Title: Complex Analysis

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Abstract:

Complex functions

$U \subset \mathbb{C}$, $f : U \to \mathbb{C}$

1. Explicit $f_1 = z^n$

$f_2 = z^n + \bar{z}^n$

2. $f(z) = e^z$

$f = u(x,y) + iv(x,y)$

$z = x + iy$

$\longrightarrow \mathbb{C}$

$+ \bar{z}^n$

2. $f(z) = e^z$

$f = u(x,y) + iv(x,y)$

$z = x+iy$ $\quad (z, \bar{z})$ and (x,y)

$\bar{z} = x-iy$

Think of complex

Think of complex functions as fns of $z \neq \bar{z}$.

$$f_3 = \mathrm{Re}\, z = \frac{z + \bar{z}}{2}$$

$$f_+ = \frac{e^{iz} + e^{-iz}}{2} = \cos z \sim \text{holomorphic fns}$$

Examples:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$|z| = 1$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

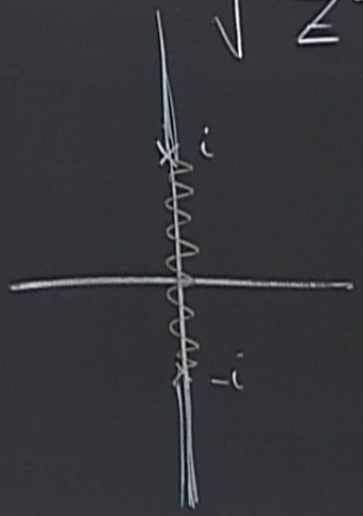
Inverse functions

1. $z^2 \longrightarrow \sqrt{z}$

2. $e^z \longrightarrow \log z = \log|z| + i\arg(z)$
$= \log|z| + i(\theta_p + 2\pi R)$

$$\sinh^{-1} z = \log\left[z + \sqrt{z^2+1}\right]$$

$$\sqrt{z^2+1} = \sqrt{(z-i)(z+i)}$$



$$z = \frac{1}{t}, \quad t = 0$$

Complex differentiability

$f = u + iv$

u is differentiable at (x_0, y_0) if $\exists\ A, B$ s.t. in a small nbhd of (x_0, y_0)

$$u(x_0+h, y_0+k) = u(x_0, y_0) + hA + kB + r(x_0, y_0, h, k)$$

$$\left| \frac{r}{\sqrt{h^2+k^2}} \right| \to 0 \text{ as } (h,k) \to 0$$

$$\vartheta(x_0+h, y_0+k) = \vartheta(x_0,y_0) + hC + kD + \varepsilon(x_0,y_0,h,k)$$

$$A = \frac{\partial u(x_0,y_0)}{\partial x}$$

$$B = \frac{\partial u(x_0,y_0)}{\partial y}$$

$$C = \frac{\partial v(x_0,y_0)}{\partial x}$$

$$D = \frac{\partial v}{\partial y}(x_0,y_0)$$

$$f(z_0+\omega) = f(z_0) + \omega E + \sigma(z_0,\omega)$$

$$E = \alpha + i\beta$$

$$\omega = h + ik$$

$$\left|\frac{\sigma}{\omega}\right| \to 0 \quad \text{as} \quad \omega \to 0$$

$$u(x_0+h, y_0+k) = u(x_0, y_0) + h\alpha - k\beta + \mathrm{Re}\,\sigma$$

$$v(x_0+h, y_0+k) = v(x_0, y_0) + h\beta + k\alpha + \mathrm{Im}\,\sigma$$

$$\alpha = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\beta = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$\sim$ Cauchy-Riemann conditions

If function f satisfies:

- x f is real differentiable
- x Satisfies Cauchy-Riemann.

at point z then: f is <u>analytic</u> at z.

If f is analytic $U \rightarrow f$ is holomorphic in U.

If function f satisfies:

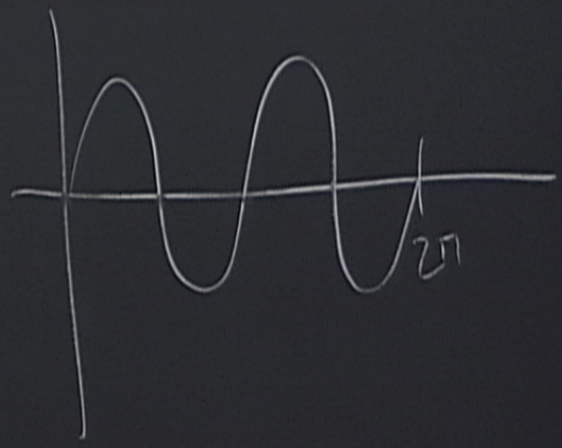
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at point z. then: f is analytic at z.

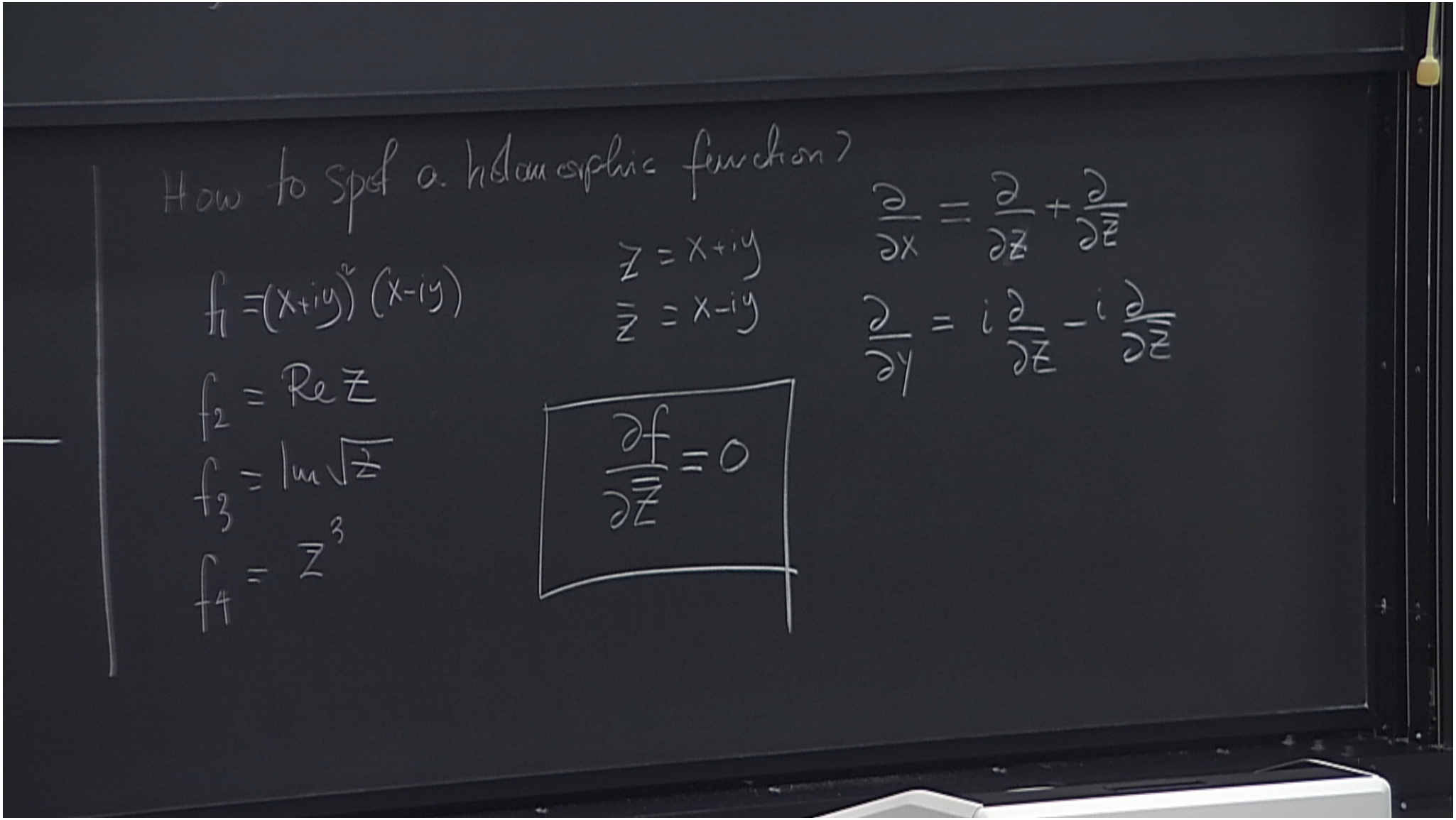
If f is analytic U → f is holomorphic in U.

$$f(x,y) = \begin{cases} 0 & \text{at } (x,y)=(0,0) \\ \dfrac{xy}{x^2+y^2} & \text{otherwise} \end{cases}$$

$\rightarrow \dfrac{\sin 2\theta}{2}$



$\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$


How to spot a holomorphic function?
$f_1 = (x+iy)^2 (x-iy)$
$f_2 = \mathrm{Re}\, z$
$f_3 = \mathrm{Im}\, z$
$f_4 = \bar{z}^3$
$z = x+iy$
$\bar{z} = x-iy$
$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$
$\frac{\partial}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar{z}}$
$\frac{\partial f}{\partial \bar{z}} = 0$

$f = f(z)$

$f = u + iv$

$\underline{T} = \begin{pmatrix} -u \\ v \end{pmatrix}$

Cauchy-Riemann $\Rightarrow$

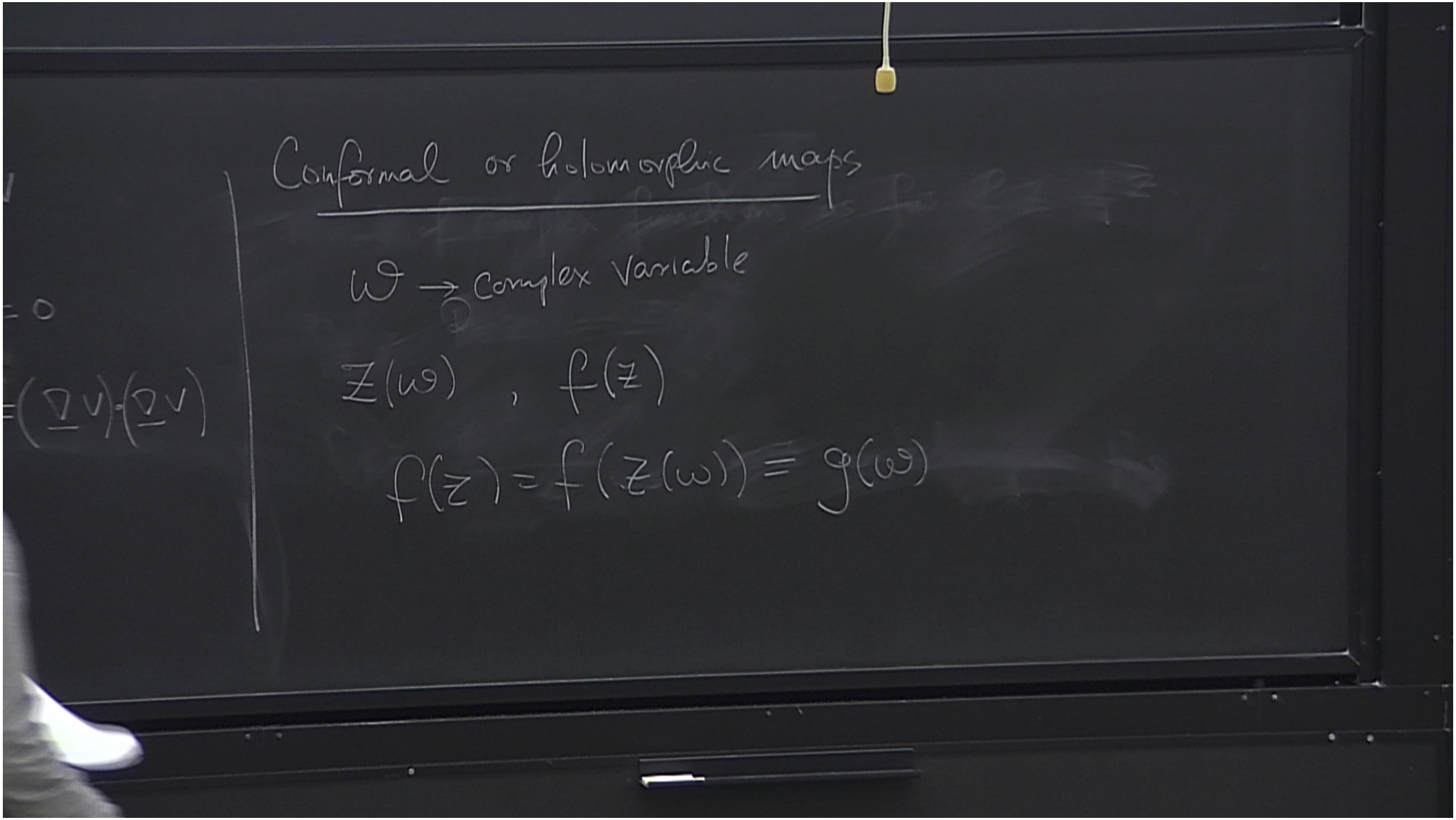
$\underline{\nabla} \cdot \underline{T} = 0$

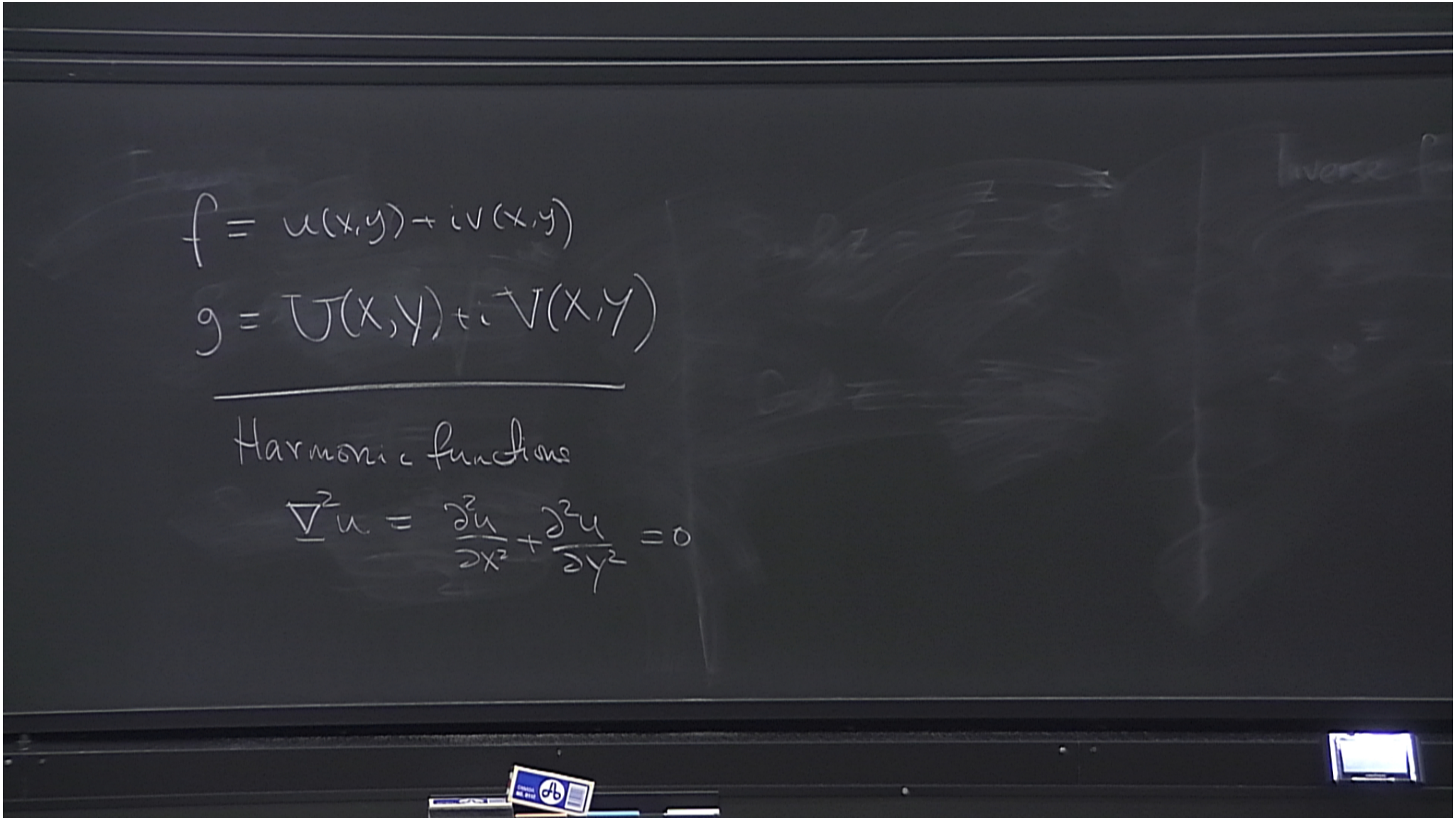
$\underline{\nabla} \times \underline{T} = 0$

$\underline{\nabla} u \perp \underline{\nabla} v$

$\underline{\nabla} u \cdot \underline{\nabla} v = 0$

$(\underline{\nabla} u) \cdot (\underline{\nabla} u) = (\underline{\nabla} v) \cdot (\underline{\nabla} v)$


Conformal or holomorphic maps
$w \rightarrow$ Complex variable
$z(w)$, $f(z)$
$f(z) = f(z(w)) = g(w)$


f = u(x,y) + iv(x,y)
g = U(x,y) + iV(x,y)
Harmonic functions
$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Integration of Complex functions

Contour Integrals

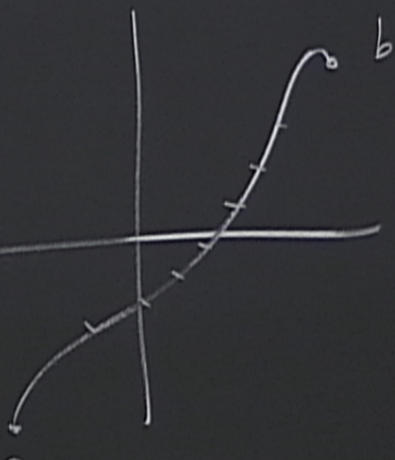
$$\int_{\Gamma} f(z)\, dz$$

$\Gamma \rightsquigarrow$ Simple curve or Jordan curve.

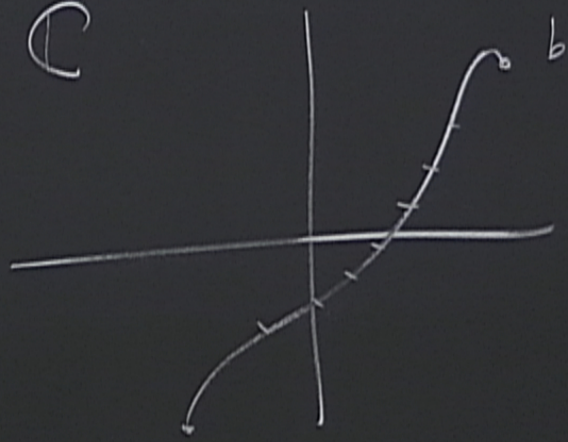
$\Gamma : [a,b] \longrightarrow U \subset \mathbb{C}$

$\Gamma(t)$

$$\int_\Gamma f(z)\,dz := \int_a^b f(\Gamma(t))\,\Gamma'(t)\,dt$$



$\Gamma : [a,b] \longrightarrow U \subset \mathbb{C}$



$\Gamma(t)$

$$\int_\Gamma f(z)\,dz := \int_a^b f(\Gamma(t))\,\Gamma'(t)\,dt$$

$s = s(t)$

$$f(z) = z^n$$

$$\Gamma(t) = r e^{it}$$

$$\int_\Gamma f(z)\,dz = \int_\Gamma z^n\,dz$$

$$= \int_0^{2\pi} r^n e^{int}\, ir e^{it}\,dt$$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t}\,dt$$

$$= \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

tal Theorem of calculus

$\Gamma : [a,b] \to U$, $f : U \to \mathbb{C}$ be continuous

$F : U \to \mathbb{C}$, $F'(z) = f(z)$

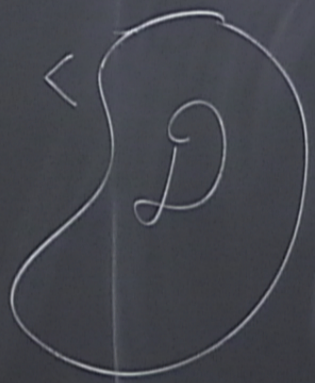
$$\int_\Gamma f(z)\,dz = F(\Gamma(b)) - F(\Gamma(a))$$

$$\int_\Gamma \frac{1}{z}\,dz = 2\pi i \neq 0$$

$\underline{A}$ → 2d vector field with continuous 1st derivative in $\mathcal{D}$ which is si

$\Gamma = \partial\mathcal{D}$

$\pi_1(\mathcal{D}) = 1$



$$\oint_\Gamma \underline{A} \cdot d\underline{\ell} = \iint_{\mathcal{D}} (\nabla \times \underline{A}) \cdot d\underline{a}$$

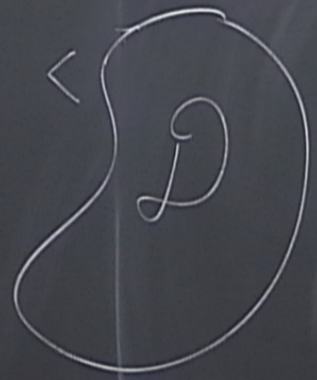
$$\oint_\Gamma A_x dx + A_y dy = \iint_{\mathcal{D}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx\,dy$$

$u(x_0, y_0) + h\alpha - k\beta + Re\,\sigma$

If function f satisfies:

field with continuous 1st derivative in $\mathcal{D}$ which is simply connected

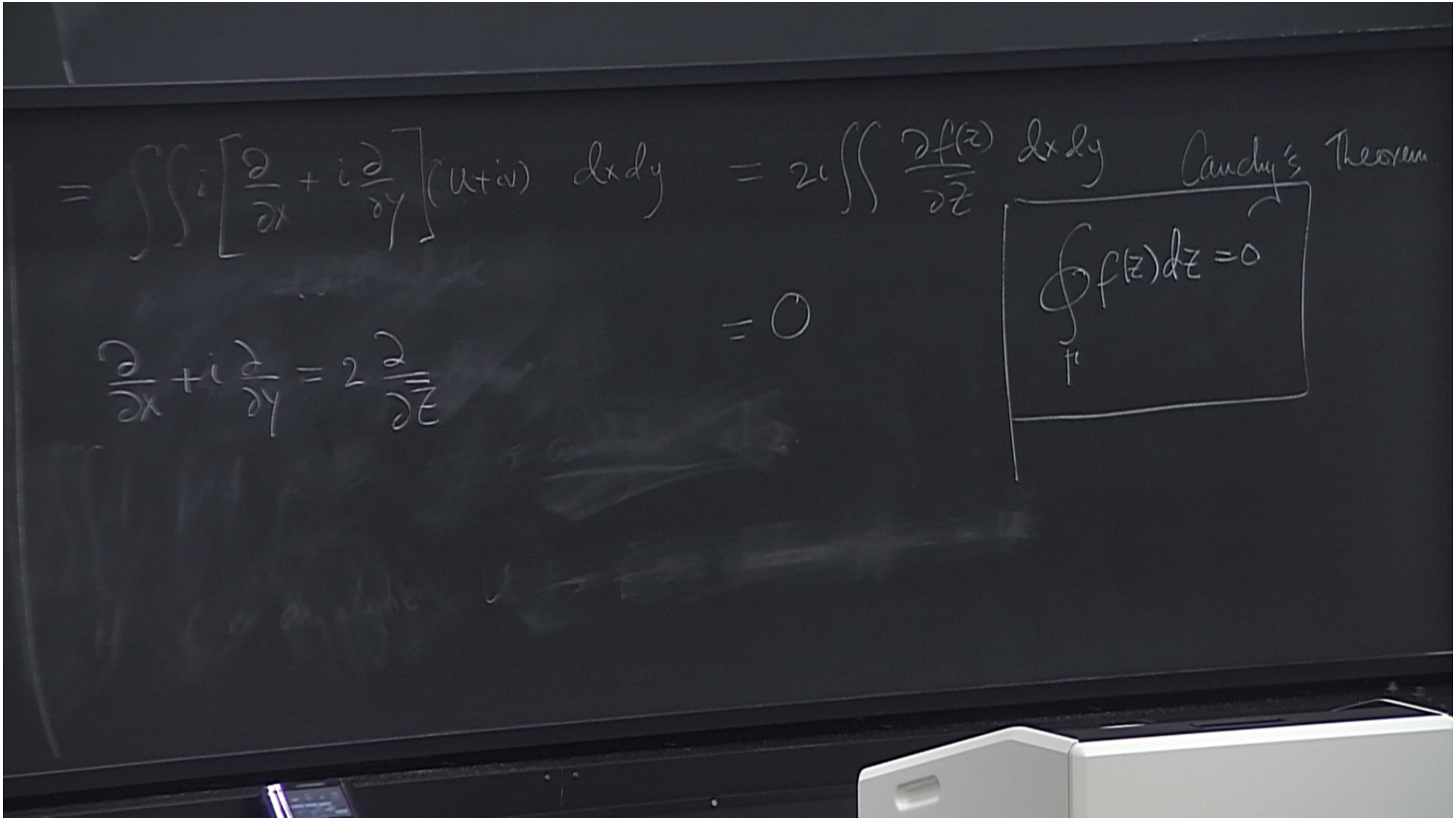
$\Gamma = \partial\mathcal{D}$



$$\iint_{\mathcal{D}} (\nabla \times \underline{A}) \cdot d\underline{a}$$

$$= \iint_{\mathcal{D}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx\,dy$$

If function f satisfies:


$= \iint i\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right](u+iv)\, dx\,dy = 2i\iint \frac{\partial f(z)}{\partial \bar{z}}\, dx\,dy$
Cauchy's Theorem
$\oint_\Gamma f(z)\,dz = 0$
$= 0$
$\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \bar{z}}$

$$\oint_\Gamma f(z)\,dz = \oint_\Gamma (u+iv)(dx+i\,dy)$$

$$= \oint_\Gamma [(u+iv)\,dx + (iu-v)\,dy]$$

$$= \iint \left[\frac{\partial (iu-v)}{\partial x} - \frac{\partial (u+iv)}{\partial y}\right] dx\,dy$$