Title: Information complementarity: A new paradigm for decoding quantum incompatibility

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Abstract: <p>The existence of observables that are incompatible or not jointly measurable is a characteristic feature of quantum mechanics, which is the root of a number of nonclassical phenomena, such as uncertainty relations, wave--particle dual behavior, Bell-inequality violation, and contextuality.</p>

<p>However, no intuitive criterion is available for determining the compatibility of even two (generalized) observables, despite the overarching importance of this problem and intensive efforts of many researchers over more than 80 years.</p>

<p>Here we introduce an information theoretic paradigm together with an intuitive geometric picture for decoding incompatible observables,</p>

<p>starting from two simple ideas: Every observable can only provide</p>

<p>limited information and information is monotonic under data</p>

<p>processing. By virtue of quantum estimation theory, we introduce a family of universal criteria for detecting incompatible observables and a natural measure of incompatibility, which are applicable to arbitrary number of arbitrary observables. Based on this framework, we derive a family of universal measurement uncertainty relations, provide a simple information theoretic explanation of quantitative wave--particle duality, and offer new perspectives for understanding Bell nonlocality, contextuality, and quantum precision limit.</p>

Outline

Notions of compatibility

Let $\mathbf{A} = \{A_\xi\}$ and $\mathbf{B} = \{B_\zeta\}$ be two generalized observables (or POVMs).

- Commutativity (Com): $A_\xi B_\zeta = B_\zeta A_\xi$.
- Nondisturbance (ND): **A** does not disturb **B** if there exists an instrument $\mathcal{I}_\mathbf{A}$ satisfying $\mathrm{tr}(\mathcal{I}_\mathbf{A}(\rho)B_\zeta) = \mathrm{tr}(\rho B_\zeta)$ or $\mathcal{I}_\mathbf{A}^\dagger(B_\zeta) = B_\zeta$.
- Joint measurability (JM): exist observable $\mathbf{M} = \{M_{\xi\zeta}\}$ satisfying $\sum_\zeta M_{\xi\zeta} = A_\xi, \quad \sum_\xi M_{\xi\zeta} = B_\zeta$
- Coexistence (CE): exist observable **G** such that $\mathrm{Range}(\mathbf{A}), \mathrm{Range}(\mathbf{B}) \subset \mathrm{Range}(\mathbf{G})$.

All four notions are equivalent if one of the observables is sharp.

In general $\mathrm{Com} \Rightarrow \mathrm{ND} \Rightarrow \mathrm{JM} \Rightarrow \mathrm{CE}$ but not vice versa.

Example

Consider noisy von Neumann observables $\mathbf{A} = \{(1 \pm \eta_x \sigma_x)/2\}$ and $\mathbf{B} = \{(1 \pm \eta_z \sigma_z)/2\}$ with $0 \leq \eta_x, \eta_z \leq 1$.

- **A** and **B** are noncommuting and mutually disturbing as long as $\eta_x, \eta_z > 0$.
- **A** and **B** are jointly measurable or coexistent if and only if (Busch86)

$$\eta_x^2 + \eta_z^2 \leq 1.$$

Goal

Propose an information theoretic paradigm for decoding incompatible observables. Offer an intuitive geometric picture.

- Introduce a family of universal criteria for detecting incompatible observables, applicable to arbitrary number of arbitrary observables.
- Introduce a natural measure of incompatibility.
- Derive a family of universal measurement uncertainty relations.
- Provide a simple information theoretic explanation of quantitative wave-particle duality.
- Offer new perspectives for understanding Bell nonlocality, EPR steering, contextuality, and quantum precision limit.

Fisher information

- The Fisher information matrix characterizes the amount of information provided by an observation or measurement,

$$I_{jk}(\theta) = \sum_{\xi} p(\xi|\theta) \frac{\partial \ln p(\xi|\theta)}{\partial \theta_j} \frac{\partial \ln p(\xi|\theta)}{\partial \theta_k}.$$

- Cramér-Rao bound: The inverse Fisher information sets a lower bound for the mean square error of any unbiased estimator, which can be saturated asymptotically by the maximum likelihood estimator.

Fisher information complementarity chamber

- Complementarity chamber: The set of Fisher information matrices $I(\theta)$ for all possible measurements, denoted by $\mathscr{C}(\theta)$.
- The Complementarity chamber is convex and closed.
- If $I(\theta)$ belongs to $\mathscr{C}(\theta)$, then so does any positive semidefinite matrix bounded by $I(\theta)$.
- If maximal Fisher information matrix $I_{\max}(\theta)$ is unique, then $\mathscr{C}(\theta)$ is an intersection of two opposite cones: $0 \leq I(\theta) \leq I_{\max}(\theta)$.
- Additional constraints on the complementarity chamber reflect subtle information tradeoff among different observables, a manifestation of the complementarity principle.
- These constraints may be understood as epistemic restrictions.

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Quantum Fisher information and Cramér–Rao bound

- Given a quantum state $\rho(\theta)$ and a measurement $\{\Pi_\xi\}$, the probability of obtaining the outcome ξ is $p(\xi|\theta) = \mathrm{tr}\{\rho(\theta)\Pi_\xi\}$.

$$I_{\Pi,jk}(\theta) = \sum_\xi \frac{1}{p(\xi|\theta)} \mathrm{tr}\left\{\frac{\partial\rho(\theta)}{\partial\theta_j}\Pi_\xi\right\} \mathrm{tr}\left\{\frac{\partial\rho(\theta)}{\partial\theta_k}\Pi_\xi\right\}.$$

- Quantum Fisher information

$$J_{jk}(\theta) = \frac{1}{2}\mathrm{tr}\{\rho(L_jL_k + L_kL_j)\},$$

where L_j is the symmetric logarithmic derivative (SLD),

$$\frac{\partial\rho(\theta)}{\partial\theta_j} = \frac{1}{2}[\rho(\theta)L_j(\theta) + L_j(\theta)\rho(\theta)].$$

- SLD bound: $I(\theta) \leq J(\theta)$.

- In the one-parameter setting, the SLD bound can be saturated by measuring the SLD $L(\theta)$. The complementarity chamber $\mathscr{C}(\theta)$ is a line segment determined by $0 \leq I(\theta) \leq J(\theta)$.
- In the multiparameter setting, the SLD bound generally cannot be saturated except when the SLDs L_j are compatible. The complementarity chamber is usually a small subset of the set of hypothetical Fisher information matrices satisfying the SLD bound.
- This difference is the main reason why multiparameter quantum estimation problems are difficult and poorly understood.
- We shall turn this difference into a powerful tool for studying the complementarity principle, uncertainty relations and, in particular, the joint measurement problem.

Gill–Massar inequality

- Gill–Massar (GM) inequality

$$\mathrm{tr}\{\tilde{I}(\theta)\} = \mathrm{tr}\{J^{-1}(\theta)I(\theta)\} \le d-1,$$

where $\tilde{I} = J^{-1/2} I J^{-1/2}$ is metric-adjusted Fisher information.

- The upper bound is saturated for any rank-one measurement when the number of parameters to be estimated is equal to d^2-1.

- GM bound for the scaled weighted mean square error (WMSE):

$$\mathcal{E}_W^{\mathrm{GM}} = \frac{\left(\mathrm{tr}\sqrt{J^{-1/2}WJ^{-1/2}}\right)^2}{d-1}.$$

- The bound can be saturated if and only if the hypothetical Fisher information matrix

$$I_W = (d-1)J^{1/2}\frac{\sqrt{J^{-1/2}WJ^{-1/2}}}{\mathrm{tr}\sqrt{J^{-1/2}WJ^{-1/2}}}J^{1/2}$$

belongs to the complementarity chamber.

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Information complementarity illustrated

- In the case of a qubit, the GM inequality is both necessary and sufficient. Moreover, any Fisher information matrix saturating the GM inequality can be realized by three mutually unbiased measurements.
- In terms of the components of the Bloch vector $\boldsymbol{s}$, the inverse quantum Fisher information matrix reads

$$J^{-1}(\boldsymbol{s}) = 1 - \boldsymbol{s}\boldsymbol{s}.$$

- When $s = 0$ and thus $J = 1$, the complementarity chamber is a cone that is isomorphic to the state space of subnormalized states for the three-dimensional real Hilbert space.
- When $s \neq 0$, the complementarity chamber is a distorted cone. The metric-adjusted chamber $\tilde{\mathscr{C}}(\boldsymbol{s}) := J^{-1/2}(\boldsymbol{s})\mathscr{C}(\boldsymbol{s})J^{-1/2}(\boldsymbol{s})$ has the same size and shape irrespective of the parameter point.

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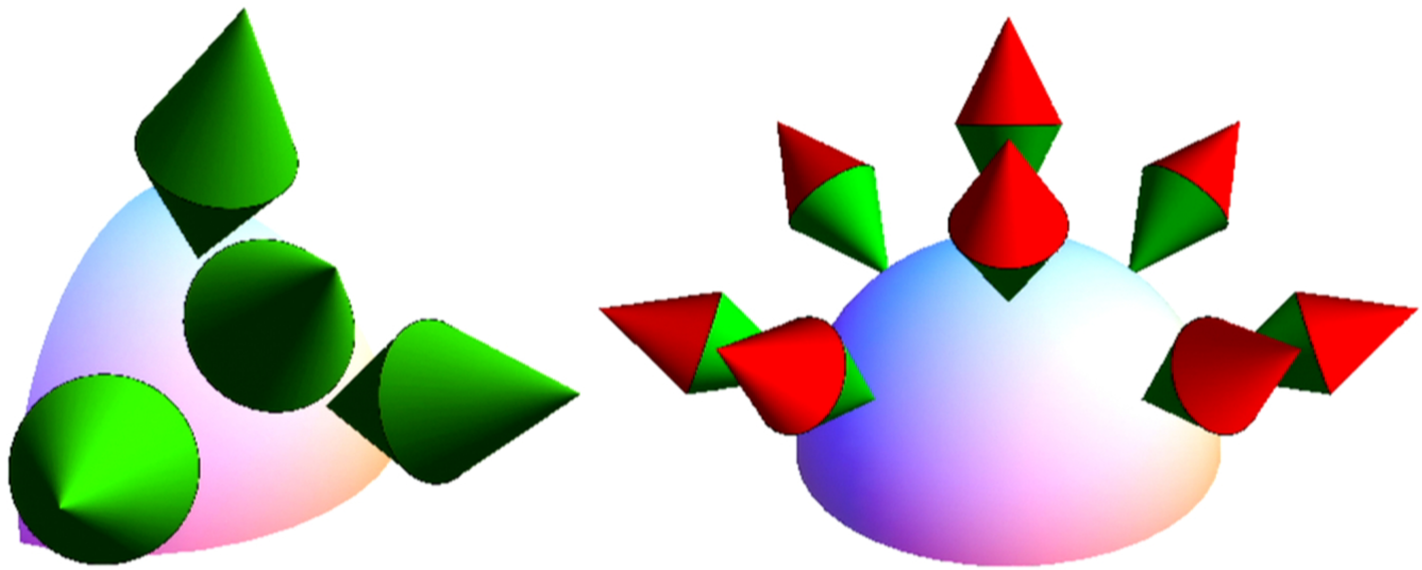
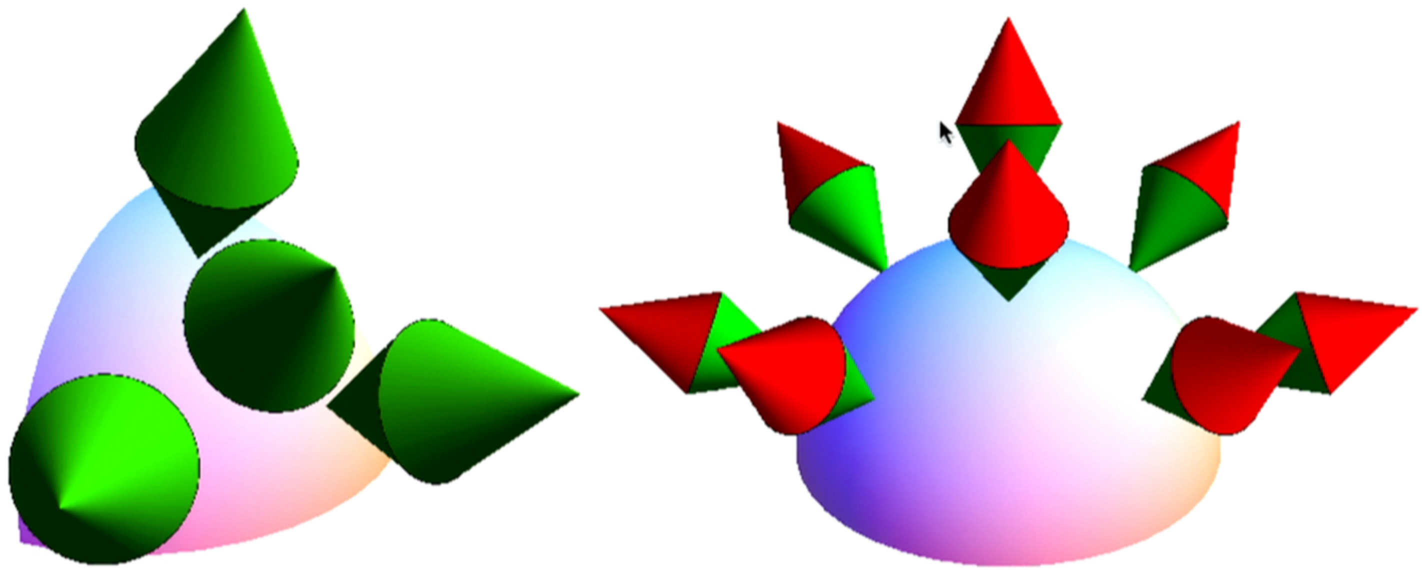


Figure : Metric-adjusted complementarity chambers $\tilde{\mathscr{C}} := J^{-1/2}\mathscr{C}J^{-1/2}$ (green cones) on the probability simplex with respect to the Fisher-Rao metric and those on the state space of the real qubit with respect to the quantum Fisher information metric. Red cones contain hypothetical Fisher information matrices satisfying the SLD bound but excluded by the Gill-Massar inequality.



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The joint measurement problem

Two observables $\mathbf{A} = \{A_\xi\}$ and $\mathbf{B} = \{B_\zeta\}$ are compatible or jointly measurable if they admit a joint observable $\mathbf{M} = \{M_{\xi\zeta}\}$,

$$\sum_\zeta M_{\xi\zeta} = A_\xi, \quad \sum_\xi M_{\xi\zeta} = B_\zeta.$$

A and **B** are called marginal observables of **M**.

The joint measurement problem

Given a set of observables $\mathbf{A}_j$, determine whether they are compatible or not.

- Closely related to measurement uncertainty relations, wave-particle dual behavior, Bell-inequality violation, and contextuality.
- No simple criterion is known. Only a few special cases have been solved, such as compatibility of von Neumann observables or that of two binary qubit observables.

Universal criteria for detecting incompatible observables

Observation: The joint observable **M** of a set of observables $\mathbf{A}_j$ is more informative than each marginal observable: $I_{\mathbf{M}}(\theta) \geq I_{\mathbf{A}_j}(\theta)$ according to the Fisher information data processing inequality (Zamir98).

Geometrically, $I_{\mathbf{M}}(\theta)$ lies in the cone $\mathscr{V}_{\mathbf{A}_j}(\theta) := \{I | I \geq I_{\mathbf{A}_j}(\theta)\}$ of hypothetical Fisher information matrices.

Universal compatibility criterion

The intersection of $\cap_j \mathscr{V}_{\mathbf{A}_j}(\theta)$ and the complementarity chamber $\mathscr{C}(\theta)$ is nonempty.

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A simpler criterion based on the GM inequality

Define

$$t(\{\tilde{I}_{\mathbf{A}_j}(\theta)\}) := \min\{\operatorname{tr}\tilde{I} \,|\, \tilde{I} \geq \tilde{I}_{\mathbf{A}_j} \text{ for all } j\}, \quad \tilde{I}_{\mathbf{A}_j} = J^{-1/2} I_{\mathbf{A}_j} J^{-1/2}.$$

$t(\{\tilde{I}_{\mathbf{A}_j}(\theta)\})$ bounds the GM trace of any hypothetical joint measurement.

Universal compatibility criterion

$$t(\{\tilde{I}_{\mathbf{A}_j}(\theta)\}) \leq d - 1.$$

- A whole family of criteria upon varying the parameter point.
- Easy to verify via semidefinite programming.
- Implications of violation: Any hypothetical joint measurement enables estimating certain parameters with error at least $t(\{\tilde{I}_{\mathbf{A}_j}\})/(d-1)$ times smaller than allowed by the quantum theory.

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Measures of incompatibility

Requirements

1. Unitary invariance: $\tau(U\mathbf{A}U^\dagger, U\mathbf{B}U^\dagger) = \tau(\mathbf{A},\mathbf{B})$;
2. Monotonicity under coarse graining: $\tau(\mathbf{C},\mathbf{D}) \leq \tau(\mathbf{A},\mathbf{B})$ if $\mathbf{C} \preceq \mathbf{A}$ and $\mathbf{D} \preceq \mathbf{B}$.

Order on observables (Martens and de Muynck 90)

- If the relation $C_\xi = \sum_\zeta \Lambda_{\xi\zeta} A_\zeta$ holds for some stochastic matrix $\Lambda_{\xi\zeta}$, then $\mathbf{C}$ is a coarse graining of $\mathbf{A}$, denoted by $\mathbf{C} \overset{\Lambda}{\preceq} \mathbf{A}$.
- Two observables $\mathbf{A}$ and $\mathbf{C}$ are equivalent if $\mathbf{C} \preceq \mathbf{A}$ and $\mathbf{A} \preceq \mathbf{C}$.

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Incompatibility measure inspired by quantum estimation theory

- $$\tau(\{\mathbf{A}_j\}) := t(\{\bar{\mathcal{G}}_{\mathbf{A}_j}\}) = t(\{\mathcal{G}_{\mathbf{A}_j}\}) - 1,$$
  where
  $$\mathcal{G}_{\mathbf{A}} = \frac{1}{d}\mathcal{F}_{\mathbf{A}}\Big(\rho = \frac{1}{d}\Big) = \sum_{\xi} |A_\xi\rangle\!\rangle \frac{1}{\operatorname{tr}(A_\xi)} \langle\!\langle A_\xi|,$$
  $$\bar{\mathcal{G}}_{\mathbf{A}} = \sum_{\xi} |\bar{A}_\xi\rangle\!\rangle \frac{1}{\operatorname{tr}(A_\xi)} \langle\!\langle \bar{A}_\xi|, \quad \bar{A}_\xi = A_\xi - \frac{\operatorname{tr}(A_\xi)}{d}.$$
- The threshold of $\tau(\cdot)$ is $d-1$. Monotonic functions of τ, such as $\max\{\tau-(d-1), 0\}$, may be considered if necessary.
- Connection with robustness
  $$R(\mathbf{A},\mathbf{B}) \geq \sqrt{\frac{\tau(\mathbf{A},\mathbf{B})}{d-1}} - 1, \quad R_{\mathrm{L}}(\mathbf{A},\mathbf{B}) \geq \frac{1}{2}\ln\frac{\tau(\mathbf{A},\mathbf{B})}{d-1}.$$

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Universal measurement uncertainty relations

For incompatible observables, any approximate joint measurement entails certain noisiness.

Noise model: $\mathbf{A}(\Lambda) := \{A_\xi(\Lambda) = \sum_\zeta \Lambda_{\xi\zeta} A_\zeta\}$. Special case:
$\mathbf{A}(\eta) = \{\eta A_\xi + (1-\eta)\,\mathrm{tr}(A_\xi)/d\}$; reduction of Fisher information $I_{\mathbf{A}(\eta)} = \eta^2 I_{\mathbf{A}}$.

- Measurement uncertainty relations

$$t(\{\tilde{I}_{\mathbf{A}_j(\Lambda_j)}\}) \leq d-1.$$

- A special case

$$\tau(\{\mathbf{A}_j(\Lambda_j)\}) = t(\{\bar{\mathcal{G}}_{\mathbf{A}_j(\Lambda_j)}\}) \leq d-1.$$

- When the noise on each observable $\mathbf{A}_j$ is characterized by a single parameter η_j, $\bar{\mathcal{G}}_{\mathbf{A}_j(\eta_j)} = \eta_j^2 \bar{\mathcal{G}}_{\mathbf{A}_j}$, we have $t(\{\eta_j^2 \bar{\mathcal{G}}_{\mathbf{A}_j}\}) \leq d-1$. If $\eta_j = \eta$, then $\tau(\{\mathbf{A}_j(\eta)\}) = \eta^2 \tau(\{\mathbf{A}_j\})$,

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Universal measurement uncertainty relations

For incompatible observables, any approximate joint measurement entails certain noisiness.

Noise model: $\mathbf{A}(\Lambda) := \{A_\xi(\Lambda) = \sum_\zeta \Lambda_{\xi\zeta} A_\zeta\}$. Special case: $\mathbf{A}(\eta) = \{\eta A_\xi + (1-\eta)\operatorname{tr}(A_\xi)/d\}$; reduction of Fisher information $I_{\mathbf{A}(\eta)} = \eta^2 I_{\mathbf{A}}$.

- Measurement uncertainty relations

$$t(\{\tilde{I}_{\mathbf{A}_j(\Lambda_j)}\}) \le d-1.$$

- A special case

$$\tau(\{\mathbf{A}_j(\Lambda_j)\}) = t(\{\bar{\mathcal{G}}_{\mathbf{A}_j(\Lambda_j)}\}) \le d-1.$$

- When the noise on each observable $\mathbf{A}_j$ is characterized by a single parameter η_j, $\bar{\mathcal{G}}_{\mathbf{A}_j(\eta_j)} = \eta_j^2 \bar{\mathcal{G}}_{\mathbf{A}_j}$, we have $t(\{\eta_j^2 \bar{\mathcal{G}}_{\mathbf{A}_j}\}) \le d-1$. If $\eta_j = \eta$, then $\tau(\{\mathbf{A}_j(\eta)\}) = \eta^2 \tau(\{\mathbf{A}_j\})$,

$$\eta^2 \le \frac{d-1}{\tau(\{\mathbf{A}_j\})}.$$

Coexistence of qubit effects

- Consider the compatibility of two noisy von Neumann observables $\mathbf{A} = \{A, 1 - A\}$ and $\mathbf{B} = \{B, 1 - B\}$ in the case of a qubit, where $A = (1 + \boldsymbol{a} \cdot \boldsymbol{\sigma})/2$ and $B = (1 + \boldsymbol{b} \cdot \boldsymbol{\sigma})/2$. This problem is equivalent to the coexistence problem of the two effects A and B.
- Degree of incompatibility

$$\tau(\mathbf{A}, \mathbf{B}) = \frac{1}{2}\left[a^2 + b^2 + \sqrt{(a^2 + b^2)^2 - 4(\boldsymbol{a} \cdot \boldsymbol{b})^2}\,\right].$$

- The compatibility criterion $\tau(\mathbf{A}, \mathbf{B}) \leq 1$ is equivalent to the necessary and sufficient criterion $\|\boldsymbol{a} + \boldsymbol{b}\| + \|\boldsymbol{a} - \boldsymbol{b}\| \leq 2$ (Busch86).

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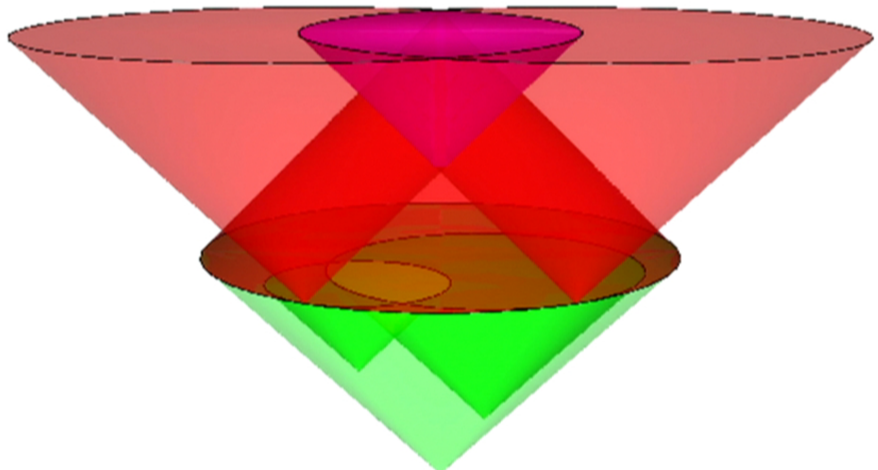
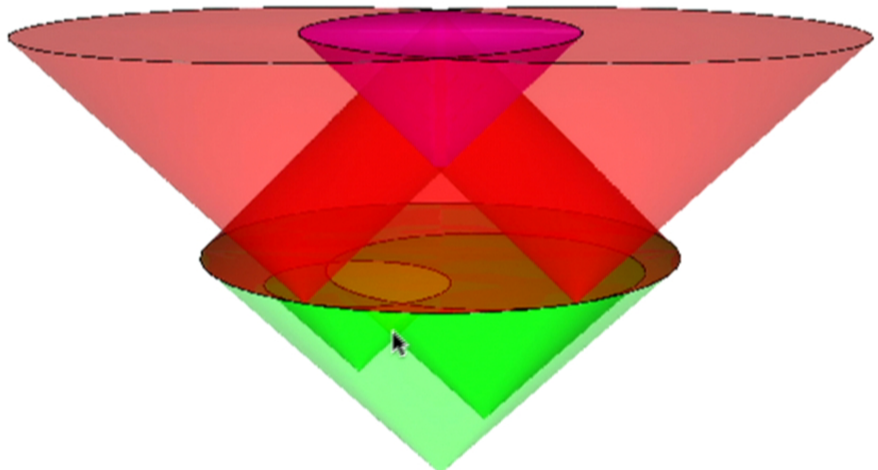


Figure : Information geometry of qubit observables. The largest green cone represents the complementarity chamber. The two upperward red cones represent the sets of hypothetical Fisher information matrices lower bounded by the Fisher information matrices of two sharp von Neumann observables. The two observables are incompatible, but their noisy versions corresponding to the tips of the two smaller green cones are compatible.



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Measurement uncertainty relations of two von Neumann observables

- Degree of incompatibility

$$\tau(\mathbf{A},\mathbf{B}) = \sum_{j=1}^{d-1}\left(1+\sqrt{1-s_j^2}\right),$$

where the s_j are singular values of $\bar{\mathcal{G}}_{\mathbf{A}}\bar{\mathcal{G}}_{\mathbf{B}}$ arranged in decreasing order. The minimum $d-1$ is attained when $\mathbf{A}=\mathbf{B}$, while the maximum $2(d-1)$ is attained when $\mathbf{A}$ and $\mathbf{B}$ are complementary.

- Measurement uncertainty relations

$$\tau(\mathbf{A}(\lambda),\mathbf{B}(\mu)) = \frac{1}{2}\sum_{j=1}^{d-1}\left(\lambda^2+\mu^2+\sqrt{(\lambda^2+\mu^2)^2-4\lambda^2\mu^2 s_j^2}\right) \le d-1.$$

Complementary observables and quantitative wave-particle duality

- The complementarity principle states that quantum systems possess properties that are equally real but mutually exclusive.
- In the the double-slit experiment, the photons can exhibit either particle behavior or wave behavior, but the sharpening of the particle behavior is necessarily accompanied with the blurring of the wave behavior, and vice versa.
- Tradeoff between path predictability and fringe visibility (GY88,JSV95,Englert96):

$$P^2 + V^2 \leq 1.$$

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- The wave-particle dual behavior is a manifestation of the impossibility of measuring simultaneously complementary observables, say σ_x and σ_z.
- Any attempt to acquire information about both observables is restricted by a measurement uncertainty relation. Observables $\mathbf{A} = \{(1 \pm \eta_x \sigma_x)/2\}$ and $\mathbf{B} = \{(1 \pm \eta_z \sigma_z)/2\}$ are compatible if and only if (Busch86)

$$\eta_x^2 + \eta_z^2 \le 1.$$

  This is an immediate consequence of the general inequality $\tau(\mathbf{A}, \mathbf{B}) \le 1$.
- Measurement uncertainty relations for generic complementary observables $\mathbf{A}_j(\eta_j)$,

$$\sum_j \eta_j^2 \le 1.$$

- Generalization for $\mathbf{A}_j(\Lambda_j)$ with Λ_j doubly stochastic,

$$\tau(\{\mathbf{A}_j(\Lambda_j)\}) = \sum_j \operatorname{tr}\Big(\Lambda_j - \frac{1}{d}K\Big)^2 \le d - 1,$$

  where K is a matrix with all entries equal to 1.

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Connection with CHSH inequality

- Given two ± 1 valued observables A, B for party 1 and C, D for party 2, state ρ satisfies the CHSH inequality if and only if $|\langle \mathbb{B} \rangle_\rho| \leq 1$ (Landau87),

$$\mathbb{B} = \frac{1}{2}[A \otimes (C + D) + B \otimes (C - D)], \quad \mathbb{B}^2 = 1 + \frac{1}{4}[A, B] \otimes [C, D].$$

- Given A and B, the maximal violation of the CHSH inequality is attained when C and D are anticommuting Pauli matrices,

$$\max_{\rho, C, D} |\langle \mathbb{B} \rangle_\rho| = \sqrt{1 + \frac{1}{2} \| [A, B] \|}.$$

- If $A = \mathbf{a} \cdot \sigma$ and $B = \mathbf{b} \cdot \sigma$, then

$$\max_{\rho, C, D} |\langle \mathbb{B} \rangle_\rho| = \sqrt{1 + \| (\mathbf{a} \times \mathbf{b}) \cdot \sigma \|} = \sqrt{\tau(A, B)}.$$

The maximum is equal to the square root of the degree of incompatibility $\tau(A, B)$.

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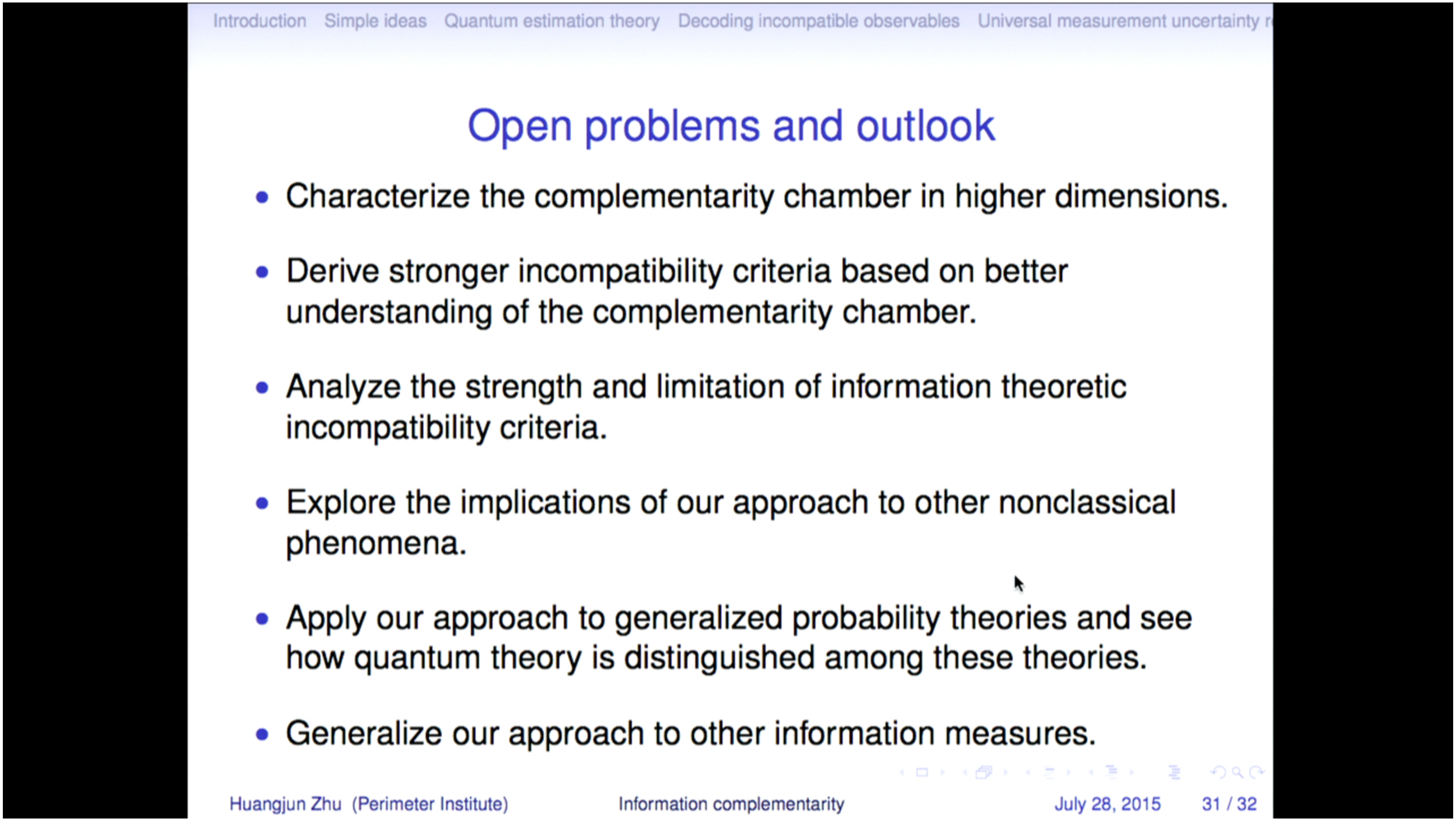
Summary

We have proposed an information theoretic paradigm together with an intuitive geometric picture for decoding incompatible observables.

- Introduced a family of universal criteria for detecting incompatible observables, applicable to arbitrary number of arbitrary observables.
- Introduced a natural measure of incompatibility.
- Derived a family of universal measurement uncertainty relations.
- Offered new perspectives for understanding quantitative wave-particle duality, Bell nonlocality, steering, contextuality, and quantum precision limit.


Introduction Simple ideas Quantum estimation theory Decoding incompatible observables Universal measurement uncertainty r
Open problems and outlook
• Characterize the complementarity chamber in higher dimensions.
• Derive stronger incompatibility criteria based on better understanding of the complementarity chamber.
• Analyze the strength and limitation of information theoretic incompatibility criteria.
• Explore the implications of our approach to other nonclassical phenomena.
• Apply our approach to generalized probability theories and see how quantum theory is distinguished among these theories.
• Generalize our approach to other information measures.
Huangjun Zhu (Perimeter Institute)
Information complementarity
July 28, 2015
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Notions of compatibility

Let $\mathbf{A} = \{A_\xi\}$ and $\mathbf{B} = \{B_\zeta\}$ be two generalized observables (or POVMs).

- Commutativity (Com): $A_\xi B_\zeta = B_\zeta A_\xi$.
- Nondisturbance (ND): $\mathbf{A}$ does not disturb $\mathbf{B}$ if there exists an instrument $\mathcal{I}_\mathbf{A}$ satisfying $\mathrm{tr}(\mathcal{I}_\mathbf{A}(\rho)B_\zeta) = \mathrm{tr}(\rho B_\zeta)$ or $\mathcal{I}_\mathbf{A}^\dagger(B_\zeta) = B_\zeta$.
- Joint measurability (JM): exist observable $\mathbf{M} = \{M_{\xi\zeta}\}$ satisfying $\sum_\zeta M_{\xi\zeta} = A_\xi, \quad \sum_\xi M_{\xi\zeta} = B_\zeta$
- Coexistence (CE): exist observable $\mathbf{G}$ such that $\mathrm{Range}(\mathbf{A}), \mathrm{Range}(\mathbf{B}) \subset \mathrm{Range}(\mathbf{G})$.

All four notions are equivalent if one of the observables is sharp.

In general $\mathrm{Com} \Rightarrow \mathrm{ND} \Rightarrow \mathrm{JM} \Rightarrow \mathrm{CE}$ but not vice versa.