Title: Complex Analysis

Date: Aug 13, 2015 09:00 AM

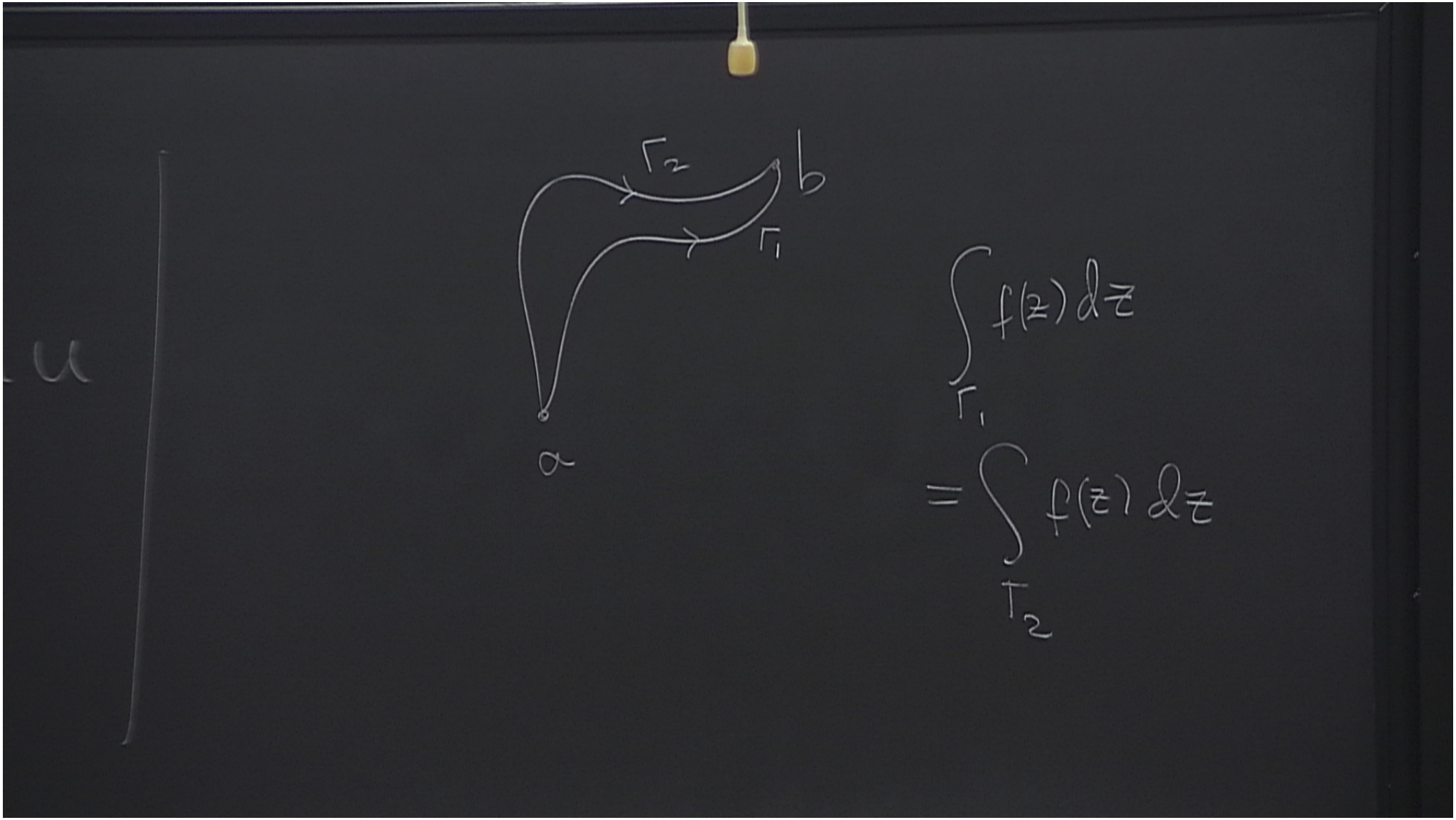
URL: http://pirsa.org/15080090

Abstract:

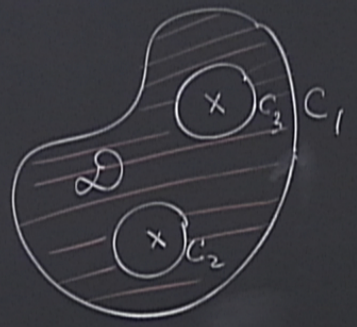
Cauchy's Theorem

U is simply connected $f(z)$ is holomorphic on U. Γ is a simple closed curve in U

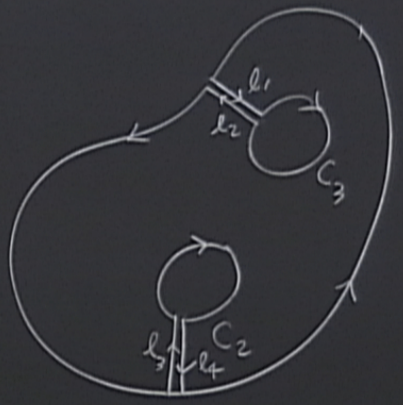
$$\oint_{\Gamma} f(z)\,dz = 0$$


Γ_2
b
Γ_1
a
$\int_{\Gamma_1} f(z)\,dz$
$= \int_{\Gamma_2} f(z)\,dz$

$f(z)$ is hdlo in $\mathcal{D}$



$\equiv$



$$\oint f(z)\,dz = 0$$

$\equiv$

$$\oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz - \oint_{C_3} f(z)\,dz = 0$$

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz$$



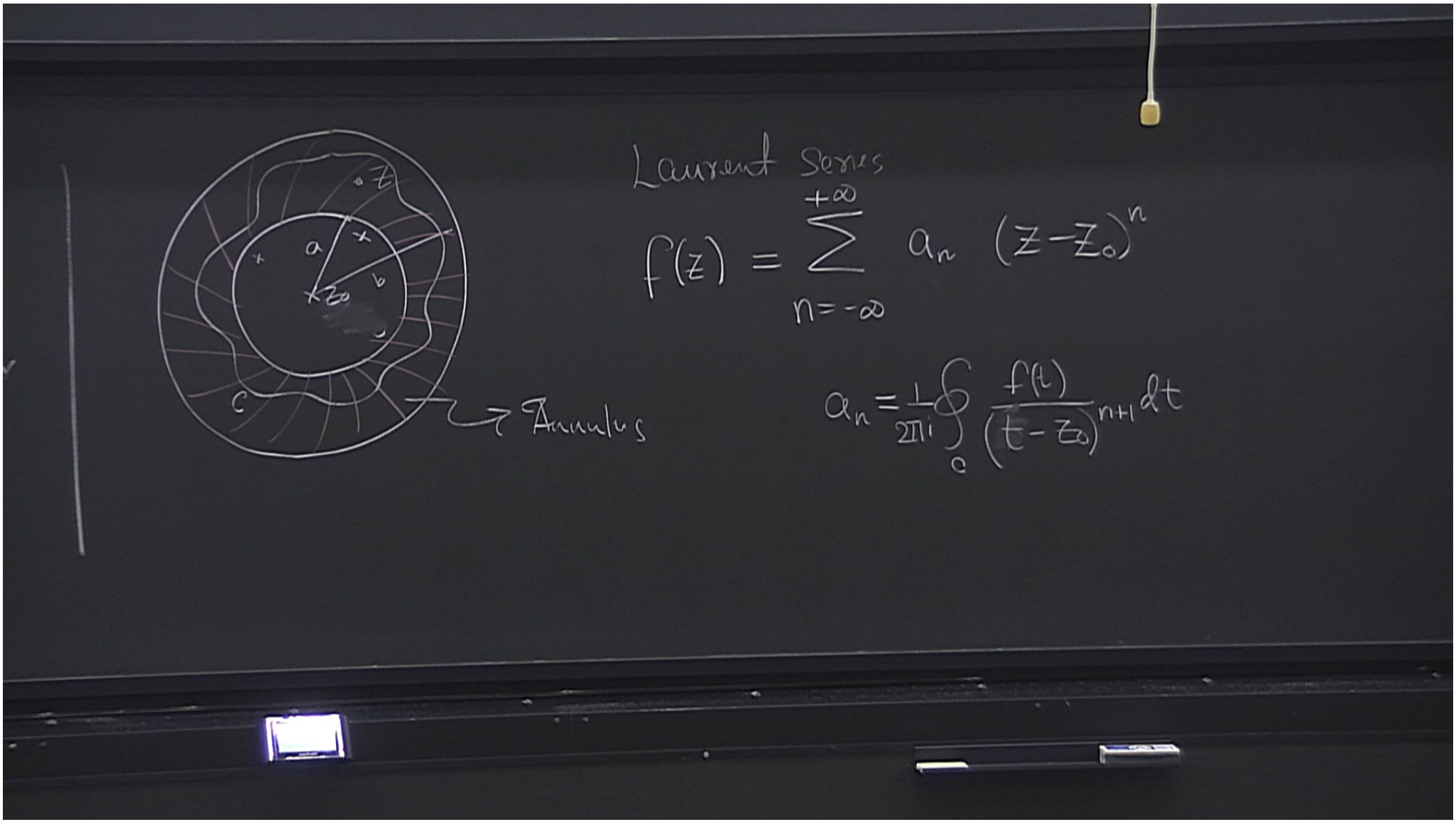
$$f(a) = \oint_C \frac{f(z)}{(z-a)} \frac{dz}{2\pi i}$$

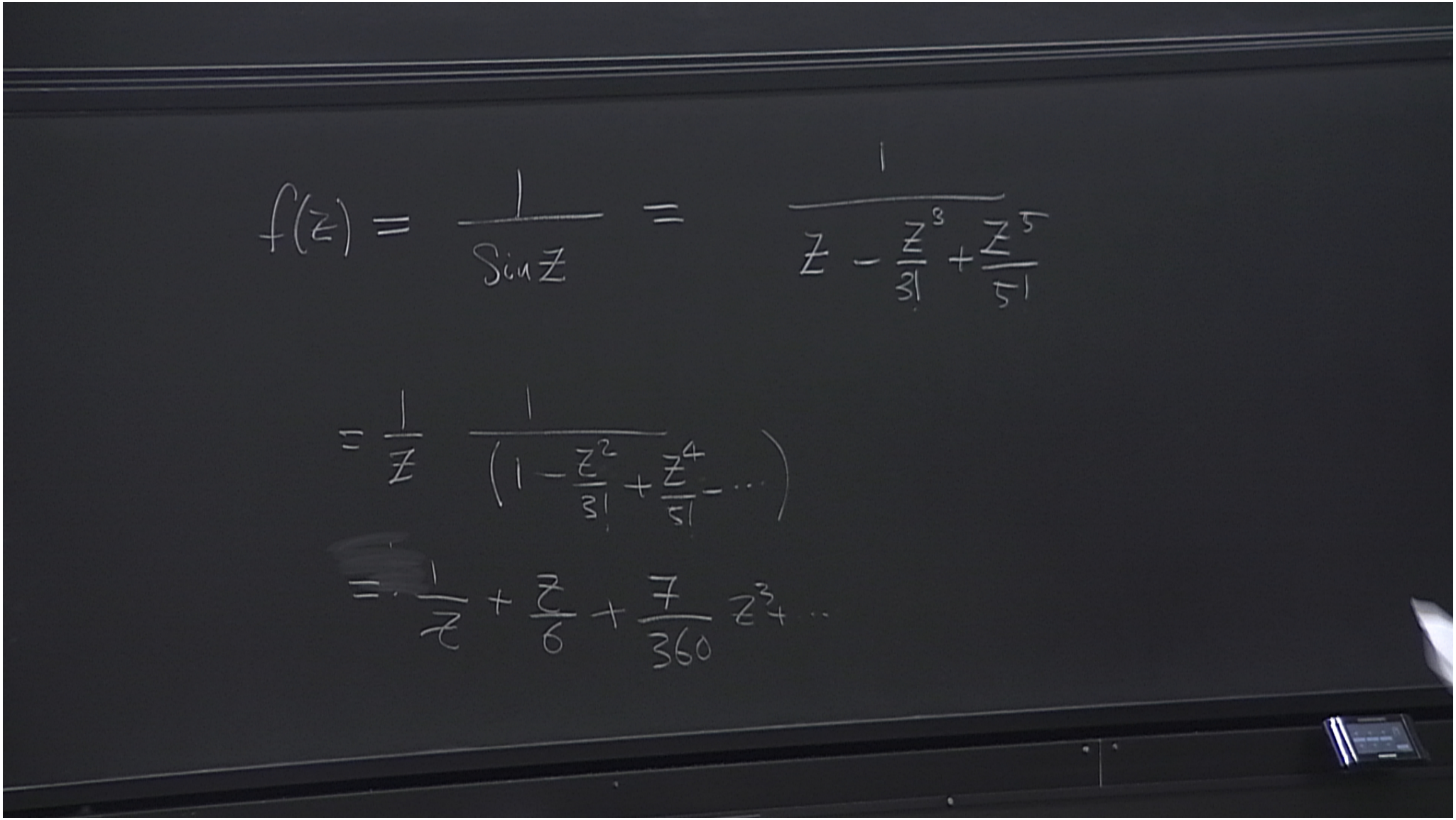
Cauchy's Integral formula.

$= 0$

$$\frac{\partial^n f(z)}{\partial z^n} := f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(t)}{(t-z)^{n+1}}\, dt$$

$$f(a+z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} f^{(n)}(a) \sim \text{Taylor series for holo fns}$$

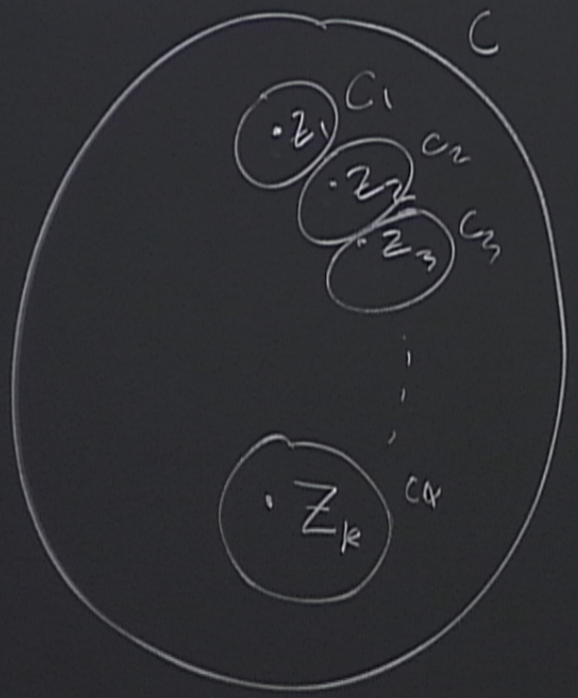

Laurent Series
$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n$
$a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t - z_0)^{n+1}} dt$
Annulus
a
b
c
z_0


$f(z) = \frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!}}$
$= \frac{1}{z} \frac{1}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)}$
$= \frac{1}{z} + \frac{z}{6} + \frac{7}{360} z^3 + \cdots$

Cauchy's Residue Theorem

$$\oint_C z^n \, dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

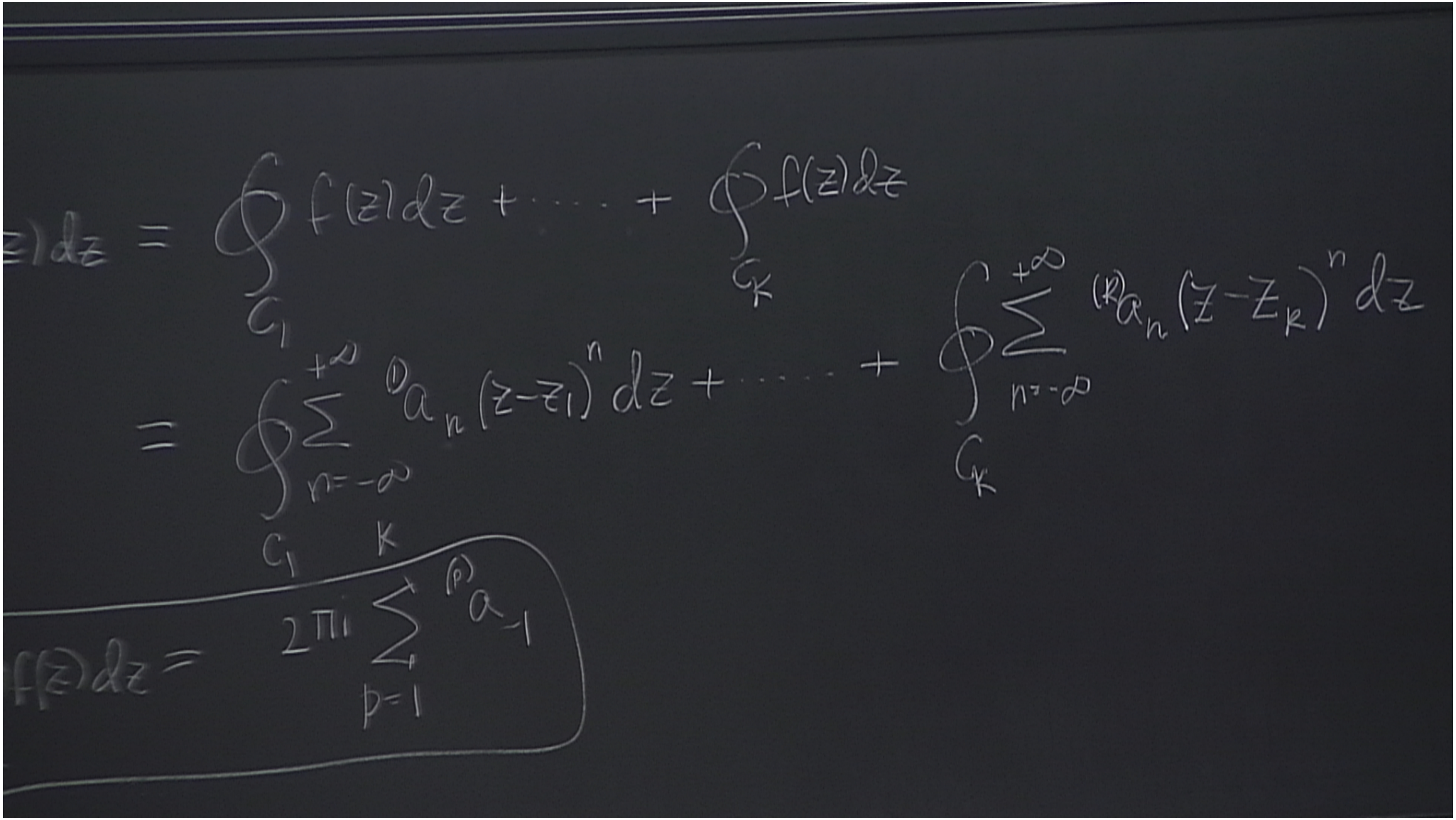
$a_{-1} \rightarrow$ Residue of $f(z)$ at $z = z_0$



$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz$$

$$= \oint_{C_1} \sum_{n=-\infty}^{+\infty} a_n (z-z_1)^n\,dz + \cdots$$

$$\oint_C f(z)\,dz = 2\pi i \sum_{p=1}^{k} a_{-1}^{(p)}$$


$\oint_{C_1} f(z)dz + \cdots + \oint_{C_K} f(z)dz$
$= \oint_{C_1} \sum_{n=-\infty}^{+\infty} {}^{(1)}a_n (z-z_1)^n dz + \cdots + \oint_{C_K} \sum_{n=-\infty}^{+\infty} {}^{(K)}a_n (z-z_K)^n dz$
$\oint f(z)dz = 2\pi i \sum_{p=1}^{k} {}^{(p)}a_{-1}$

Singularities of holo functions

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-a) + \cdots$$

$n \rightarrow$ positive integer

Principal part.

$z_0 \rightarrow$ Pole singularity

of holo functions

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-a) + \cdots$$

Principal part. $g(z) = (z-z_0)^n f(z)$

→ Pole singularity

powers, then z_0 is an essential singularity.

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots\cdots$$

$$f(z) = \frac{a_{-1}}{(z-z_0)} + a_0 + \cdots$$

$z_0 \rightarrow$ Simple pole.

$$f(z) = \frac{a_n}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{(z-z_0)} + \cdots$$

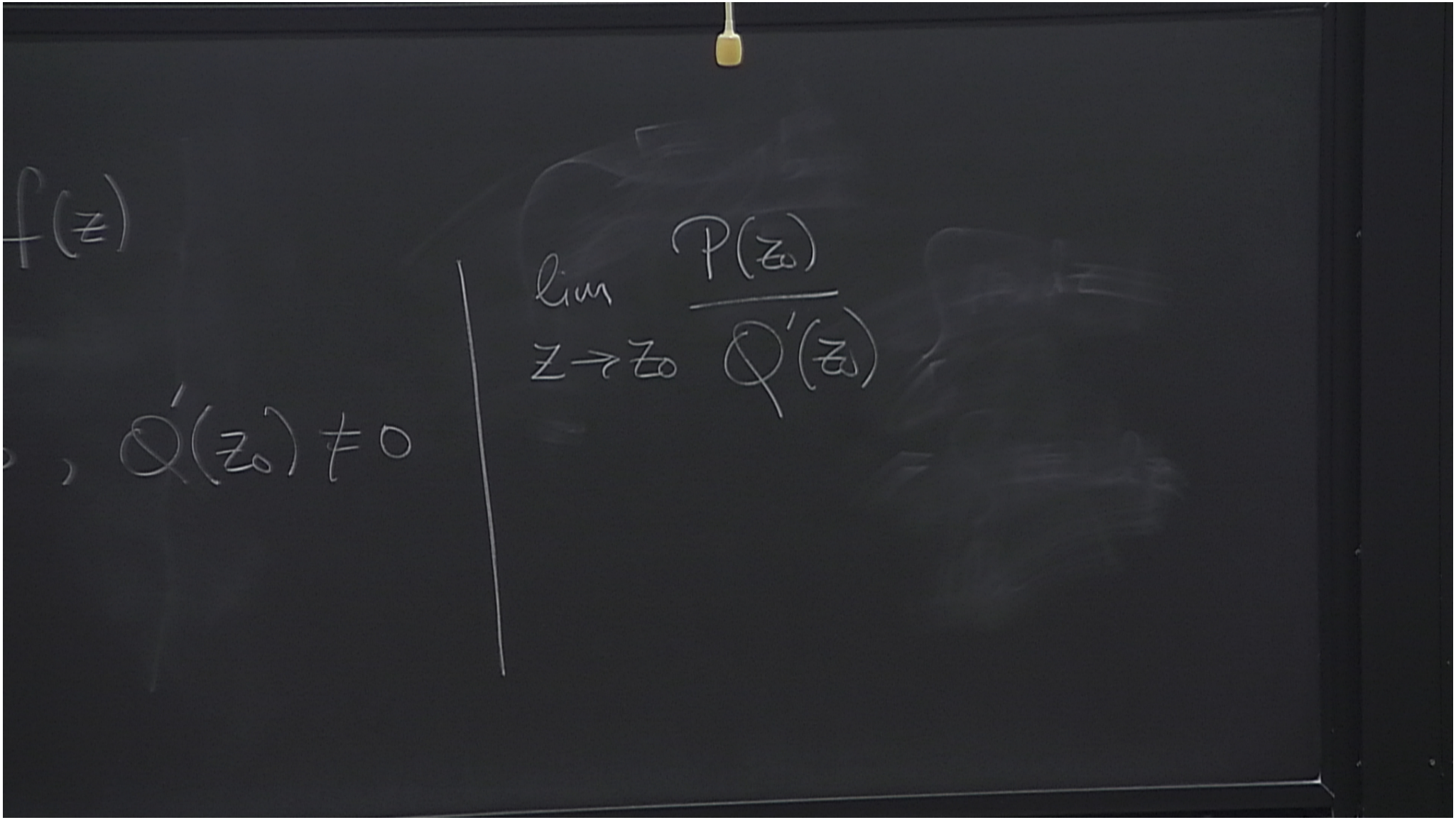
multiple pole

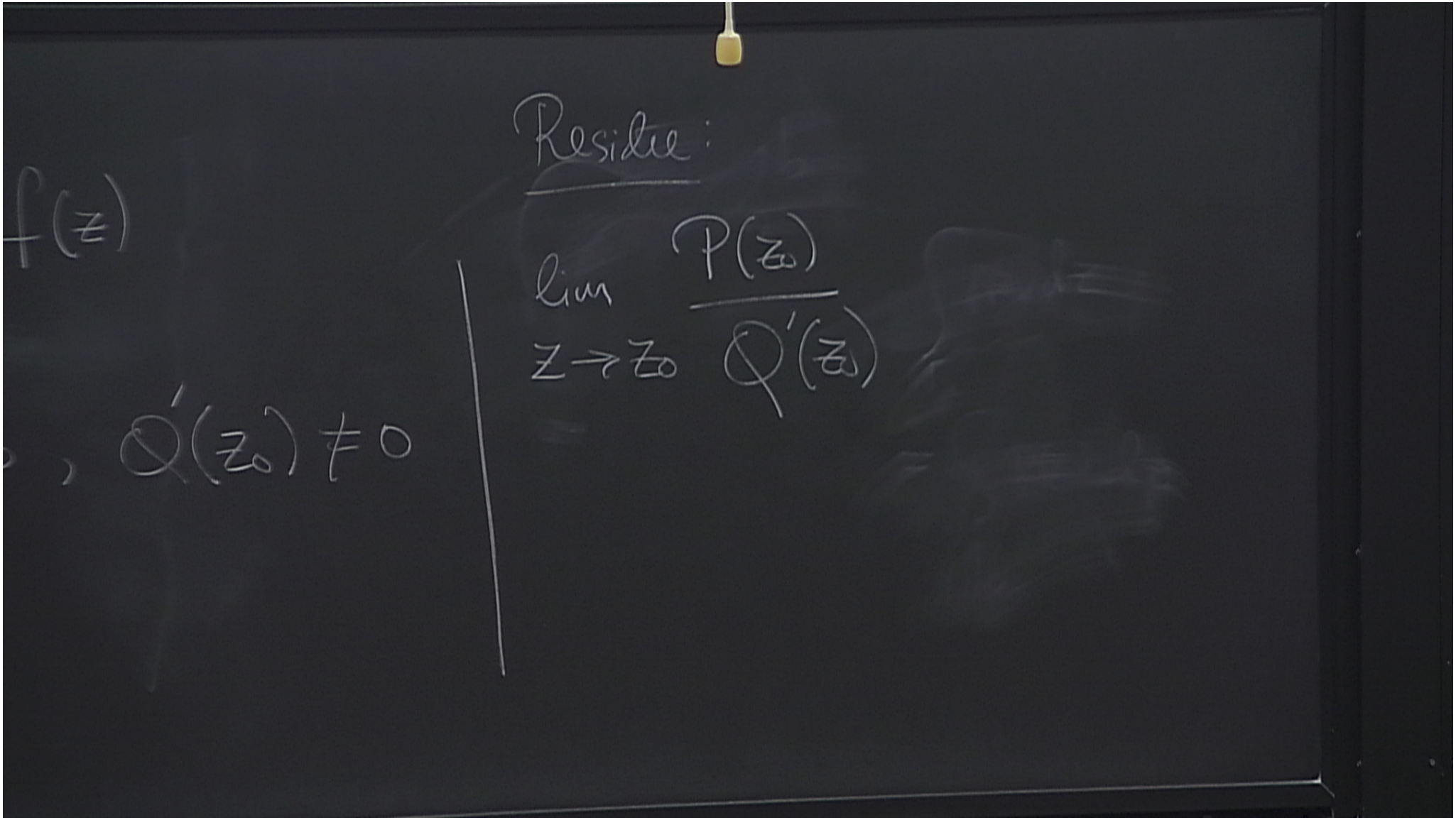
a_{-1}

$$a_{-1} = \lim_{(z \to z_0)} \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial z^{n-1}} (z - z_0)^n f(z)$$

$$f(z) = \frac{P(z)}{Q(z)} \qquad Q(z_0) = 0 \,, \; Q'(z_0) \neq 0$$

$$P(z_0) \neq 0$$


f(z)
, Q'(z₀) ≠ 0
lim P(z₀) / Q'(z₀)
z→z₀


f(z)
, Q'(z₀) ≠ 0
Residue:
lim P(z₀) / Q'(z₀)
z→z₀

$$f_1(z) = \frac{1}{(a^2+z^2)^2} = \frac{1}{(z+ia)^2(z-ia)^2}$$

$\pm ia$ $\quad z_0 = +ia$

$z = x + ia$

$$= \frac{1}{(x+2ia)^2 x^2}$$

$$= \frac{-1}{4a^2} \frac{1}{x^2} \frac{1}{\left(1+\frac{x}{2ia}\right)^2}$$