Title: PSI 2015/2016 Classical Mechanics 1 - David Kubiznak

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Abstract:

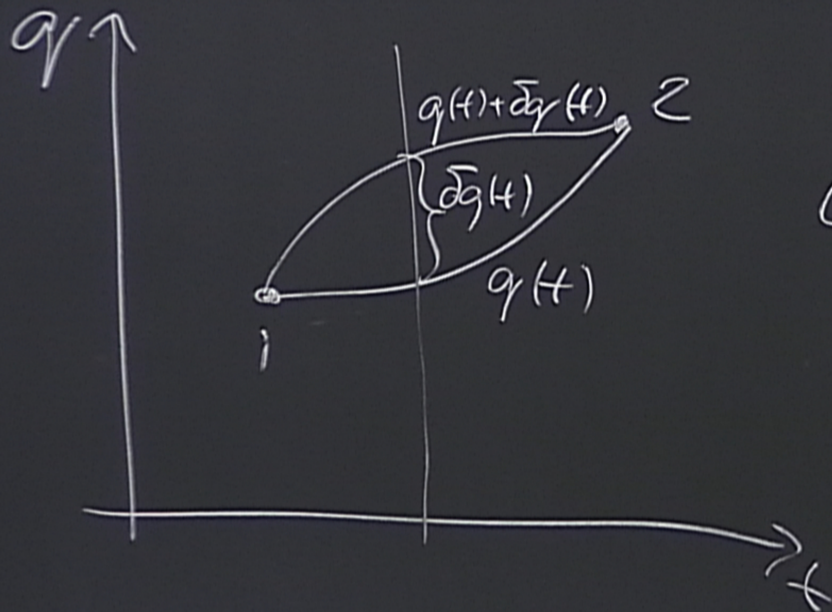
(HAMILTON (1805 – 1865))

TIME INTERVAL $t \in (t_1, t_2)$ (1833–34)

HE ACTION FUNCTIONAL

... DEFINITE F. CHAR. SYSTEM

- FIXED END POINTS $\delta q(t_1) = 0 = \delta q(t_2)$



EXTREMUM:

$$\delta S = S[q(t)+\delta q(t)] - S[q(t)] = 0$$

$$\delta S = \int_{t_1}^{t_2} \delta L \, dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = \left| \delta \frac{d}{dt} = \frac{d}{dt} \delta \right|$$

$$= \int_{t_1}^{t_2} \underbrace{\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right]}_{0} \delta q \, dt + \left[\underbrace{\frac{\partial L}{\partial \dot{q}} \delta q}_{0} \right]_1^2 = 0$$

$$\boxed{\frac{\partial L}{\partial q^I} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^I} \right) = 0} \quad (E\text{-}L)$$

EULER-LAGRANGE EQUATION

CHOOSE PROP.

3 REMARKS:

i) $L = L(q, \dot{q}, t)$ $\Rightarrow$ (E-L) .. 2ND ORDER. EOM

ii) FOR MECHANICAL SYSTEMS $L = \underset{\text{KIN}}{T} - \underset{\text{POT}}{V}$ (E.G. $L \propto L$)

iii) FREEDOM IN L:

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \int \frac{d\Lambda(q, t)}{dt}$$

$\Rightarrow$ THE SAME EOM.

b) INTEGRALS OF MOTION

SOLVING (E-L) → NUMERICALLY

SOLVING (E-L) → APPROXIMATELLY

SOLVING (E-L) → ANALYTIC (USING 1st INTEGRALS OF MOTION)

DEF: INTEGRAL OF MOTION $\equiv$ FONCTION $I(q, \dot{q}, t)$ S.T.

$$\frac{dI}{dt} = 0 \quad \text{FOR ANY } q(t) \text{ SOLVING EOM}$$

NO

$\frac{}{dt} = 0$ FOR $q(t)$ SOLVING EOM

NOETHER'S THEOREM (E. NOETHER (1882-1935))

FOR EVERY GLOBAL CONTINUOUS SYMMETRY OF THE SYSTEM, THERE IS A CORRESPONDING INTEGRAL OF MOTION

"GEOMETRIZATION OF PHYSICS"

2 SIMPLE EXAMPLES:

i) $L \neq L(t)$ EXPL. $\Rightarrow$ $\boxed{E = \frac{\partial L}{\partial \dot{q}^I}\dot{q}^I - L}$ GENER. ENERGY

E.S.C

PROOF: $$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \cancel{\frac{\partial L}{\partial \dot{q}}\ddot{q}} - \frac{\partial L}{\partial q}\dot{q} - \cancel{\frac{\partial L}{\partial \dot{q}}\ddot{q}} \overset{(E-L)}{=} 0.$$

ii) LET $L \neq L(q_c)$ $\Rightarrow$ $\boxed{p = \frac{\partial L}{\partial \dot{q}_c}}$ GEN. MOMENTUM

CYCLIC

2 REMARKS ON TERMINOLOGY:

i) ON-SHELL $\equiv$ PROVIDED EOM ARE SATISFIED

OFF-SHELL $\equiv$ IRRESPECTIVE OF EOM.

ii) TYPES OF SYMMETRIES

CONSIDER: A TRANSF:

$$\boxed{\begin{aligned} t &\to t' = t + \tilde{\delta} t \\ q &\to q'_{t'} = q_t + \tilde{\delta} q_t \end{aligned}} \qquad (\tilde{\delta})$$

AND WRITE

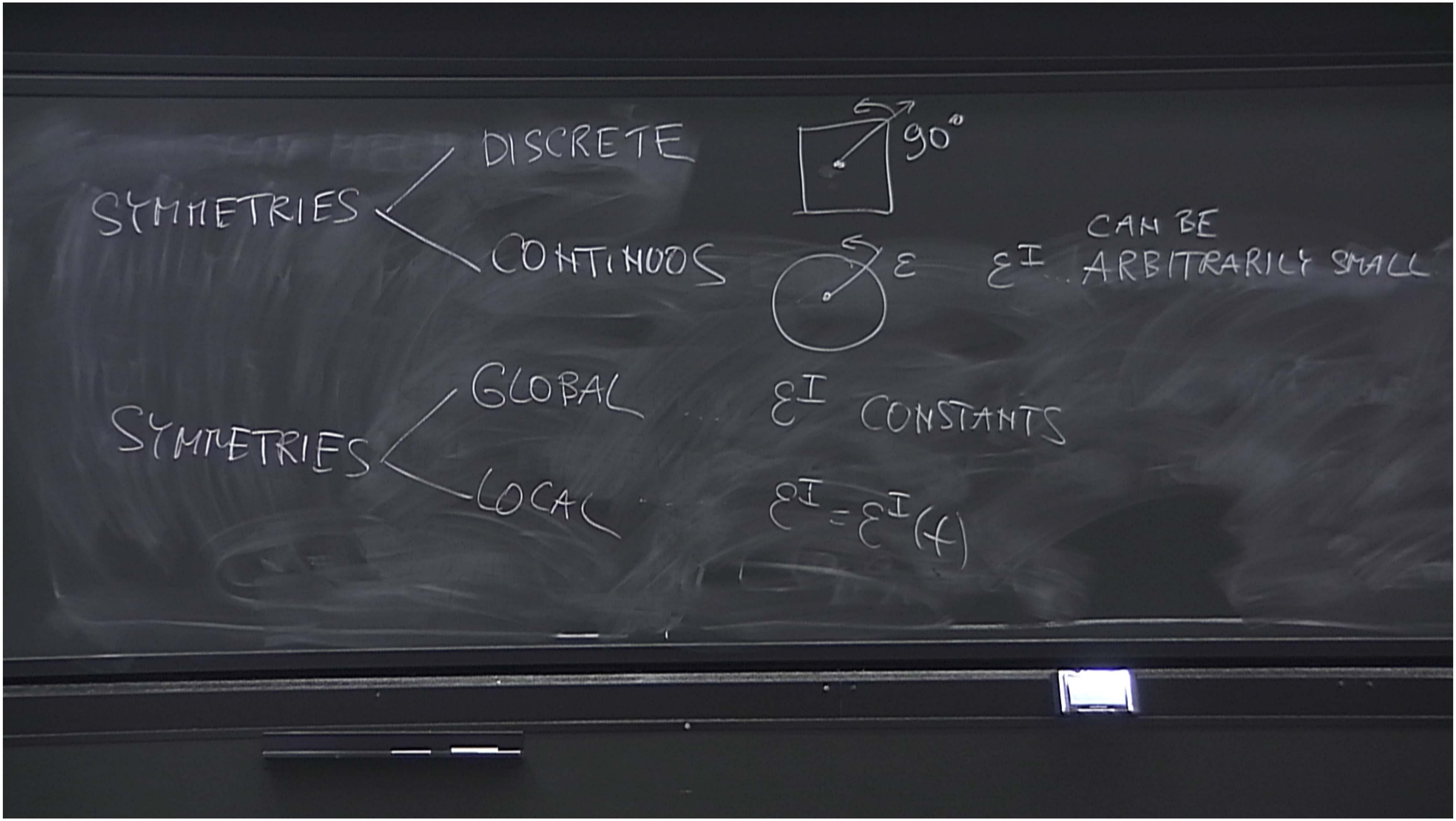
$$\boxed{\tilde{\delta} q = \varepsilon^I \tilde{\delta}_I q} \quad , \quad \tilde{\delta} t = \varepsilon^I \tilde{\delta}_I t$$

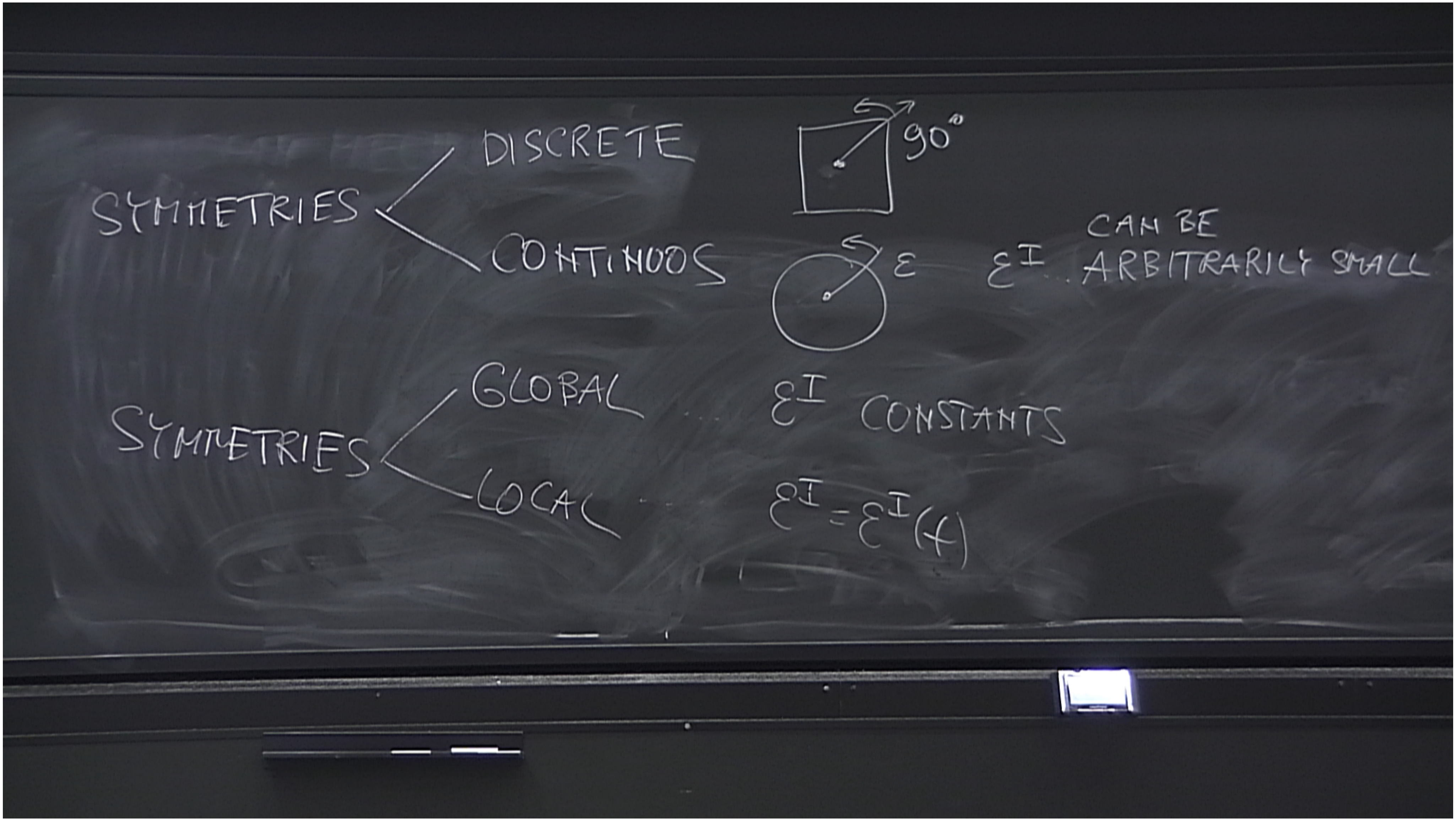
PARAMETER ("HOW BIG" THE ACTION IS)

"GENERATOR" (SPECIFIES TRANSF.)

<u>EG:</u> GEN... ROTATION

PARAM... φ


SYMMETRIES
DISCRETE
CONTINUOUS
90°
ε
ε^I ... CAN BE ARBITRARILY SMALL
SYMMETRIES
GLOBAL
ε^I CONSTANTS
LOCAL
$\varepsilon^I = \varepsilon^I(x)$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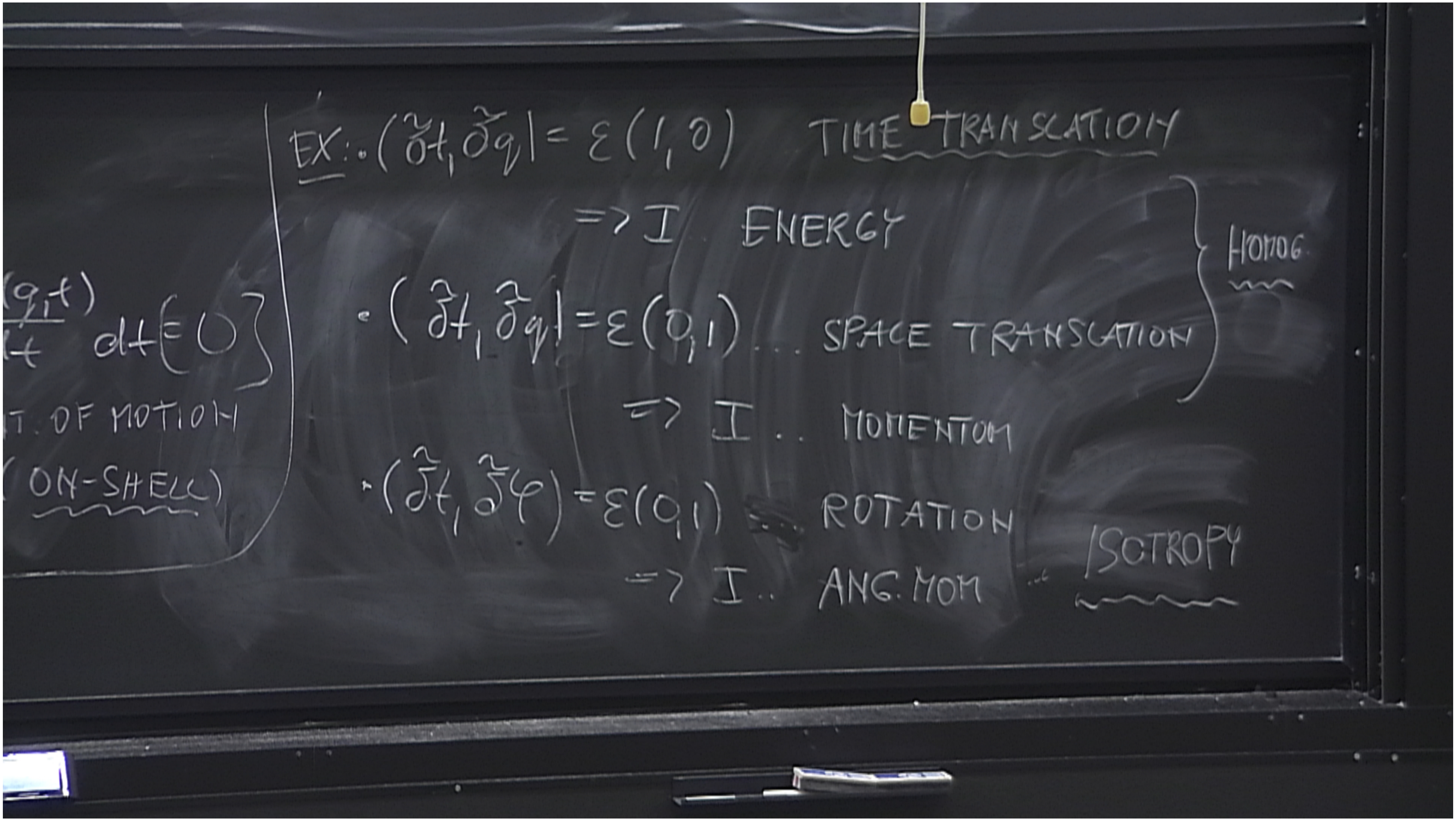

EX: $\cdot\,(\tilde{\partial}t, \tilde{\partial}q) = \varepsilon(1,0)$ TIME TRANSLATION
$\Rightarrow I$ ENERGY
$\cdot\,(\tilde{\partial}t, \tilde{\partial}q) = \varepsilon(0,1)$ … SPACE TRANSLATION
$\Rightarrow I$ .. MOMENTUM
HOMOG.
$\cdot\,(\tilde{\partial}t, \tilde{\partial}\varphi) = \varepsilon(0,1)$ ROTATION
$\Rightarrow I$ .. ANG. MOM
ISOTROPY
(q,t)
dt
dt ∈ ()
T. OF MOTION
(ON-SHELL)

PROOF: • VARIATIONS δ & $\tilde{\delta}$:

$$\cdot\ \tilde{\delta}\, dt = dt' - dt = \frac{d\tilde{\delta}t}{dt}\, dt$$

$$\cdot\ \tilde{\delta} q(t) \equiv q'(t') - q(t) \simeq q'(t) + \tilde{\delta}t\, \frac{dq'}{dt} + \dots - q(t)$$

$$= \delta q(t) + \tilde{\delta}t\, \frac{dq}{dt}$$

$$\boxed{\tilde{\delta} = \delta + \tilde{\delta}t\, \frac{d}{dt}}$$

$$\ldots\delta\dot{q} + \frac{\partial L}{\partial q}\delta q = \frac{d}{dt}\Big(\underline{\frac{\partial L}{\partial \dot{q}}\delta q}\Big) - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big)\delta q + \frac{\partial L}{\partial q}\delta q \Big|$$

$$\int \Big\{ \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\delta q + L\tilde{\delta} t\Big) + \Big[\underbrace{\frac{\partial L}{\partial q} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big)}_{\emptyset}\Big]\delta q \Big\} dt = \int \frac{d\Lambda}{dt} dt$$

ON-SHELL

$$\boxed{\frac{\partial L}{\partial \dot{q}}\delta q + L\tilde{\delta} t - \Lambda}$$

$$\int_1^2 \frac{dI}{dt} dt = 0$$

=> I..

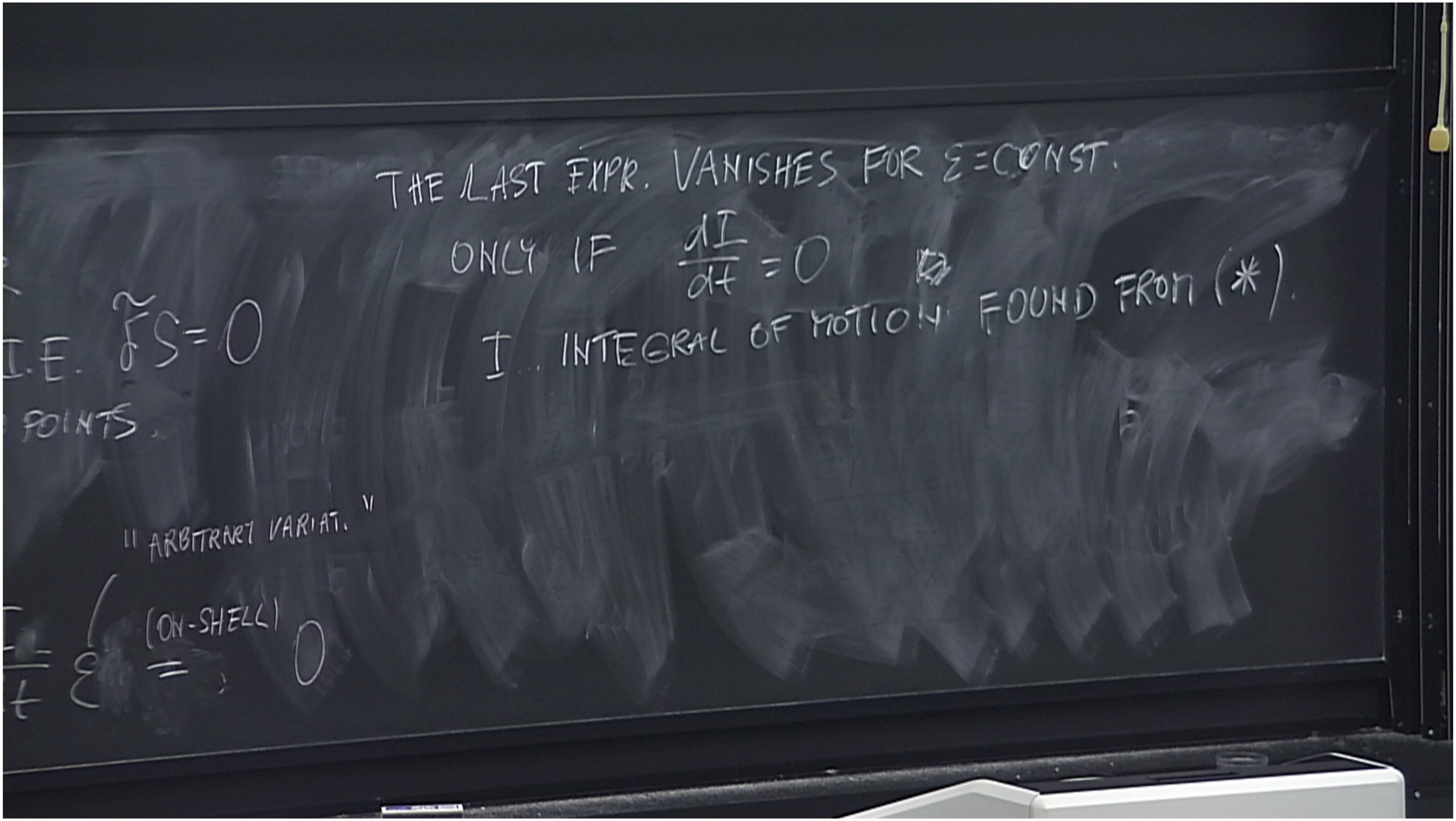
VERSION 2: NOETHER TH. (SNEAKY RECIPE)

1) OBSERVE THAT $\underline{S}$ IS INVARIANT UNDER A GLOBAL CONT. TRANSF (ε^I.. CONSTANTS), I.E. $\tilde{\delta} S = 0$

2) PROMOTE ε^I TO $\varepsilon^I(t)$ WITH FIXED END POINTS.

$$\Rightarrow \boxed{\tilde{\delta} S = \int dt \; I \; \dot{\varepsilon}} \quad (*) \qquad \text{" ARBITRARY VARIAT. "}$$

3) INTEGRATE BY PARTS $\quad \tilde{\delta} S = -\int dt \, \frac{dI}{dt} \, \varepsilon \overset{\text{(ON-SHELL)}}{=} 0$


THE LAST EXPR. VANISHES FOR ε = CONST.
ONLY IF dI/dt = 0
I ... INTEGRAL OF MOTION FOUND FROM (*).
I.E. δ̃S = 0
POINTS.
"ARBITRARY VARIAT."
(ON-SHELL)
= 0

$$\Rightarrow I = \frac{\partial L}{\partial \dot{q}} \tilde{\delta q} + \left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right)\tilde{\delta t} - \Lambda$$

INT. OF MOTION

C) MANIFOLDS & TENSORS

• MANIFOLD

DEF: A MANIFOLD IS A SET OF POINT[S] ... OF SUBSETS $\{O_\alpha\}$ S.T.:

i) EACH $p \in M$ LIES IN AT LEA[ST] ...

ii) FOR EACH α. THERE IS 1-1, ON[TO] ... WHERE U_α IS AN OPEN SUBSE[T] ...

iii) IF ANY TWO SETS O_α AND O_β ...

$$\frac{\partial L}{\partial \dot{q}} \delta q + L \delta t - \Lambda \qquad \int \frac{dI}{dt}\, dt = 0$$

INT. OF MOTION

$\Rightarrow I$ .. MOMENTUM

... SPACE TRANSLATION)

... IS A SET OF POINTS TOGETHER WITH A COLLECTION

...S $\{O_\alpha\}$ S.T.:

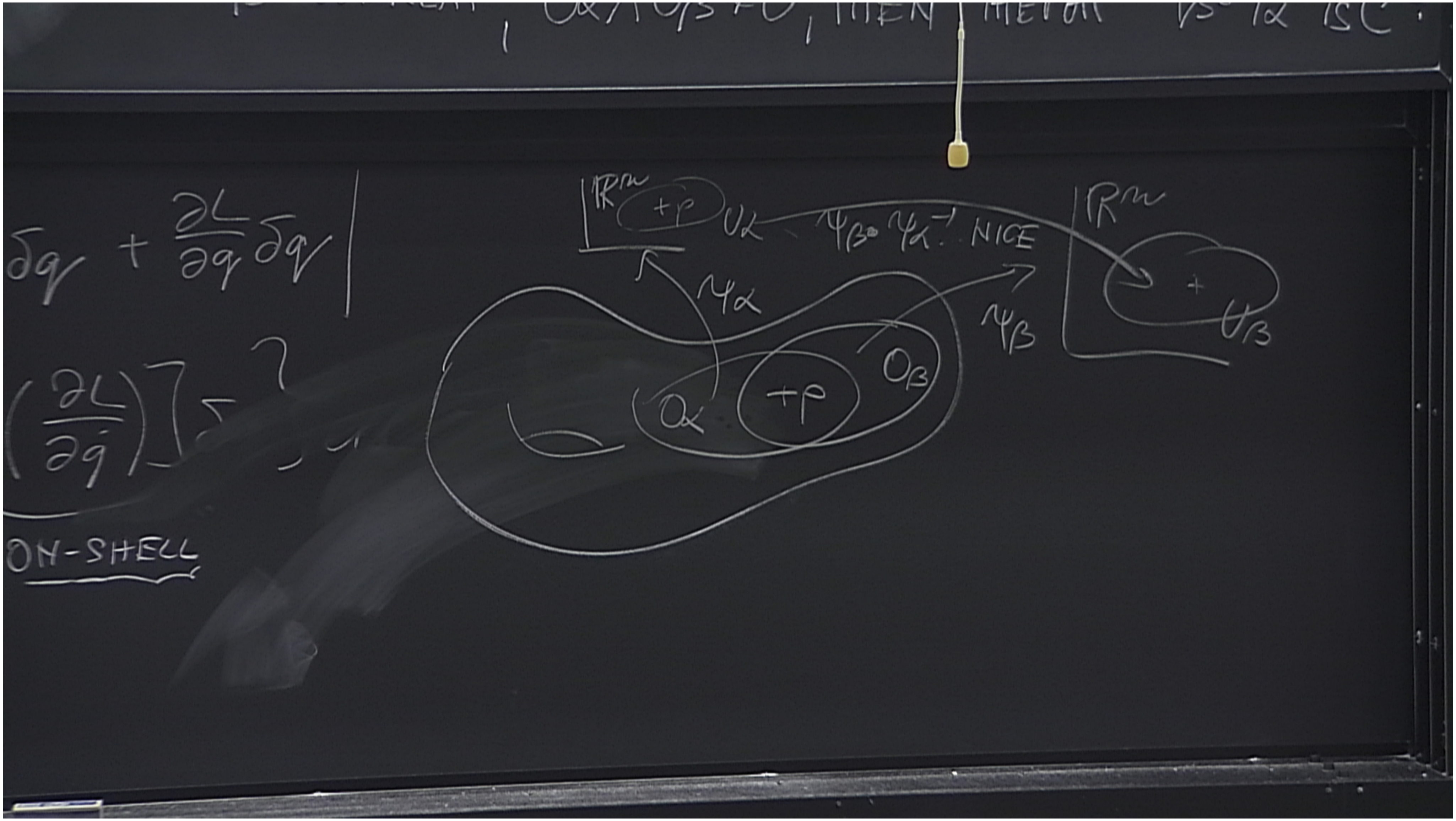
LIES IN AT LEAST ONE O_α, $\{O_\alpha\}$ COVER M.

THERE IS 1-1, ONTO, MAP $\psi_\alpha : O_\alpha \to U_\alpha$

IS AN OPEN SUBSET OF $\mathbb{R}^n$. (UNION OF OPEN BALLS)

... SETS O_α AND O_β OVERLAP, $O_\alpha \cap O_\beta \neq 0$, THEN THE MAP $\psi_\beta \circ \psi_\alpha^{-1}$ IS C^∞.

$dt = 0$


ON-SHELL
$\mathbb{R}^n$
$+p$
U_α
$\psi_\beta \circ \psi_\alpha^{-1}$: NICE
ψ_α
ψ_β
O_α
O_β
U_β
$\mathbb{R}^n$

1) LAGRANGIAN MECHANICS

a) HAMILTON'S PRINCIPLE OF LEAST ACTION (HAMILTON (1805 - 1865))

MOTIONS OF THE MECHANICAL SYSTEM IN TIME INTERVAL $t \in (t_1, t_2)$ COINCIDE WITH THE EXTREMALS OF THE ACTION FUNCTIONAL

$$S = S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

L .. DEFINITE F. CHAR. SYSTEM