Title: Area terms in entanglement entropy

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Abstract: We discuss area terms in entanglement entropy and show that a recent formula by Rosenhaus and Smolkin is equivalent to the Adler-Zee sum rule for the renormalization of the Newton constant in terms of correlator of traces of the stress tensor. We elaborate on how to fix the ambiguities in these formulas: Improving terms for the stress tensor of free fields, boundary terms in the modular Hamiltonian, and contact terms in the Euclidean correlation functions. We make computations for free fields and show how to apply these calculations to understand some results for interacting theories which have been studied in the literature. We check the sum rule holographicaly. We also discuss an application to the F-theorem.

Area terms in entanglement entropy

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Plan of the talk

Based on HC, F.D. Mazzitteli, E. Teste (2014)
H.C., E. Teste, G. Torroba, to appear

Entropy and area: BH entropy and area terms in entanglement entropy

Mutual information calculation for free fields

Adler-Zee formula for the renormalization of Newton's constant

QFT derivation of area terms in EE

Relations to c-theorems

Subtleties for free scalar fields

Interacting fields: Ising model, some holographic results

General holographic calculation

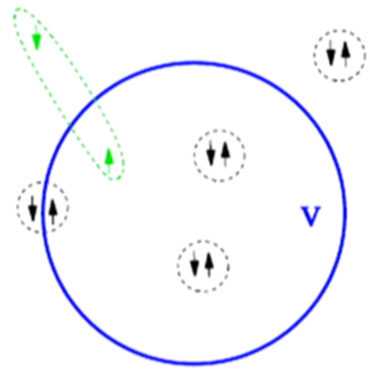
Renormalization of newton constant

Entropy and area:

Black holes and entanglement entropy S=A/(4G)? Bombelli, Koul, Lee, Sorkin (86)

Entanglement entropy in Minkowski space

Srednicki (1993)



Subleading terms contain information of the QFT (2000)

$$S = \mu \text{Area} + c_{d-3} R^{d-3} + \ldots$$

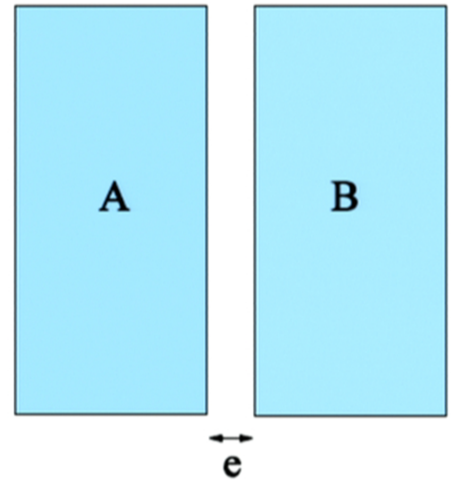
For a region large with respect to all scales in the theory

$$\mu \; = \; \left(\frac{k_{d-2}}{\epsilon^{d-2}} + k_{d-3} \frac{m}{\epsilon^{d-3}} + \ldots + k_0 m^{d-2} \log(m\epsilon) + k_0' m^{d-2} \right)$$

Hertzberg, Wilczek (2011)

The area term renormalizes from small to large regions.
Any relation with the renormalization of G?
How to compute the area term in terms of the operators of the theory?

Mutual information (universality?)



$$I(A,B) = S(A) + S(B) - S(A\cup B)$$

$$S(A,\epsilon) \sim \frac{1}{2} I(A_{\epsilon/2}, A^c_{\epsilon/2})$$

For free fields dimensional reduction to d=2 fields

$$\begin{aligned} H &= \frac{1}{2}\int d^{d-1}x\, \left(\pi^2(x) + (\nabla\phi(x))^2\right) + m^2\phi^2(x)) \\ &= \left(\frac{L}{2\pi}\right)^{d-2} \int d^{d-2}p_{\parallel} \frac{1}{2} \int dx^1 \left(\pi^2_{p_\parallel}(x) + (\partial_{x^1}\phi_{p_\parallel}(x))^2 + (m^2 + p_\parallel^2)\phi_{p_\parallel}(x)\right) \end{aligned}$$

The result can be expressed in terms of the d=2 c-function $C(r) = rS'(r)$ $\quad C^{(n)} = \int_0^\infty dy\, y^n\, C(y)$

Even d

$$I(\epsilon,m) = L^{d-2}\left(\sum_{i=0}^{d/2-2} \frac{(-1)^i k}{d-2i-2} \binom{\frac{d}{2}-2}{i} C^{(d-3-2i)} \frac{m^{2i}}{\epsilon^{d-2-2i}} + (-1)^{\frac{d}{2}} \frac{kC(0) m^{d-2}\log(m\epsilon)}{d-2}\right)$$

Odd d

$$I(L,m) = L^{d-2}\left(\sum_{i=0}^{d/2-3/2} \frac{(-1)^i k}{d-2i-2} \binom{\frac{d}{2}-2}{i} C^{(d-3-2i)} \frac{m^{2i}}{\epsilon^{d-2-2i}} + (-1)^{\frac{d-1}{2}} \frac{k\pi C(0) m^{d-2}}{2(d-2)}\right)$$

$$\kappa = \frac{d-2}{2^{d-2}\pi^{\frac{d-2}{2}}\Gamma[d/2]} \qquad C(0) = 1/3$$

Extra loglog term for the scalar $\quad \frac{(-1)^{d/2} m^{d-2}}{2(d-2)} \log(-\log(m\epsilon))$

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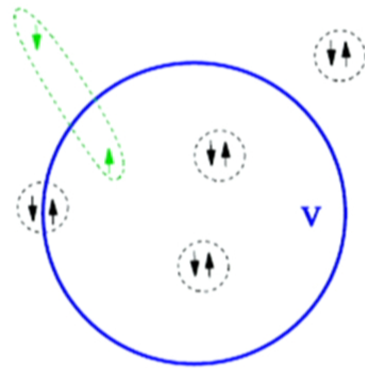

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The result for the universal part reproduces the Herzberg-Wilczek results (heat kernel+replica)

$$\mu^S_{\rm univ} = (-1)^{\frac{d-1}{2}} \frac{\pi}{2^d 3 \pi^{\frac{d-2}{2}} \Gamma[d/2]} m^{d-2}$$

$$\mu^S_{\rm univ} = (-1)^{\frac{d}{2}} \frac{1}{2^{d-1} 3 \pi^{\frac{d-2}{2}} \Gamma[d/2]} m^{d-2} \log(m\epsilon)$$

$$\mu_S = \frac{m^{d-2}\Gamma[1-d/2]}{3\pi^{d/2-1}2^d}$$

$$\mu_F = \frac{d_\Psi}{2}\mu_S$$

How does Newton's constant renormalizes: Adler-Zee formula (1982)

Effective action for gravity integrating other fields, in an expansion of small curvature

$$e^{iS_{\text{eff}}[g_{\mu\nu}]} = \int d\{\phi\} e^{iS[g_{\mu\nu},\{\phi\}]}$$

$$S_{\text{eff}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R-2\Lambda) \quad + \ldots$$

$$\Delta((4G)^{-1}) = -\frac{\pi}{d(d-1)(d-2)}\int d^dx\; x^2\,\langle 0|\Theta(0)\Theta(x)|0\rangle + \frac{4\pi}{d-2}\langle \mathcal{O}\rangle \qquad \Theta(x) = T_\mu^\mu(x)$$

$\mathcal{O} = \delta\Theta/\delta R$ depends on curvature couplings in the Lagrangian

Universal pieces of the entanglement entropy in Minkowski should not depend on curvature couplings nor on contact terms (by definition!). Proposed equation:

$$\mu = -\frac{\pi}{d(d-1)(d-2)}\int_{|x|>\epsilon} d^dx\; x^2\,\langle 0|\Theta(0)\Theta(x)|0\rangle$$

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What stress tensor use for theories that have more than one?

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Derivation in QFT (modifyed from V. Rosenhaus, M. Smolkin 2014)

First law $\delta S = \mathrm{tr}(\delta\rho K)$ $\delta S = S(\rho) - S(\rho_0),\ \delta\rho = \rho - \rho_0$ $\rho_0 \sim e^{-K}$

$$S = L^{d-2}\mu \longrightarrow L\frac{dS}{dL} = (d-2)S$$ (half space)

This can be traded to a change in the coordinates $x \to \lambda x$ in the path integral representation of the density matrix, keeping all mass scales and L fixed. The scale transformation pulls down a trace of stress tensor

$$S = \frac{1}{(d-2)}\int dx^d \,\langle \Theta(x) K\rangle = -\frac{2\pi}{d-2}\int d^d x \int_{y^1>0} d^{d-2}y\, y^1 \langle \Theta(x) T_{00}(y)\rangle$$

$$\mu = -\frac{2\pi}{d-2}\int d^d x \int_{y^1>0} dy^1\, y^1 \langle \Theta(x) T_{00}(y)\rangle$$

Using a spectral representation of the two point stress tensor correlators we get

$$\mu = -\frac{\pi}{d(d-1)(d-2)}\int d^d x\; x^2 \,\langle 0|\Theta(0)\Theta(x)|0\rangle$$

We need to compute the action on shell for the perturbations to second order with boundary condition $\delta A(x, r_{UV}) = \delta A_0(x)$

A_0 is the source of θ. The correlator is obtained from the second derivative of the on shell action

$$\langle \Theta(-p)\Theta(p)\rangle = \frac{\delta^2 S^{(2)}_{\text{on-shell}}}{\delta(\delta A^0_p)\delta(\delta A^0_{-p})}$$

$$\int d^d x\, x^2 \langle \Theta(0)\Theta(x)\rangle = -\nabla^2_p \langle \Theta(-p)\Theta(p)\rangle\big|_{p=0} \longrightarrow$$

We only need the low momentum correlator up to order p^2

$dr = -a(z)dz\ ,\ e^{A(r)} = a(z)$ other radial variable

$$S^{(2)}_{\text{on-shell}} = -\frac{d-1}{16\pi G}\int \frac{d^d p}{(2\pi)^d}\, a^{d-2}(z_{UV})\delta A(-p, z_{UV})\left[\varepsilon(z_{UV})a(z_{UV})\partial_z + \frac{p^2}{H(z_{UV})}\right]\delta A(p, z_{UV}) \qquad H = \dot{A}(r) \qquad \varepsilon = -\frac{\dot{H}}{H^2}$$

$$\lim_{z\to 0}\phi(z) \approx \phi^0_{UV} z^{d-\Delta_{UV}} \qquad \Delta_{UV} < d \qquad \lim_{z\to\infty}\phi(z) \approx \phi^0_{IR} z^{-(\Delta_{IR}-d)} \qquad \Delta_{IR} > d$$

$$\begin{aligned}\delta A_{UV}(z) &= (pz)^{\alpha_{UV}}\left(h_0 K_{\alpha_{UV}}(pz) + h_1 I_{\alpha_{UV}}(pz)\right)\\ \delta A_{IR}(z) &= (pz)^{\alpha_{IR}} h_2\, K_{\alpha_{IR}}(pz)\,,\end{aligned} \qquad \alpha \equiv \Delta - \frac{d}{2}$$

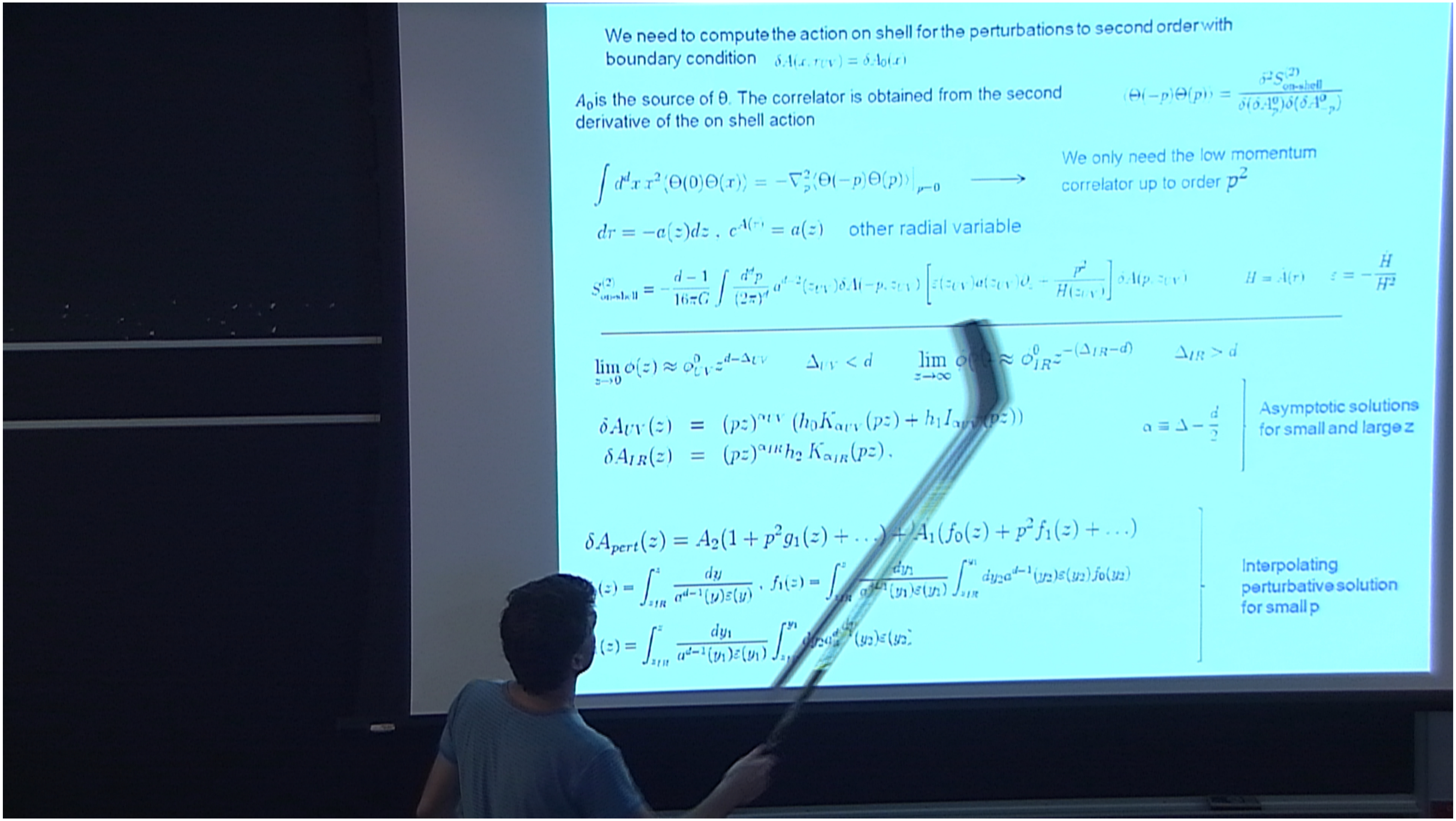
Asymptotic solutions for small and large z

$$\delta A_{pert}(z) = A_2(1 + p^2 g_1(z) + \ldots) + A_1(f_0(z) + p^2 f_1(z) + \ldots)$$

$$f_0(z) = \int_{z_{IR}}^{z} \frac{dy}{a^{d-1}(y)\varepsilon(y)}\ ,\ f_1(z) = \int_{z_{IR}}^{z} \frac{dy_1}{a^{d-1}(y_1)\varepsilon(y_1)}\int_{z_{IR}}^{y_1} dy_2 a^{d-1}(y_2)\varepsilon(y_2) f_0(y_2)$$

$$g_1(z) = \int_{z_{IR}}^{z} \frac{dy_1}{a^{d-1}(y_1)\varepsilon(y_1)}\int_{z_{IR}}^{y_1} dy_2 a^{d-1}(y_2)\varepsilon(y_2)$$

Interpolating perturbative solution for small p


We need to compute the action on shell for the perturbations to second order with boundary condition
A0 is the source of θ. The correlator is obtained from the second derivative of the on shell action
We only need the low momentum correlator up to order p2
other radial variable
Asymptotic solutions for small and large z
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Putting all together we get

$$\int d^dx\, x^2 \langle \Theta(x)\Theta(0)\rangle = \frac{d(d-1)}{\pi(4G)} \frac{e^{(d-2)A(r)}}{\dot{A}(r)}\Big|_{r_{IR}}^{r_{UV}} - \frac{d(d-1)(d-2)}{\pi(4G)} \int dr\, e^{(d-2)A(r)}$$

For d=2 this reproduces Zamolodchikov's sum rule

$$3\pi \int d^2x\, x^2 \langle \Theta(x)\Theta(0)\rangle = \frac{3}{2G}\frac{1}{\dot{A}(r)}\Big|_{r_{IR}}^{r_{UV}} = \frac{3}{2G}(L_{UV} - L_{IR}) = C_{UV} - C_{IR}$$

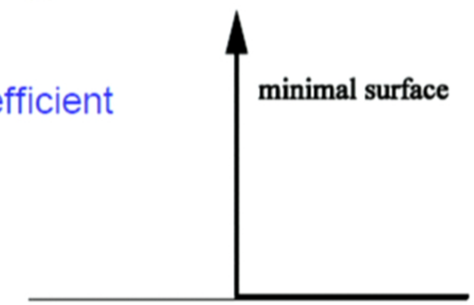
For d>2, we get

except for a UV term that cancel the most divergent part of the entopy and ensure both sides of the equation are finite for $(d-2)/2 < \Delta < (d+2)/2$

$$-\frac{\pi}{d(d-1)(d-2)} \int d^2x\, x^2 \langle \Theta(x)\Theta(0)\rangle = \frac{1}{4G}\int dr\, e^{(d-2)A(r)}$$

According to Ryu-Takayanagi prescription this is the coefficient of the area in the entropy of half space!

$$S = \mu L^{d-2} = \frac{\left(\int dr\, e^{(d-2)A(r)}\right)}{4G} L^{d-2}$$



Renormalization of newton constant

How to evaluate AZ formula such that it coincides with for example heat kernel calculation of renormalization of G? How it differs from EE?

$$\Delta((4G)^{-1}) = -\frac{\pi}{d(d-1)(d-2)}\int d^dx\; x^2\,\langle 0|\Theta(0)\Theta(x)|0\rangle + \frac{4\pi}{d-2}\langle\mathcal{O}\rangle$$

Full AZ formula

$$\mathcal{A} = -\int d^dx\,\sqrt{-g}\,\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2 + \xi R\phi^2\right)$$

Non minimally coupled scalar

$$\longrightarrow\quad \Theta(x) = -\left(\frac{d}{2} - 1 - 2(d-1)\xi\right)(\nabla\phi)^2 - \frac{d}{2}m^2\phi^2(x) + 2(d-1)\xi\phi\nabla^2\phi - \frac{(d-2)\xi}{2}\phi^2 R$$

$$-\frac{\pi}{d(d-1)(d-2)}\int d^dx\; x^2\,\langle\Theta(0)\Theta(x)\rangle = \mu_S$$

Independent of coupling to curvature, but contact terms are important

New term $\mathcal{O} = \frac{\partial\Theta}{\partial R} = -\frac{(d-2)\xi}{2}\phi^2 \quad\longrightarrow\quad \frac{4\pi}{d-2}\langle\mathcal{O}\rangle = -2\pi\xi\langle\phi^2\rangle = -2\pi\xi\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2+m^2} = (-6\xi)\frac{m^{d-2}\Gamma[1-d/2]}{3\pi^{d/2-1}2^d}$

Full result coincides with heat kernel calculations As an EE is unphysical for example for d=3 and $\xi > 1/6$

$$\longrightarrow\quad \Delta(4G)^{-1} = (1-6\xi)\frac{m^{d-2}\Gamma[1-d/2]}{3\pi^{d/2-1}2^d} = (1-6\xi)\mu_S$$

Contact terms can be eliminated with equations of motion at the expense of changing the second term. The result is unchanged

$$(-\nabla^2 + m^2 + \xi R)\phi = 0$$