALENCE)

ii) TENSOR PRODUCT

$$(T\otimes S)^{\alpha\ldots\mu\ldots}{}_{\gamma\ldots\chi\ldots} = T^{\alpha\ldots}{}_{\gamma\ldots}\, S^{\mu\ldots}{}_{\chi\ldots}$$

iii) CONTRACTION

$$(T\cdot S)^{\gamma\ldots\delta\ldots}{}_{\chi\ldots\mu\ldots} = T^{\overbrace{\alpha\beta}^{\text{DUMMY}}\overset{\text{FREE}}{\gamma}\ldots}{}_{\chi\ldots}\, S^{\delta\ldots}{}_{\alpha\beta\,\mu\ldots}$$



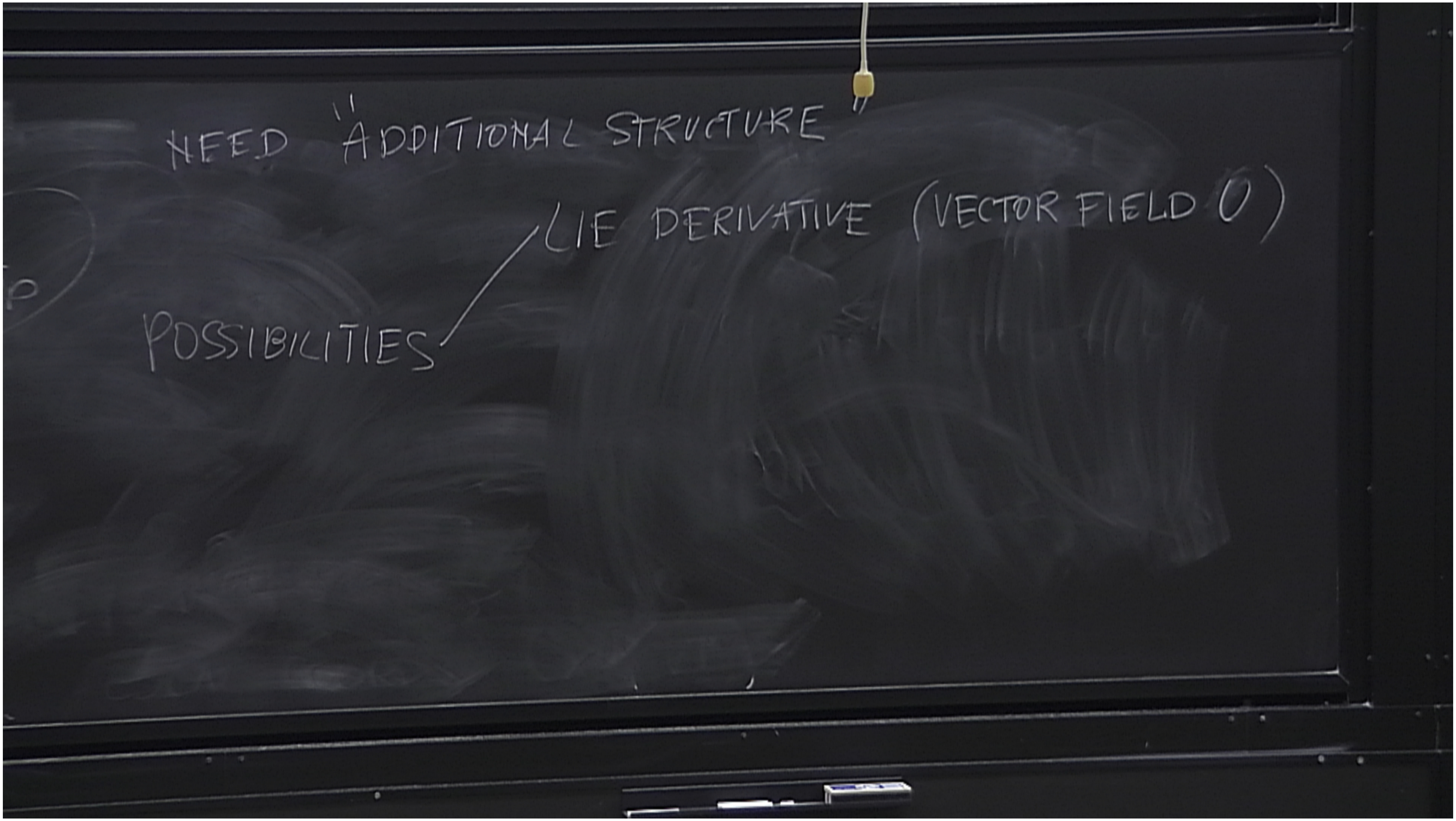
## d) LIE DERIVATIVE

- DIFFERENTIATION OF TENSORS ON $\underline{M}$ IS "PROBLEMATIC"

$$\frac{df}{dt}\Big|_{t_0} = \lim_{s\to 0} \frac{f(t_0+s)-f(t_0)}{s}$$

ON $\underline{M}$ - REPLACE $\underline{t_0}$ BY $p \in M$. HOW TO "ADD $\underline{s}$ to $p$"?

HOW TO COMPARE A VECTOR AT "$p+\delta p$" TO A VECTOR AT $p$?



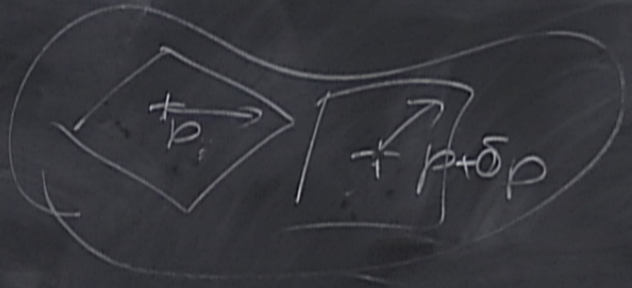
NEED "ADDITIONAL STRUCTURE"

LIE DERIVATIVE (VECTOR FIELD 0)

POSSIBILITIES

KS ON $\underline{M}$

$$\frac{\ldots) - f(t_0)}{\delta}$$

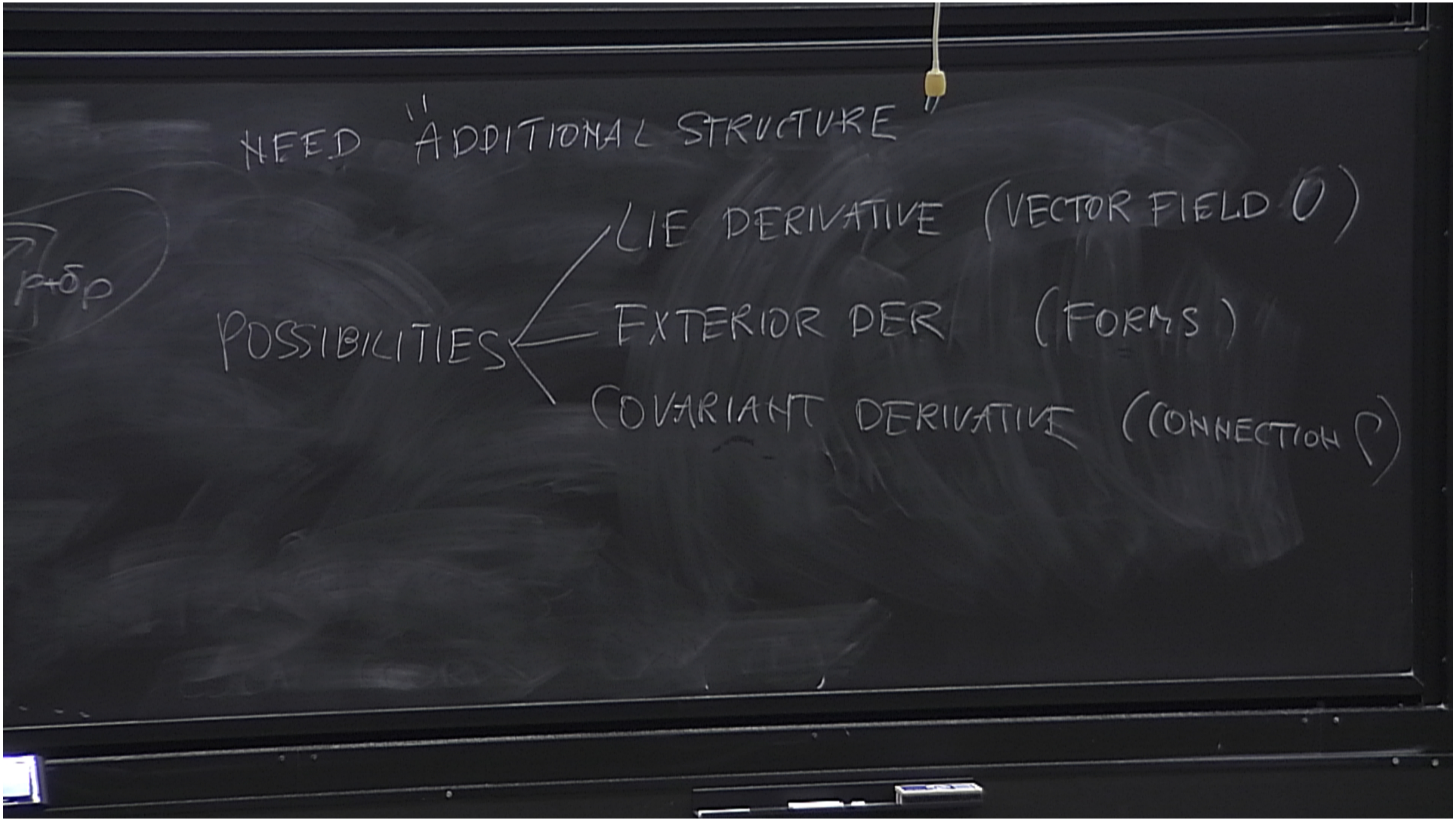


NEED "ADDITIONAL STRUCT

POSSIBILITIES — LIE DER

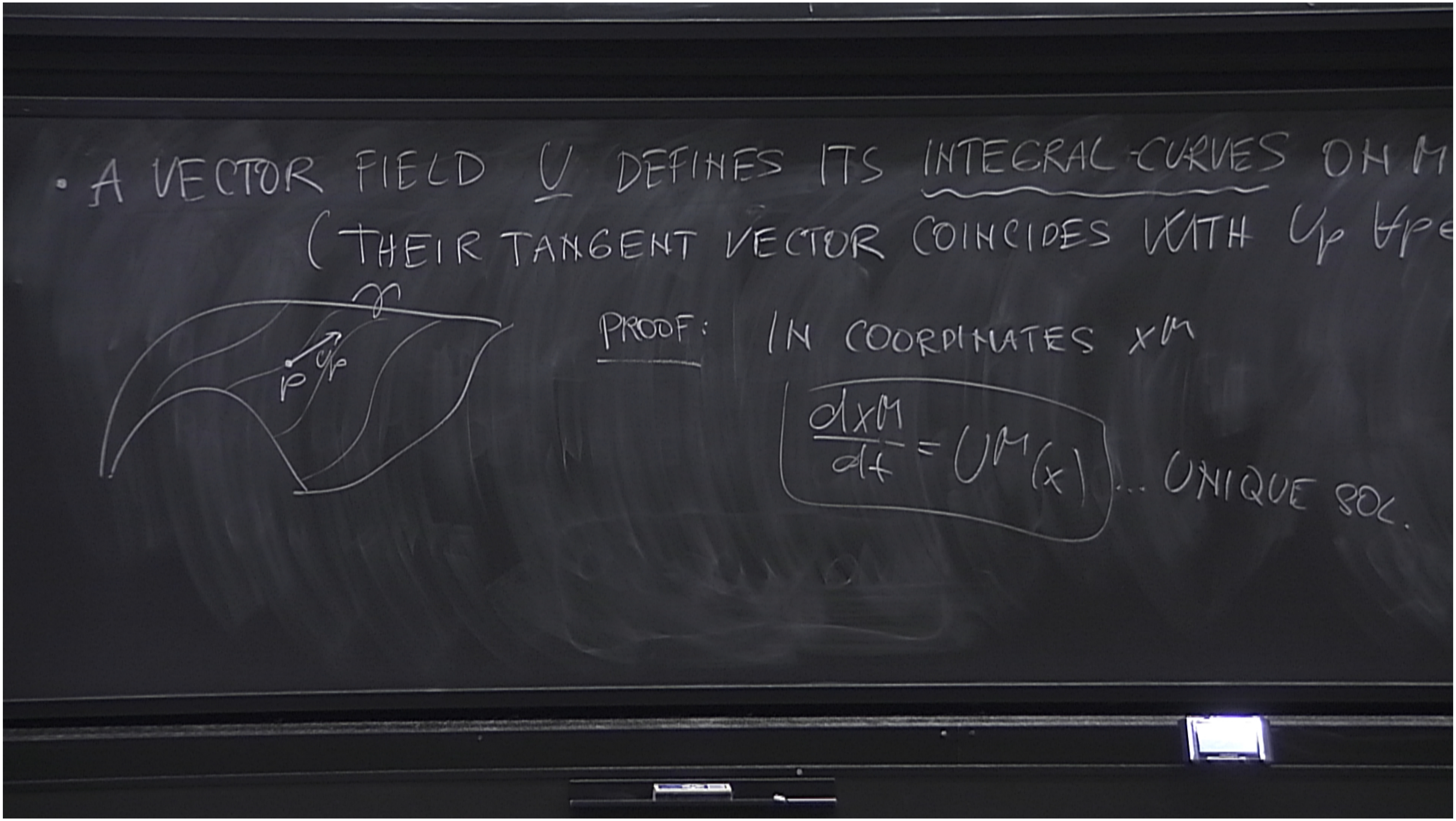
h. HOW TO "ADD $\underline{v}$ to P"?

TOR AT "P+δP" TO A VECTOR AT P?



NEED "ADDITIONAL STRUCTURE"

POSSIBILITIES
- LIE DERIVATIVE (VECTOR FIELD $0$)
- EXTERIOR DER (FORMS)
- COVARIANT DERIVATIVE (CONNECTION $\Gamma$)

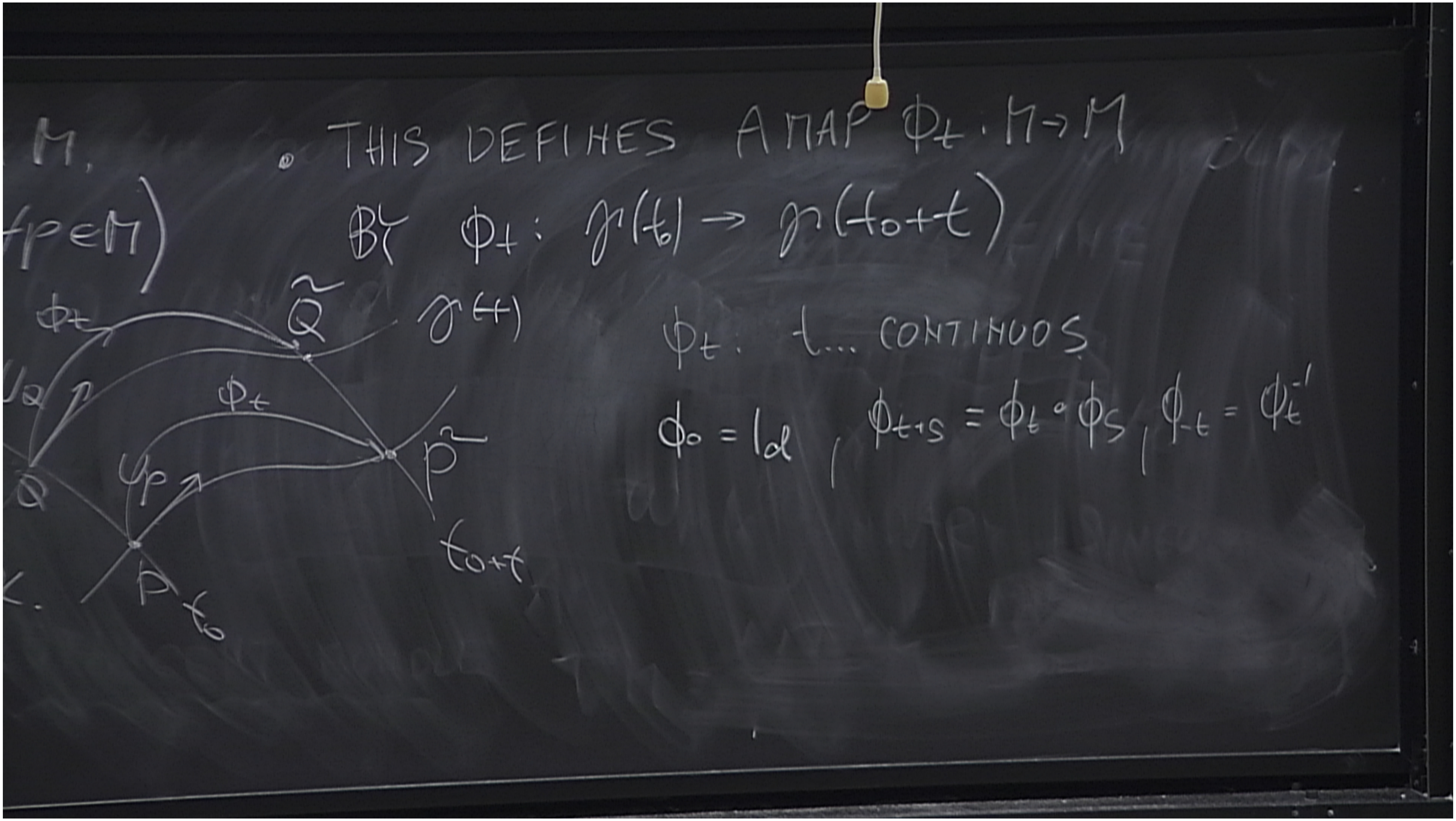


- A VECTOR FIELD $\underline{U}$ DEFINES ITS INTEGRAL CURVES ON M

(THEIR TANGENT VECTOR COINCIDES WITH $U_p$ $\forall p \in$

PROOF: IN COORDINATES $x^\mu$

$$\boxed{\frac{dx^\mu}{dt} = U^\mu(x)} \ldots \text{UNIQUE SOL.}$$



M,

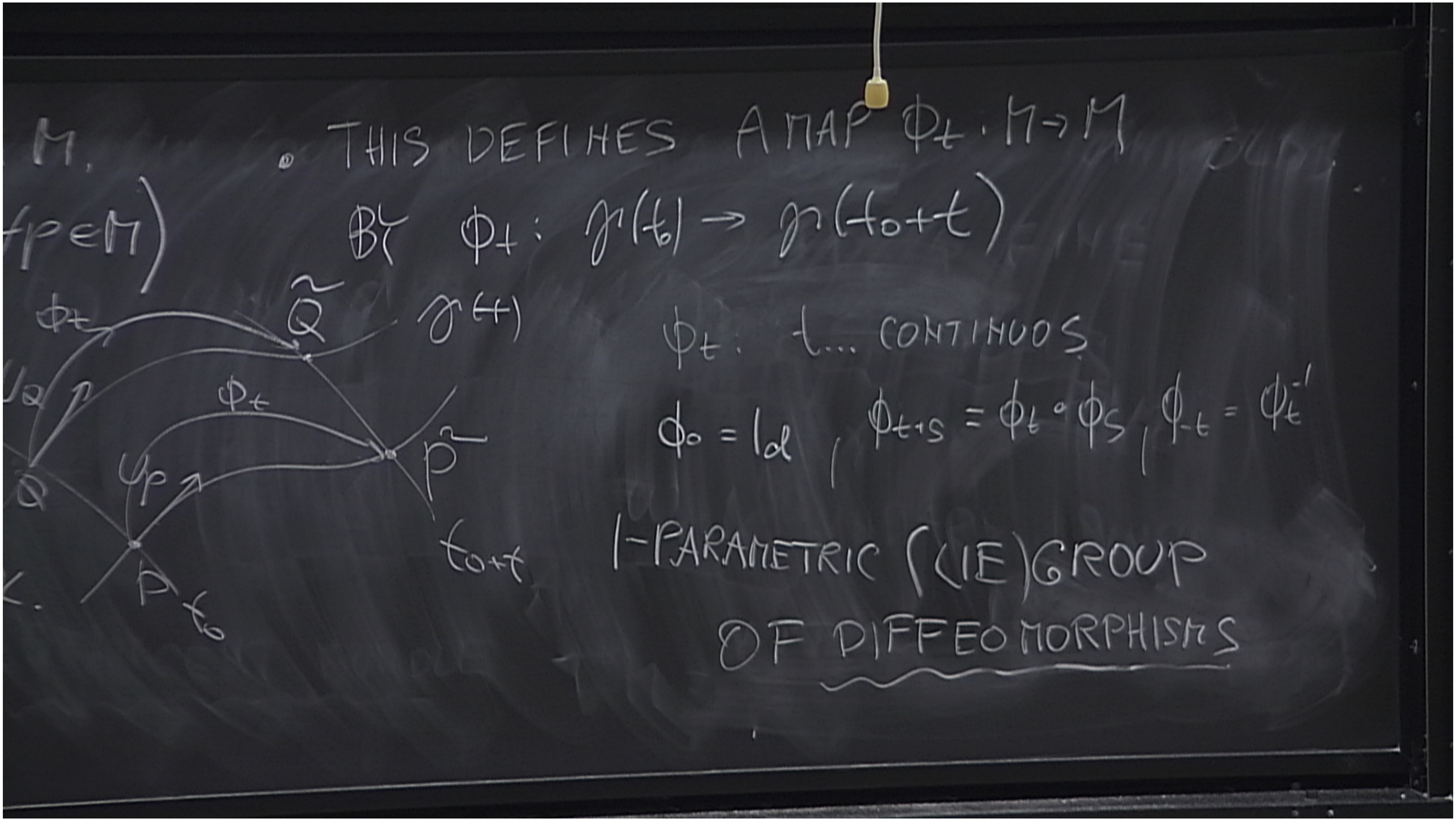
- THIS DEFINES A MAP $\phi_t : M \to M$

$p \in M)$

BY $\phi_t : \gamma(t_0) \to \gamma(t_0+t)$

$\phi_t$ : t... CONTINUOS

$\phi_0 = Id$ , $\phi_{t+s} = \phi_t \circ \phi_s$ , $\phi_{-t} = \phi_t^{-1}$



M,

∀p∈M)

k.

- THIS DEFINES A MAP $\phi_t : M \to M$

BY $\phi_t : \gamma(t_0) \to \gamma(t_0+t)$

$\phi_t$ : t... CONTINUOS

$\phi_0 = Id$ , $\phi_{t+s} = \phi_t \circ \phi_s$ , $\phi_{-t} = \phi_t^{-1}$

1-PARAMETRIC (LIE) GROUP

OF DIFFEOMORPHISMS