Title: TBA

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Abstract: TBA

3D Holography: from discretum to continuum

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very recent wip
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QI in QG II
PI, Aug 2015

Motivation

Setting:
3D (Euclidean) quantum gravity.

Aim:
Provide a concrete model for holography, that connects non-perturbative and / or discrete approaches to the continuum.

Strategy:
Work with an explicit realization of 3D (quantum) gravity.

Puzzle:
3D gravity is topological. Degrees of freedom can be captured by discretization, in a discretization independent way. How does on obtain a (conformal) field theory from this theory?

Further motivation:
AdS/CFT in non-perturbative approaches? Need to develop a strategy.

3D gravity - CFT correspondence

- [86 Brown-Henneaux] central extension of asymptotic charge algebra (AdS)
- [Maldacena 97, …] AdS-CFT
- in 3D in particular application to BTZ black holes [Banados-Teitelboim-Zanelli 92, Strominger 97, …]

- [Giombi-Maloney-Yin 08] one-loop partition function reproduces character of Virasoro algebra

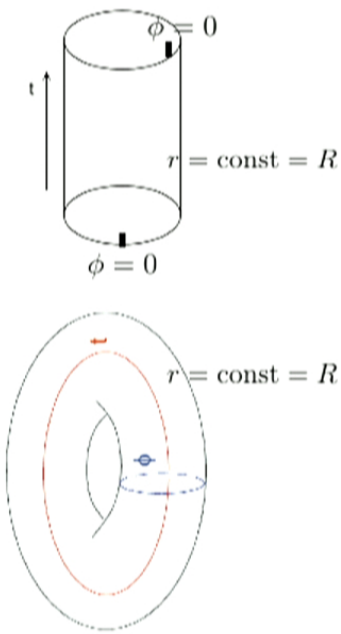
Extension to flat space?

- [06 Barnich-Compere, … ,Barnich et al] central extension of asymptotic charge algebra for flat case (BMS group at null infinity)

- [15 Barnich-Gonzalez-Maloney-Oblak] one-loop partition function reproduces character of BMS group [15 Oblak]

$\rightarrow$ review of continuum computation of one-loop partition function

Thermal spinning flat space



[Pictures: Giombi]

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \ = dt^2 + dr^2 + r^2 d\phi^2$$

periodic identification:

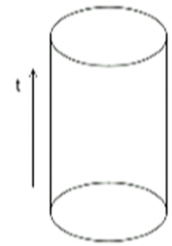
$$(t, r, \phi) \sim (t + \beta, r, \phi + \alpha)$$

inverse temperature and angular potential:
moduli parameter for the (boundary) torus

Want to study partition function:

$$Z(\beta, \alpha) = \int \mathcal{D}g \, \exp(-\frac{1}{\hbar} S_E)$$

Classical limit: Action evaluated on solution



$$
\begin{aligned}
S &= -\frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{g}R - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{h}K \\
&\underset{\text{sol}}{=} -\frac{1}{8\pi G}\mathrm{Ar}(\text{torus})_{|r=1} = -\frac{\beta}{4G} \qquad \text{no}\,\alpha\text{ dependence}
\end{aligned}
$$

One loop correction

[15 Barnich-Gonzalez-Maloney-Oblak]

- although topological theory, one loop contribution might be non-vanishing, as graviton loops might not be completely canceled by ghost loops

$$S^{(1)} = -\frac{1}{2}\ln\det\Delta^{(2)} + \ln\det\Delta^{(1)} - \frac{1}{2}\ln\det\Delta^{(0)}$$

Laplacians for spin 2, spin 1 and spin 0

$$-\ln\det(\Delta - m^2) \quad = \quad \int_0^\infty \frac{dt}{t}\int d^3x\, K(t,x,x)$$

- evaluate determinants by heat kernel approach (regulate with mass term)
- use method of images to get heat kernel on quotient space from heat kernels on R^3 (summation over images of the action of the group)

Summing the three contributions together, and taking mass to zero:

$$e^{S^{(1)}} = \prod_{k=2}^{\infty}\frac{1}{|1-q^k|^2} \qquad \text{with} \qquad q = e^{i\alpha}$$

Depends only on angular potential.

Remarks

- the calculation does not make use of the topological nature of the theory

- Can we change that?
- Understand better contributing degrees of freedom and structure of the result? (Why $|k| > 1$?)

- making use of topological nature:
  first step to consider the full non-perturbative path integral

- address more directly the puzzle: how to get field theory (on boundary) from topological theory?

Regge calculus

(at least in 3D) classical limit of spin foams

Regge calculus [61 Regge]

classical version of the Ponzano Regge model [68 Ponzano, Regge]
(first spin foam model, first topological phase, first 3D quantum gravity)

- discretization of general relativity
- use an (arbitrary) triangulation
- assign length variables to edges of the triangulation: defines (piecewise flat) geometry

- dynamics defined by the (here 3D) Regge action:

$$S_{\text{Regge}} = -\sum_{\text{bulk edge}} l_{\text{edge}}\epsilon_{\text{edge}} - \sum_{\text{bdry edge}} l_{\text{edge}}\omega_{\text{edge}}$$

$$\epsilon_{\text{edge}} = 2\pi - \sum_{\text{tetrahedra}} \theta^{\text{tetrahedron}}_{\text{edge}}$$

dihedral angle

$$S_{\text{Eucl}} = -\frac{1}{2}\int d^3x\sqrt{g}R - \int d^2x\sqrt{h}K$$

boundary

$$\omega_{\text{edge}} = \pi - \sum_{\text{tetrahedra}} \theta^{\text{tetrahedron}}_{\text{edge}}$$

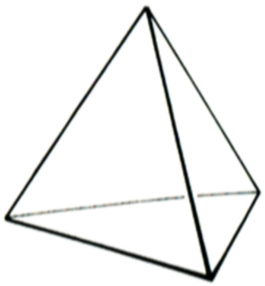
Regge calculus: Hamilton-Jacobi functions

$$S_{\text{Regge}} = -\sum_{\text{bulk edge}} l_{\text{edge}}\epsilon_{\text{edge}} - \sum_{\text{bdry edge}} l_{\text{edge}}\omega_{\text{edge}}$$

$$= -\sum_{\text{edge}} l_{\text{edge}} n_{\text{edge}} \pi + \sum_{\text{tetrahedra}} S_{\text{tetrahedron}}$$

$n_{\text{edge}} = 1 \text{ or } 2$

$$S_{\text{tetrahedron}} = \sum_{\text{edge}} l_{\text{edge}}\, \theta^{\text{tetrahedron}}_{\text{edge}} \qquad \sim \int d^2x \sqrt{h} K$$



This is the Hamilton-Jacobi function of continuum GR for tetrahedral boundary data.

Regge calculus: Hamilton-Jacobi functions

$$S_{\text{Regge}} = -\sum_{\text{bulk edge}} l_{\text{edge}}\epsilon_{\text{edge}} - \sum_{\text{bdry edge}} l_{\text{edge}}\omega_{\text{edge}}$$

Schlaefli identity for tetrahedron:

$$\sum_e l_{\text{edge}}\delta\theta^{\text{tetrahedron}}_{\text{edge}} = 0$$

continuum analogue:

contraction of the variation of Ricci tensor gives total divergence

Variation of action:

$$\delta S_{\text{Regge}} = -\epsilon_{\text{edge}}\,\delta l_{\text{edge}} \stackrel{!}{=} 0$$

EOM:

3D gravity is flat.

$$S_{\text{Regge}|\text{sol}} = -\sum_{\text{bdry edge}} l_{\text{edge}}\omega_{\text{edge}}$$

Hamilton-Jacobi function (of continuum GR) for piecewise flat boundary data.

Independent of choice of bulk triangulation.

(Together with local form of action shows topological nature of the theory.)

Quantum Regge calculus

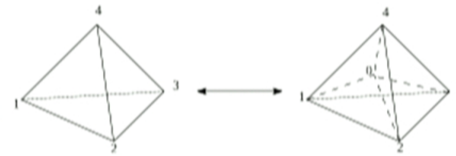
[reviews by Hamber, Williams]

$$Z_{\text{Regge}} = \int \mathcal{D}\mu(l)\ \exp(-\frac{1}{\hbar}S_{\text{Regge}}(l))$$

- 'heated' debate on the correct measure [Lund-Regge, Hamber-Williams, Menotti et al, ...]

- [11 BD, Steinhaus]: Measure should be chosen such so that (at least the linearized) theory is independent of choice of (bulk) triangulation.

  → If such a (local) measure exist, it gives the one-loop correction of the continuum theory for tetrahedral boundary data.

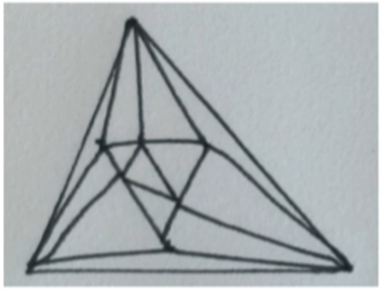


- [11 BD, Steinhaus]: Key point:
  The (3D) Hessian for Pachner move configurations (local changes of the triangulation) factorizes over tetrahedra and edges.
  This allows to construct a triangulation invariant measure!

$$\mu(l) = \frac{\prod_e \frac{l_e}{\sqrt{12\pi}}}{\prod_\tau \sqrt{V_\tau}}$$

- function of background (flat) solution
- matches exactly asymptotics of Ponzano Regge amplitude for one tetrahedron

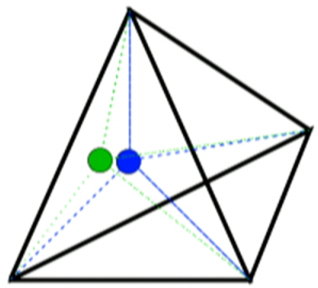
One loop correction via Regge calculus



We can start with a very finely subdivided tetrahedron and integrate out bulk edges.
But we know that the result is (bulk) triangulation independent, so choosing the empty bulk triangulation we obtain the (continuum) result

$$Z_{\text{tetrahedron}} = \frac{1}{(12\pi)^3} \frac{1}{\sqrt{V_{\text{tetrahedron}}}} \prod_e l_e \;\; \exp(\sum_e l_e(\pi - \theta_e^{\text{tetrahedron}}))$$

Diffeomorphism symmetry in Regge calculus



3D Regge EOM impose flatness. Given a flat solution with bulk vertices, the position of any of these vertices can be changed as long as the geometry remains flat. Thus one expects three gauge modes per vertex.
In the linearized theory this results in three null modes per vertex. [Roceck-Williams 81]

This 'vertex translation' symmetry is the discrete remnant of the continuum diffeomorphism symmetry and is responsible for the topological nature of the theory.

[Bahr, BD 09]: This diffeomorphism symmetry is broken for 4D Regge solutions with curvature. This has severe repercussions (e.g. triangulation dependence) and necessitates a continuum limit.

Remarks

Include cosmological constant

- The following (type of) calculation can be easily generalized to dS or AdS background.
- One has to use Regge calculus with homogeneously curved tetrahedra, only this will give triangulation independence. [Bahr, BD 09a, Bahr, BD 09b]

Full non-perturbative version?

- The full (non-linear) quantum version of Regge calculus is given by the Ponzano Regge (spin foam) model. [Ponzano, Regge 68]
  Corresponds to topological phase (but with SU(2) group) and quantization of BF theory. Related to two copies of Chern-Simons.
- Using (3D) spherical tetrahedra corresponds to the Turaev Viro model, a "quantum deformation" of the Ponzano Regge model (uses SU(2)q quantum group).
- Using (3D) hyperbolic tetrahedra: analytical continuation of Turaev Viro (results)? [eg Geiller, Noui 13]

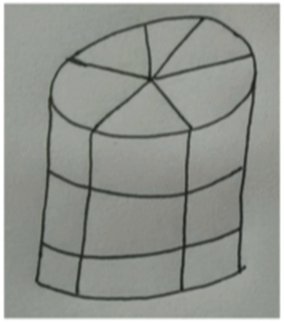
The solid torus partition function
a la Regge

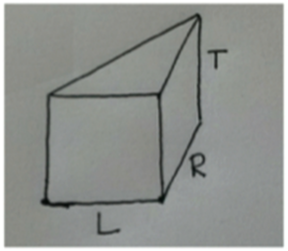
Questions to answer

- Can we take a boundary at finite radius?
  (actually for flat space there is a simple scaling argument)
- Do we need to take the continuum limit on the boundary?
  In angular and in time direction?
- Does it work if we take the coarsest bulk triangulation possible?

Set-up

- Choose a triangulation of the solid torus. Allow a very fine triangulation on the boundary but choose bulk triangulation as simple as you can think of.



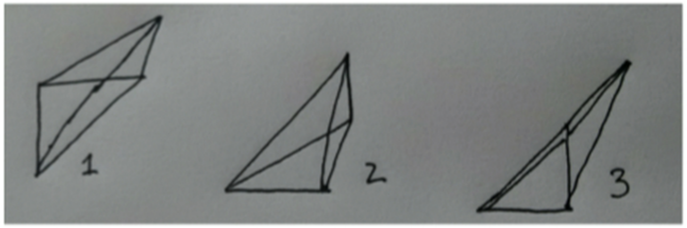


$$V_1 = V_2 = V_3 =: V$$

To get flatness (around time-like axis):

$$a := \frac{L^2}{2R^2} \stackrel{!}{=} 1 - \cos \frac{2\pi}{N}$$

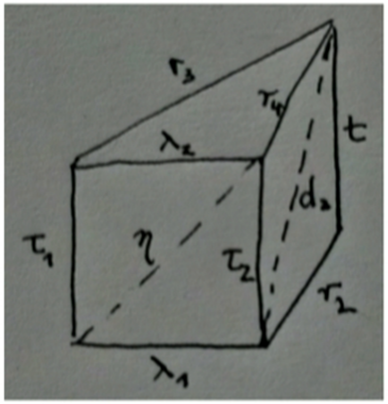
number of prisms in a time slice



- Evaluation of classical Regge action on solution is straightforward and works with simplest triangulation (even just using three tetrahedra).
  Reproduces continuum result. No continuum limit (on boundary) necessary.

Hessian of the Regge action I

- The one-loop correction is determined by (determinant of) the Hessian of the Regge action



$$H_{ee'} = \frac{\partial S_{\text{Regge}}}{\partial l_e \partial l_{e'}} = \sum_{\text{tetra}} \frac{\partial \theta_e^{\text{tetra}}}{\partial l_{e'}}$$

$$H_{ee'} = \frac{l_e l_{e'}}{12V} M_{ee'}$$

dimension free, only depends on $a = \frac{L^2}{2R^2}$

factors (partially) cancelled by measure term

$$M^{\text{prism}} = \begin{array}{cc} \begin{pmatrix} 2(-1+a) & 2 & 0 & 0 & 2(1-a) & -2 & 2a & 0 & 0 & 0 & 0 & 0 \\ 2 & 2(-1+a) & 0 & 0 & -2 & 2(1-a) & 2a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2a & 0 & 2a & 0 & 0 & 2(1-a) & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -2a & 0 & 2a & 0 & -2 & -2a & -2 & 0 & 2 \\ 2(1-a) & -2 & 2a & 0 & -2 & 2 & 0 & 0 & -2 & 0 & -2 & 2 \\ -2 & 2(1-a) & 0 & 2a & 2 & -2 & 0 & 0 & 2 & 2 & 0 & -2 \\ 2a & 2a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2(1-a) & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2a & -2 & 2 & 0 & 0 & -2 & -1 & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 & 2 & -2 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 & 2 & -2 & 0 & 0 & 2 & 1 & 1 & -2 \end{pmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ d_1 \\ d_2 \\ t \\ \tau_1 \\ \tau_2 \\ \lambda_1 \\ \lambda_2 \\ \eta \end{matrix} \end{array}$$

- Only ratio L/R matters: do not need to go to infinite radius if we take (instead) N to infinity.
- Two null vectors corresponding to overall scaling symmetry and scaling in time direction only.

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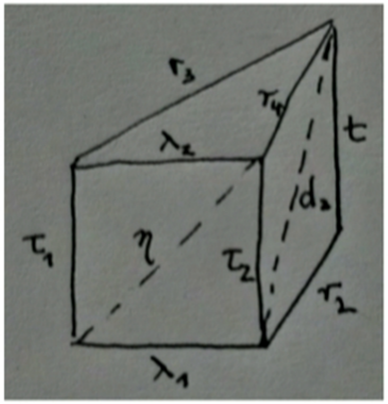

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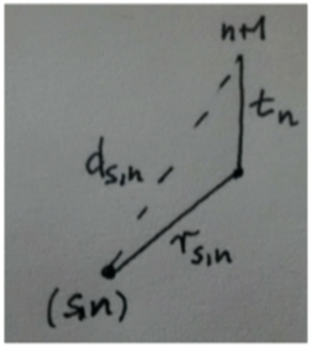
Hessian of the Regge action II

Hessian for the full triangulation

- resort (bulk) variables by associating them to vertices (s=angular step, n=time step) on the bdry
- variable transformation to absorb scaling factors
- Fourier trafo in angular direction only: $x_{k,n} = \sum_s e^{-\frac{2\pi}{N} k\cdot s}\, x_{s,n}$



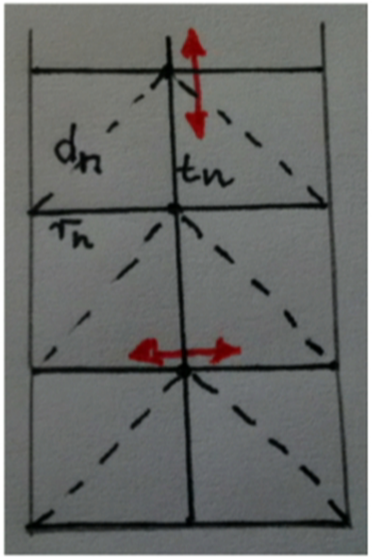
$$S^{(2)}_{\text{rescaled}} = \frac{1}{2N} \sum_{k,n,n'} S(k,n,n')$$

$$\begin{aligned} S(k,n,n') = \; & r_{k,n} r_{-k,n'} \Delta(k)\, \delta(n,n') + d_{k,n} d_{-k,n'} \Delta(k)\, \delta(n,n') + \\ & (r_{k,n} d_{-k,n'} + d_{k,n} r_{-k,n'})(-\Delta(k) + 2a)\, \delta(n,n') - \\ & r_{k,n} d_{-k,n'}\; 2a\, \delta(n, n'+1)\, (1 + \delta_{n,0}(e^{-ik\alpha} - 1)) - \\ & d_{k,n} r_{-k,n'}\; 2a\, \delta(n, n'-1)\, (1 + \delta_{n',0}(e^{ik\alpha} - 1)) - \\ & N\, (t_n r_{0,n'} + r_{0,n} t_{n'}) 2a\, \delta(n,n') \end{aligned}$$

Lattice Laplacian in angular direction $\quad \Delta(k) = 2 - 2\cos(\frac{2\pi}{N} k)$

$$2a = 2 - 2\cos(\tfrac{2\pi}{N})$$

Null vectors: diffeomorphism symmetry



Null vectors (for each time step) correspond to vertex displacements:

- involving $d(k=0)$ and t variables: displacement of bulk vertex along axis
- involving $r(k=+1)$, $d(k=+1)$ variables: displacement of bulk vertex in radial direction
- involving $r(k=-1)$, $d(k=-1)$ variables: displacement of bulk vertex in (some other) radial direction

Only modes $k=0,+1,-1$ affected by gauge symmetry
(3 null vectors per bulk vertex)

We can integrate out the r-variables for $k>0$.
The $k=0$ mode contribution vanishes (with t integration).

After integrating out r-variables

lattice Laplacian in time direction with angular twist

$$S_{dd}(k,n,n') = d_{k,n} d_{-k,n'}\, 2a \left(1 - \frac{2a}{\Delta(k)}\right) \Big(2\delta(n,n') - \delta(n,n'+1) - \delta(n,n'-1) + \\ - \delta(n',-1)\delta(n,0)(e^{-ik\alpha}-1) - \delta(n,-1)\delta(n',0)(e^{ik\alpha}-1)\Big)$$

- vanishes for k=+1 and k=-1 modes as $\Delta(k) = 2 - 2\cos(\frac{2\pi}{N}k)$ and $2a = 2 - 2\cos(\frac{2\pi}{N})$

$$\det \begin{pmatrix} 2 & -1 & 0 & \cdots & -e^{-ik\alpha} \\ -1 & 2 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -e^{ik\alpha} & 0 & \cdots & -1 & 2 \end{pmatrix} = 2 - 2\cos(k\alpha)$$

vanishes for $\alpha = 0$

from previous integration

Finally: (for N odd)

$$\frac{1}{\sqrt{\det(M_{\text{red}})}} = \frac{1}{N} \prod_{2\le k\le (N-1)/2} \frac{1}{2a}\left(1-\frac{2a}{\Delta(k)}\right)^{-1} \frac{1}{(1-e^{ik\alpha})(1-e^{-ik\alpha})}$$

$$= \frac{1}{N}\sqrt{4-2a} \prod_{2\le k\le (N-1)/2} \frac{1}{2a} \frac{1}{(1-e^{ik\alpha})(1-e^{-ik\alpha})}$$

$$= \frac{2}{N}(1+\mathcal{O}(a)) \prod_{2\le k\le (N-1)/2} \frac{1}{2a} \frac{1}{(1-e^{ik\alpha})(1-e^{-ik\alpha})}$$

factors get cancelled by measure term

Features

$$e^{S^{(1)}} = \prod_{k=2}^{\infty} \frac{1}{|1-q^k|^2} \qquad \text{with} \qquad q = e^{i\alpha}$$

- much simpler than continuum calculation:
  no gauge fixing, no (further) regularization necessary, at finite boundary
- makes direct use of topological nature of theory: get gauge modes only at k=0,+1,-1
- identifies contributing degrees of freedom and explains structure of the result
- need to take continuum limit in angular direction:
  for getting correction term small and for getting the product over all modes
- result is independent of number of time slices! [free particle in a 'perfect discretization': Bahr, BD, Steinhaus 11]
- computation can be generalized to AdS and dS case
- can try to consider this in the Ponzano Regge model,
  asymptotics for the cases needed is known

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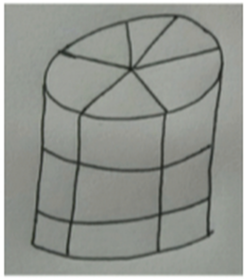
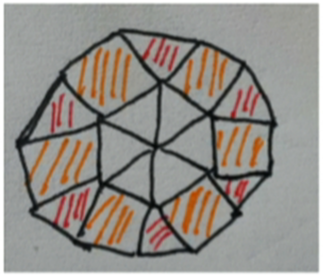
Boundary fluctuations



- next step: consider boundary fluctuations:
  (how) do we get a conformal field theory at the boundary
- 'holographic map' given by radial evolution
  (can use periodiziation in time variable)

realization of MERA?



- Hamiltonian analysis of asymptotic charges in (Ponzano-) Regge realization: symplectic structure on boundaries [recent paper: Andrade, Marolf]
- Do we get realization of Virasoro algebra?

[canonical Regge calculus: and Hamiltonian constraints: 09 Bahr, BD, 12, 13 BD, Hoehn;
constraint algebra of 3D constraints in the discrete: 13 BD, Bonzom;
quantum Hamiltonians and algebra of quantum Hamiltonians: 09 Bonzom, Freidel; 14 BD, Esterlis]

Conclusions

- in 3D discrete models provide elegant method to evaluate partition functions
- way to provide an explicit model for holography (based on topological field theory)
- shows the power of 'perfect discretizations' and more generally the consistent boundary formulation — a framework for renormalization in background independent theories

  [Bahr, BD 09; Bahr, BD, Steinhaus 11, BD12; BD, Steinhaus 13, BD 14, BD to appear]
- provides toy model to understand continuum limit in background independent theories

  [BD 14]

Many further directions to explore

- dynamics of the boundary fluctuations in flat case and AdS, canonical analysis
- (BTZ) black holes: identify degrees of freedom responsible for entropy (at the horizon) [Carlip]
- applications to dS?
- Ponzano Regge/ Turaev-Viro partition functions?
- Applications to 4D: need to take full continuum limit.
- By comparing to corresponding continuum computation, can we fix the measure for 4D Regge calculus?

[14 BD, Kaminski, Steinhaus: Discretization independence implies non-locality in 4D gravity]

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