Title: Complex Analysis

Date: Aug 14, 2015 09:00 AM

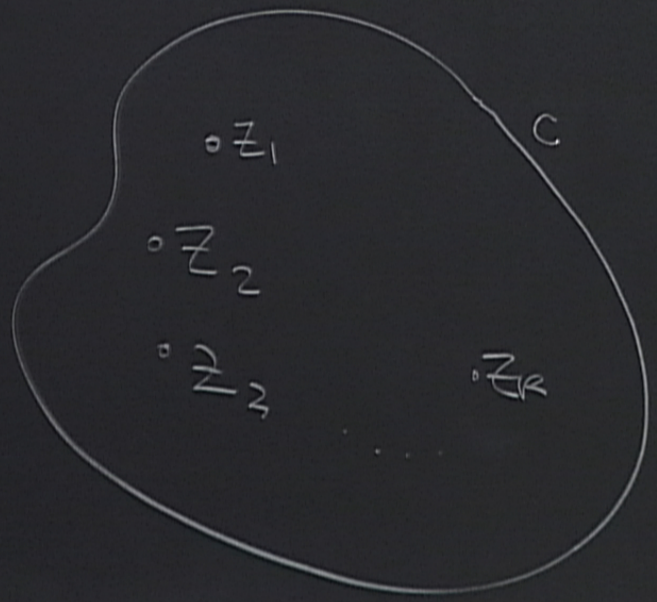
URL: http://pirsa.org/15080091

Abstract:

$$\oint_C f(z)\,dz = 2\pi i \sum_{i=1}^{k} \text{Res of } f(z) \text{ at } z = z_i$$

Residue Theorem

$f(z)$



$$\oint_C f(z)\,dz = 2\pi i \sum_{i=1}^{R} \text{Re}$$

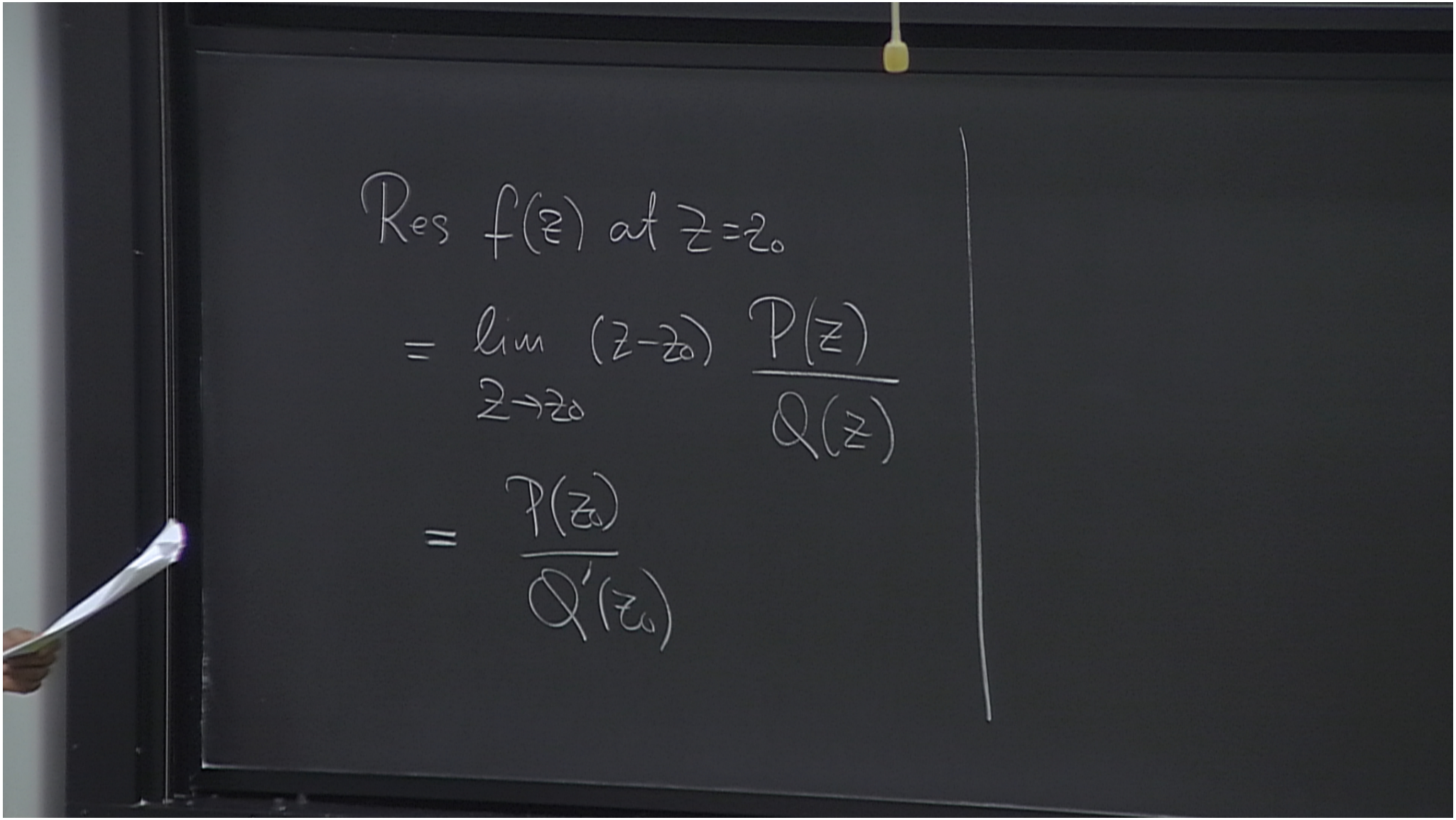
$f(z)$ at $z = z_0$

$$f(z) = \frac{P(z)}{Q(z)}$$

$Q(z) \rightarrow$ Polynomial

$Q(z_0) = 0$, $Q'(z_0) = 0$

$P(z_0) \neq 0$


Res f(z) at z=z₀
= lim (z−z₀) P(z)/Q(z)
z→z₀
= P(z₀)/Q'(z₀)

$$I = \int_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta \quad , \quad a > b > 0 \, , \, (a,b) \in \mathbb{R}$$

$$z = e^{i\theta} \quad , \quad \cos\theta = \frac{z + 1/z}{2}$$

$$I = -2i \oint_C \frac{dz}{2za + b + bz^2} \quad , \text{ where } C \text{ is } |z| = 1$$

$, (a,b) \in \mathbb{R}$

$P(z) = bz^2 + 2za + b$

$$z_{\pm} = -\frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}$$

C is $|z| = 1$

$$\lim_{z \to z_+} (z - z_+) \frac{-2i}{b(z - z_-)(z - z_+)}$$

$$= \frac{-2i}{b(z_+ - z_-)} = \frac{-1}{\sqrt{a^2 - b^2}}$$

$$I = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Semi-circular contour

$f(z)$ s.t. $|f(z)|$

Semi-circular contours

$f(z)$ s.t. $|f(z)| \leq \frac{M}{R^R}$ on Γ parametrized by $z = Re^{i}$

$R > 1$, $M > 0$ $\quad \left| \int_\Gamma f(z)\,dz \right| \leq \frac{M\pi}{R^{R-1}}$

Vanishes as $R \to \infty$

$$I = \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$$

$$\frac{1}{2} \frac{}{(a^2+z^2}$$

UHP $z =$

$$I = 2\pi i \frac{1}{2} \left[\lim_{z \to i} \right.$$

$$= \frac{\pi}{2ab}$$

$$\frac{1}{2}\,\frac{1}{(a^2+z^2)(b^2+z^2)}$$

UHP $\quad z = +ia, +ib$

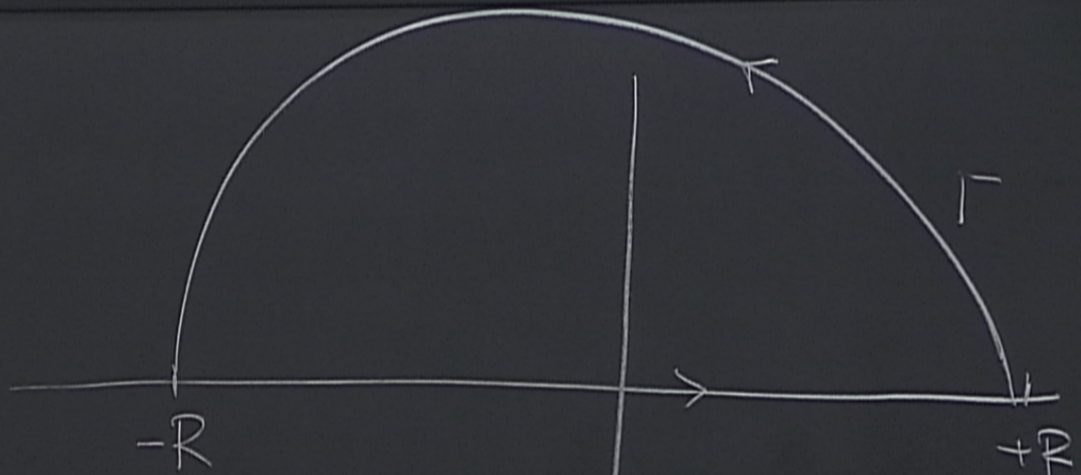
$$I = 2\pi i\,\frac{1}{2}\left[\lim_{z\to ia}\frac{1}{(z+ia)(z^2+b^2)} + \lim_{z\to ib}\frac{1}{(z^2+a^2)(z+ib)}\right]$$

$$= \frac{\pi}{2ab}\,\frac{1}{(a+b)}$$

if

$a = b$

$\frac{\pi}{4a^3}$



Jordan's Lemma

$$\int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx$$

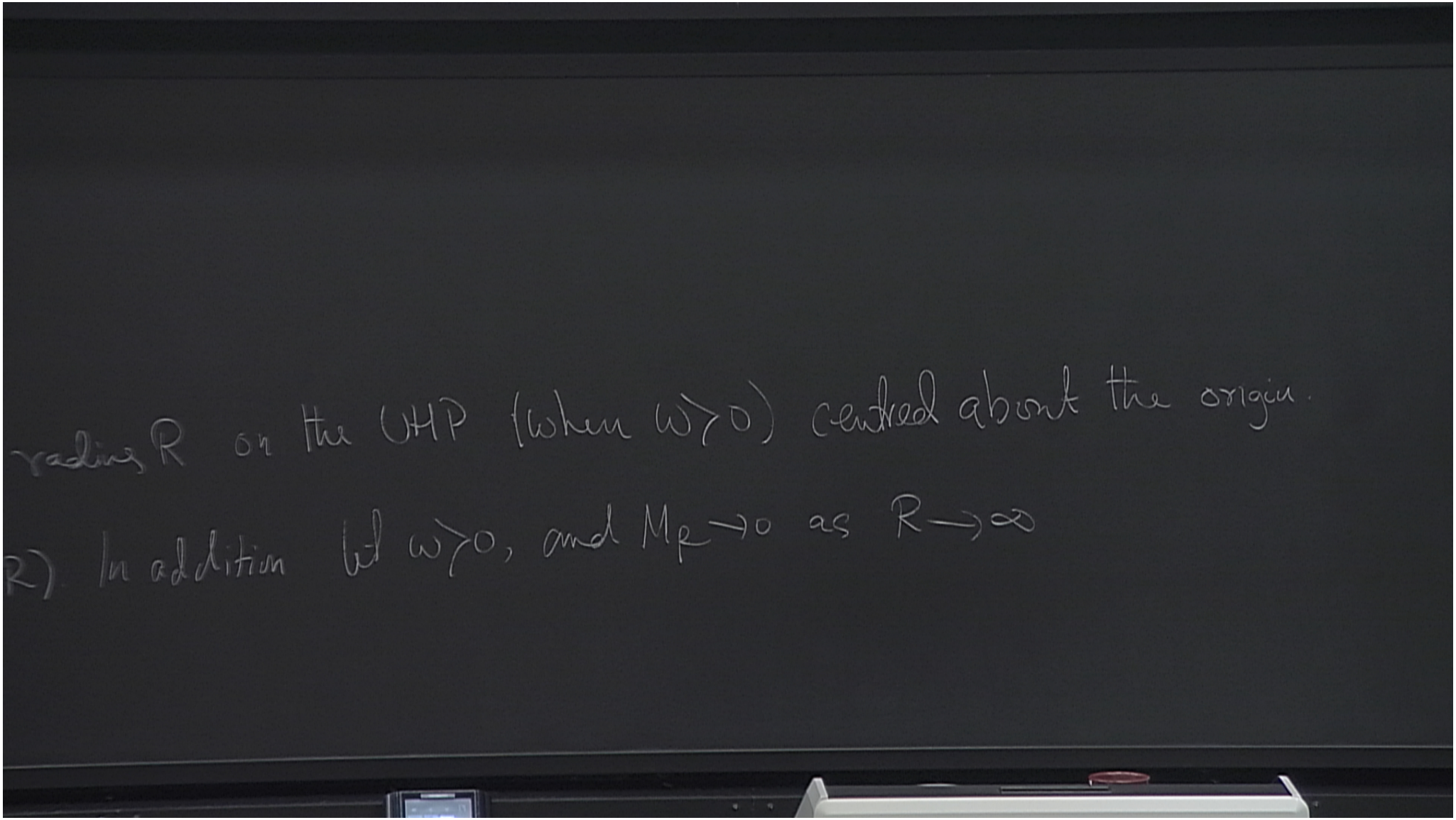
Suppose $\Gamma(R)$ is a semicircular contour of radius R o

Jordan's Lemma

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Suppose $\Gamma(R)$ is a semicircular contour of radius R on the

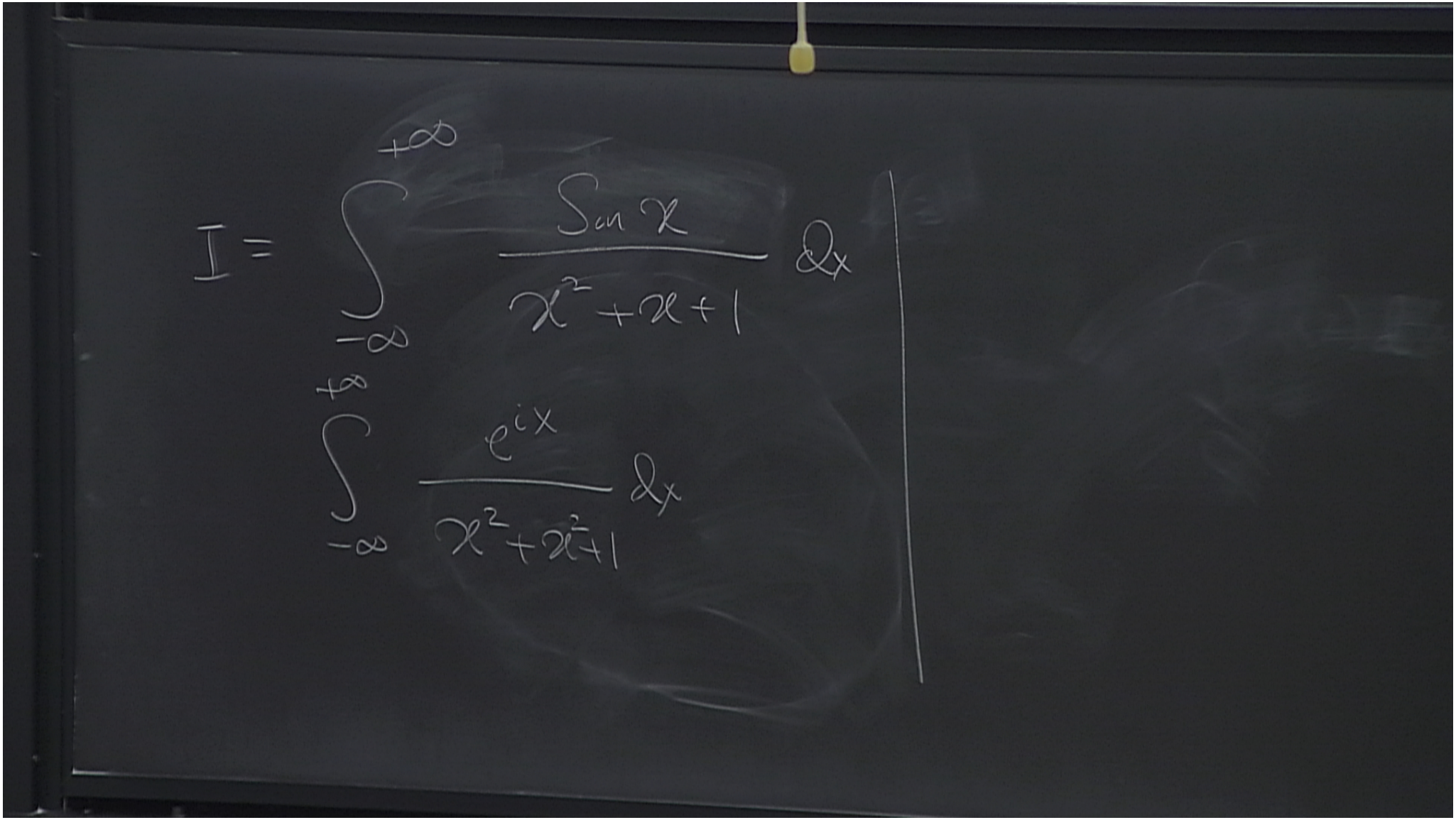
Let M_R be the maximum of $|f(z)|$ on $\Gamma(R)$. In addition w


radius R on the UHP (when ω>0) centred about the origin.
R). In addition to ω>0, and Mp→0 as R→∞

$$\int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx = 2\pi i \sum_{UHP} \text{Res}\left[f(z) e^{i\omega z}\right]$$

If $\omega < 0$ then

$$\int_{-\infty}^{+\infty} \cdots\cdots = -2\pi i \sum_{LHP} \text{Res}\left[f(z) e^{i\omega z}\right]$$

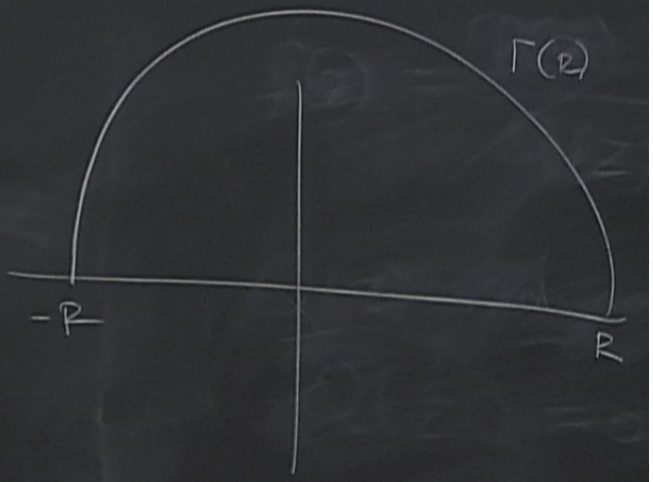

$I = \int_{-\infty}^{+\infty} \frac{\sin x}{x^2+x+1}\,dx$
$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+x+1}\,dx$

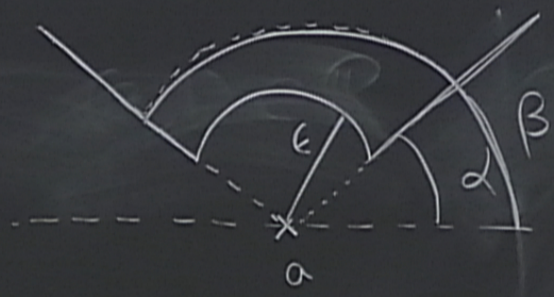
$$\int_{C(R)} \frac{e^{iz}}{(z^2+z+1)}\,dx$$

$R \to \infty \quad e^{i(x+iy)} = e^{ix}\,e^{-y}$

Poles $z_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,i$

$$I = -\frac{2e^{-\sqrt{3}/2}\,\pi\,\sin\frac{1}{2}}{\sqrt{3}}$$





a with residue σ

Proof:

$$g(z) + \frac{\sigma}{(z-a)} = f(z)$$

$$\lim_{\epsilon \to 0} \int_{\Phi_\epsilon} f(z)\,dz$$

-a) iσ

Mousehole Contours

$f(z)$ which has a simple pole at $z=a$ with residue σ

$$\phi_\epsilon(t) = a + \epsilon e^{it}, \quad \alpha < t < \beta$$

then $$\lim_{\epsilon \to 0} \int_{\phi_\epsilon} f(z)\,dz = (\beta - \alpha)\, i\sigma$$

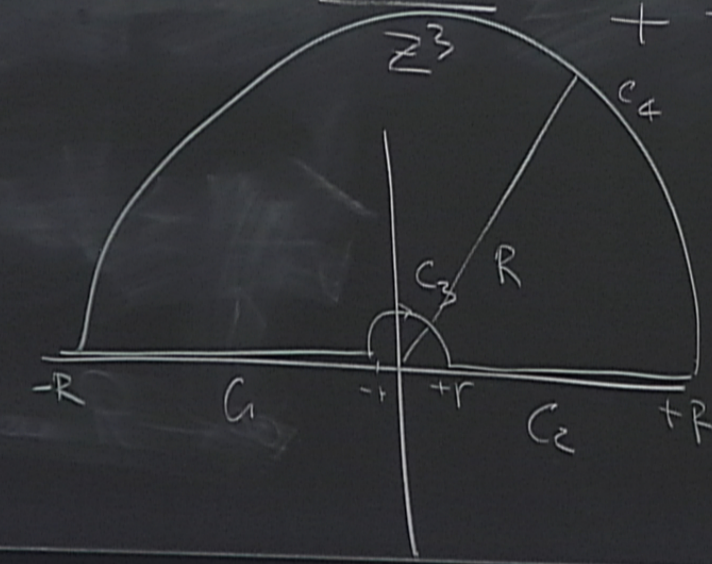
$$\int_0^\infty \frac{x - \sin x}{x^3}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x - \sin x}{x^3}\,dx$$

$$f(z) = \frac{iz - e^{iz}}{z^3}$$

$$\int_0^\infty \frac{x - \sin x}{x^3}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x - \sin x}{x^3}\,dx$$

$$f(z) = \frac{iz - e^{iz}}{z^3} = \frac{1}{z^3}\left[iz - 1 - iz + \frac{z^2}{2!} - \cdots\right]$$

$$h(z) = \frac{iz - e^{iz}}{z^3} + \frac{1}{z^3}$$



$$\left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right) h(z)\,dz$$

$R \to \infty$

$r \to 0$

$$\left(\int_{C_1} + \int_{C_2}\right) h(z)\,dz = -\int_{C_3} h(z)\,dz$$

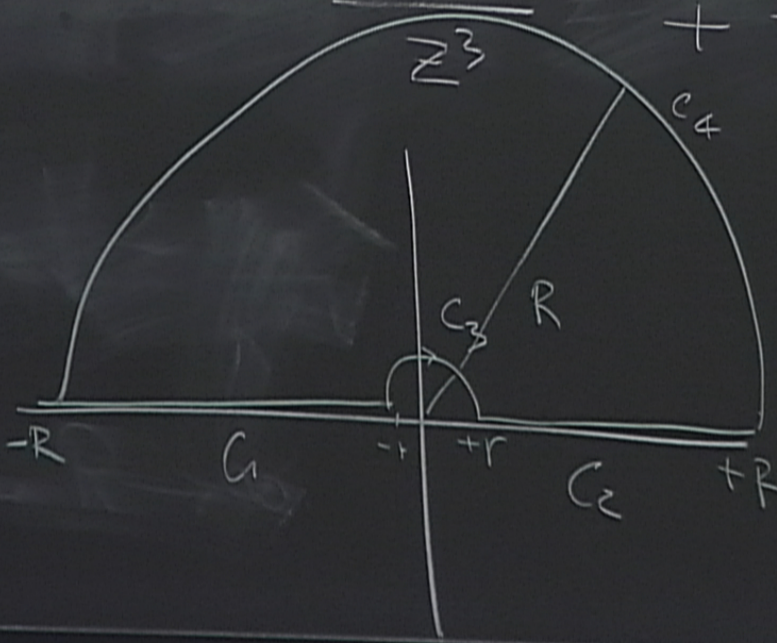
$$= \frac{(\pi - 0)\,i}{2} = \frac{i\pi}{2}$$

$$\int_0^\infty \frac{x - \sin x}{x^3}\,dx = \pi/4$$

$$\int_0^\infty \frac{x - \sin x}{x^3}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x - \sin x}{x^3}\,dx$$

$$f(z) = \frac{iz - e^{iz}}{z^3} = \frac{1}{z^3}\left[iz - 1 - iz + \frac{z^2}{2!} - \cdots\right]$$

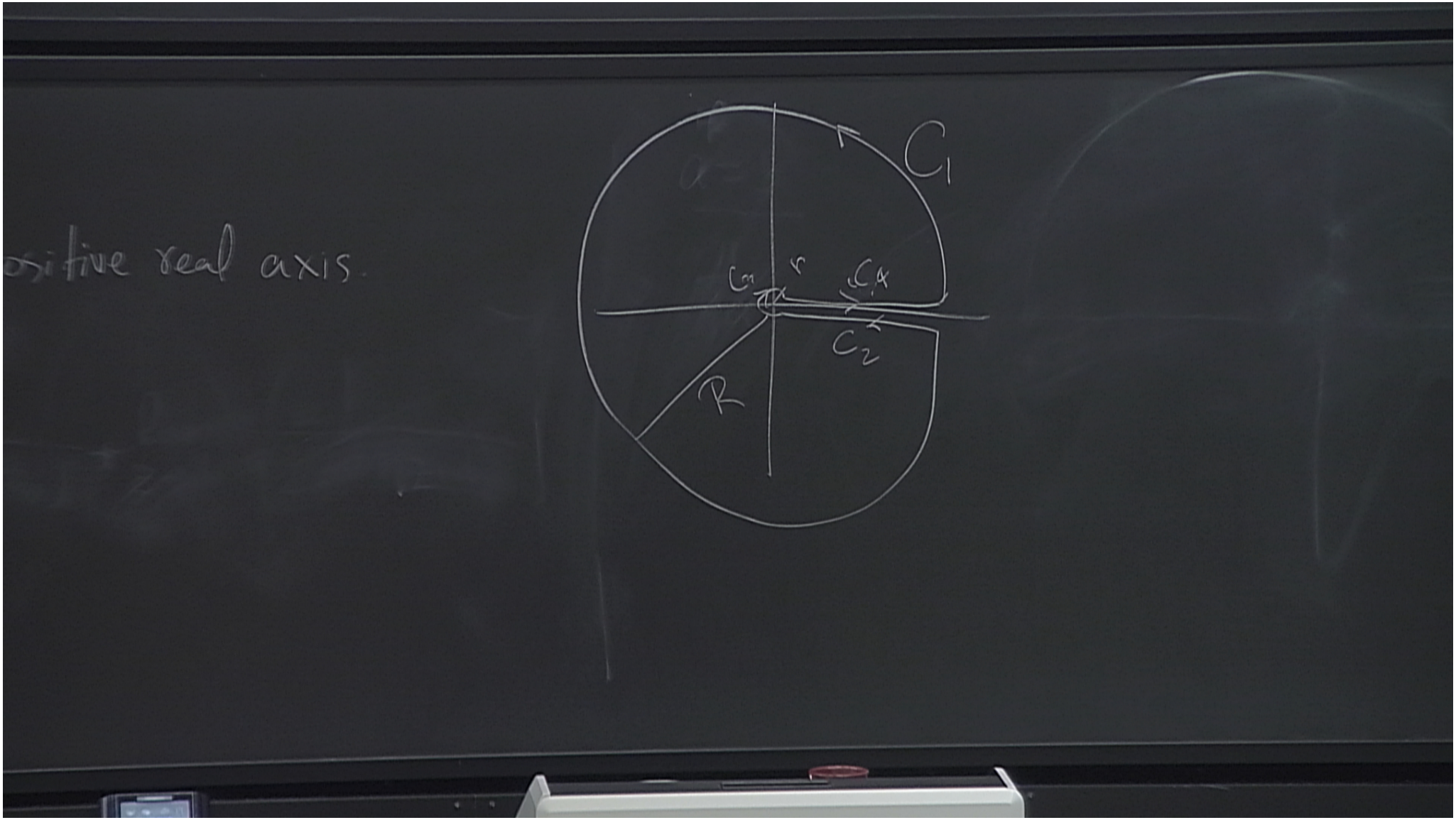
$$h(z) = \frac{iz - e^{iz}}{z^3} + \frac{1}{z^3}$$



$$\left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right) h(z)$$

$R \to \infty$

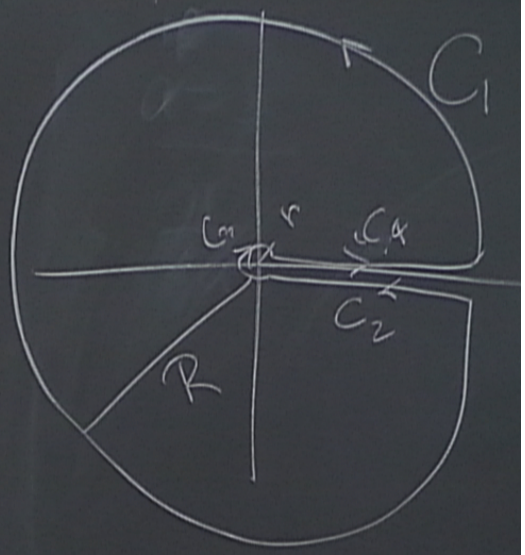
$r \to 0$


positive real axis.
C_1
C_3
C_4
C_2
R

f has no singularities on positive real axis.

$$\int_0^\infty f(x)\log x\,dx + \int_\infty^0 f(x)(\log x + 2\pi i)\,dx$$

$$= -\int_0^\infty 2\pi i f(x)\,dx$$



If when $R \to \infty$ the contribution from $C_1 \to 0$

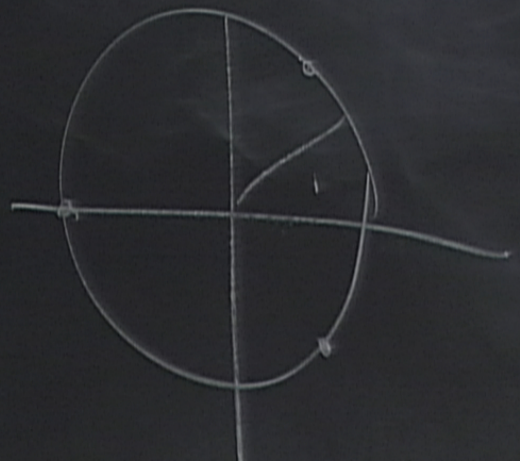
and " " $r \to 0$ " " " $C_3 \to 0$

$$\int_0^\infty f(x)\,dx = -\sum_i \text{Res of } f(z)\log(z) \text{ inside } C$$

$$\int_0^\infty \frac{1}{x^3+1}\,dx$$

$\log z$

$e^{i\pi/3}$, $e^{i\pi}$, $e^{i5\pi/3}$



$$\log z = \log e^{i\theta} = i\theta = \operatorname{Arg}(z)$$

$$\left| \frac{\log z}{z^3+1} \right| \qquad \left| \frac{i \operatorname{Arg}(z)}{3z^2} \right|$$

$= i\theta = \mathrm{Arg}(z)$

$\frac{i\,\mathrm{Arg}(z)}{3z^2}$

$\left(-\frac{i\pi}{18} + \frac{\pi}{6\sqrt{3}}\right), \frac{i\pi}{3}, \frac{-5i\pi}{18} - \frac{5\pi}{6\sqrt{3}}$

$$\int_0^\infty \frac{1}{x^3+1}\,dx = \frac{2\pi}{3\sqrt{3}}$$

$$\log z = \log e^{i\theta} = i\theta = \operatorname{Arg}(z)$$

$$\frac{\log z}{z^3+1} \qquad \frac{i\operatorname{Arg}(z)}{3z^2}$$

$$\frac{P(z_0)}{Q'(z_0)}$$

$$\left(-\frac{i\pi}{18} + \frac{\pi}{6\sqrt{3}}\right)$$

$$\int_0^\infty \frac{1}{x^3+1}\,dx$$