Title: Microlocal analysis in quantum field theory on curved spacetime and related topics

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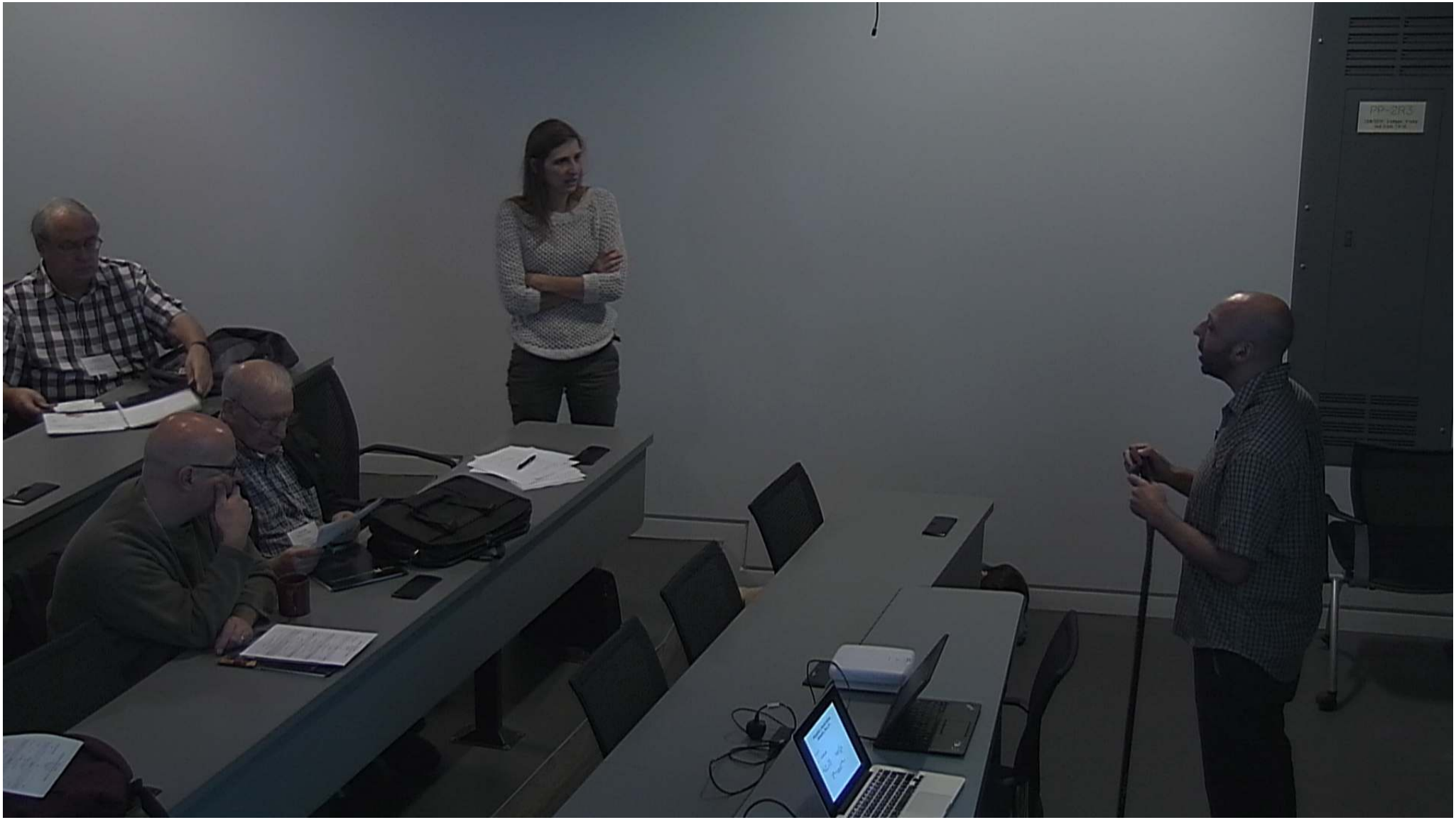
Abstract: Presented is a discussion of quantum field theory on curved spacetime and of microlocal analysis, with an emphasis on the way that these two areas connected for me personally through a specific problem, namely that of resolving Kay's singularity conjecture for two point functions of a linear scalar field on a globally hyperbolic spacetime. A particular case of this conjecture is presented, namely the translation invariant case on flat Minkowski spacetime, which does not require microlocal analysis. Next, the results of Duistermaat and Hoermander concerning distinguished parametrices of the Klein Gordon equation on a curved spacetime are described, since they lead to the notion of a wave front set (or microlocal) spectral condition, which could be viewed as a remnant of the spectral condition on flat spacetime. This condition on the wavefront set of the two point function has been employed by Brunetti, Koehler and Fredenhagen to develop a method of renormalization on a general curved spacetime, which has been developed further by Hollands and Wald. Other QFT-related topics to which microlocal methods may apply are: Lorentz symmetry breaking models and many body QM models (e.g., the free electron gas in a metal). In the case of vector or spinor models, the polarization set may be used to refine information about the singularities. Similarly, the principal symbol of the two point function, viewed as a Fourier integral operator, is a constant times a canonical half density on the natural Lagrangian submanifold associated with the Klein-Gordon operator, suggesting a tangent space Lorentz invariance property for the free model.



Microlocal Analysis in Quantum Field Theory on Curved Spacetime and Related Topics

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Presented at: Unruh Fest Workshop
Perimeter Institute

13 August, 2015

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- Kay's singularity conjecture
- Duistermaat and Hörmander's distinguished parametrices
- Wave front set (microlocal) spectral condition
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- Polarization set and principal symbol

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- Use inner product $(u,v) = i\int u^*(x)\overleftrightarrow{\frac{\partial}{\partial t}}v(x)d\mathbf{x}^3$, positive definite on positive energy solutions, since $i\frac{\partial}{\partial t} = \omega_k > 0$ on $u_k(x)$
- Above commutation relations for fields lead to comm. relns for annihilation, creation operators: $[a_k, a_l] = \left[a_k^\dagger, a_l^\dagger\right] = 0,\ \left[a_k, a_l^\dagger\right] = \delta^{(3)}(\mathbf{k}-\mathbf{l})$
- Define vacuum state (lowest energy state) in analogy with simple harmonic oscillator: $\forall \mathbf{k},\ a_k\left|0\right\rangle = 0$
- Form two-point function $\Delta^+(x,y) = \left\langle 0\right|\phi(x)\phi(y)\left|0\right\rangle = \frac{1}{(2\pi)^3}\int\frac{d\mathbf{k}^3}{2\omega_k}\mathrm{e}^{-i(\omega_k(t-s)-\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}))}$

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- Basics of quantum field theory: wish to quantize a classical field, e.g., electromagnetic A^μ, scalar ϕ, or Dirac ψ, subject to:
- Commutation relations (for bosons): $[\phi(0,\mathbf{x}),\pi(0,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y})$, $[\phi(0,\mathbf{x}),\phi(0,\mathbf{y})] = 0$ where $\pi = \frac{\partial\phi}{\partial t}$, anti-commutation relations for fermions
- Wave equation: Klein Gordon $\Box\phi + m^2\phi = 0$, Dirac $i\not{\partial}\psi - m\psi = 0$
- Ansatz for field operator: $\phi(x) = \int d\mathbf{k}^3 \left[u_k(x)a_k + u_k^*(x)a_k^\dagger\right]$, where $u_k(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} \mathrm{e}^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}$, $\omega_k = \sqrt{\mathbf{k}^2 + m^2}$

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- Feynman propagator $\langle 0|\, T\left[\phi(x)\phi(y)\right]|0\rangle = \frac{1}{(2\pi)^4}\int dk^4 \mathrm{e}^{-ik\cdot(x-y)}\frac{i}{k^2-m^2+i\epsilon}$, which is $\Delta^+(x,y)$ if $x^0 < y^0$, $\Delta^+(y,x)$ if $x^0 < y^0$
- FP propagates "positive frequencies to future (particles), negative frequencies to past (anti-particles)"
- FP transforms to nice object under Wick rotation, allowing perturbation theory, (dimensional) regularization, renormalization (under certain restrictions), Weinberg's theorem

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- What happens on curved spacetime (CST)?
  - Not easy to generalize notion of positive/negative frequencies (energies) to CST, easy if ultra-static spacetime
  - Different states have different particle interpretations, e.g., Fulling, Hawking, Unruh, see Birrell-Davies (1984)
  - No global isometry groups, in general
  - No notion of Fourier transform, by which to express spectral condition, in general

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- Algebraic approach to QFT suitable $\longrightarrow$ C^*-algebras, von Neumannn algebras
- Gårding-Wightman formalism: form vacuum expectation values $\langle 0|\, \phi(f_1) \cdots \phi(f_n)\, |0\rangle \longrightarrow$ get distributions $\langle 0|\, \phi(x_1) \cdots \phi(x_n)\, |0\rangle$, which are elements of the dual space to test function spaces
- Can give rigorous meaning to these generalized functions via distribution theory
- Wightman reconstruction theorem gives back $\mathcal{H}, \mathcal{D}, \hat{\phi}(x), |0\rangle$ satisfying appropriate relations
- $\phi(f)$ unbounded, domain problems esp. in interacting theories, constructive QFT

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- Segal field Φ maps solutions of wave equation to unbounded operators on $\mathcal{H}$
- Wightman field $\phi(x)$ maps $f \in \mathcal{C}_0^\infty(M)$ via smearing $\phi(f) = \int \phi(x) f(x)\, dx^4$
- More general nature of smooth (or Schwarz) test functions, when "peeled away" from $\phi(x)$, allow one to see singular nature of VEVs of $\phi(x)$ more clearly

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- Ingredients of QFT on CST:
  - Globally hyperbolic 4D manifold $\mathcal{M}$ with metric

$$g \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

  - Borchers algebra with $*, \times, \mathbf{1} = (1, 0, \ldots)$, locally convex topology

$$\begin{aligned} \mathcal{B}(\mathcal{M}) &= \oplus_{m=0}^{\infty} \mathcal{D}_m(\mathcal{M}) \\ \mathcal{D}_m(\mathcal{M}) &= \otimes_{n=1}^{m} \mathcal{C}_0^{\infty}(\mathcal{M}) \end{aligned}$$

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- Ingredients (cont.)
  - State $\omega \in \mathcal{B}'(\mathcal{M})$, $\omega = \times_{m=0}^{\infty} \omega_m$, $\omega_m \in \mathcal{D}'_m(\mathcal{M}), m \geq 0$
- Properties:
  - Positivity $\omega(f^* \times f) \geq 0, \quad \forall f \in \mathcal{B}(\mathcal{M})$
  - Normalized state $\omega(\mathbf{1}) = 1$
  - Local commutativity for massive or light-like particles (LC): for $g(\text{supp } f^i, \text{supp } f^{i+1}) < 0$

$$\omega_m(f^1 \otimes \cdots \otimes f^i \otimes f^{i+1} \otimes \cdots \otimes f^m) = \omega_m(f^1 \otimes \cdots \otimes f^{i+1} \otimes f^i \otimes \cdots \otimes f^m)$$

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- Specialize to linear case and ω_2:
  - Positive Type (PT) $\omega_2(\bar{f} \otimes f) \geq 0$
  - (LC) $\omega_2(f^1 \otimes f^2) = \omega_2(f^2 \otimes f^1)$ for $g(\text{supp } f^1, \text{supp } f^2) < 0$
  - Klein Gordon equation (KG) $\omega_2(Pf^1 \otimes f^2) = \omega_2(f^1 \otimes Pf^2) = 0$, $P = \Box + m^2$
  - Commutator (Com) $(\omega_2)_- = i(\Delta_A - \Delta_R) = i\Delta$, $P\Delta_A = \Delta_A P = P\Delta_R = \Delta_R P = id$ on $\mathcal{C}_0^\infty(\mathcal{M})$
  - Quasi-free $\omega_{2m+1} = 0$ and $\omega_{2m}(f^1 \otimes \cdots \otimes f^{2m}) =$

$$\sum_{\substack{\pi_1 < \pi_2, \ldots, \pi_{2m-1} < \pi_{2m} \\ \pi_1 < \pi_3 < \cdots < \pi_{2m-1}}} \omega_2(f^{\pi_1} \otimes f^{\pi_2}) \cdots \omega_2(f^{\pi_{2m-1}} \otimes f^{\pi_{2m}})$$

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- Local Hadamard condition (LH): ω_2 is (KG), x_1, x_2 in convex normal neighbourhood, $\sigma(x_1, x_2)$ signed geodesic distance squared, $t = T(x_1) - T(x_2)$

$$\begin{aligned}\omega_2(x_1, x_2) &= \lim_{\epsilon \to 0} \frac{1}{(2\pi)^2} \left[\frac{\Delta^{\frac{1}{2}}(x_1, x_2)}{\sigma(x_1, x_2) + 2i\epsilon t + \epsilon^2} \right. \\ &+ \left. v(x_1, x_2) \ln \left(\sigma(x_1, x_2) + 2i\epsilon t + \epsilon^2 \right) \right] + \mathcal{C}^\infty\end{aligned}$$

- Global Hadamard condition (GC): $\omega_2(x_1, x_2)$ has no other singularities at spacelike separated (x_1, x_2)
- Physical importance: renormalizing $T_{\mu\nu}(x)$

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- Kay's conjecture: if $\exists \mu_2$ (PT), (Com), and (GH) [also (KG)] and if ω_2 satisfies
  - (PT)
  - (Com)
  - (LH)

  then ω_2 is (GH)
- (KG) usually holds for ω_2 but not needed

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- Proof for tempered, translation invariant ω_2 on Minkowski space:
  - $\omega_2(x,y) = F(x-y)$
  - ω_2 of (PT) $\Longrightarrow$ $\hat{F}(k)$ positive-valued (consider as a function)
  - For μ_2 take vacuum

$$\begin{aligned} \Delta^+(x,y) &= \frac{1}{(2\pi)^3}\int dk^4 \mathrm{e}^{-ik\cdot(x-y)}\theta(k^0) \\ &\times \quad \delta(k^2-m^2) = F_0(x-y) \end{aligned}$$

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- Proof for translation invariant case (cont.)
  - $(\omega_2)_- = (\Delta^+)_- = i\Delta \implies F - F_0$ is real and even, $\implies b = \hat{F} - \hat{F}_0$ is real and even
  - $k^0 \leq 0 \implies b(k) = \hat{F}(k) \geq 0$
  - So $b(k) = b(-k) \implies b(k) \geq 0$ for $k^0 \geq 0$ also, thus $\forall k\ , b(k) \geq 0$
  - Hence $\omega_2 - \Delta^+$ is (PT) and locally $\mathcal{C}^\infty$
- Lemma: $v \in \mathcal{S}_2'(\mathbf{R}^4)\ , v \geq 0\ , v$ locally $\mathcal{C}^\infty \implies v \in \mathcal{C}^\infty$
- Proof of lemma uses Cauchy Schwartz inequality: $\|v(f \otimes g)\| \leq v(f \otimes f)v(g \otimes g)\ , \bar{f} = f\ , \bar{g} = g$

Duistermaat and Hörmander's distinguished parametrices

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- Wave front set of distribution: $u \in \mathcal{D}'$, $\phi \in \mathcal{C}_0^\infty$, $\Sigma(\phi u)$, $\Sigma_x(u)$, zero section $\mathbf{0}$ of $T^*(\mathcal{M})$
  WF $(u) = \{(x,k) : k \in \Sigma_x(u)\} \subset T^*(\mathcal{M}) \backslash \mathbf{0}$
- Theorem (Duistermaat-Hörmander) $P = \Box + m^2$, $(\mathcal{M}, g)$ globally hyperbolic, C_ν is one of the sets shown on next page, then $\exists$ unique, up to $\mathcal{C}^\infty$ parametrix E_ν (i.e., $PE_\nu = E_\nu P = \mathrm{id}, \mod \mathcal{C}^\infty$) such that

$$\mathrm{WF}'(E_\nu) = C_\nu \cup \Delta^*$$

- Also have propagation of singularities theorem (DH)

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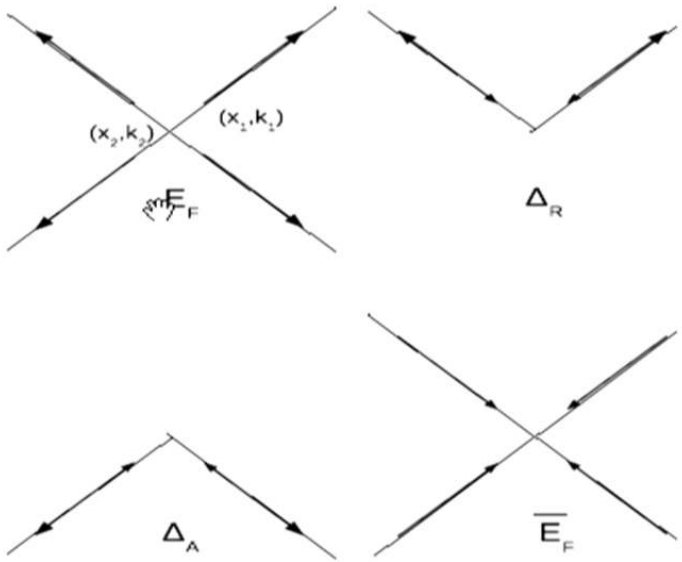

Wave front set (microlocal) spectral condition

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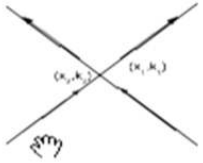
- Theorem (MJR '92): The three conditions are equivalent:
  - ω_2 is (KG) mod $\mathcal{C}^\infty$, $(\omega_2)_- = i\Delta$, and (GH)
  - $E_F = i\omega_2 + \Delta_A$ mod $\mathcal{C}^\infty$
  - ω_2 is (KG) mod $\mathcal{C}^\infty$, $(\omega_2)_- = i\Delta$, and WF $(\omega_2) =$



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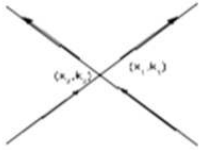

Lorentz symmetry breaking models

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- SME (Standard Model extension) Colladay-Kostelecký:

$$(i\gamma^\mu\partial_\mu - a_\mu\gamma^\mu - b_\mu\gamma_5\gamma^\mu - m)\psi = 0 \implies$$

$$[(i\partial - a)^2 - b^2 - m^2 + 2i\gamma_5\sigma^{\mu\nu}b_\mu(i\partial_\nu - a_\nu)]\psi = 0$$

- Act on distinguished parametrices of above by:

$$\mathcal{O} \equiv i\gamma^\mu\partial_\mu - a_\mu\gamma^\mu - b_\mu\gamma_5\gamma^\mu + m$$

to obtain distinguished parametrices of original Dirac-like operator $\implies$ WF (ω_2) has same form as above

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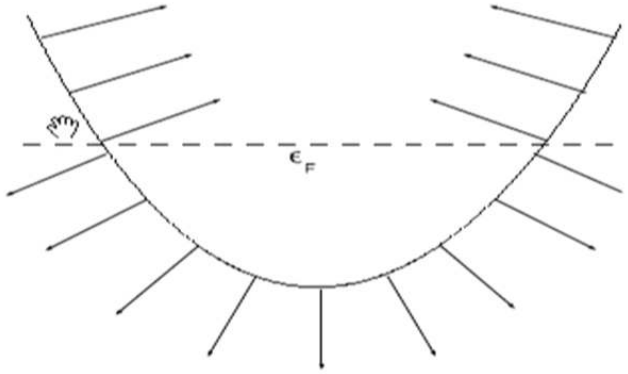

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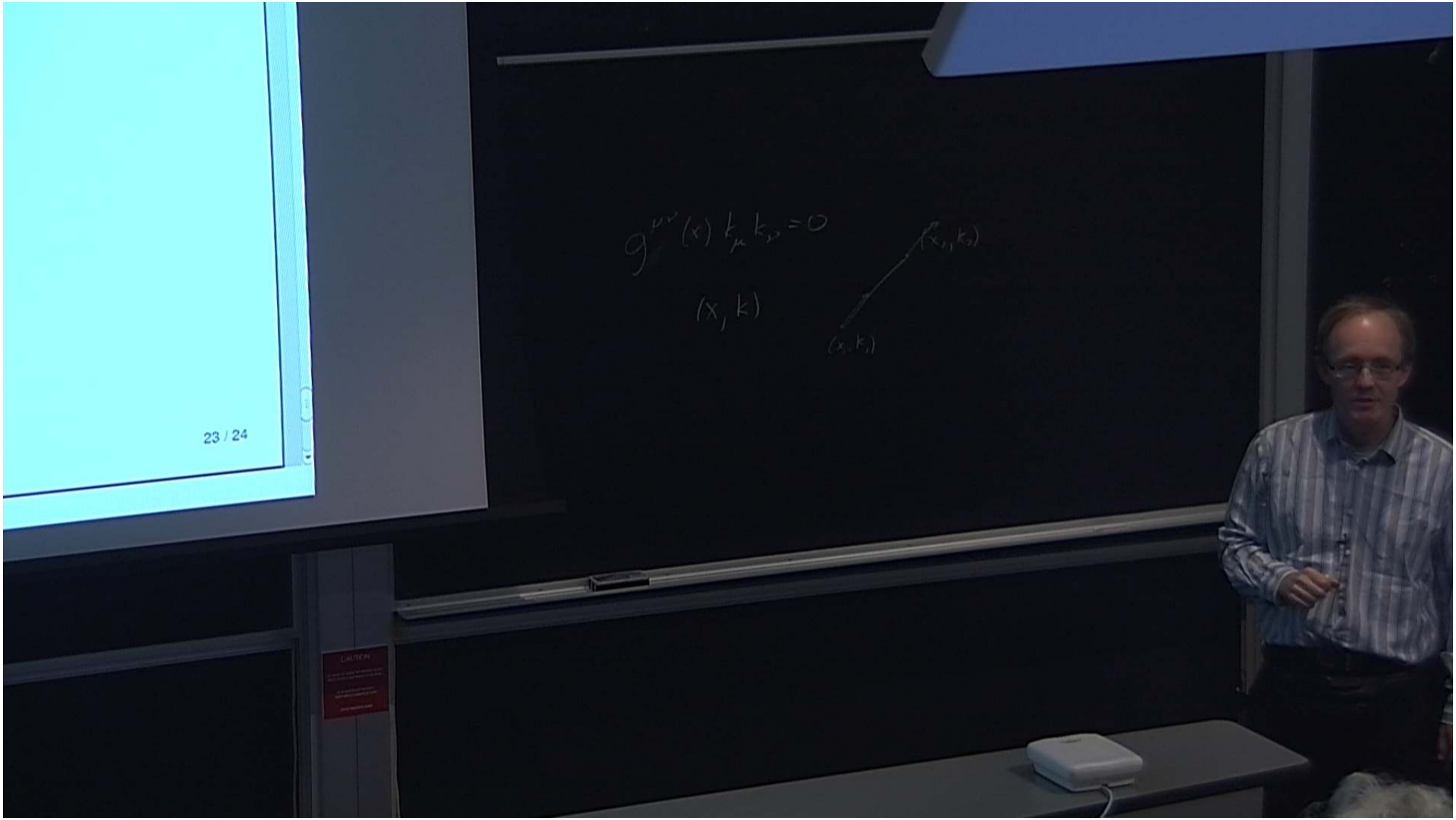
Polarization set and principal symbol

- Abstract
- Table of Contents
- Introduction
- Kay's singularity conjecture
- Duistermaat and Hörmander's distinguished parametrices
- Wave front set (microlocal) spectral condition
- Lorentz symmetry breaking models
- Many body quantum mechanics

- Polarization set: Leading order singular part of a vector- or spinor-valued distribution u^A, v^μ associated with each (x, k) of WF (u^A) or WF (v^μ)
- $\Longrightarrow$ propagation of polarization set for distributional solutions of Dirac and Maxwell's equations on CST is the usual geometric notion
- Two point function ω_2 satisfying (KG),(Com),(GH) is a Fourier integral operator with principal symbol $= 1 \times$ canonical half density
- $\Longrightarrow$ remnant of Lorentz invariance condition on the symbol of ω_2
- Connection with geometric optics: singularities propagate as classical massless particles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