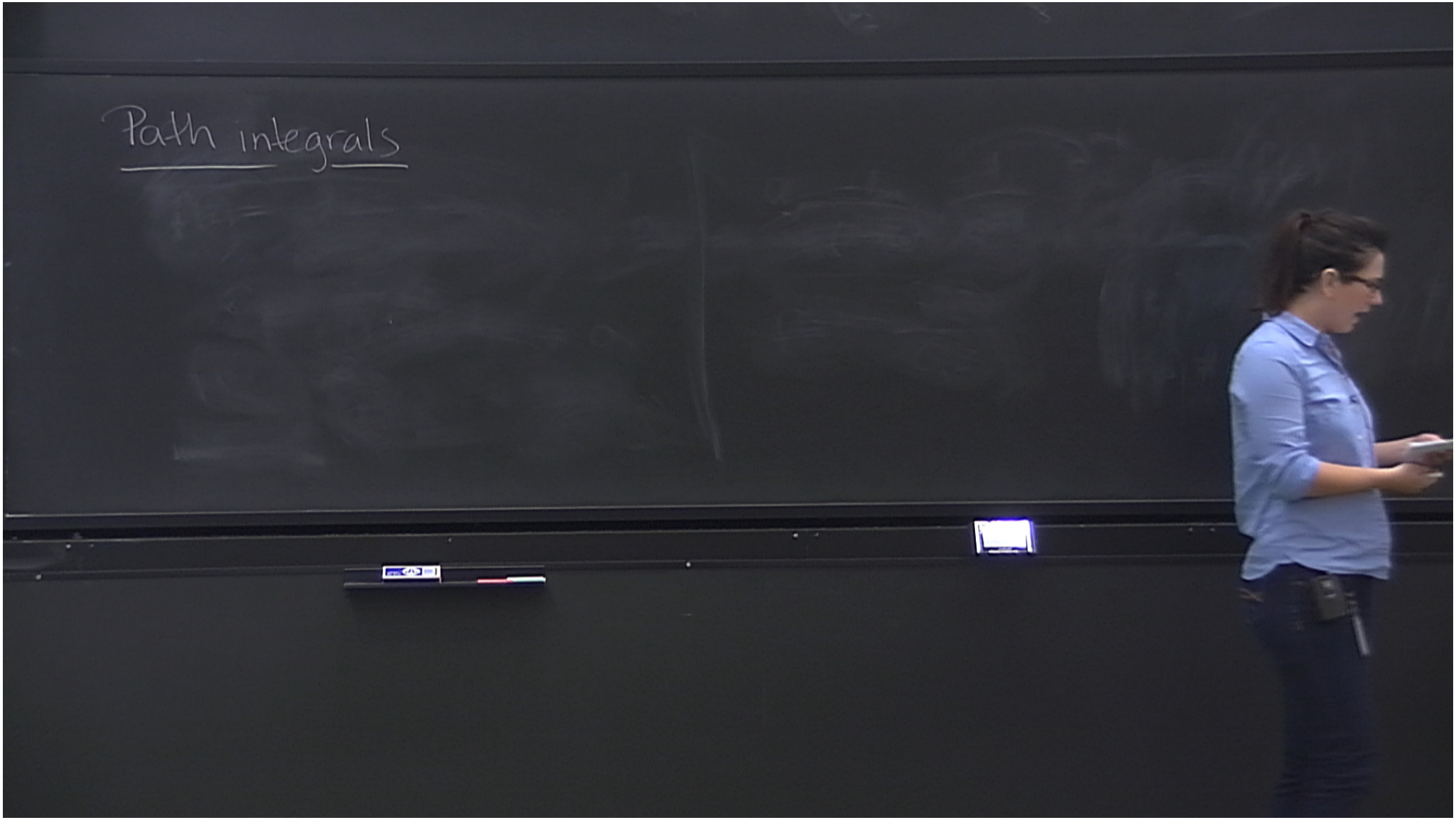
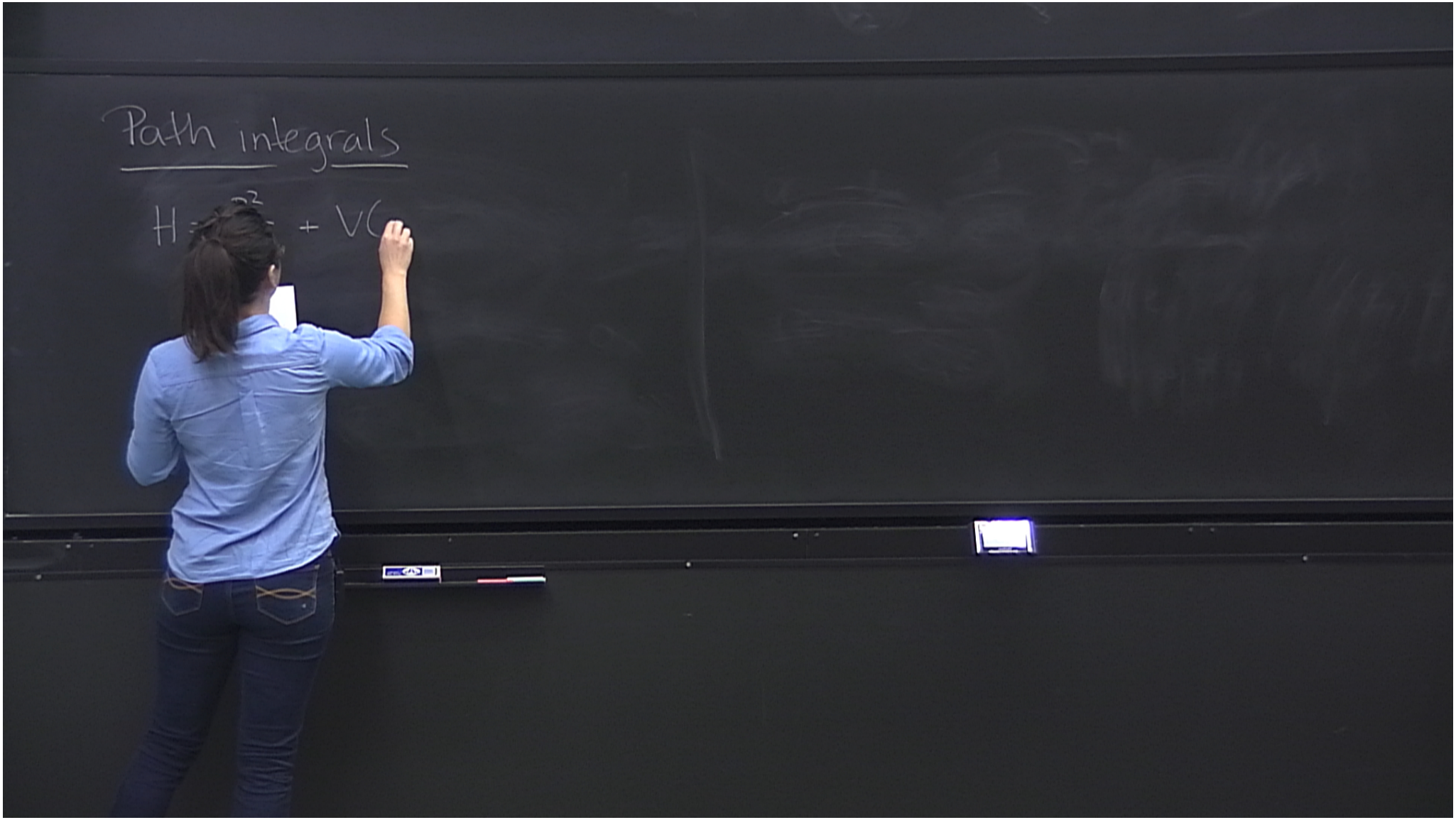
Title: Quantum Mechanics

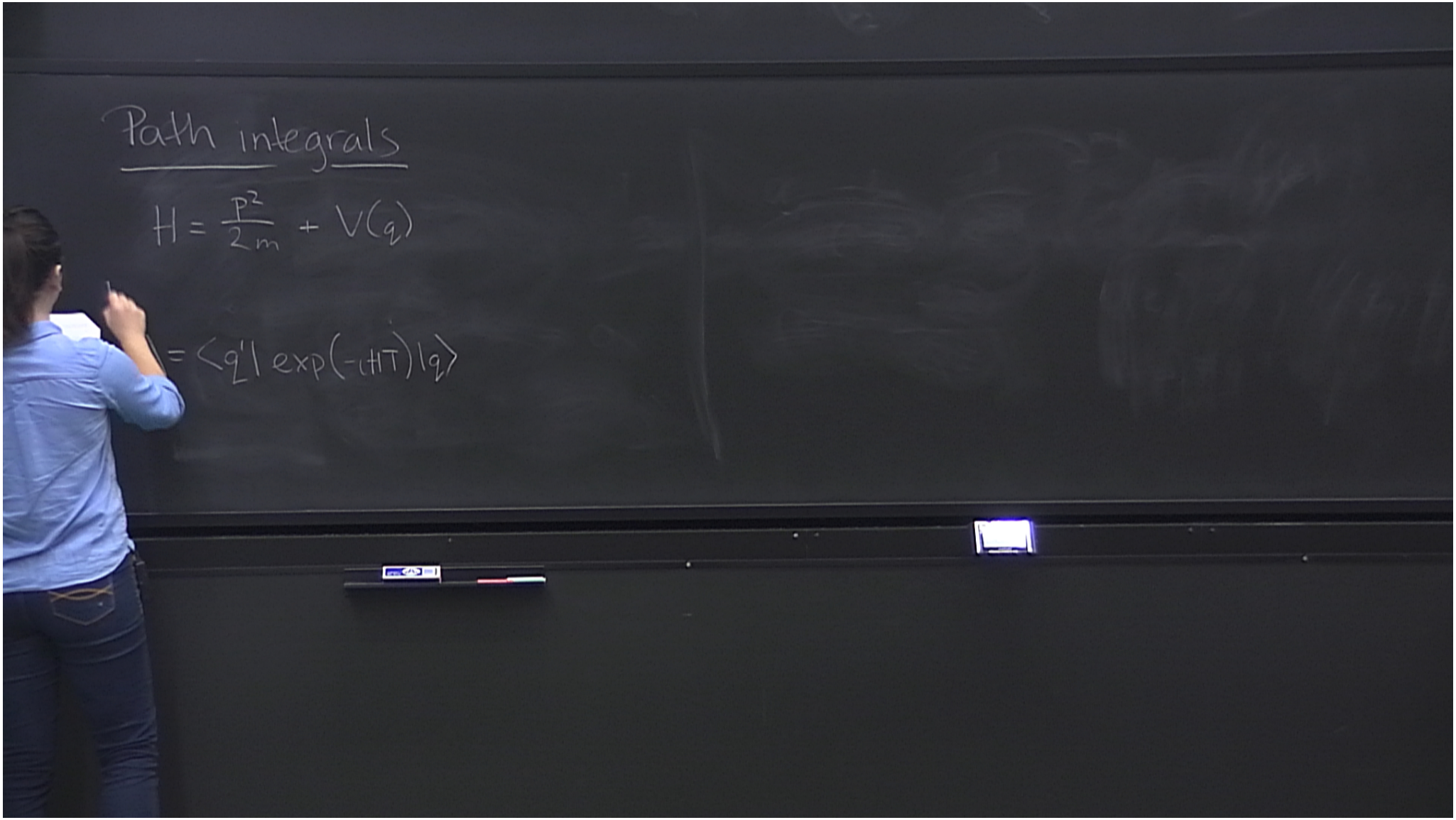
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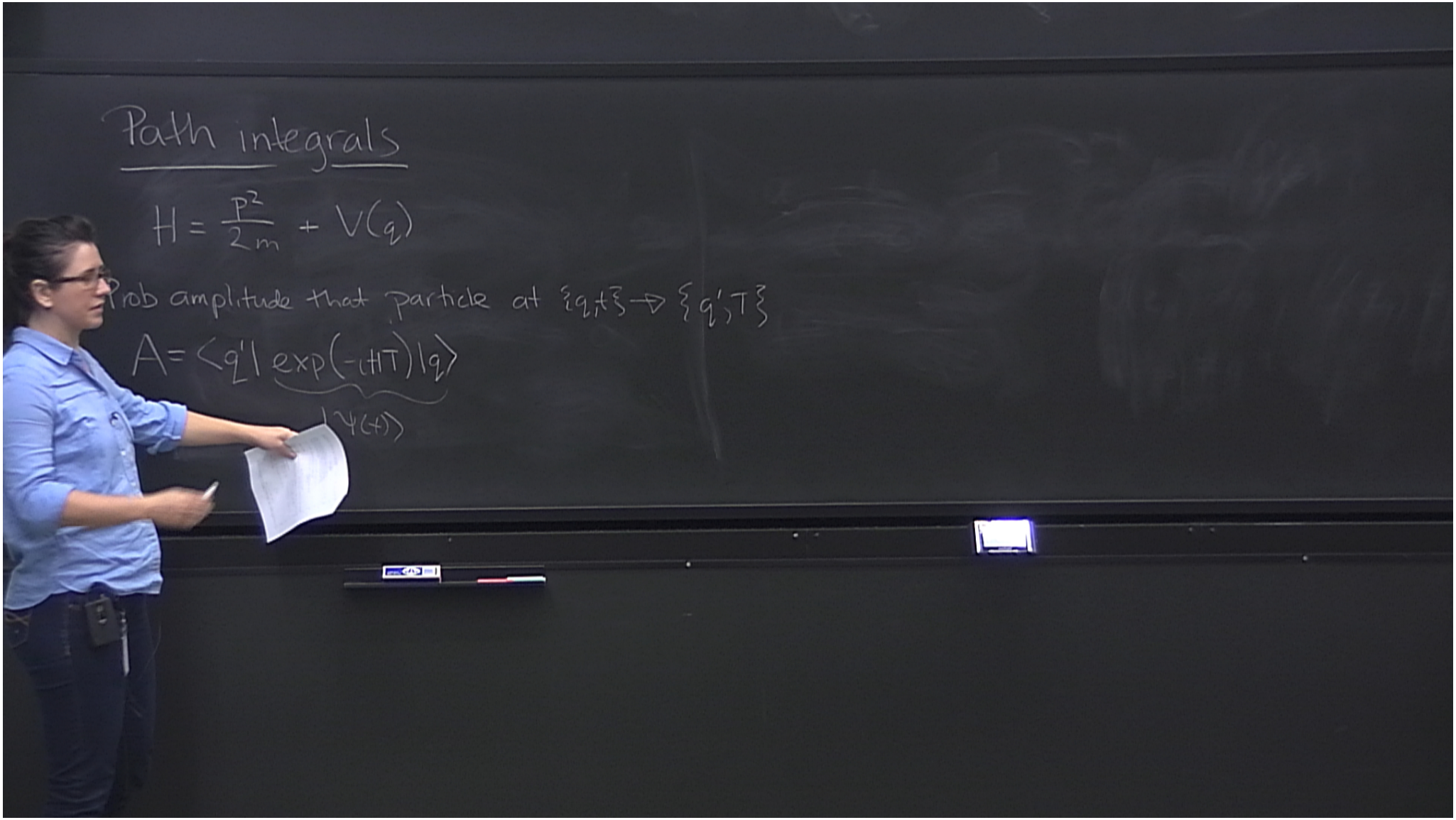
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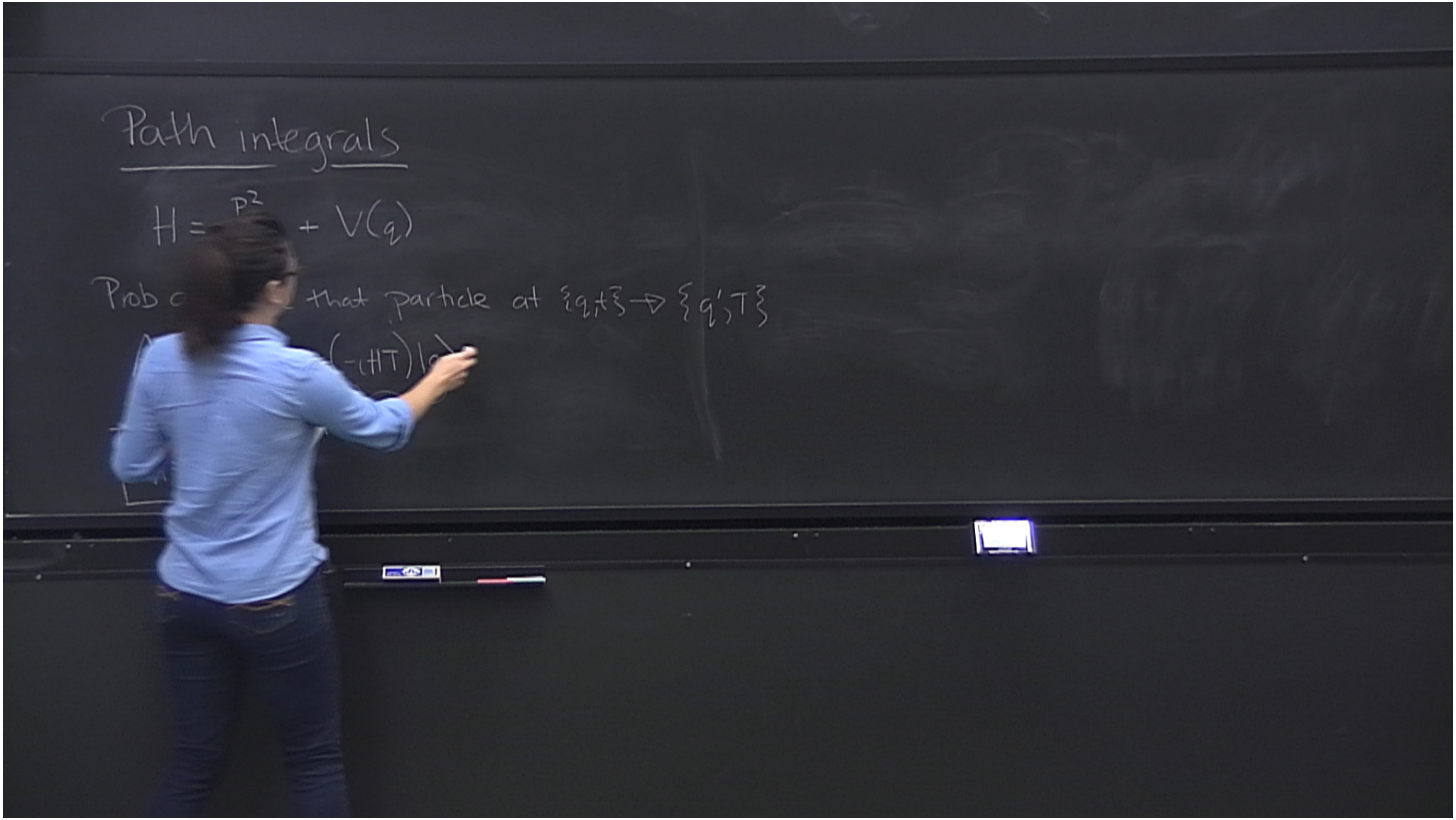
Abstract:

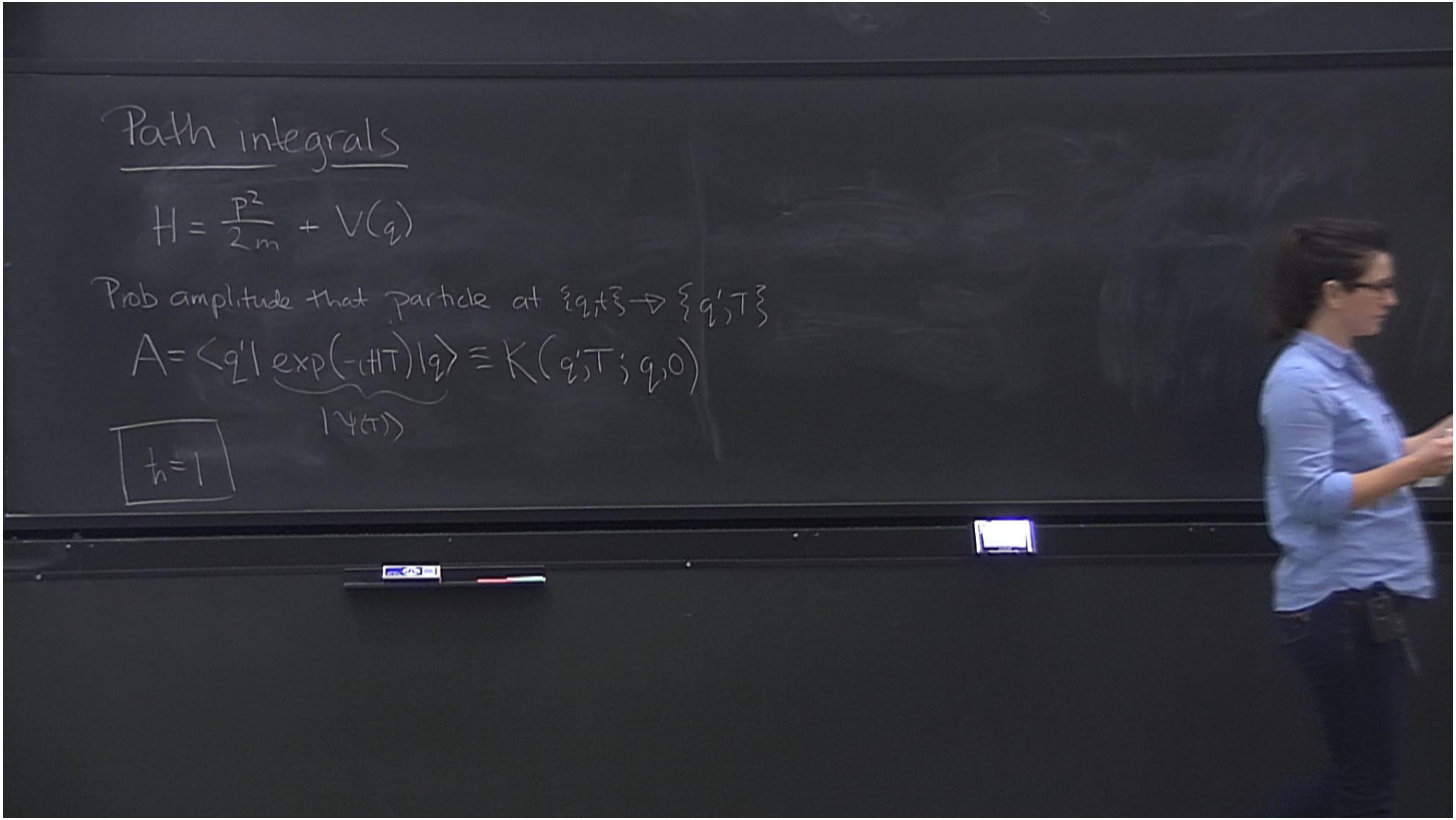

Path integrals

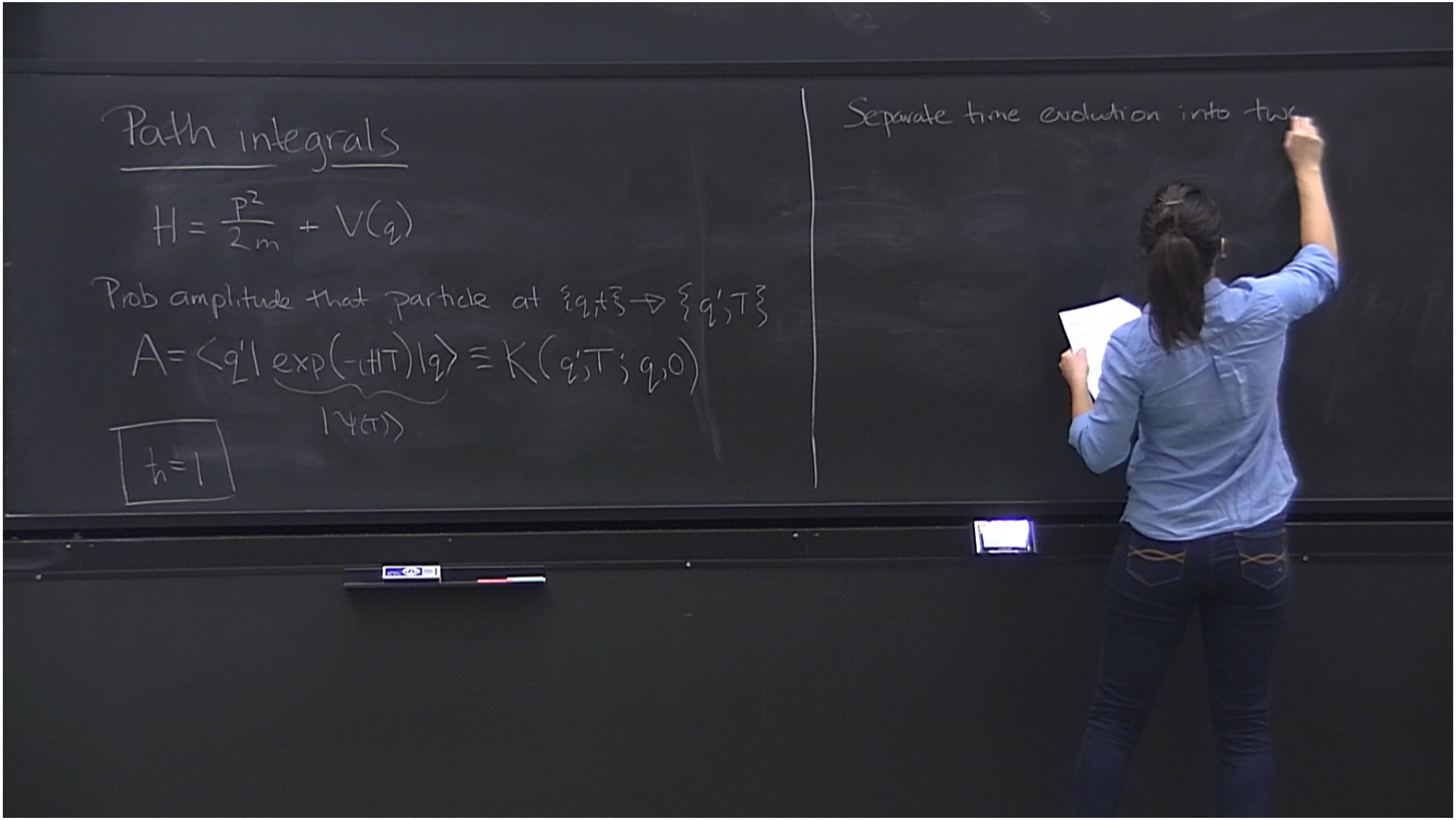

Path integrals

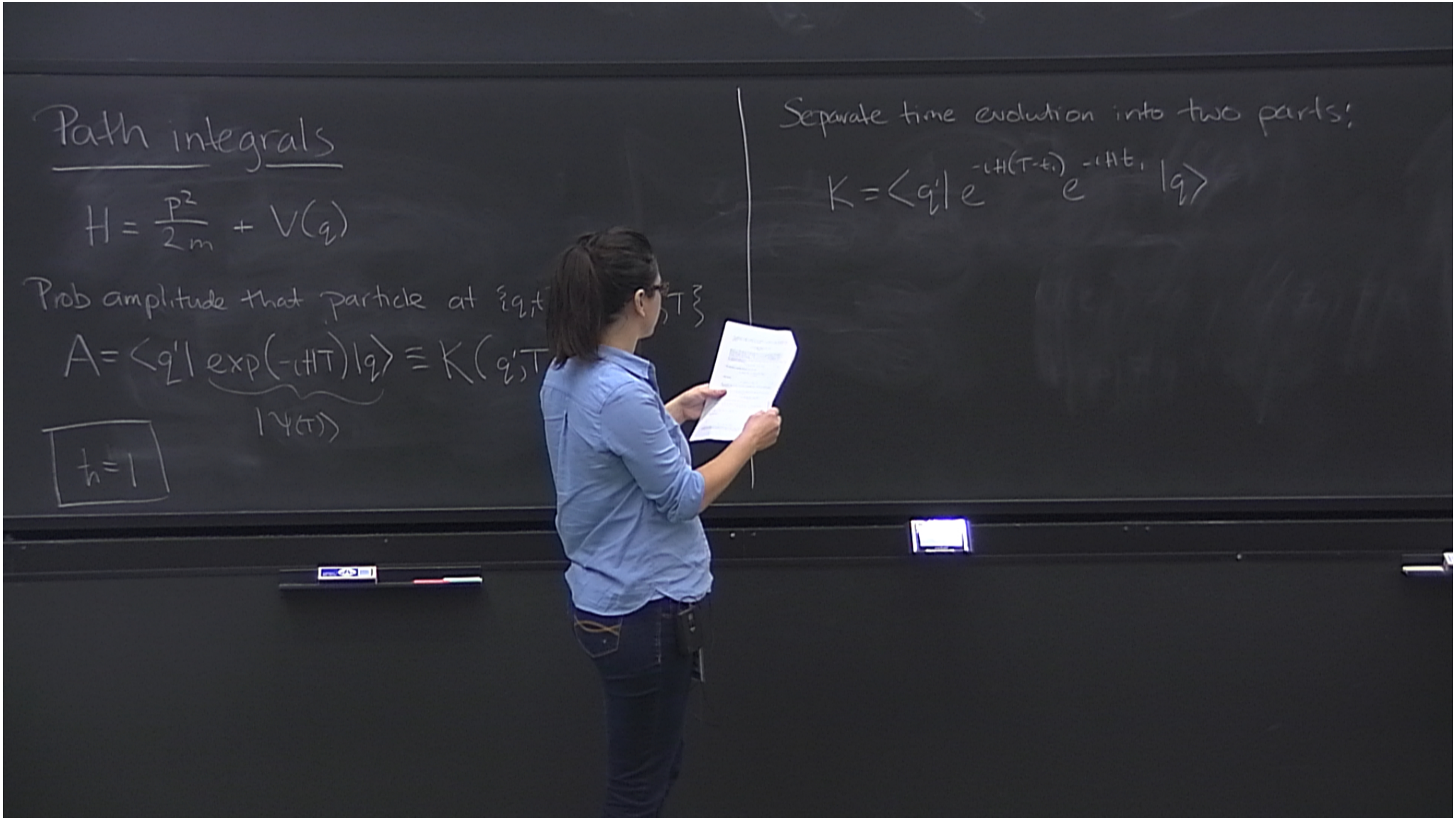

Path integrals
$H = \frac{p^2}{2m} + V(q)$
$= \langle q' | \exp(-iHT) | q \rangle$

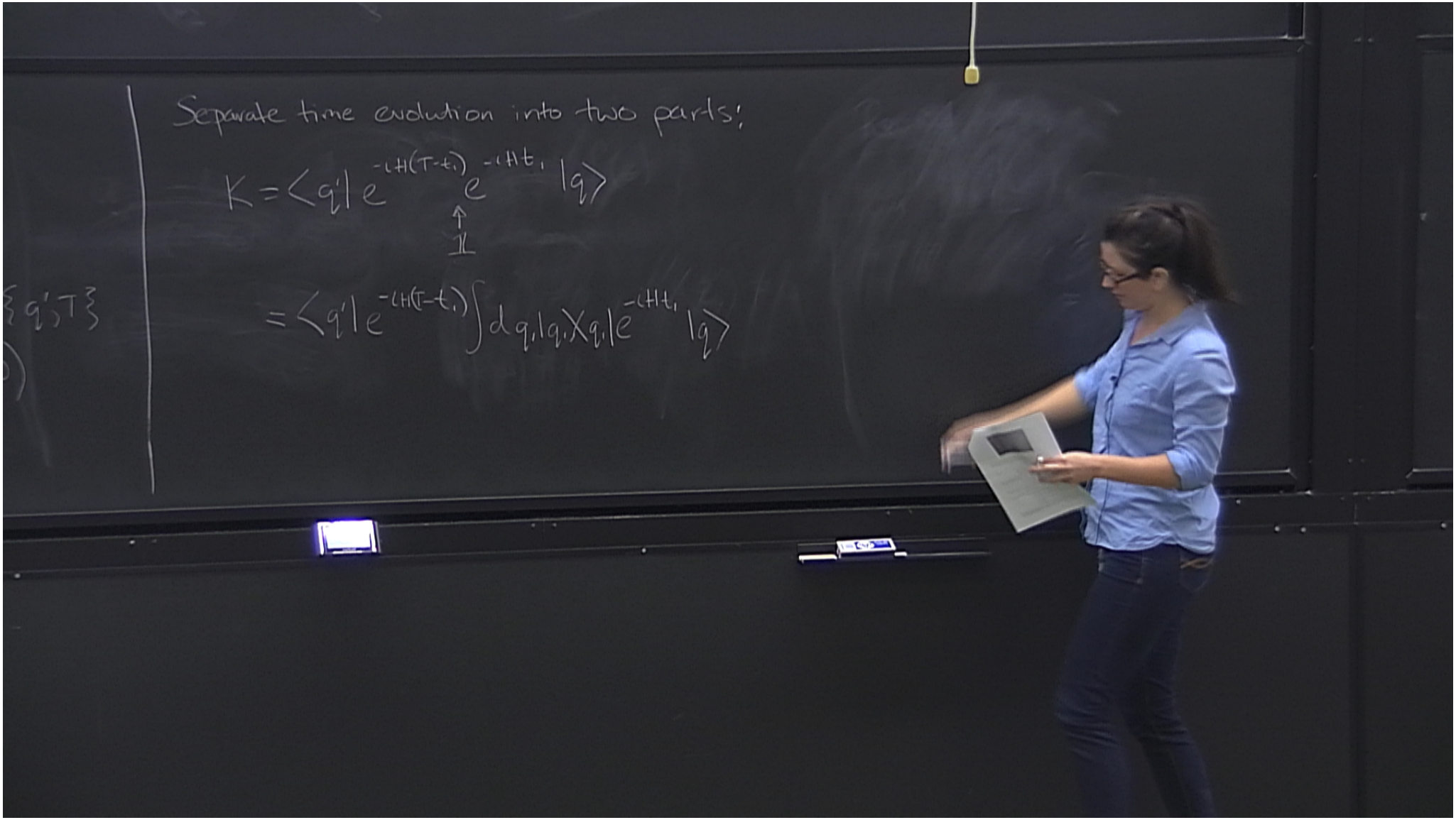

Path integrals
$H = \frac{p^2}{2m} + V(q)$
Prob amplitude that particle at $\{q,t\} \to \{q',T\}$
$A = \langle q' | \exp(-\frac{i}{\hbar}HT) | q \rangle$
$|\psi(t)\rangle$

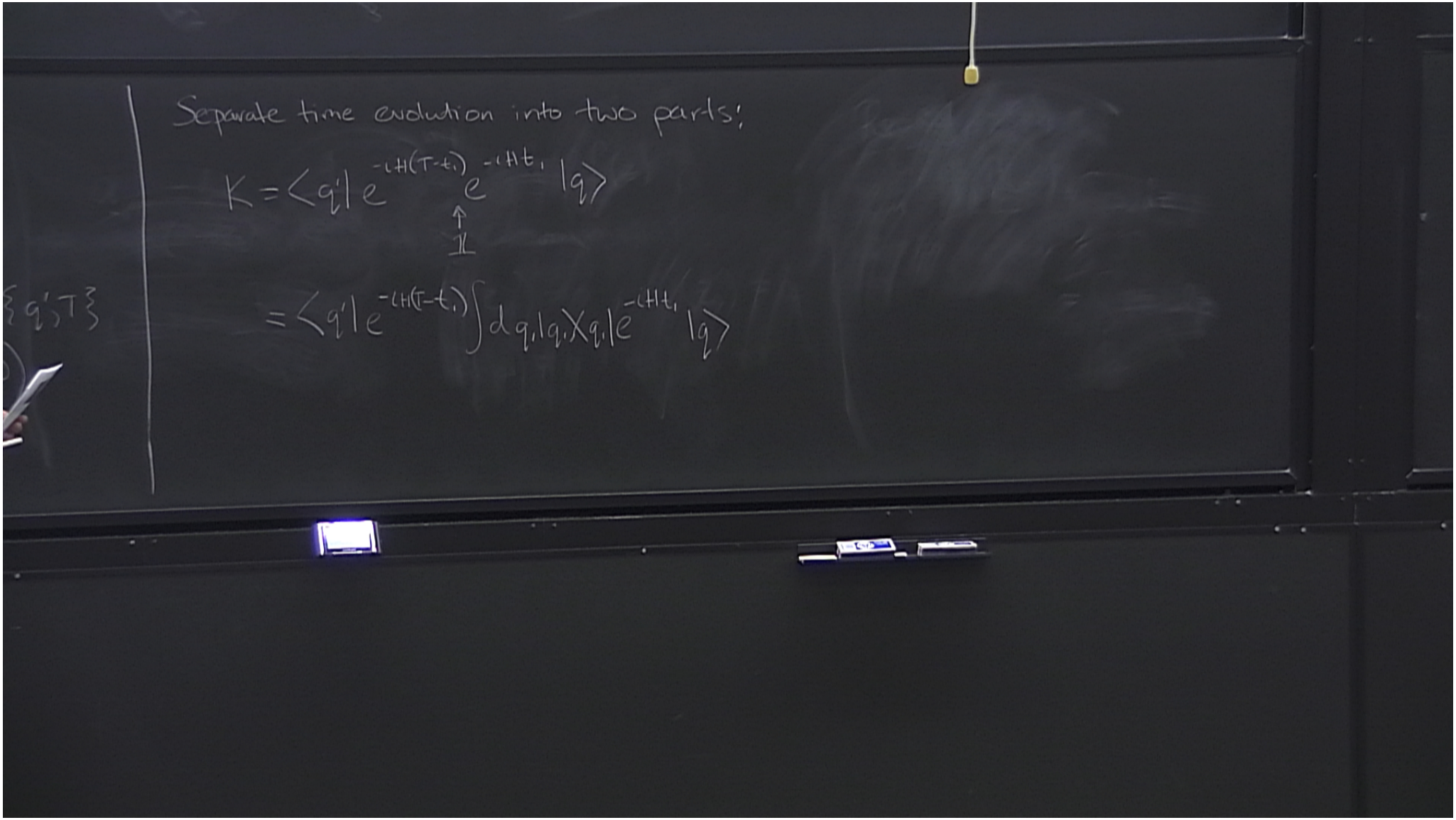

Path integrals
H = p² + V(q)
Prob that particle at {q,t} → {q',T}

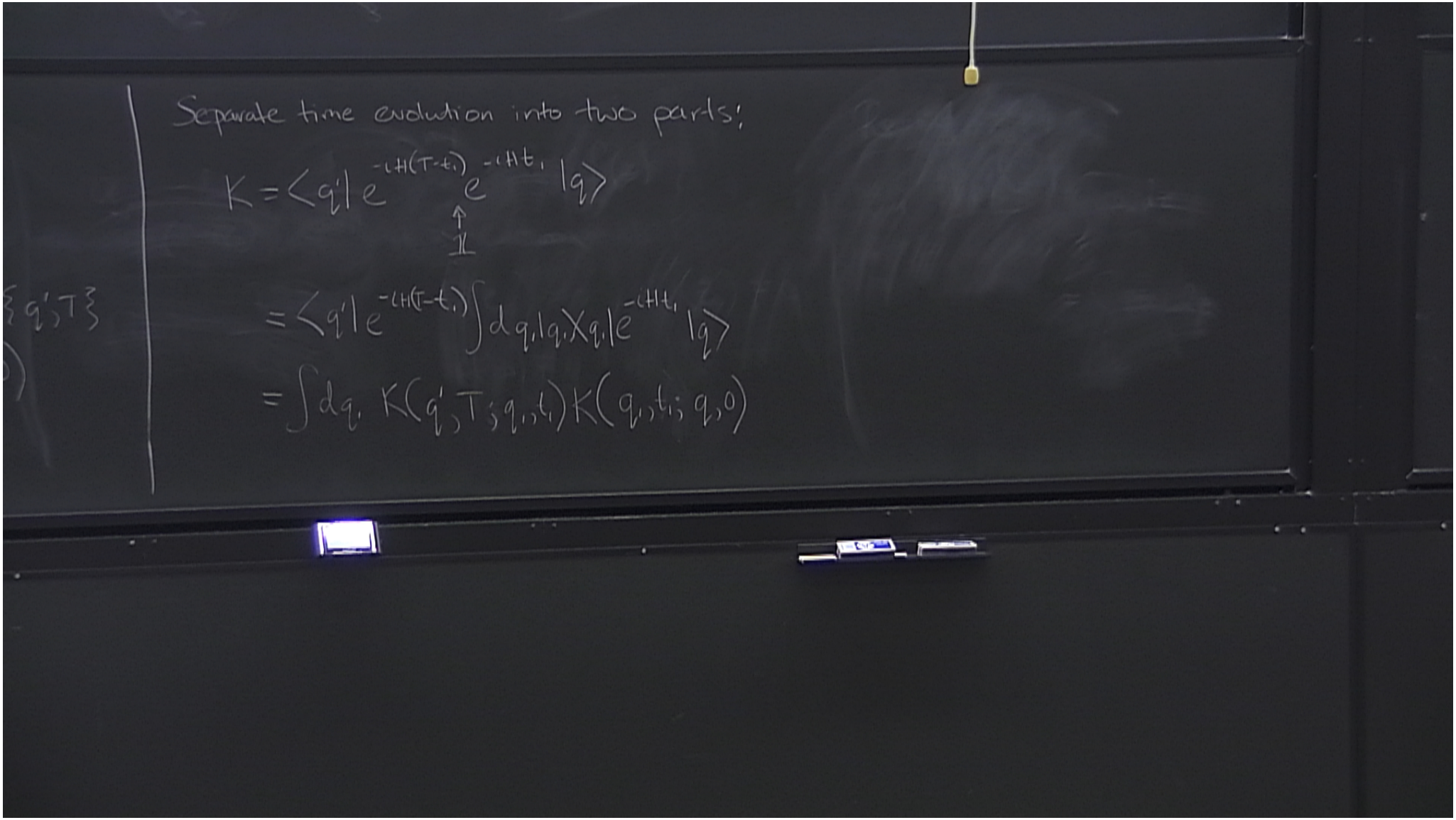

Path integrals
$H = \frac{p^2}{2m} + V(q)$
Prob amplitude that particle at $\{q,t\} \to \{q',T\}$
$A = \langle q' | \exp(-iHT) | q \rangle \equiv K(q',T;q,0)$
$|\psi(T)\rangle$
$\hbar = 1$

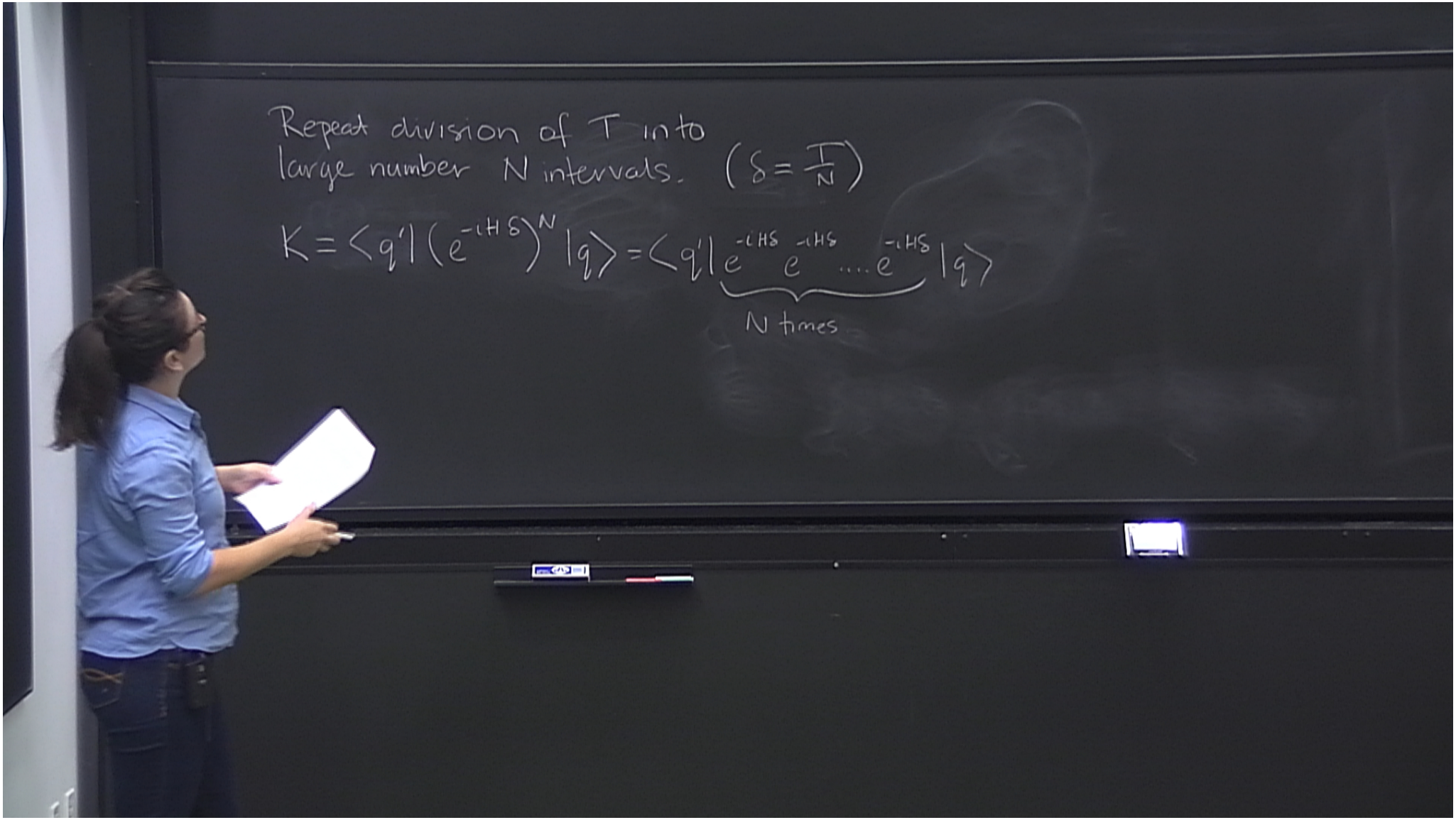

Path integrals
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$|\psi(T)\rangle$
$\hbar = 1$
Separate time evolution into two

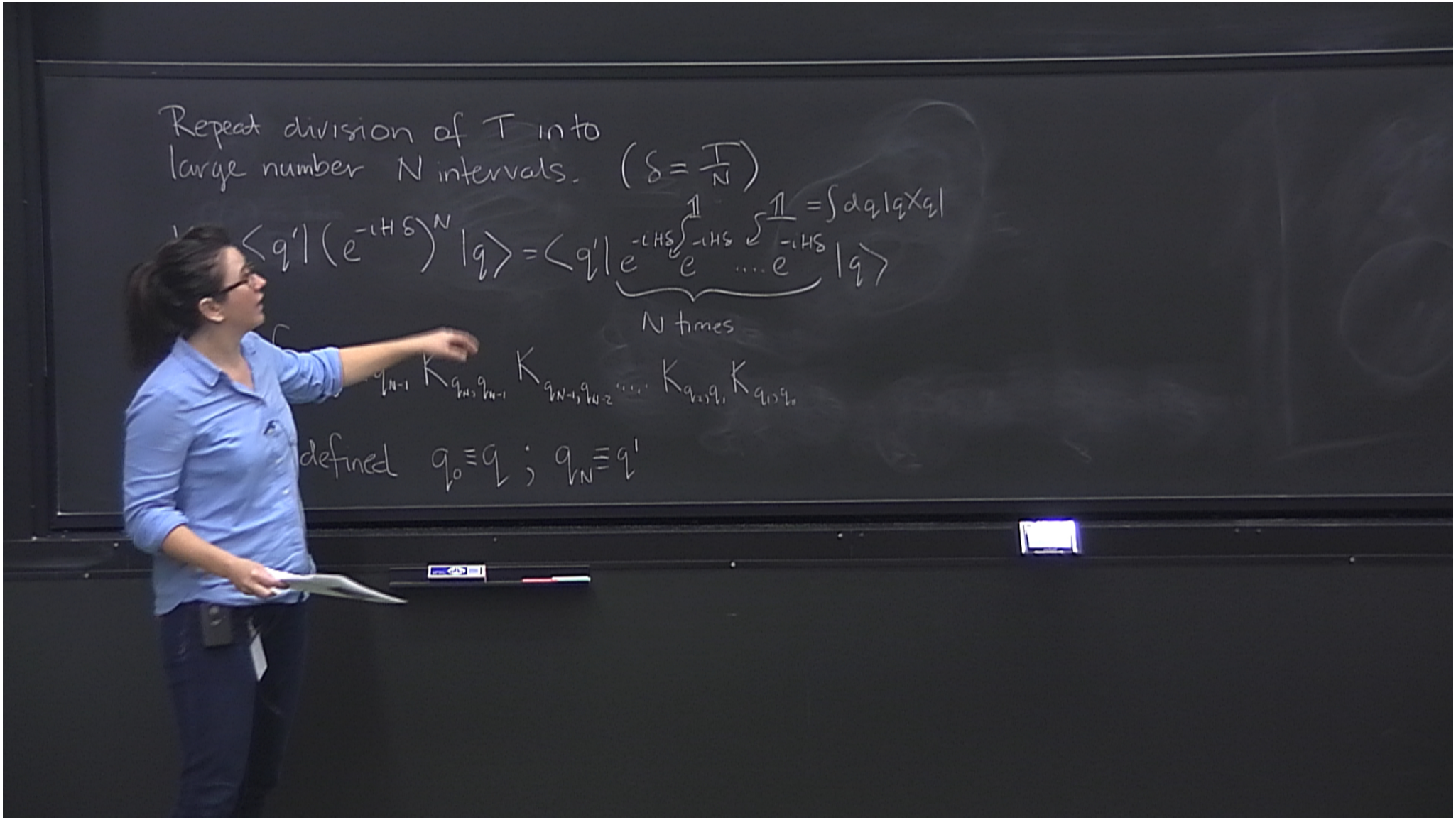

Path integrals
$H = \frac{p^2}{2m} + V(q)$
Prob amplitude that particle at
$A = \langle q' | \exp(-\imath HT) | q \rangle \equiv K(q', T$
$|\psi(T)\rangle$
$\hbar = 1$
Separate time evolution into two parts:
$K = \langle q' | e^{-\imath H(T-t_1)} e^{-\imath H t_1} | q \rangle$


Separate time evolution into two parts:
$K = \langle q' | e^{-\frac{i}{\hbar}H(T-t_1)} e^{-\frac{i}{\hbar}Ht_0} | q \rangle$
$= \langle q' | e^{-\frac{i}{\hbar}H(T-t_1)} \int dq_1 |q_1\rangle\langle q_1| e^{-\frac{i}{\hbar}Ht_0} | q \rangle$


Separate time evolution into two parts:
$K = \langle q' | e^{-iH(T-t_1)} e^{-iHt_1} | q \rangle$
$= \langle q' | e^{-iH(T-t_1)} \int dq_1 |q_1\rangle\langle q_1| e^{-iHt_1} | q \rangle$


Separate time evolution into two parts:
$K = \langle q' | e^{-iH(T-t_1)} e^{-iHt_1} | q \rangle$
$= \langle q' | e^{-iH(T-t_1)} \int dq_1 |q_1\rangle\langle q_1| e^{-iHt_1} | q \rangle$
$= \int dq_1 \, K(q', T; q_1, t_1) K(q_1, t_1; q, 0)$


Repeat division of T into
large number N intervals. ($\delta = \frac{T}{N}$)
$K = \langle q' | (e^{-iH\delta})^N | q \rangle = \langle q' | \underbrace{e^{-iH\delta} e^{-iH\delta} \cdots e^{-iH\delta}}_{N \text{ times}} | q \rangle$


Repeat division of T into
large number N intervals. ($\delta = \frac{T}{N}$)
$\langle q'|(e^{-iH\delta})^N|q\rangle = \langle q'|e^{-iH\delta}\mathbb{1}e^{-iH\delta}\mathbb{1}\cdots\mathbb{1}e^{-iH\delta}|q\rangle$
$= \int dq_1 \langle q|q_1\rangle\langle q_1|$
N times
$K_{q_N,q_{N-1}} K_{q_{N-1},q_{N-2}} \cdots K_{q_2,q_1} K_{q_1,q_0}$
defined $q_0 \equiv q$; $q_N \equiv q'$

Repeat division of T into large number N intervals. $(\delta = \frac{T}{N})$

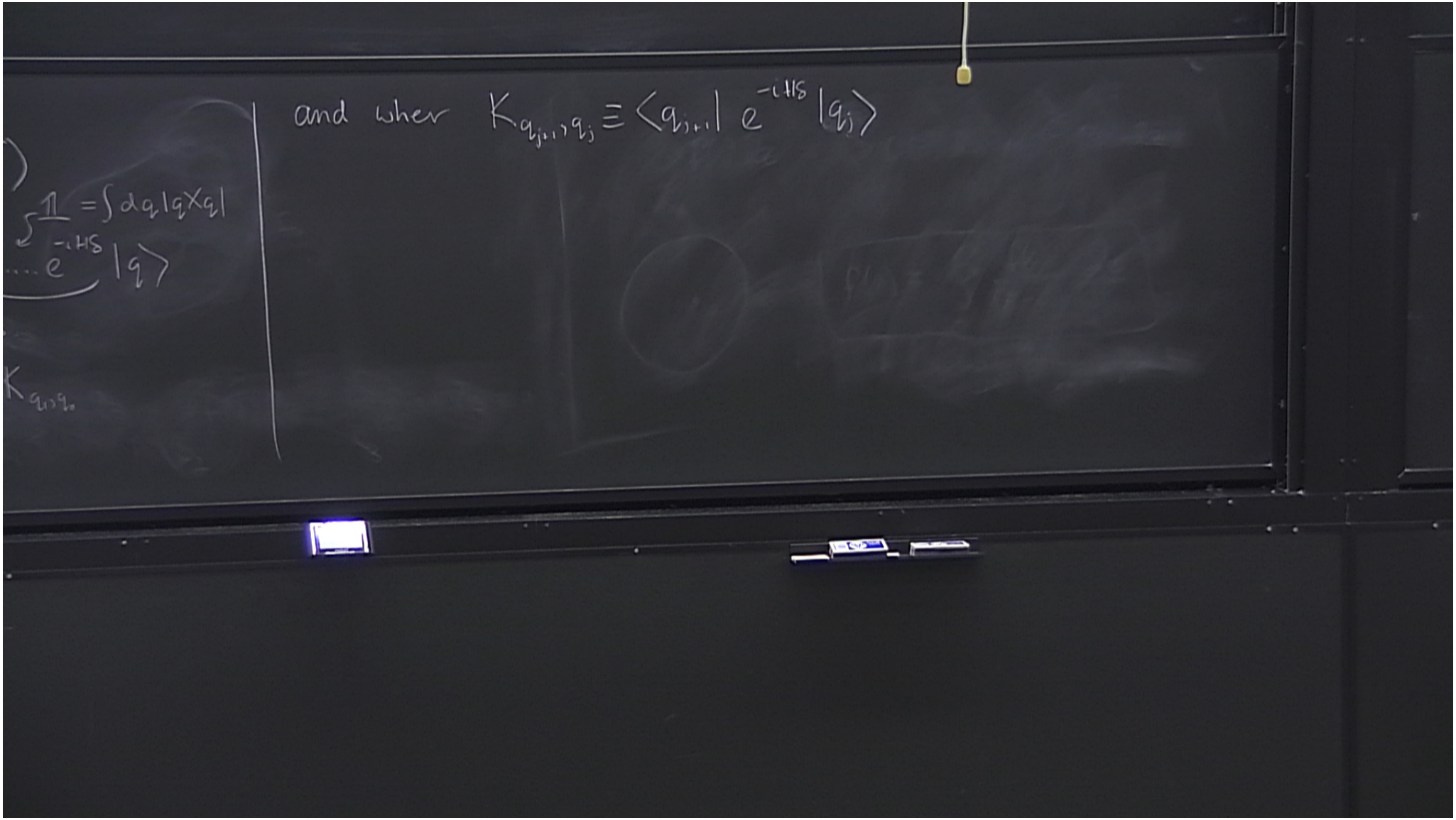
$$K = \langle q' | (e^{-iH\delta})^N | q \rangle = \langle q' | \underbrace{e^{-iH\delta} e^{-iH\delta} \cdots e^{-iH\delta}}_{N \text{ times}} | q \rangle$$

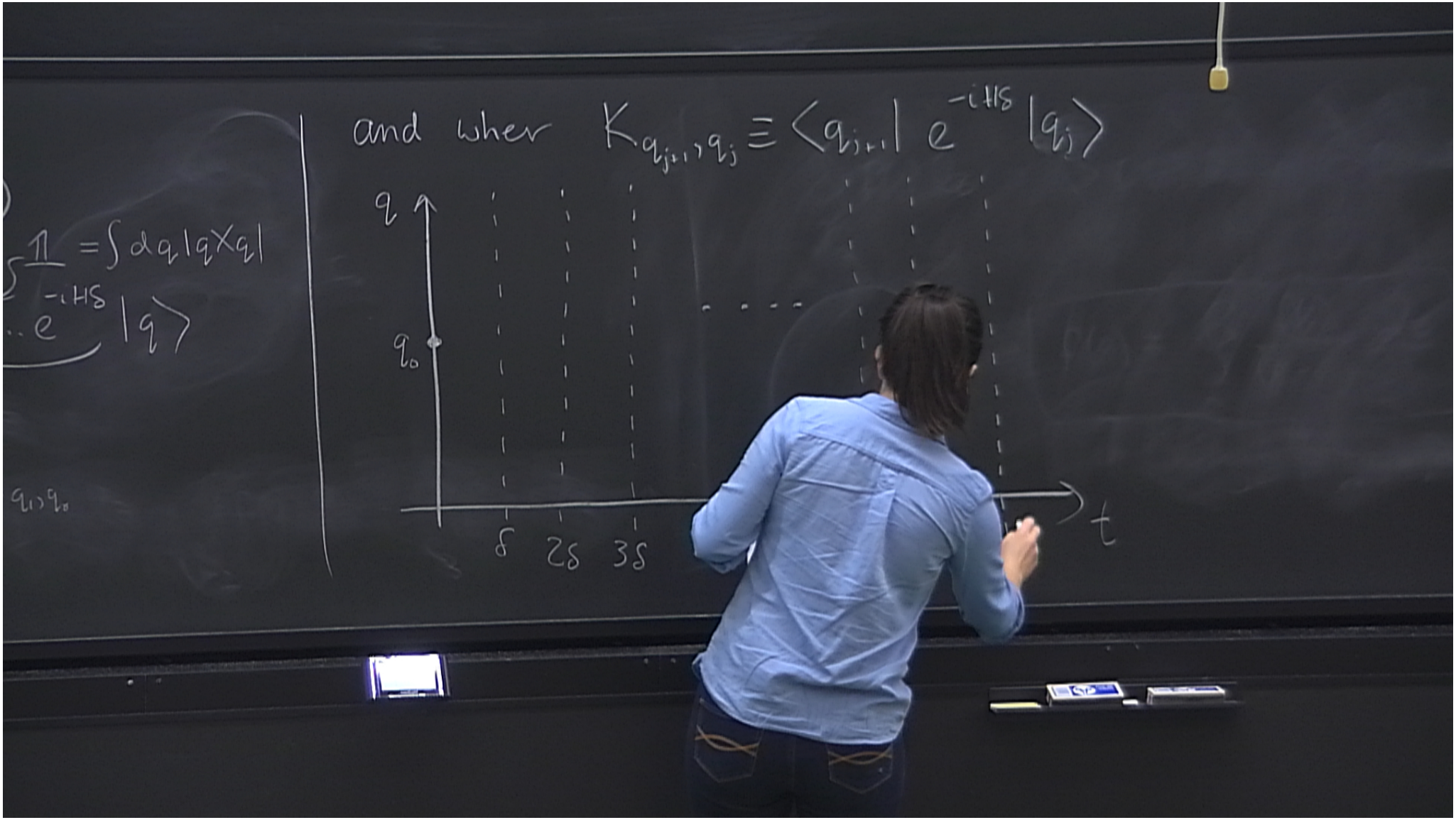
$$= \int dq_1 \cdots q_{N-1} K_{q_N, q_{N-1}} K_{q_{N-1}, q_{N-2}} \cdots K_{q_2, q_1} K_{q_1, q_0}$$

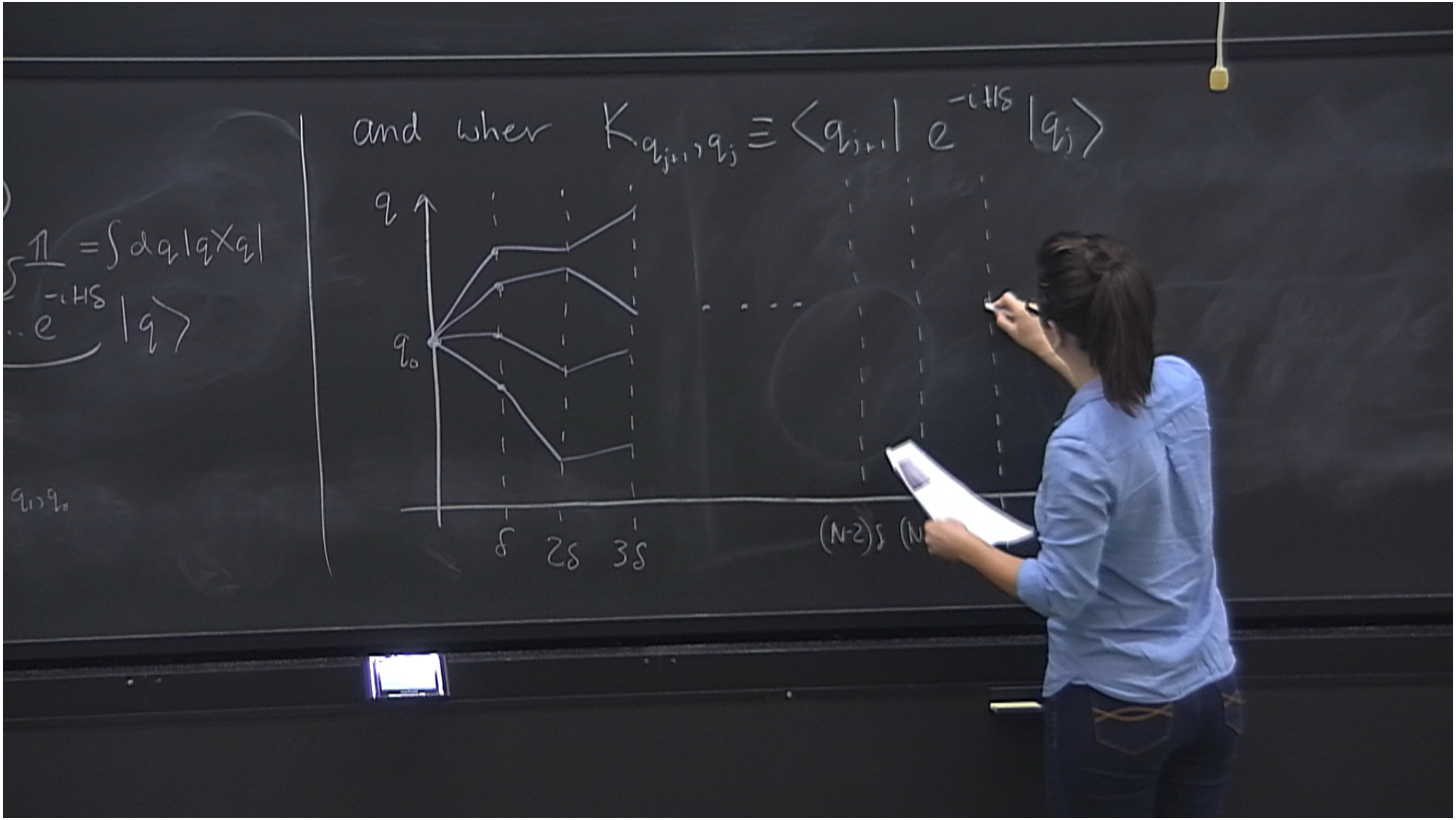
where we defined $q_0 \equiv q$; $q_N \equiv q'$

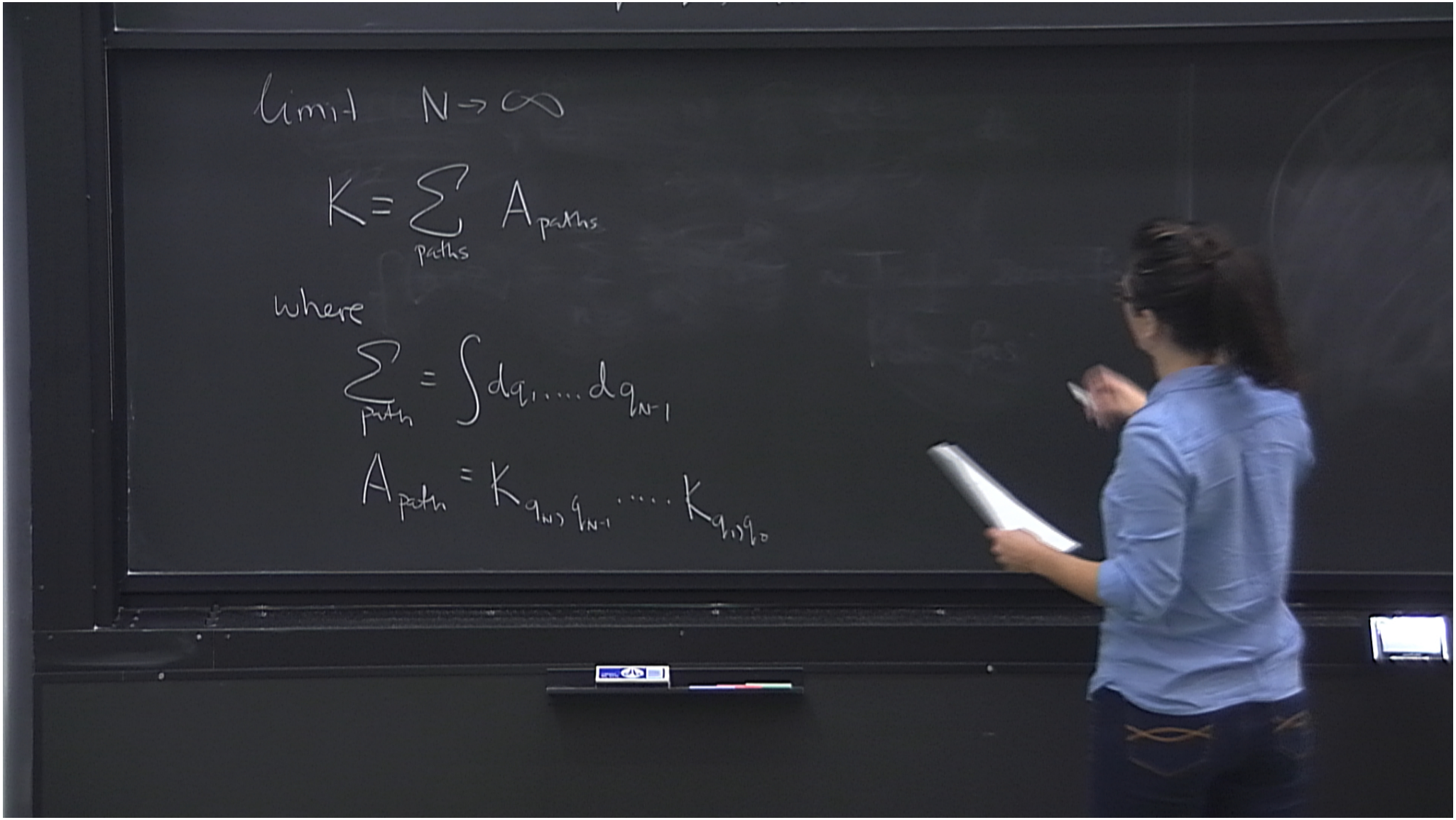
and wher $K_{q_{j+1}, q_j} \equiv \langle q$

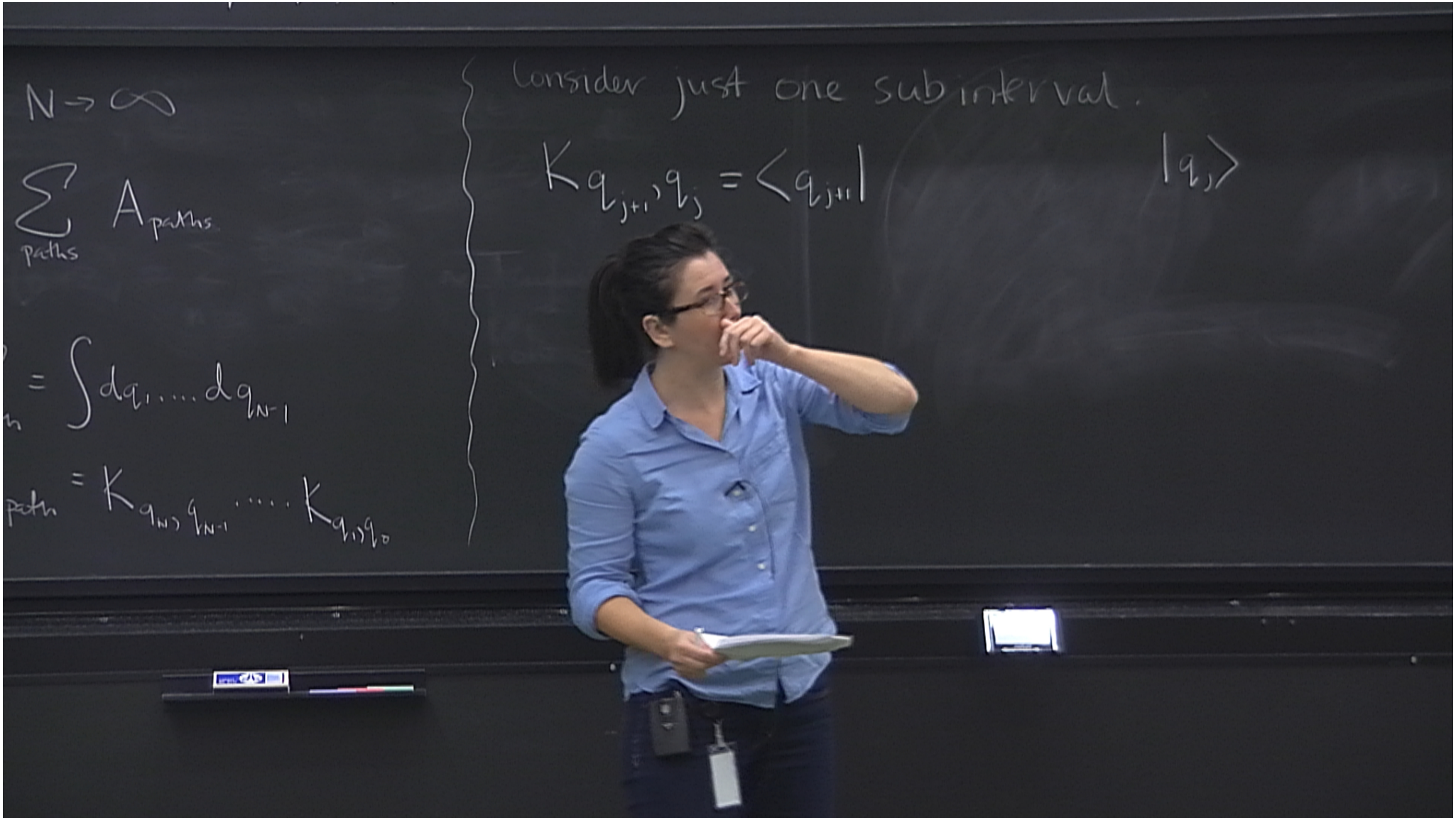


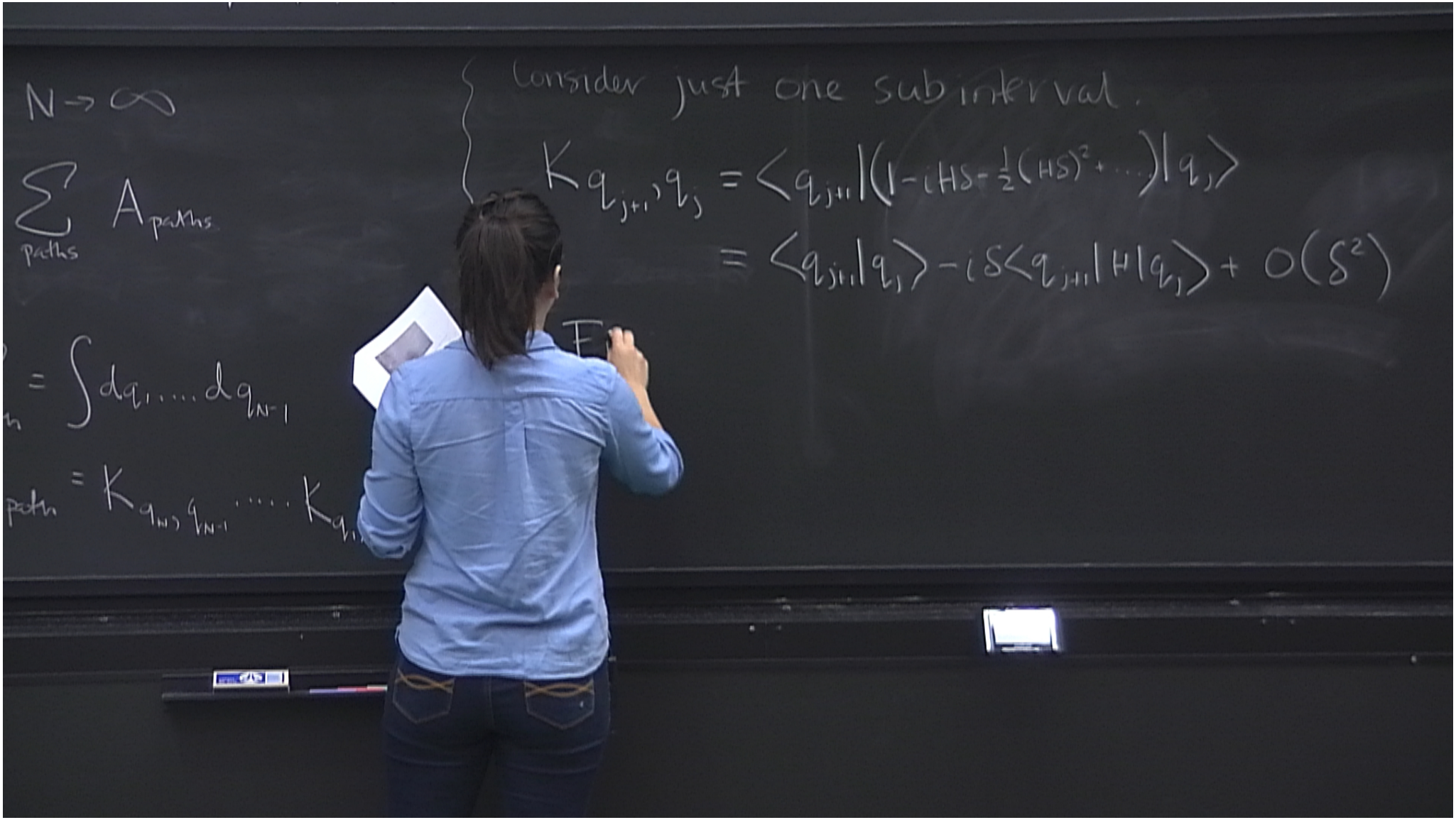

and wher $K_{q_{j+1},q_j} \equiv \langle q_{j+1}| e^{-iHs} |q_j\rangle$
$\mathbb{1} = \int dq_i |q\rangle\langle q|$
$e^{-iHs} |q\rangle$
K_{q_1,q_0}

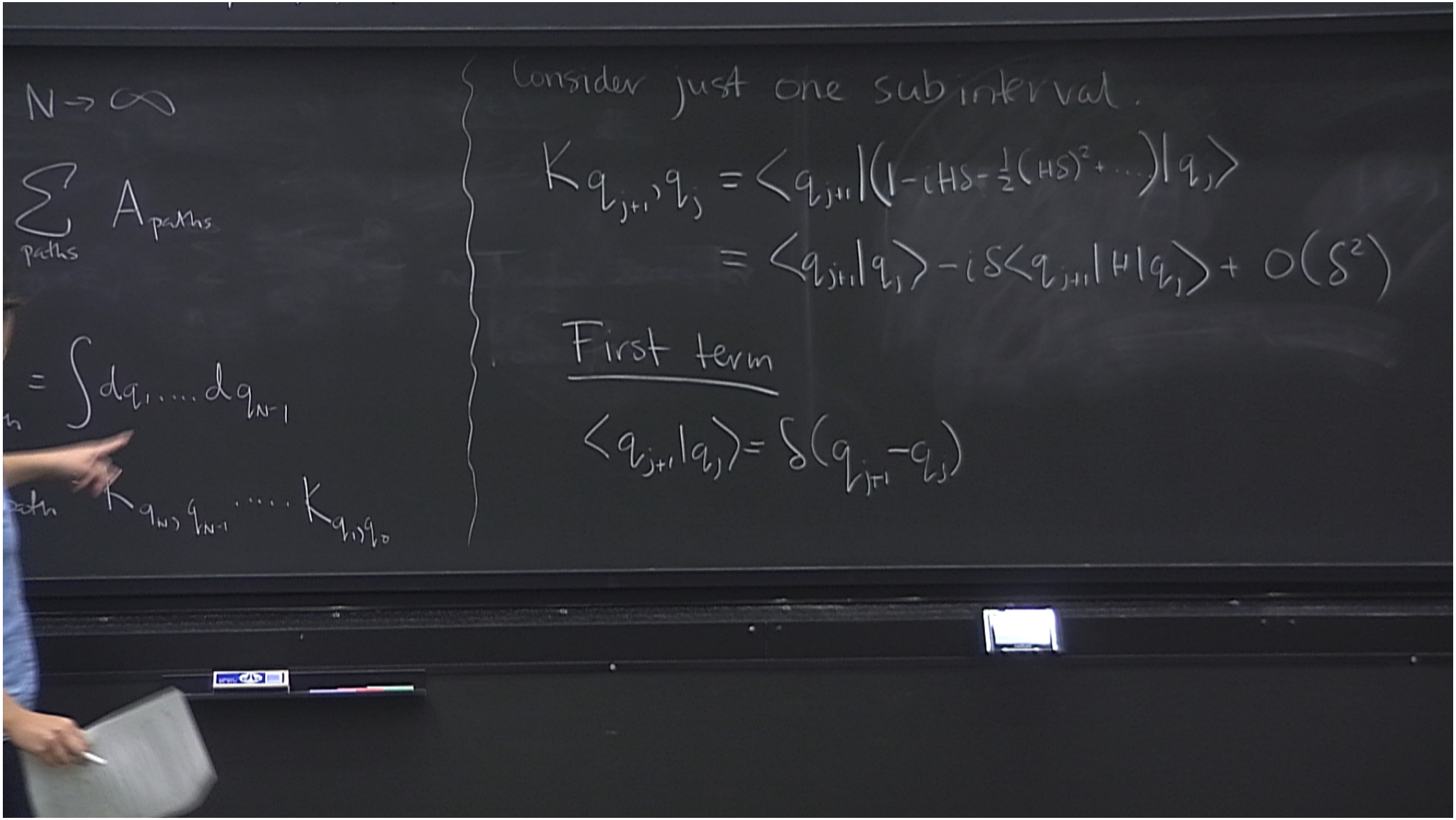

and wher $K_{q_{j+1},q_j} \equiv \langle q_{j+1}| e^{-iHs} |q_j\rangle$
$\mathbb{1} = \int dq\, |q\rangle\langle q|$
$e^{-iHs} |q\rangle$
q_1, q_0
q
q_0
s
2s
3s
t

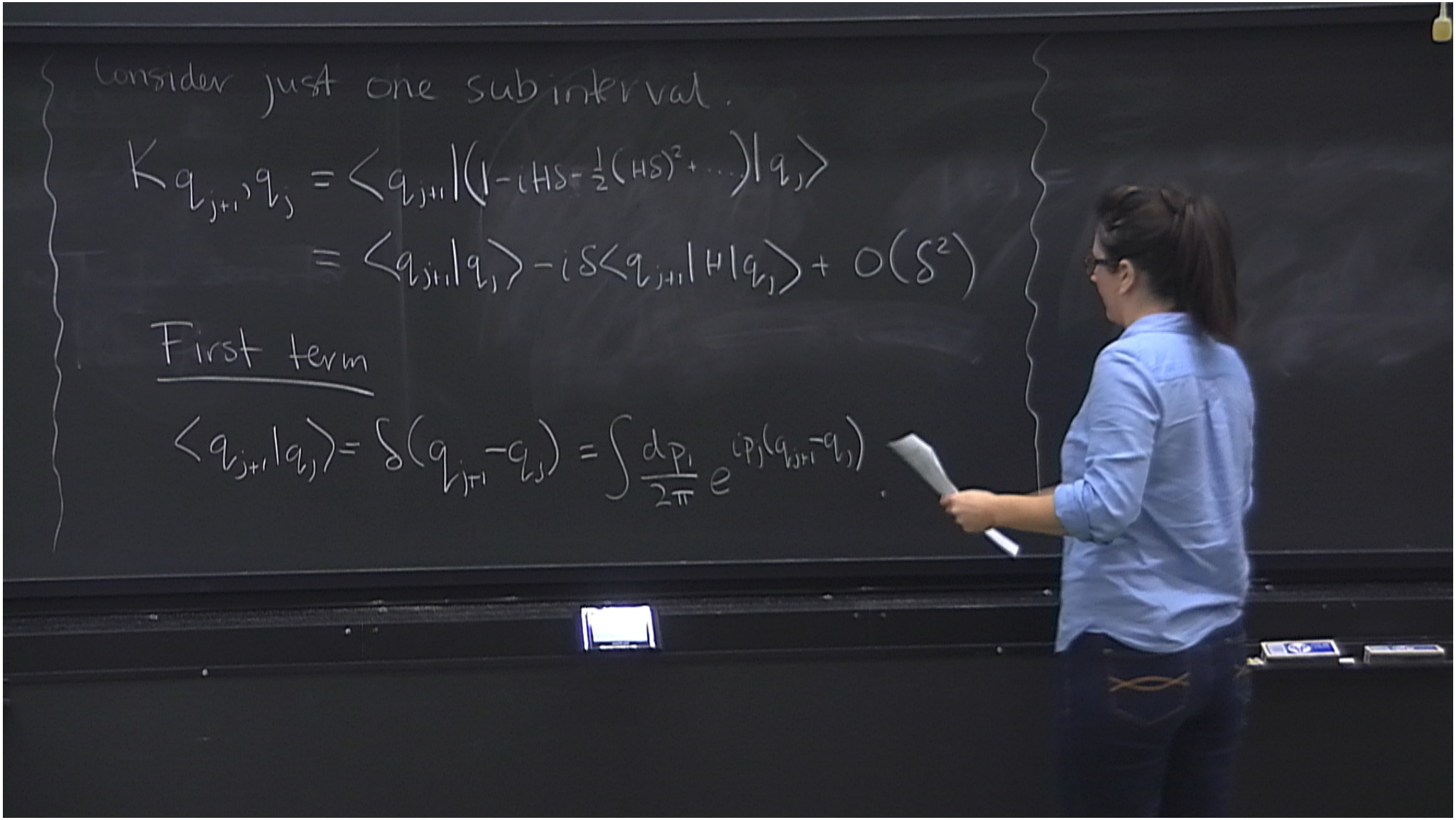

and wher $K_{q_{j+1},q_j} \equiv \langle q_{j+1} | e^{-iHs} | q_j \rangle$
$\mathbb{1} = \int dq\, |q\rangle\langle q|$
$e^{-iHs} |q\rangle$
q_1, q_0
q
q_0
s
2s
3s
(N-2)s

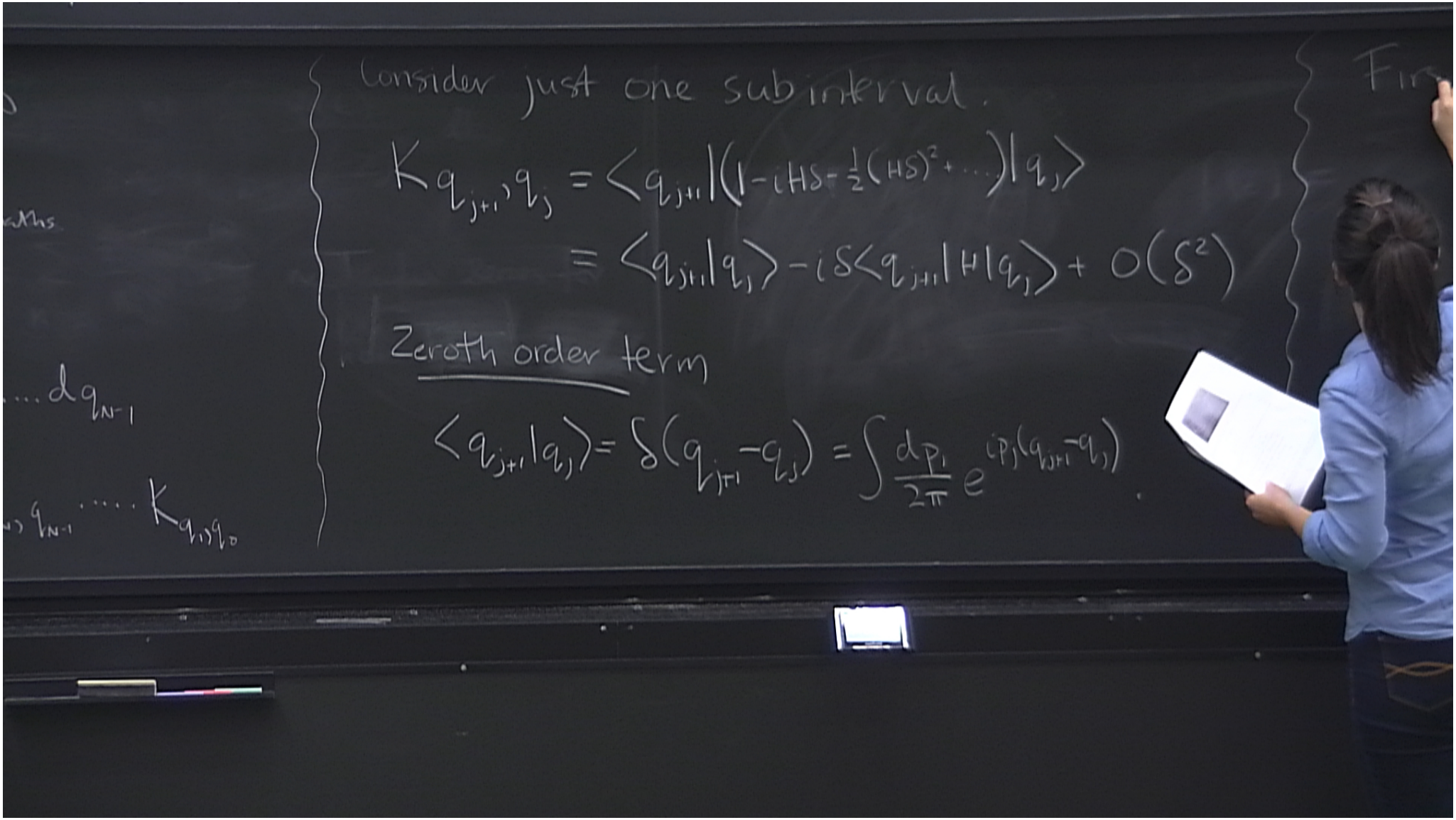

limit N → ∞
K = Σ_paths A_paths
where
Σ_path = ∫ dq_1 .... dq_{N-1}
A_path = K_{q_N, q_{N-1}} .... K_{q_1, q_0}

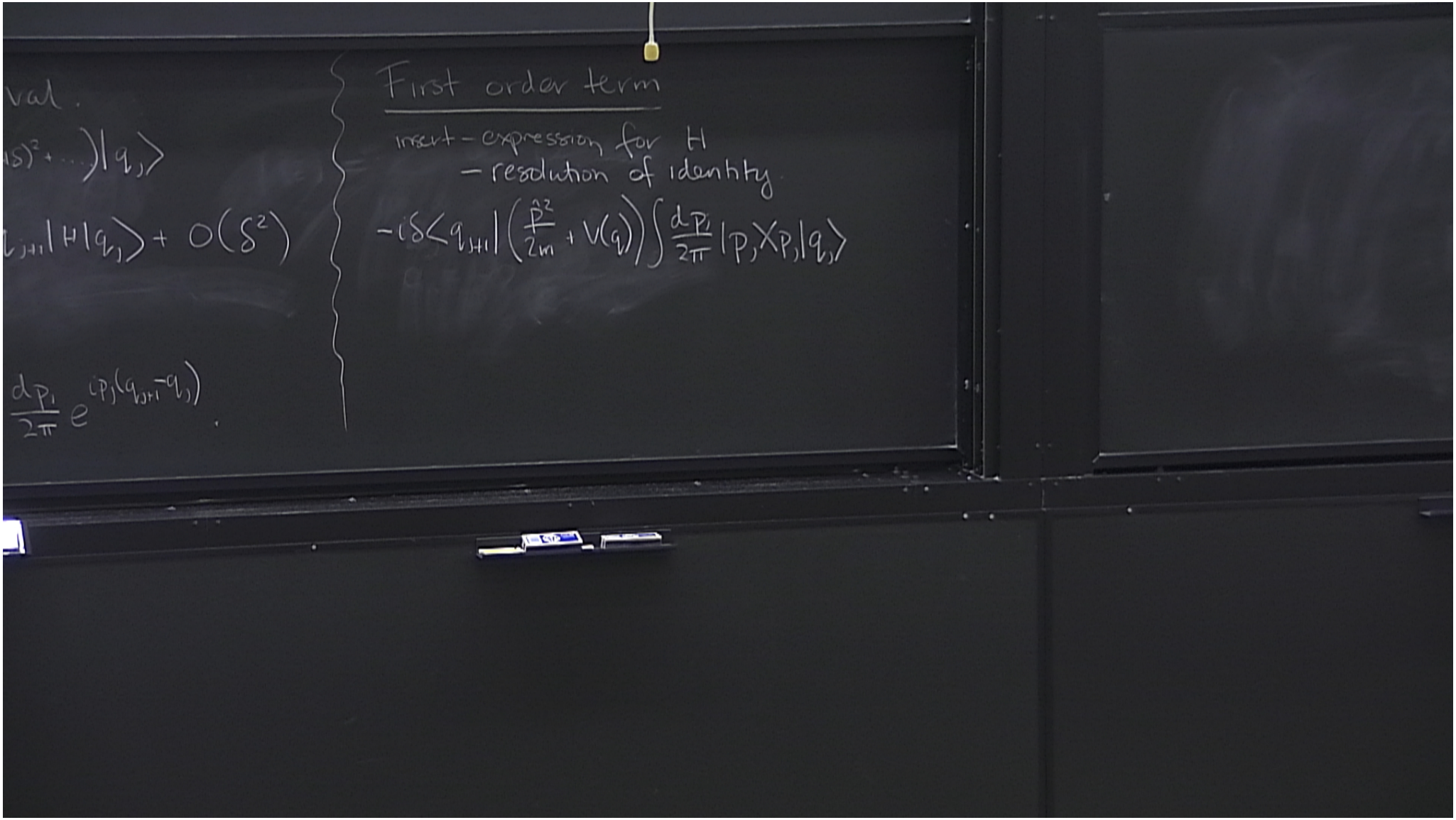

$N \to \infty$
$\sum_{\text{paths}} A_{\text{paths}}$
$= \int dq_1 \ldots dq_{N-1}$
$K_{q_N, q_{N-1}} \ldots K_{q_1, q_0}$
Consider just one subinterval.
$K_{q_{j+1}, q_j} = \langle q_{j+1} |$
$| q_j \rangle$

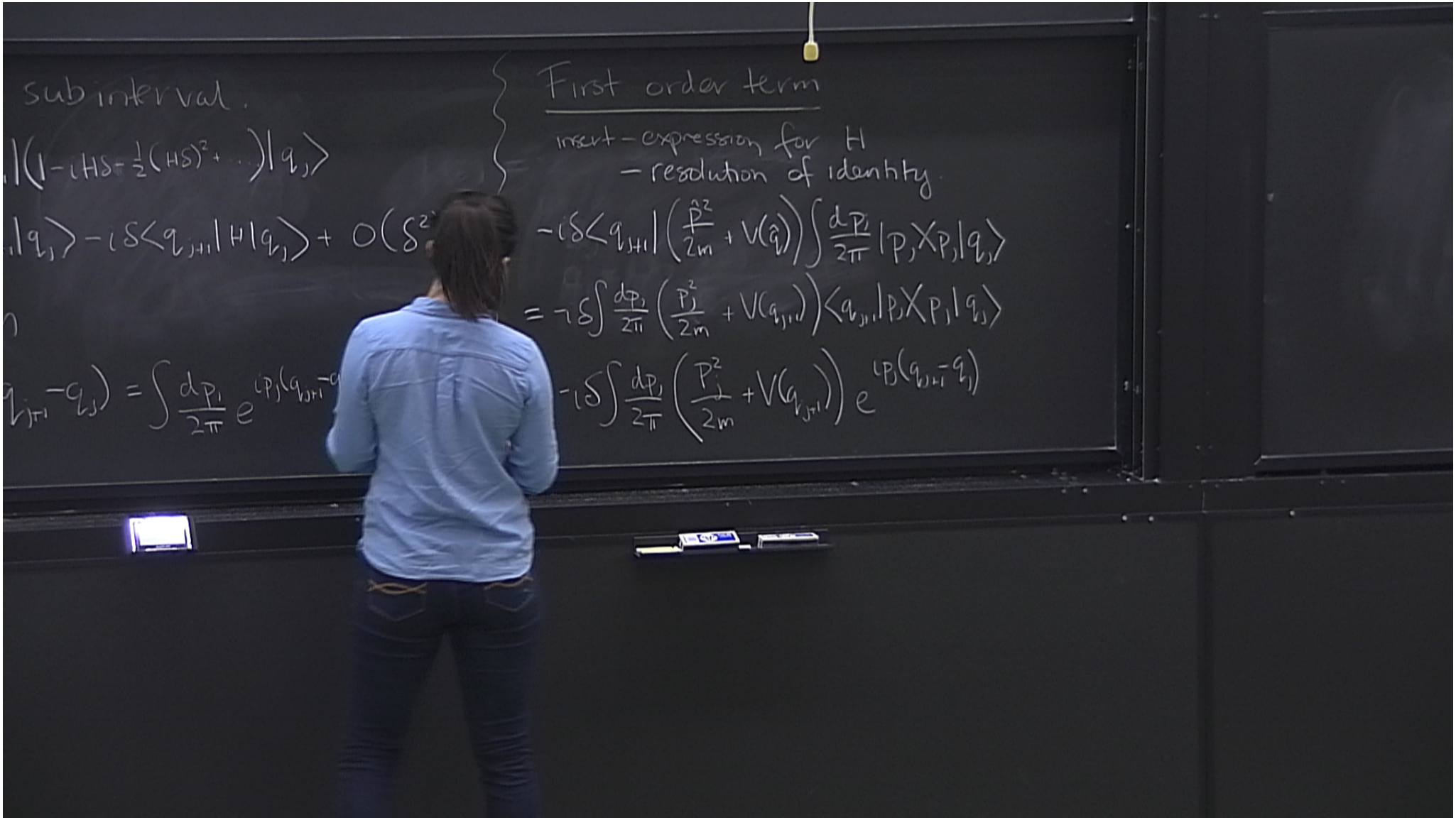

N → ∞
Consider just one sub interval.
$K_{q_{j+1},q_j} = \langle q_{j+1}|\left(1 - iHS - \frac{1}{2}(HS)^2 + \ldots\right)|q_j\rangle$
$= \langle q_{j+1}|q_j\rangle - iS\langle q_{j+1}|H|q_j\rangle + O(S^2)$
$\sum_{paths} A_{paths}$
$= \int dq_1 \ldots dq_{N-1}$
$K_{q_N,q_{N-1}} \ldots K_{q_1,}$

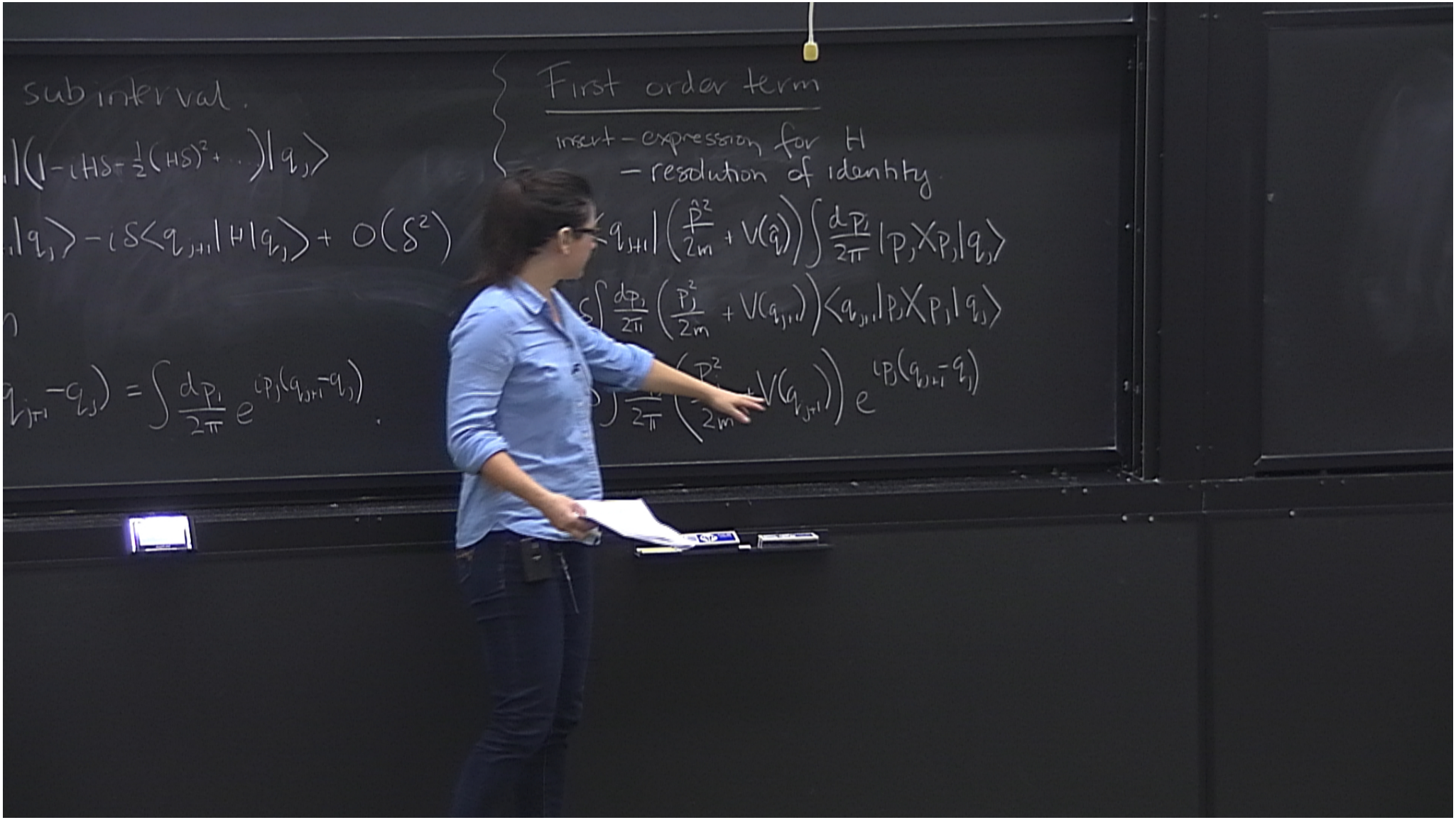

$N \to \infty$
$\sum_{\text{paths}} A_{\text{paths}}$
$= \int dq_1 \ldots dq_{N-1}$
$K_{q_N, q_{N-1}} \ldots K_{q_1, q_0}$
Consider just one sub interval.
$K_{q_{j+1}, q_j} = \langle q_{j+1} | (1 - iHS - \frac{1}{2}(HS)^2 + \ldots) | q_j \rangle$
$= \langle q_{j+1} | q_j \rangle - iS \langle q_{j+1} | H | q_j \rangle + O(S^2)$
First term
$\langle q_{j+1} | q_j \rangle = \delta(q_{j+1} - q_j)$

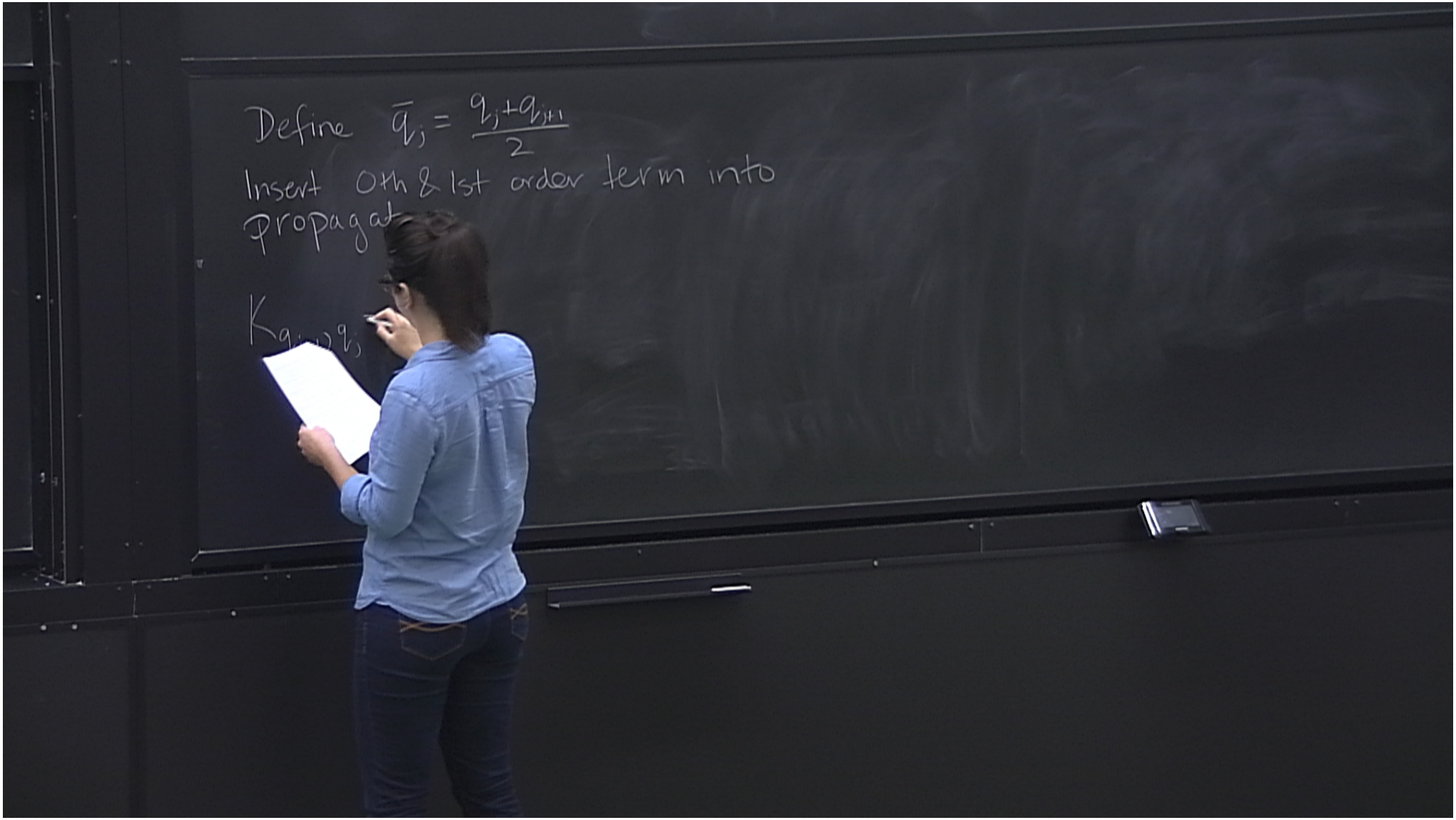

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$K_{q_{j+1},q_j} = \langle q_{j+1}|\left(1 - iHS - \frac{1}{2}(HS)^2 + \ldots\right)|q_j\rangle$
$= \langle q_{j+1}|q_j\rangle - i\delta\langle q_{j+1}|H|q_j\rangle + O(\delta^2)$
First term
$\langle q_{j+1}|q_j\rangle = \delta(q_{j+1} - q_j) = \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)}$

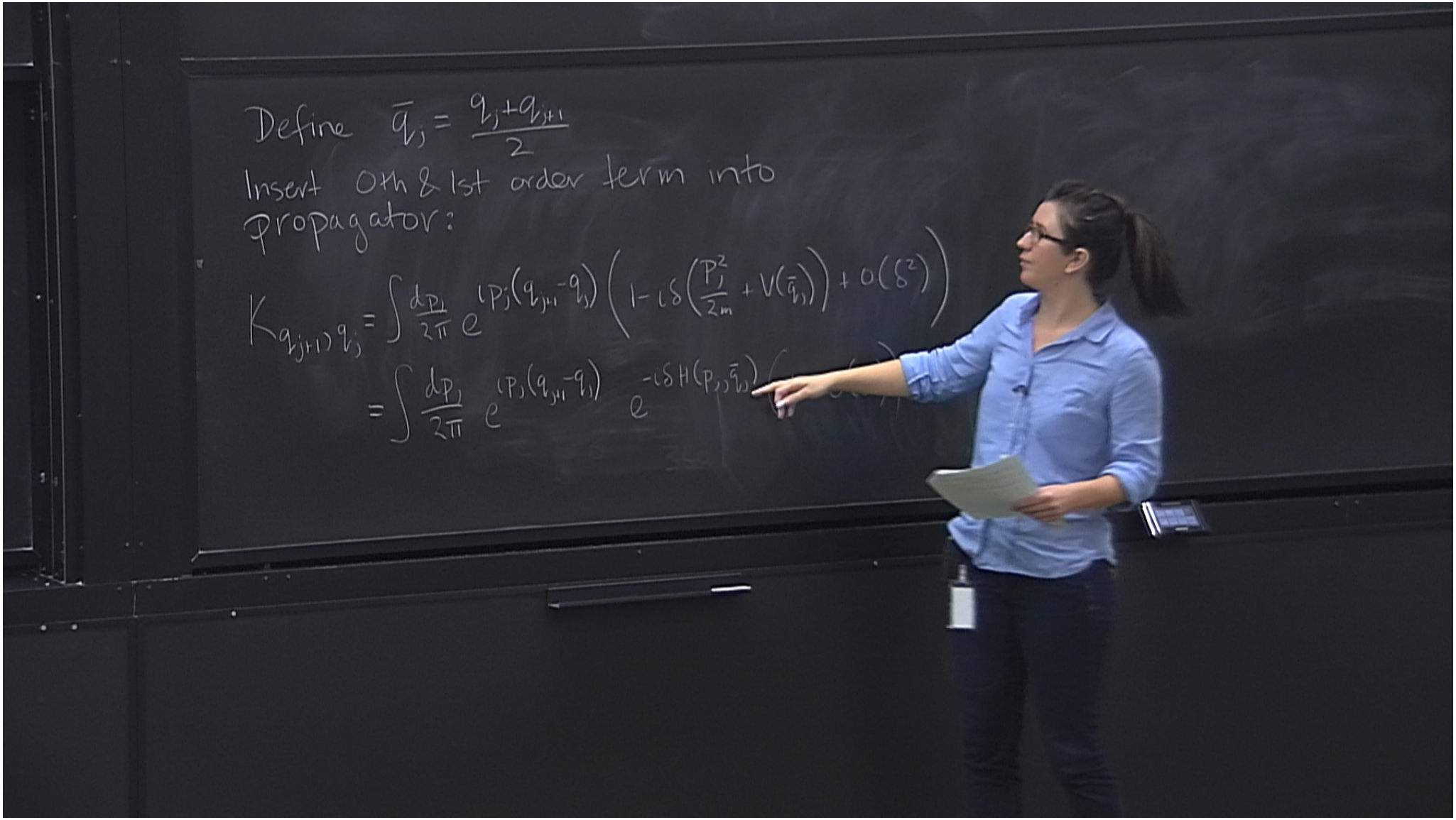

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$K_{q_{j+1},q_j} = \langle q_{j+1}|\left(1 - iHS - \frac{1}{2}(HS)^2 + \ldots\right)|q_j\rangle$
$= \langle q_{j+1}|q_j\rangle - iS\langle q_{j+1}|H|q_j\rangle + O(S^2)$
Zeroth order term
$\langle q_{j+1}|q_j\rangle = \delta(q_{j+1} - q_j) = \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)}$
$\ldots dq_{N-1}$
$q_{N-1} \ldots K_{q_1,q_0}$

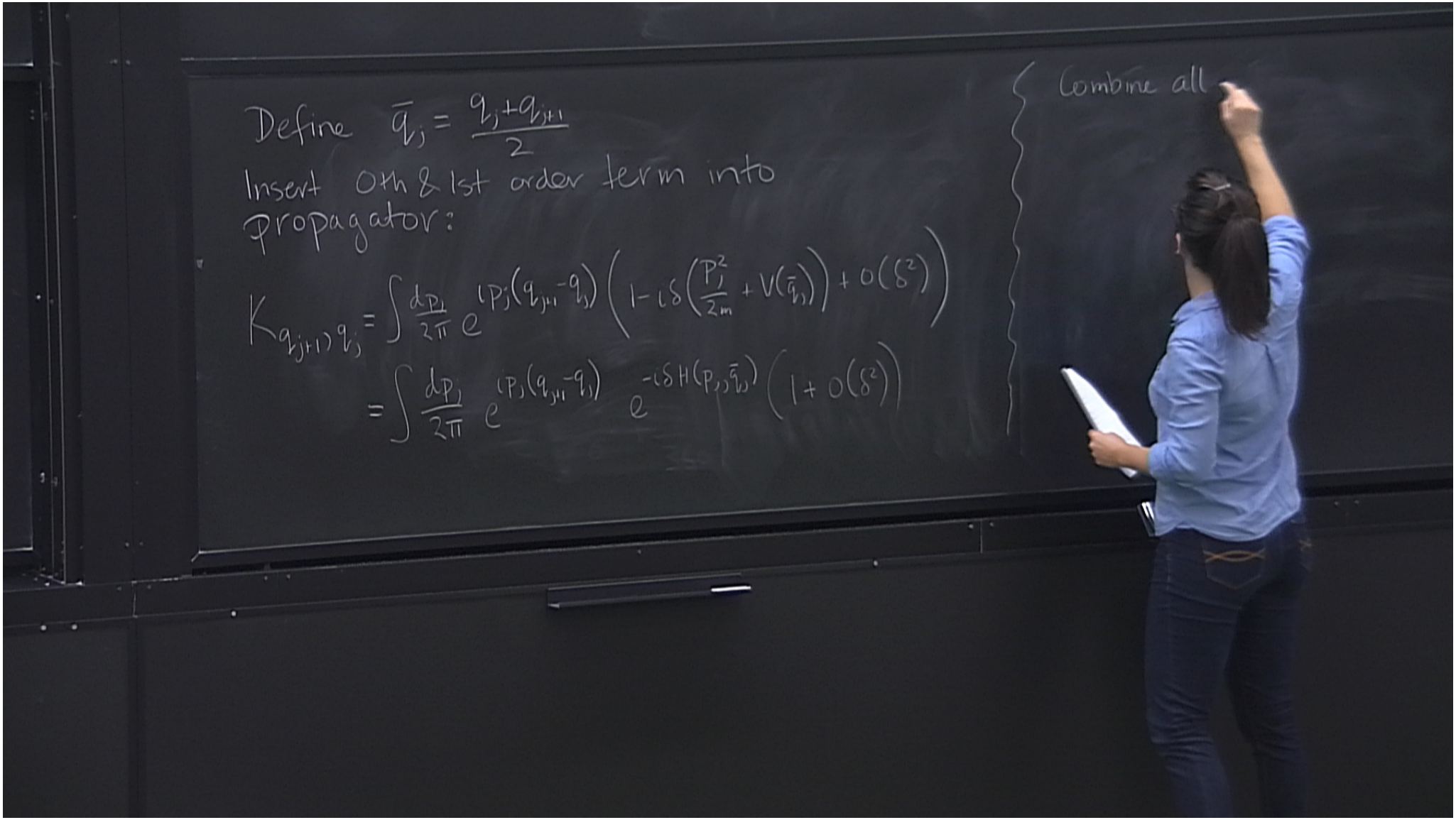

val.
$\delta)^2 + \ldots)|q_j\rangle$
$\langle q_{j+1}|H|q_j\rangle + O(\delta^2)$
$\frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)}$
First order term
insert - expression for H
- resolution of identity
$-i\delta\langle q_{j+1}|\left(\frac{\hat{p}^2}{2m} + V(q)\right)\int \frac{dp_j}{2\pi}|p_j\rangle\langle p_j|q_j\rangle$

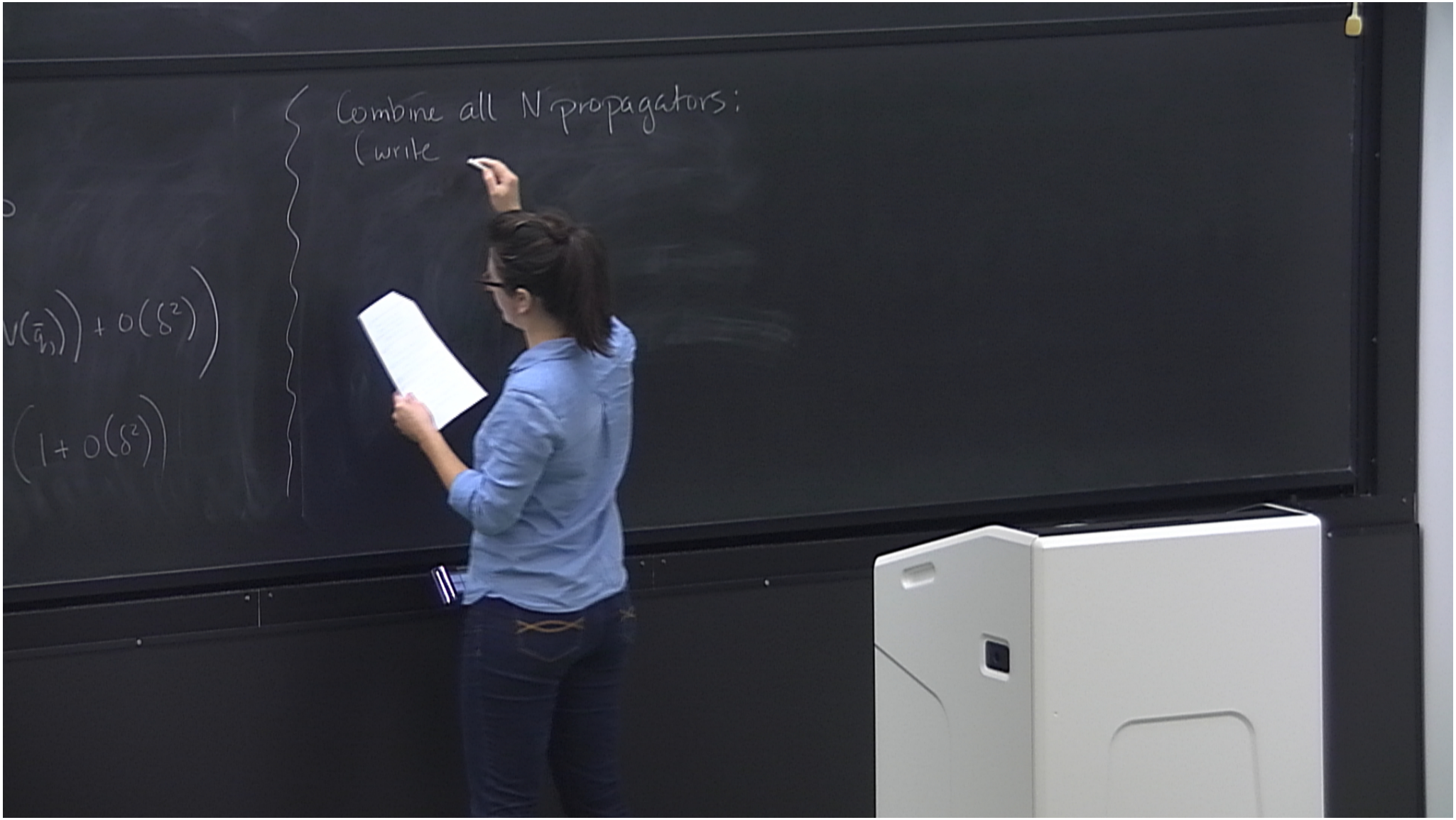

sub interval.
First order term
insert – expression for H
– resolution of identity.

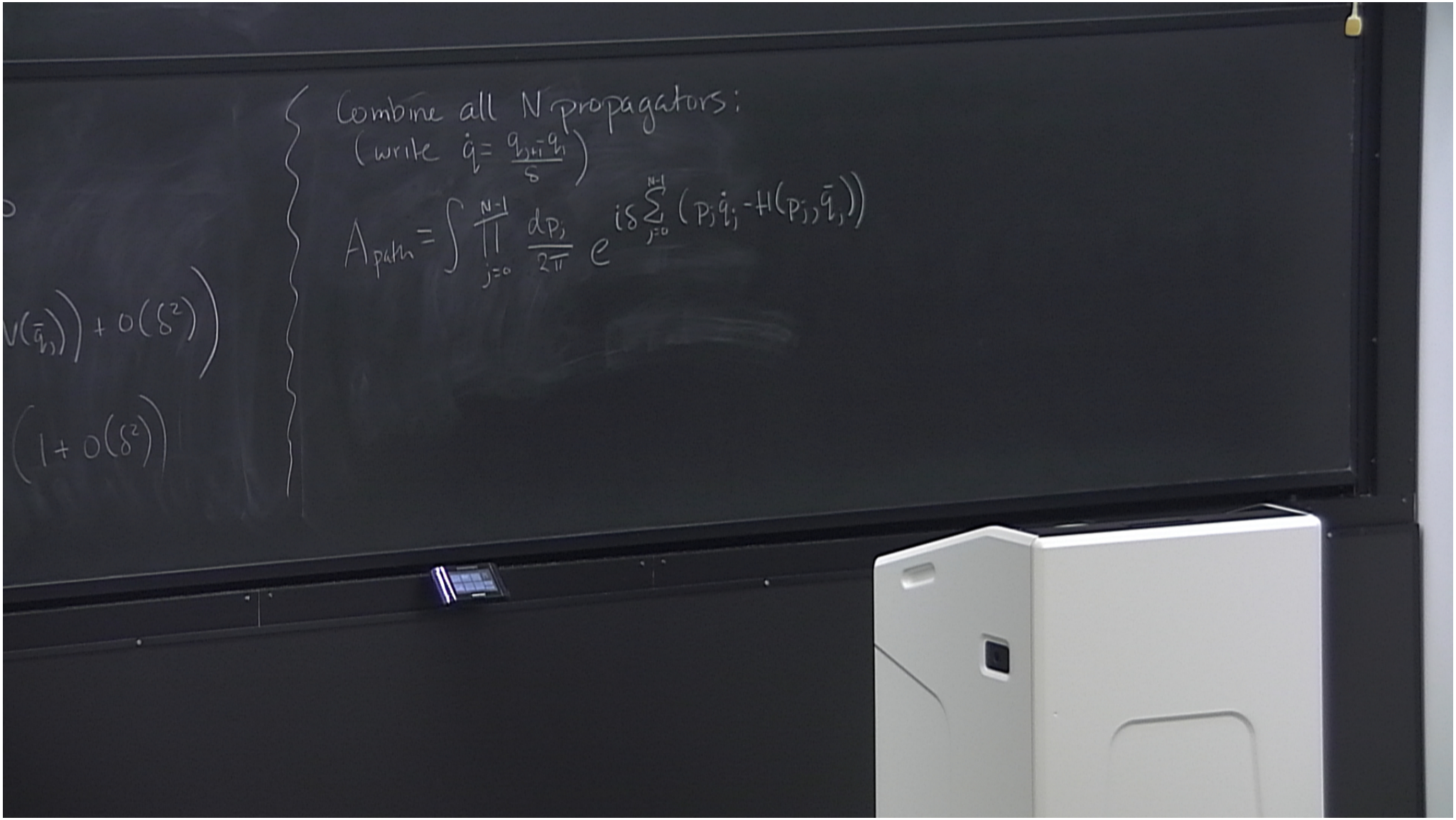

sub interval.
First order term
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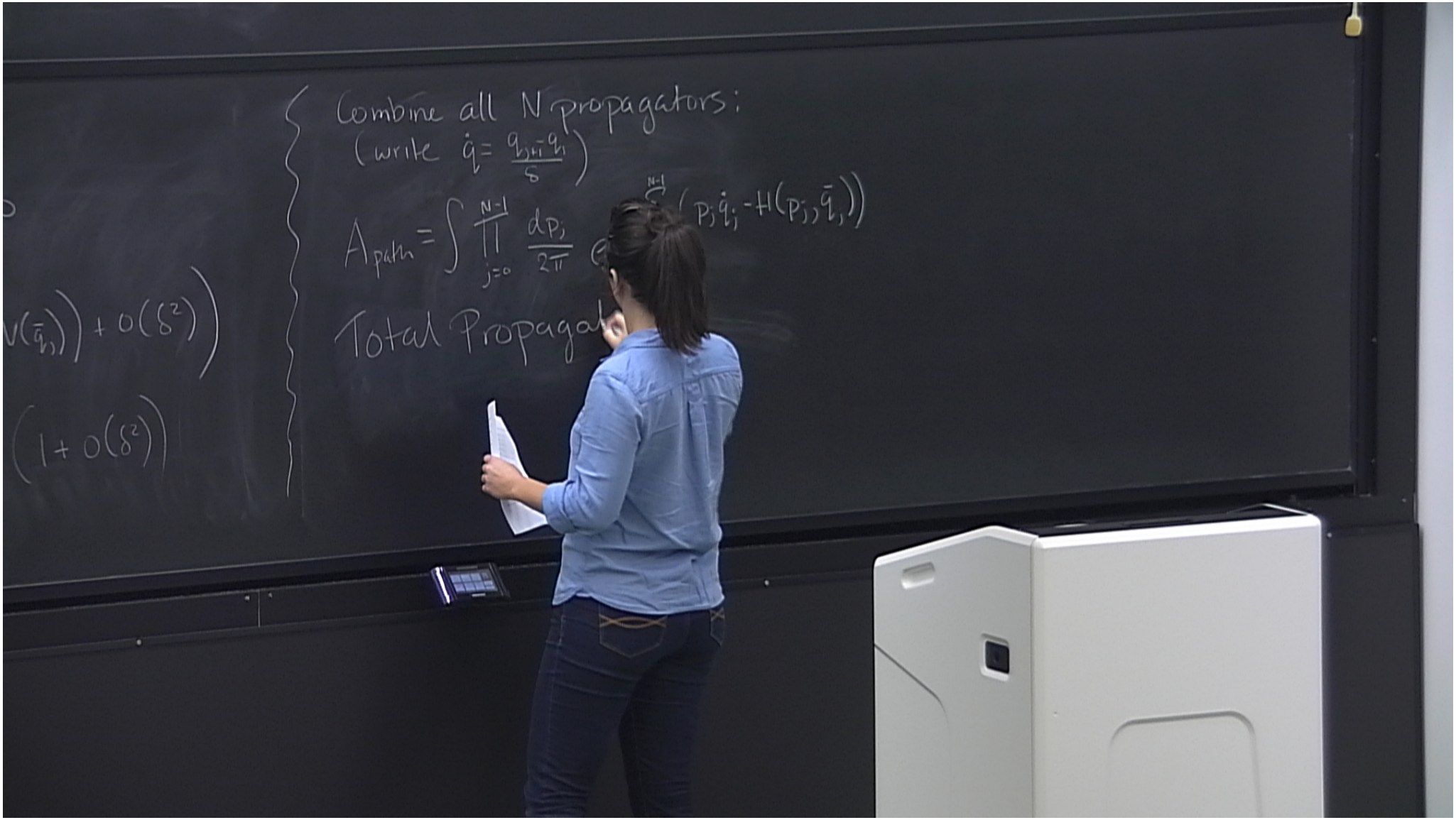

Define $\bar{q}_j = \frac{q_j + q_{j+1}}{2}$
Insert 0th & 1st order term into
propagat

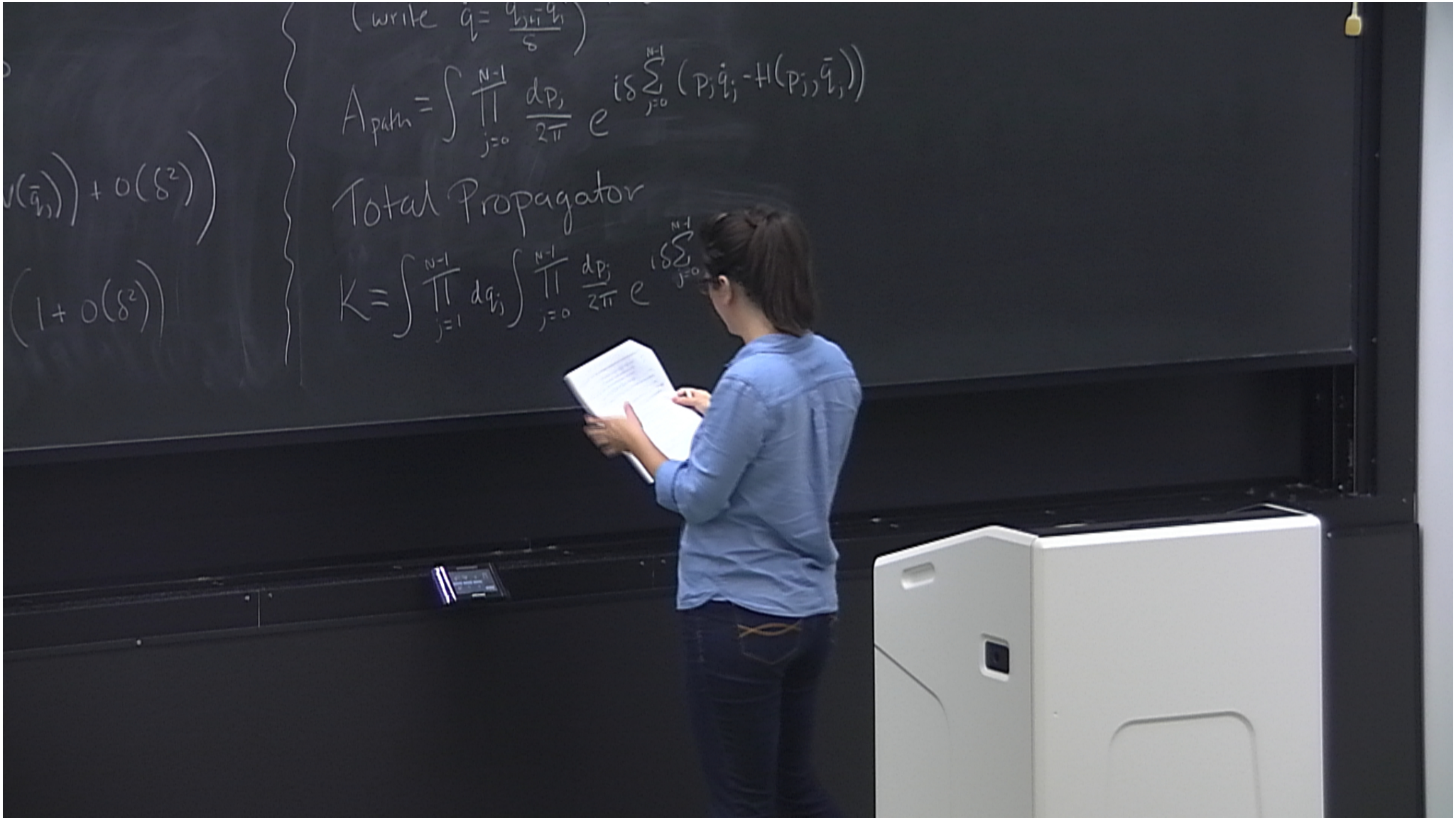

Define $\bar{q}_j = \frac{q_j + q_{j+1}}{2}$
Insert 0th & 1st order term into propagator:
$K_{q_{j+1}, q_j} = \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1} - q_j)} \left(1 - iS\left(\frac{p_j^2}{2m} + V(\bar{q}_j)\right) + O(S^2)\right)$
$= \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1} - q_j)} \; e^{-iSH(p_j, \bar{q}_j)}$

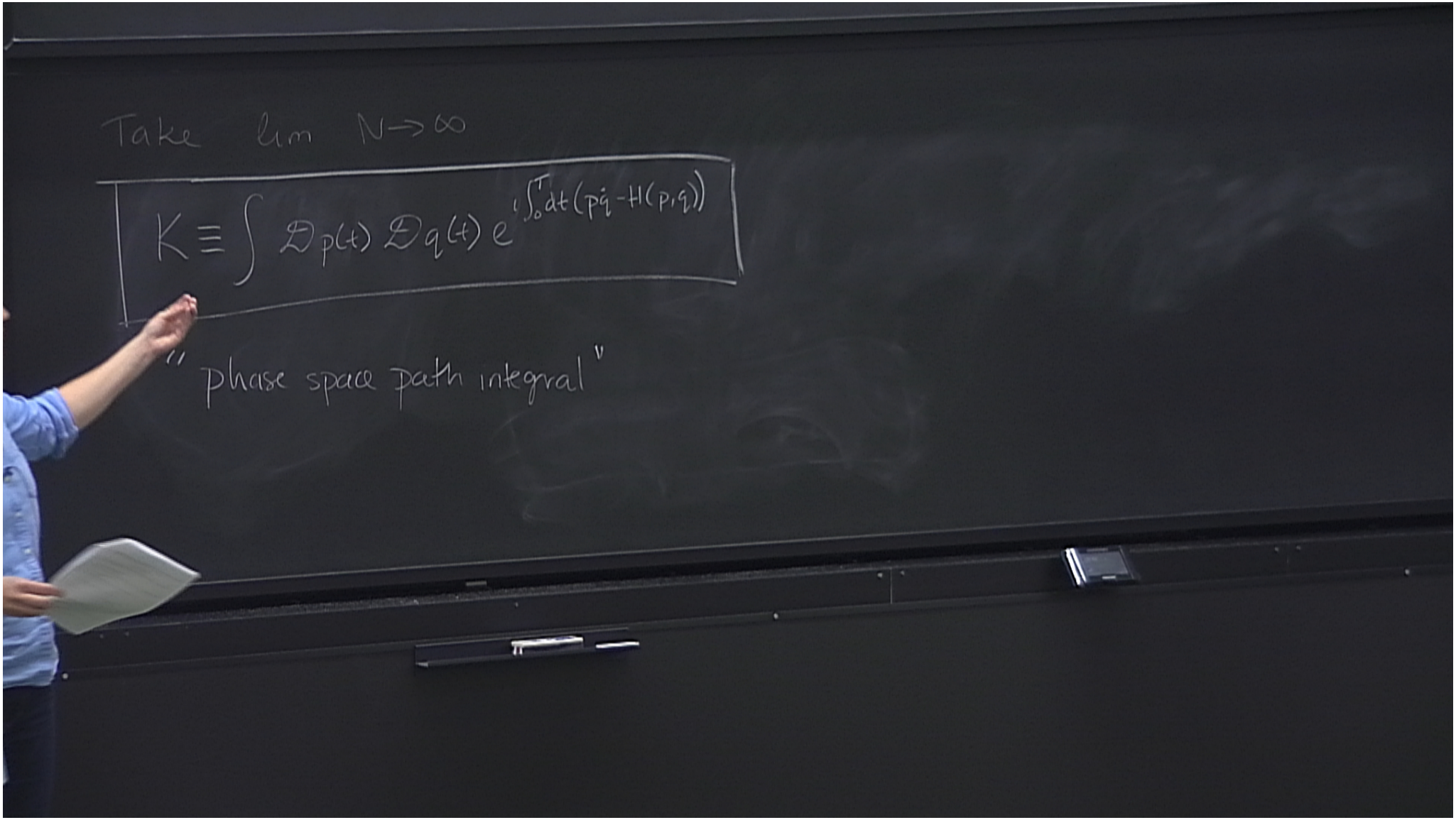

Define $\bar{q}_j = \frac{q_j + q_{j+1}}{2}$
Insert 0th & 1st order term into propagator:
$K_{q_{j+1}, q_j} = \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1} - q_j)} \left(1 - i\delta\left(\frac{p_j^2}{2m} + V(\bar{q}_j)\right) + O(\delta^2)\right)$
$= \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1} - q_j)} \; e^{-i\delta H(p_j, \bar{q}_j)} \left(1 + O(\delta^2)\right)$
Combine all

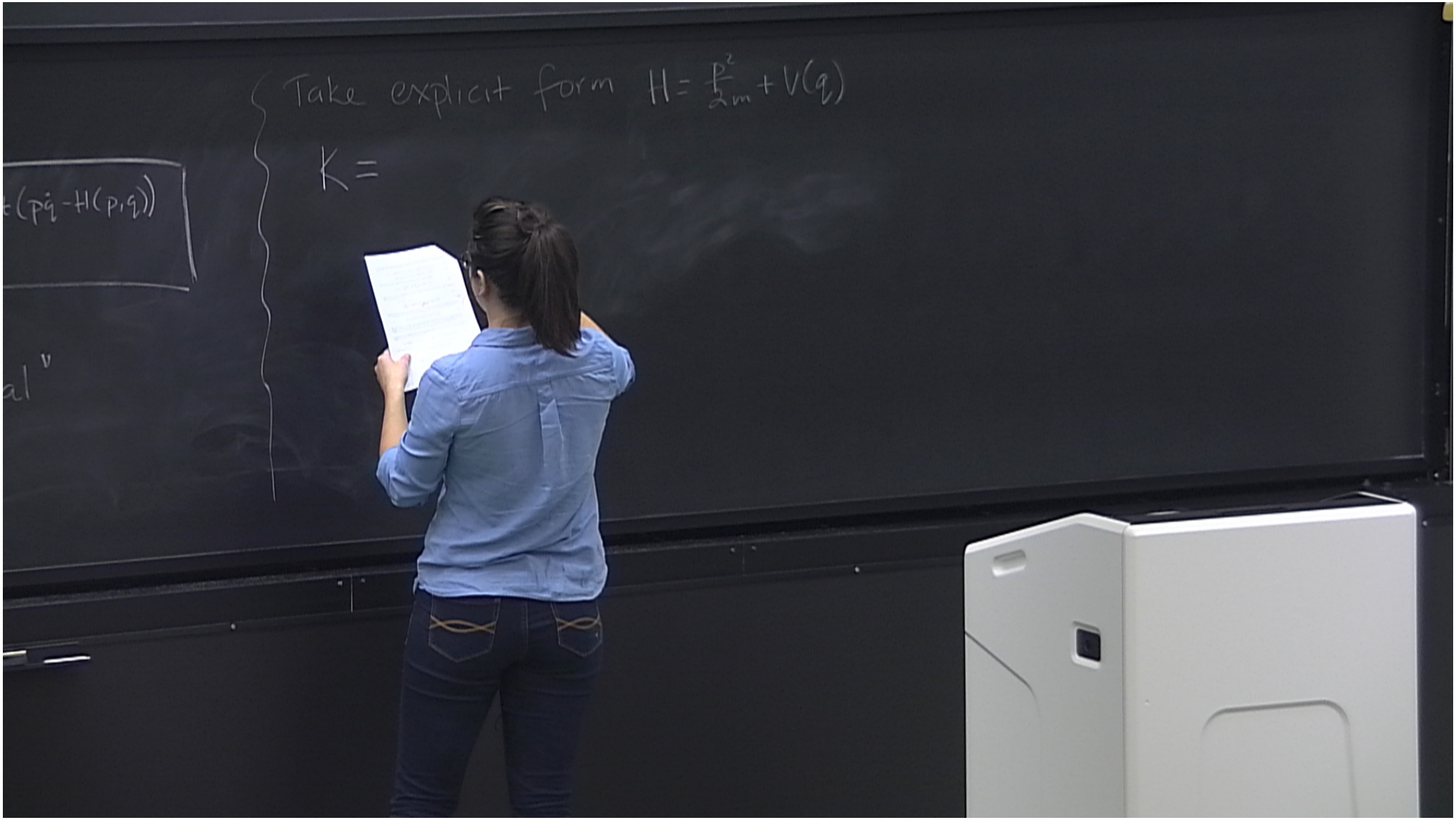

Combine all N propagators:
(write

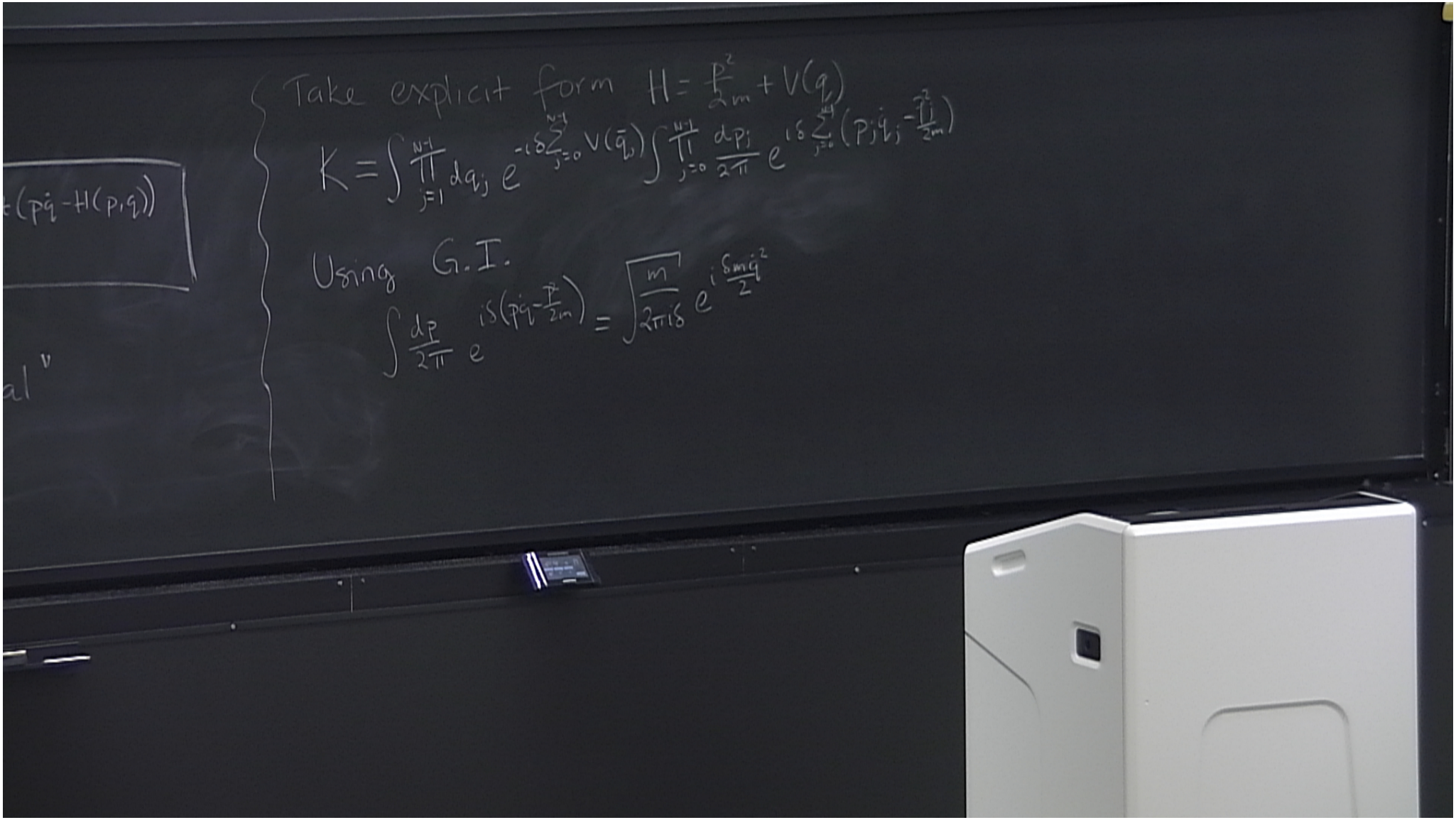

Combine all N propagators:
(write $\dot{q} = \frac{q_{j+1} - q_j}{\delta}$)
$A_{path} = \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi} e^{i\delta \sum_{j=0}^{N-1} (p_j \dot{q}_j - H(p_j, \bar{q}_j))}$
$V(\bar{q}_j)) + O(\delta^2))$
$(1 + O(\delta^2))$

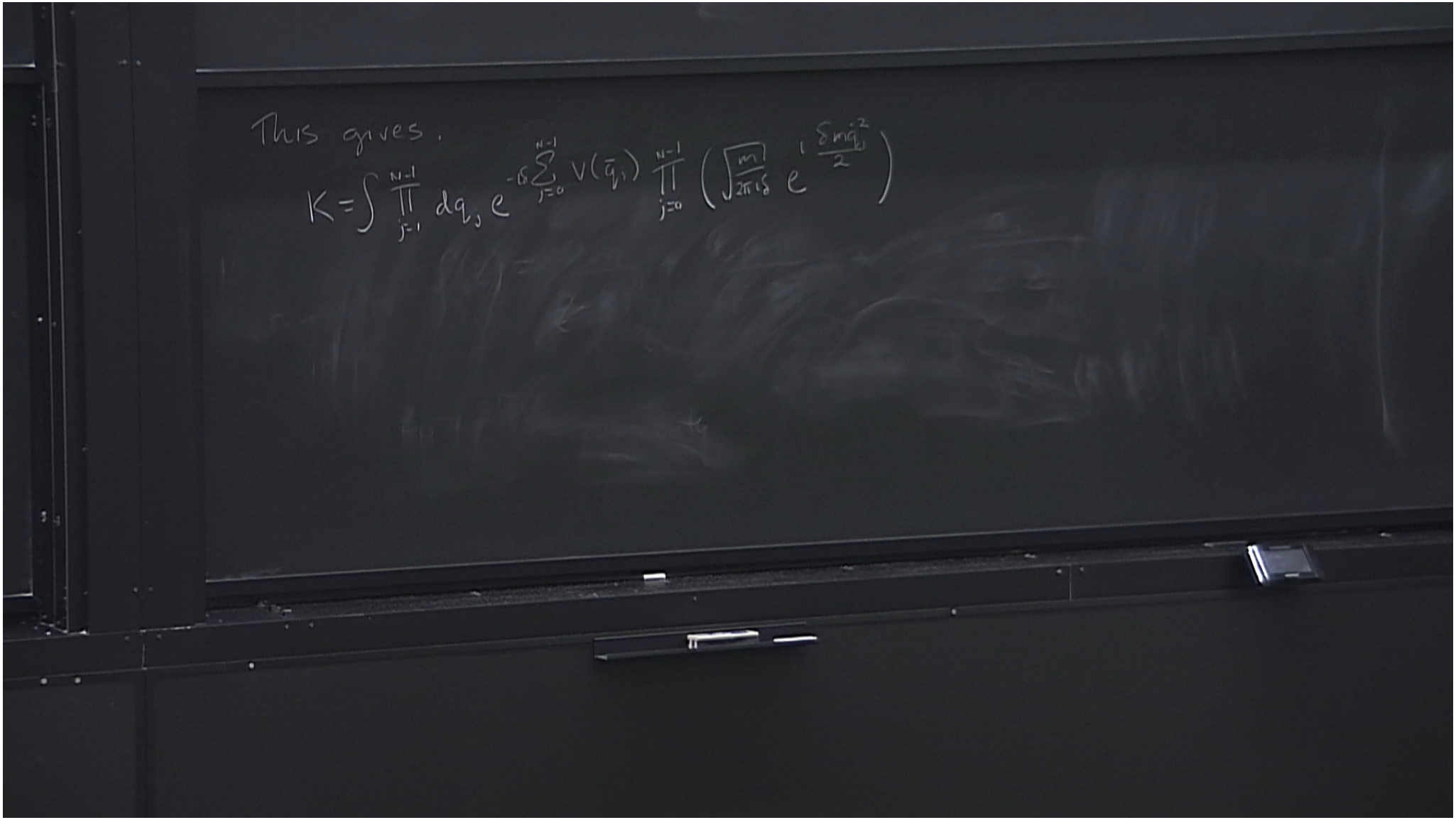

Combine all N propagators:
(write $\dot{q} = \frac{q_{j+1} - q_j}{\delta}$)
$A_{path} = \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi}$
$\sum_{j=0}^{N-1} (p_j \dot{q}_j - H(p_j, \bar{q}_j))$
Total Propaga
$V(\bar{q}_j)) + O(\delta^2))$
$(1 + O(\delta^2))$

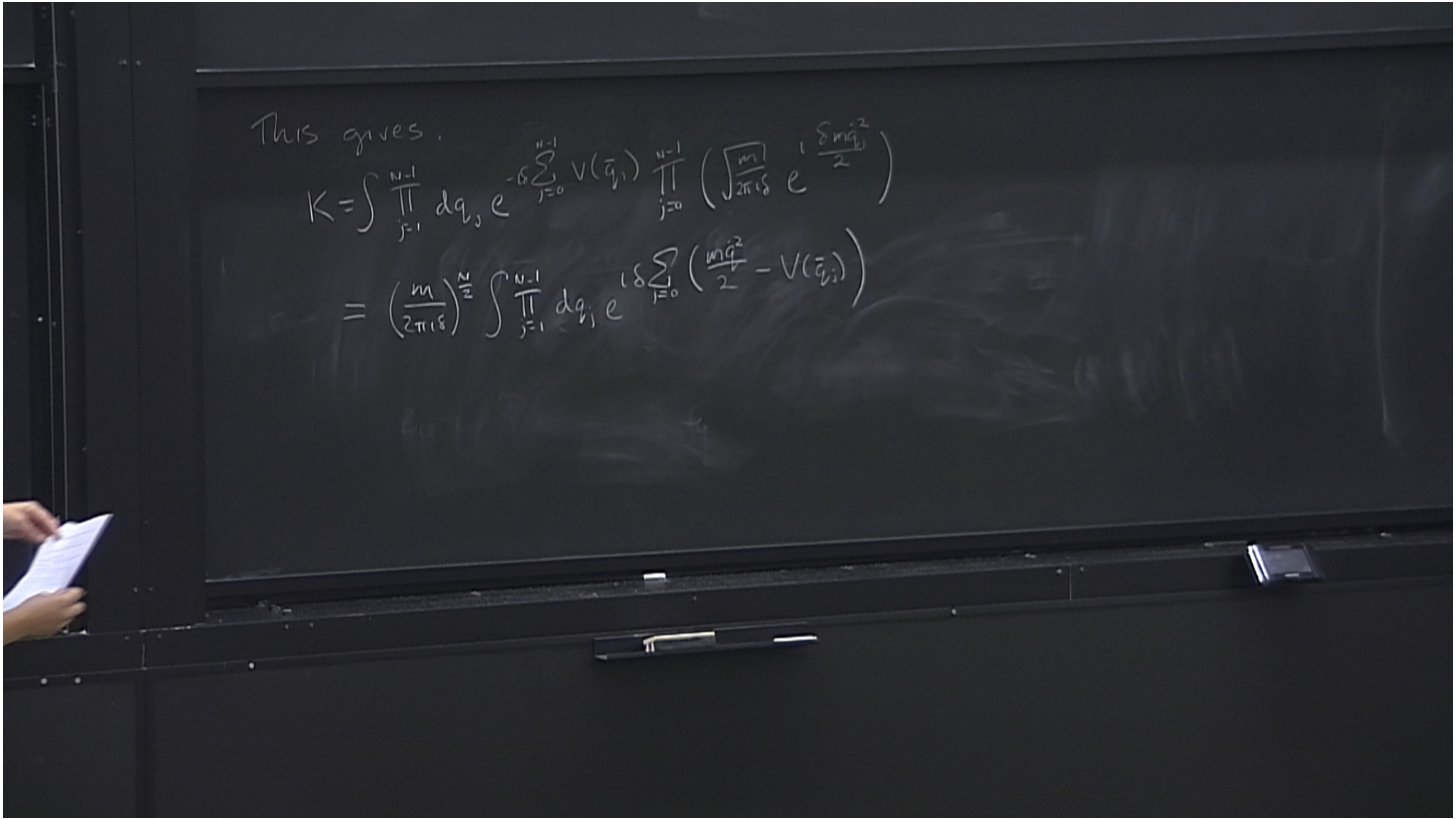

(write $\dot{q} = \frac{q_{j+1} - q_j}{\delta}$)
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Total Propagator
$K = \int \prod_{j=1}^{N-1} dq_j \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi} e^{i\delta \sum_{j=0}^{N-1}}$
$V(\bar{q}_j)) + O(\delta^2))$
$(1 + O(\delta^2))$

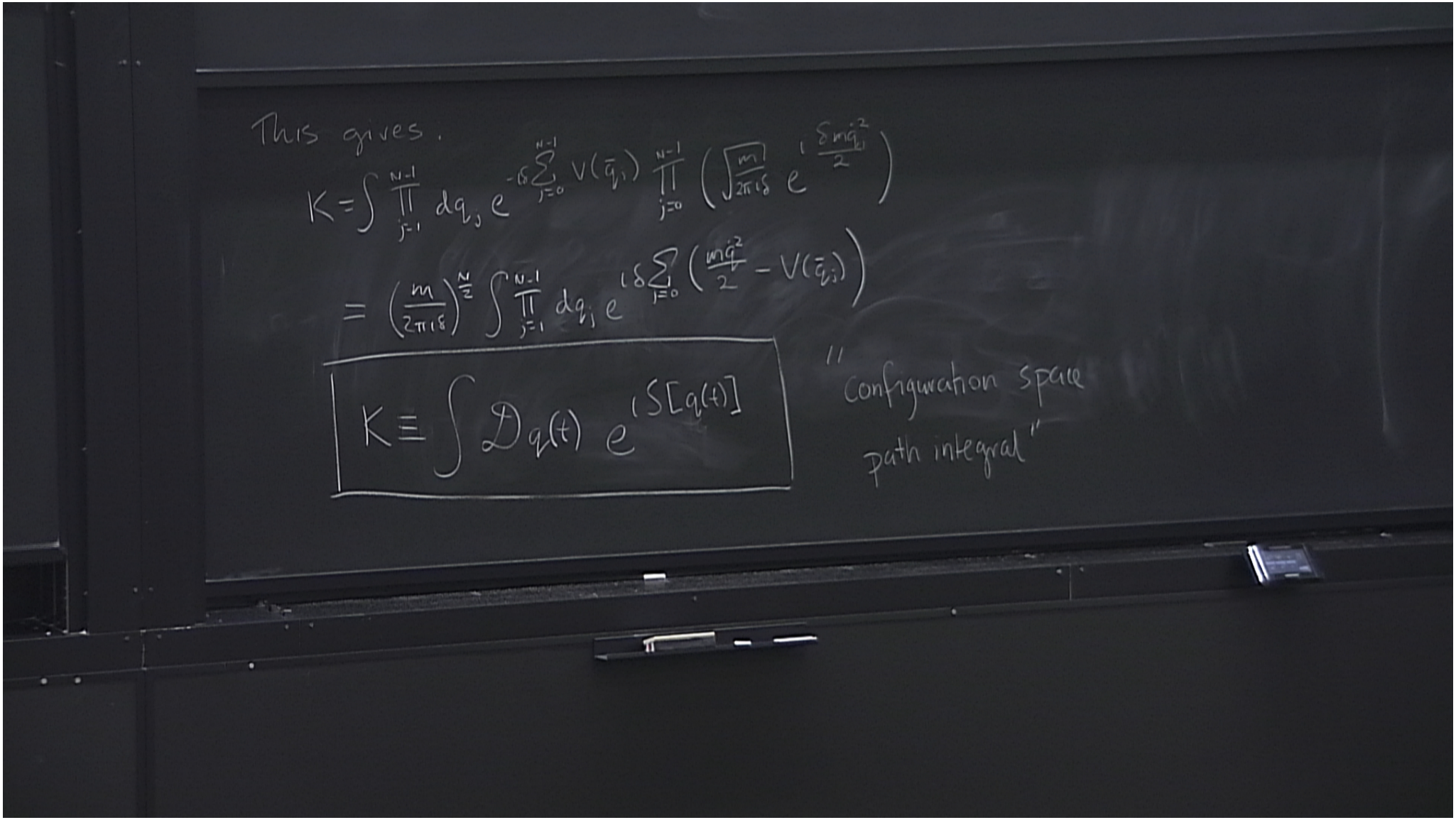

Take lim N→∞
$K \equiv \int \mathcal{D}p(t)\,\mathcal{D}q(t)\, e^{i\int_0^T dt\,(p\dot{q} - H(p,q))}$
" phase space path integral "

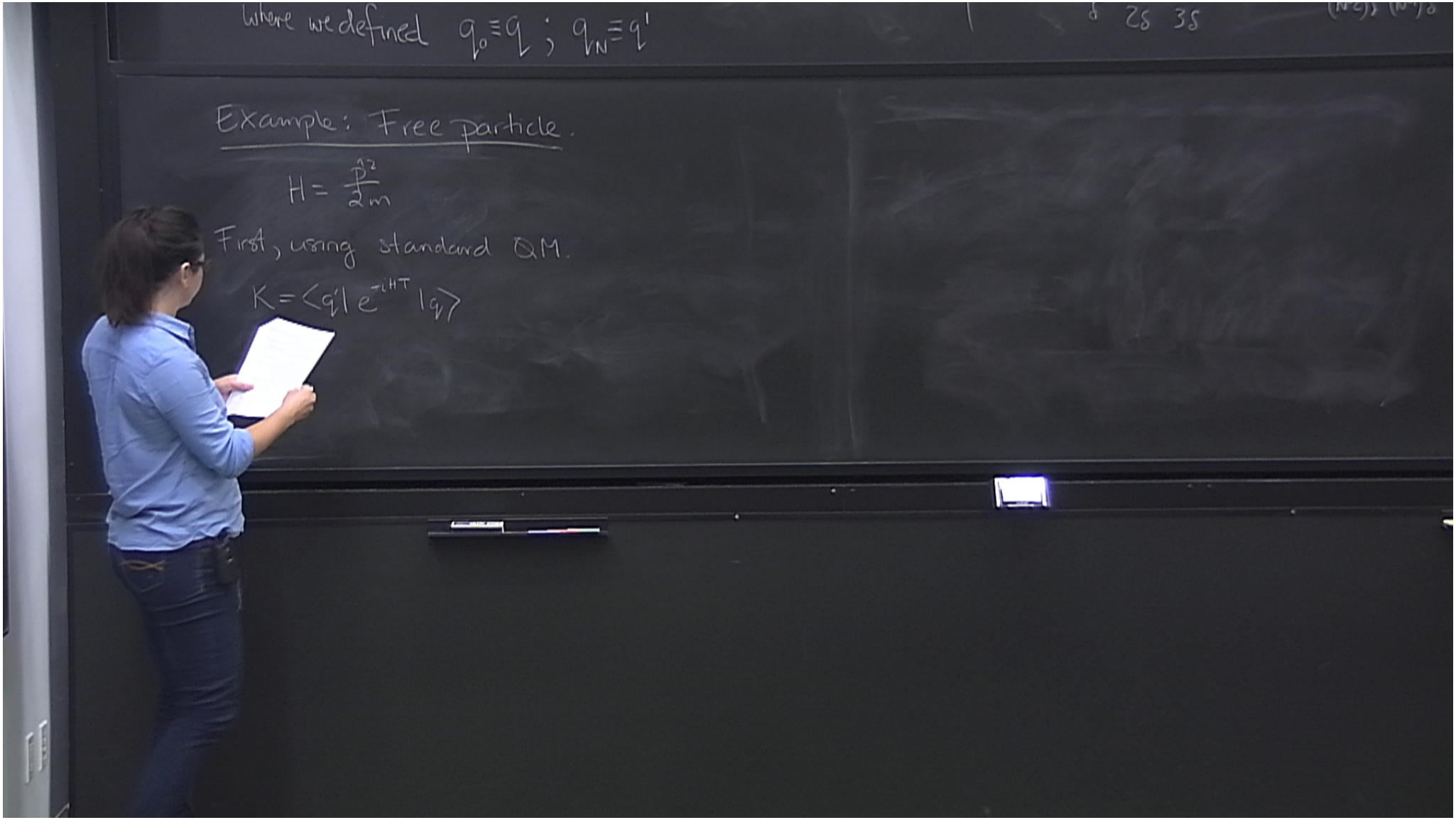

Take explicit form $H = \frac{p^2}{2m} + V(q)$
$K =$
$(p\dot{q} - H(p,q))$

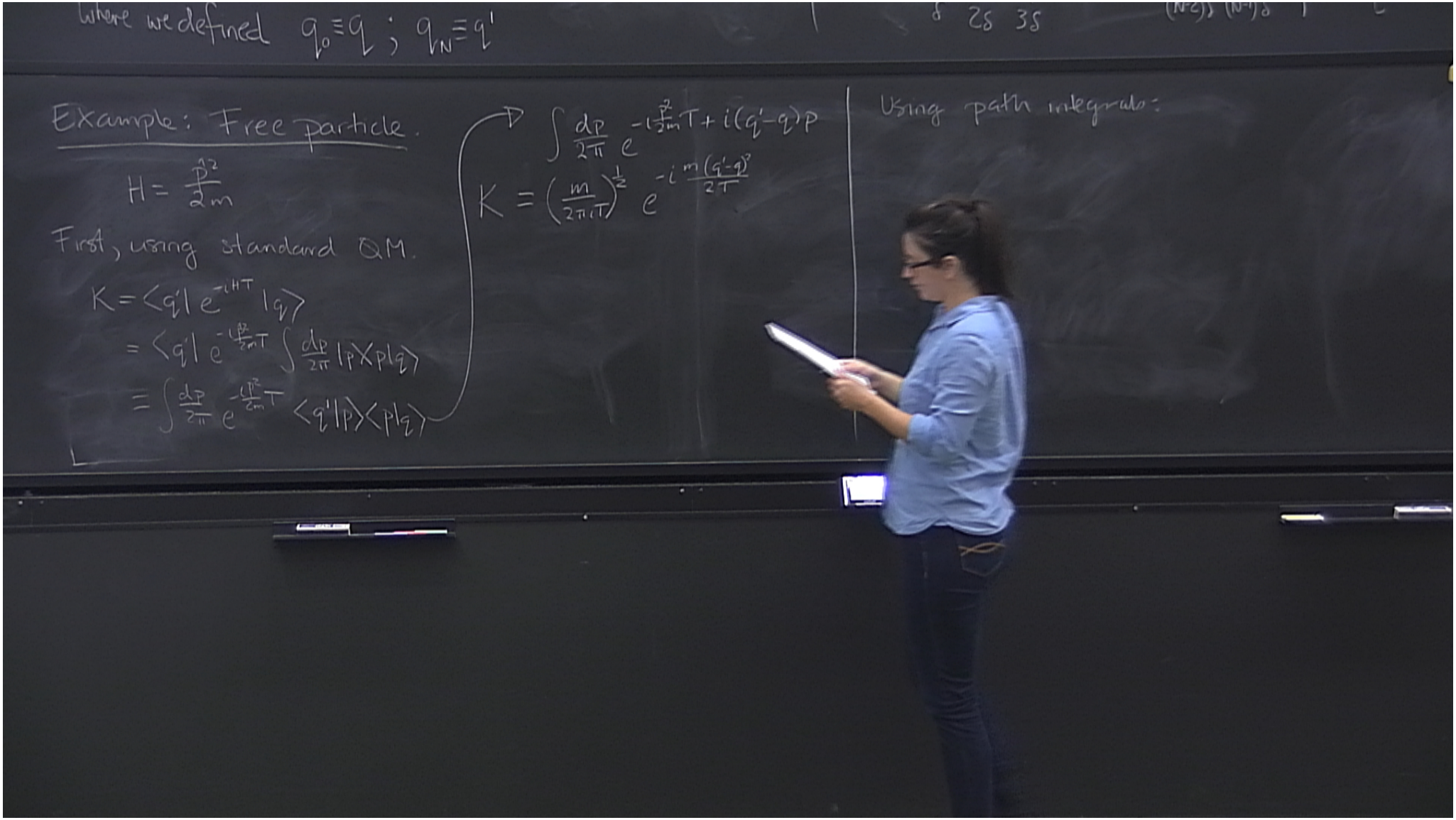

Take explicit form $H = \frac{p^2}{2m} + V(q)$
$K = \int \prod_{j=1}^{N-1} dq_j \, e^{-i\delta \sum_{j=0}^{N-1} V(\bar{q}_j)} \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi} e^{i\delta \sum_{j=0}^{N-1} (p_j \dot{q}_j - \frac{p_j^2}{2m})}$
Using G.I.
$\int \frac{dp}{2\pi} e^{is(p\dot{q} - \frac{p^2}{2m})} = \sqrt{\frac{m}{2\pi i \delta}} e^{i\frac{\delta m \dot{q}^2}{2}}$


This gives.
$K = \int \prod_{j=1}^{N-1} dq_j \, e^{-\beta \sum_{j=0}^{N-1} V(\bar{q}_j)} \prod_{j=0}^{N-1} \left(\sqrt{\frac{m}{2\pi i \delta}} \, e^{i \frac{\delta m \dot{q}_j^2}{2}} \right)$


This gives.
$K = \int \prod_{j=1}^{N-1} dq_j \, e^{-i\delta \sum_{j=0}^{N-1} V(\bar{q}_j)} \prod_{j=0}^{N-1} \left(\sqrt{\frac{m}{2\pi i\delta}} \, e^{i \frac{\delta m \dot{q}_j^2}{2}} \right)$
$= \left(\frac{m}{2\pi i\delta} \right)^{\frac{N}{2}} \int \prod_{j=1}^{N-1} dq_j \, e^{i\delta \sum_{j=0}^{N} \left(\frac{m\dot{q}^2}{2} - V(\bar{q}_j) \right)}$


This gives.
$K = \int \prod_{j=1}^{N-1} dq_j \, e^{-i\delta \sum_{j=0}^{N-1} V(\bar{q}_j)} \prod_{j=0}^{N-1} \left(\sqrt{\frac{m}{2\pi i\delta}} \, e^{i \frac{\delta m \dot{q}_j^2}{2}} \right)$
$= \left(\frac{m}{2\pi i\delta} \right)^{\frac{N}{2}} \int \prod_{j=1}^{N-1} dq_j \, e^{i\delta \sum_{j=0} \left(\frac{m\dot{q}^2}{2} - V(\bar{q}_j) \right)}$
$K \equiv \int \mathcal{D}q(t) \, e^{iS[q(t)]}$
"Configuration space path integral"


where we defined $q_0 \equiv q$; $q_N \equiv q'$
Example: Free particle.
$H = \frac{\vec{p}^2}{2m}$
First, using standard QM.
$K = \langle q' | e^{-iHT} | q \rangle$


where we defined $q_0 \equiv q$; $q_N \equiv q'$
Example: Free particle.
$H = \frac{\hat{p}^2}{2m}$
First, using standard QM.
$K = \langle q' | e^{-iHT} | q \rangle$
$= \langle q' | e^{-i\frac{\hat{p}^2}{2m}T} \int \frac{dp}{2\pi} |p\rangle\langle p| q \rangle$
$= \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}T} \langle q'|p\rangle\langle p|q\rangle$
$\int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}T + i(q'-q)p}$
$K = \left(\frac{m}{2\pi i T}\right)^{\frac{1}{2}} e^{-i\frac{m(q'-q)^2}{2T}}$
Using path integrals:

$q_0 \equiv q$; $q_N = q'$

δ 2δ 3δ $(N-2)\delta$ $(N-1)\delta$

particle.

d QM.

$\frac{dp}{2\pi} |p\rangle\langle p|q\rangle$

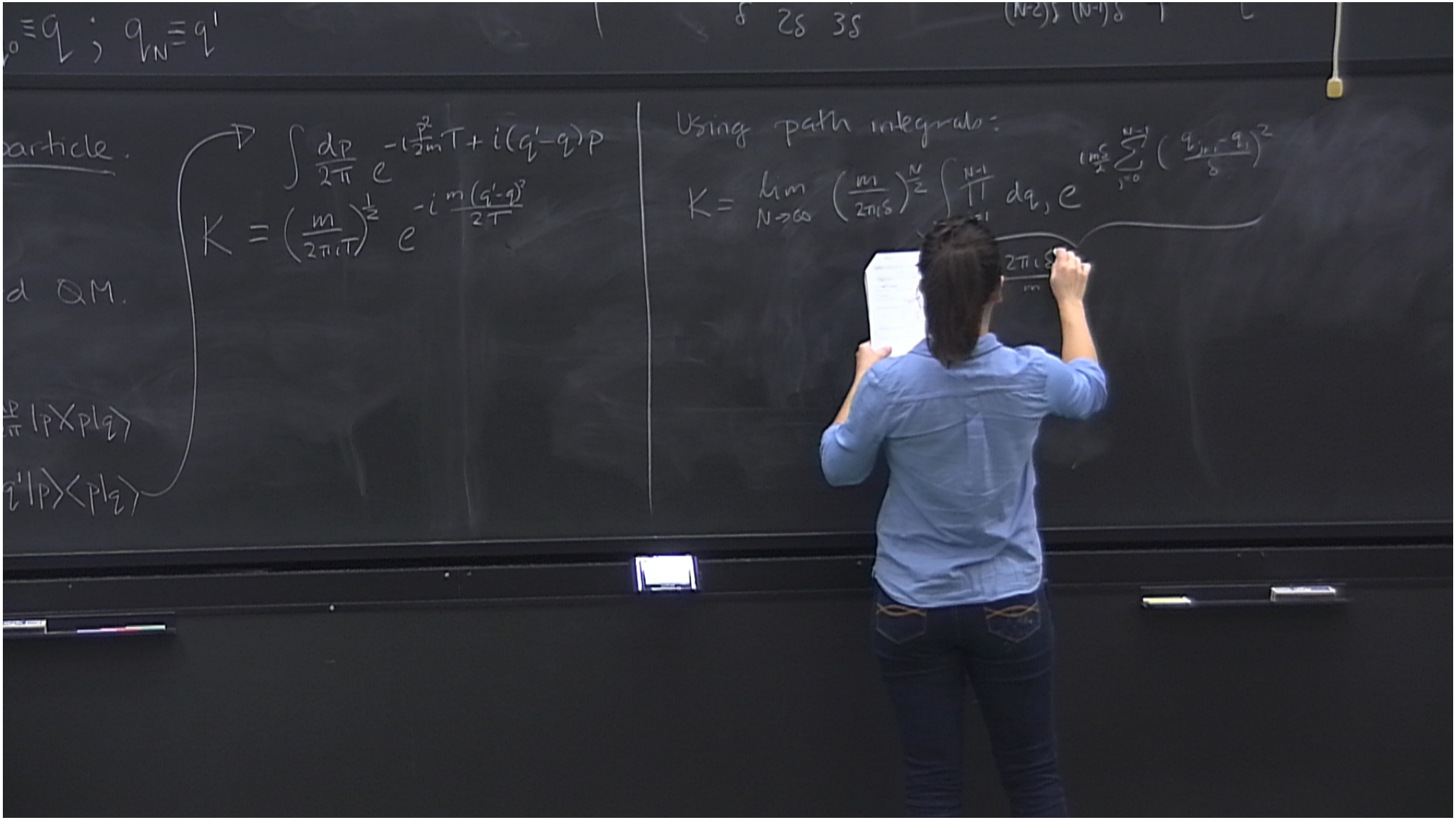
$\langle q'|p\rangle\langle p|q\rangle$

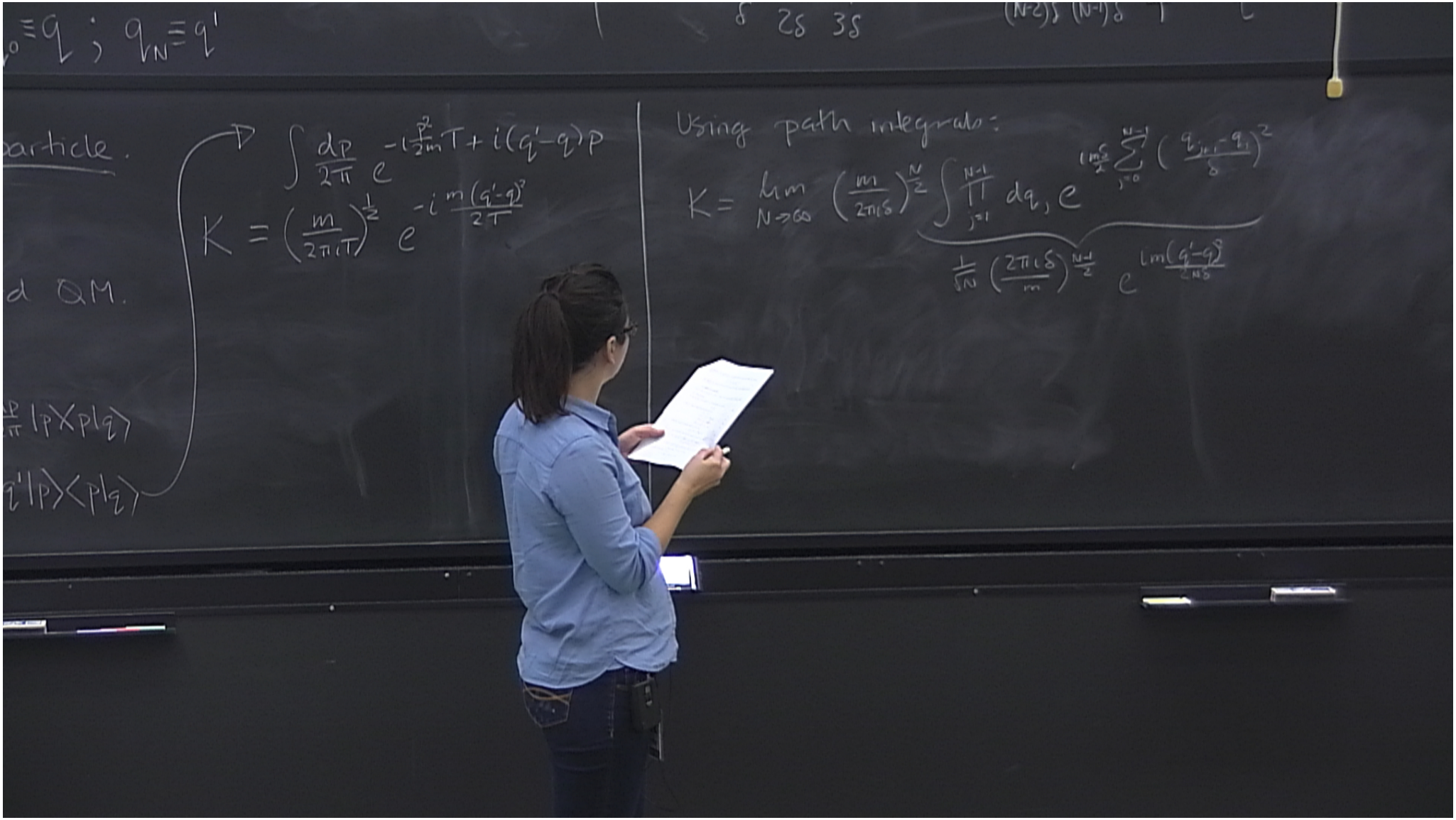
$$\int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}T + i(q'-q)p}$$

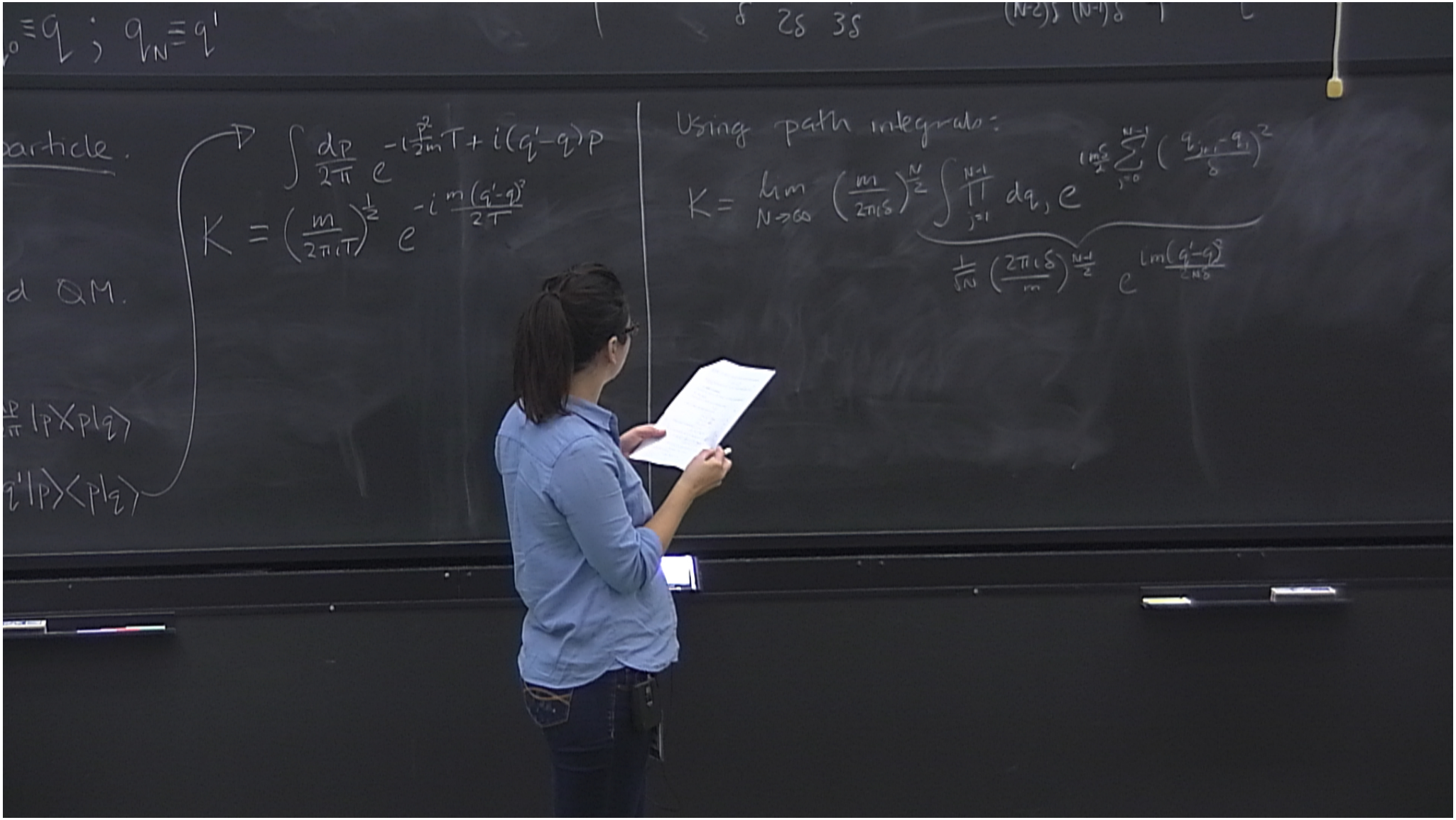
$$K = \left(\frac{m}{2\pi i T}\right)^{\frac{1}{2}} e^{-i\frac{m(q'-q)^2}{2T}}$$

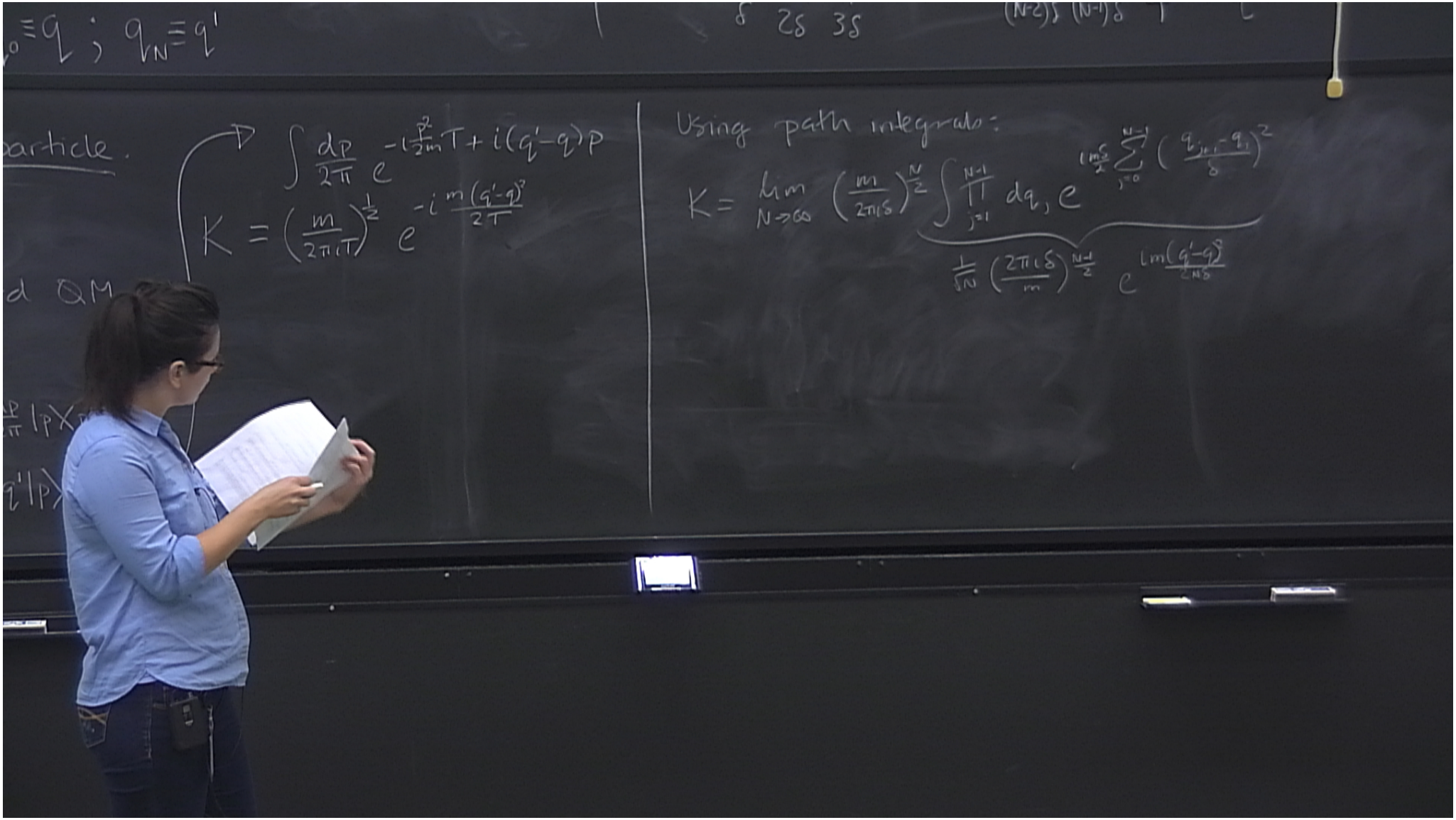
Using path integrals:

$$K = \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta}\right)^{\frac{N}{2}} \int \prod_{j=1}^{N-1} dq_j \, e^{i\frac{m\delta}{2}\sum_{j=0}^{N-1}\left(\frac{q_{j+1}-q_j}{\delta}\right)^2}$$


$q_0 \equiv q$; $q_N = q'$
particle.
d QM.
$\int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}T + i(q'-q)p}$
$K = \left(\frac{m}{2\pi i T}\right)^{\frac{1}{2}} e^{-i\frac{m(q'-q)^2}{2T}}$
Using path integrals:
$K = \lim_{N\to\infty} \left(\frac{m}{2\pi i \delta}\right)^{\frac{N}{2}} \int \prod_{j=1}^{N-1} dq_j \, e^{\frac{im\delta}{2}\sum_{j=0}^{N-1}\left(\frac{q_{j+1}-q_j}{\delta}\right)^2}$


particle.
d QM.
Using path integrals:


particle.
d QM.
Using path integrals:


particle.
Using path integrals:

$q_0 \equiv q$; $q_N = q'$

article.

d QM.

$\frac{dp}{2\pi} |p\rangle\langle p|q\rangle$

$\langle q'|p\rangle\langle p|q\rangle$

$$\int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}T + i(q'-q)p}$$

$$K = \left(\frac{m}{2\pi i T}\right)^{\frac{1}{2}} e^{-i\frac{m(q'-q)^2}{2T}}$$

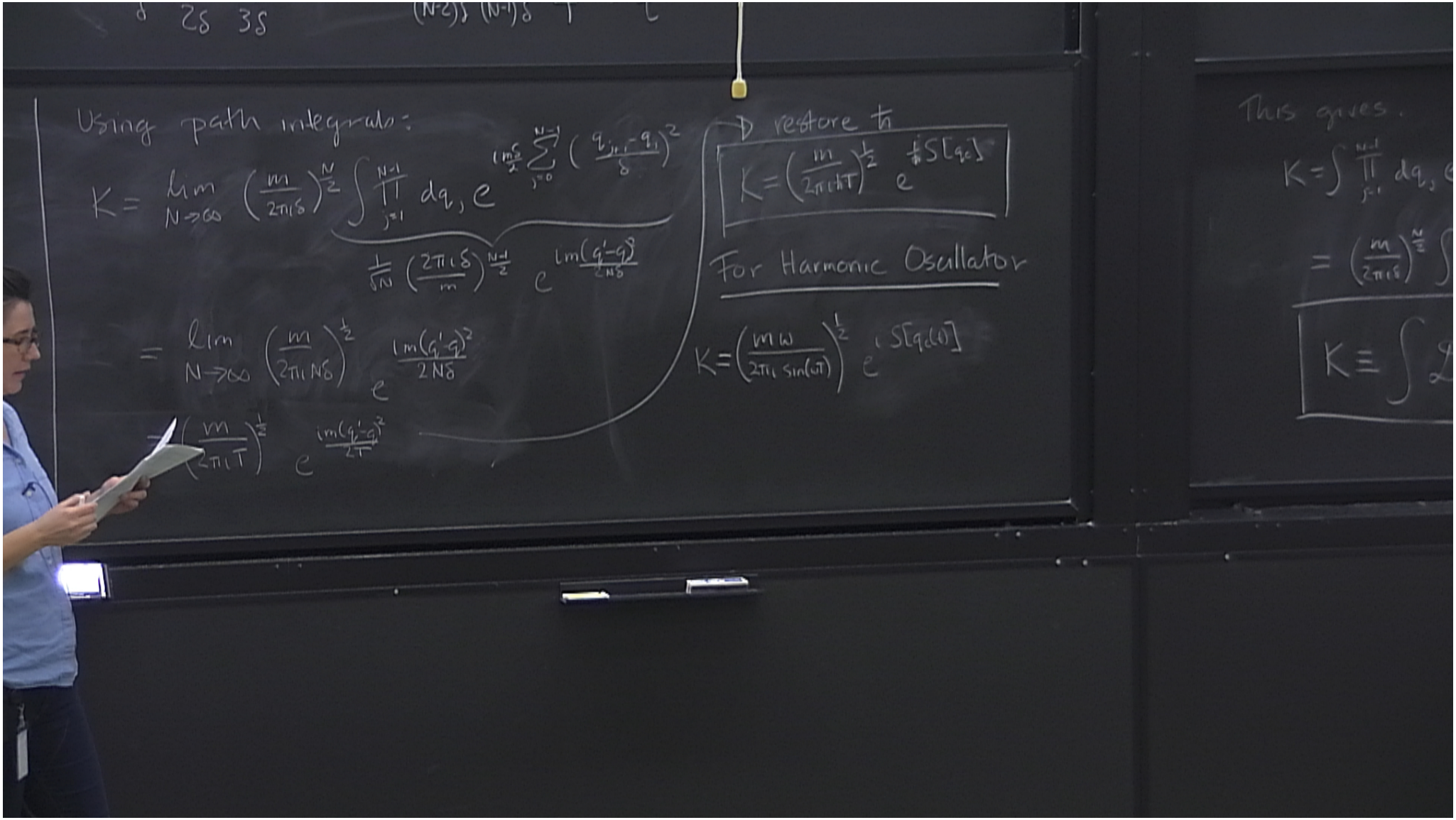
Using path integrals:

$$K = \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta}\right)^{\frac{N}{2}} \int \prod_{j=1}^{N-1} dq_j \, e^{\frac{im\delta}{2}\sum_{j=0}^{N-1}\left(\frac{q_{j+1}-q_j}{\delta}\right)^2}$$

$$\frac{1}{\sqrt{N}}\left(\frac{2\pi i\delta}{m}\right)^{\frac{N-1}{2}} e^{\frac{im(q'-q)^2}{2N\delta}}$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi i N\delta}\right)^{\frac{1}{2}} e^{\frac{im(q'-q)^2}{2N\delta}}$$

$$= \left(\frac{m}{2\pi i T}\right)^{\frac{1}{2}} e^{\frac{im(q'-q)^2}{2T}}$$


Using path integrals:
restore ħ
For Harmonic Oscillator
This gives.