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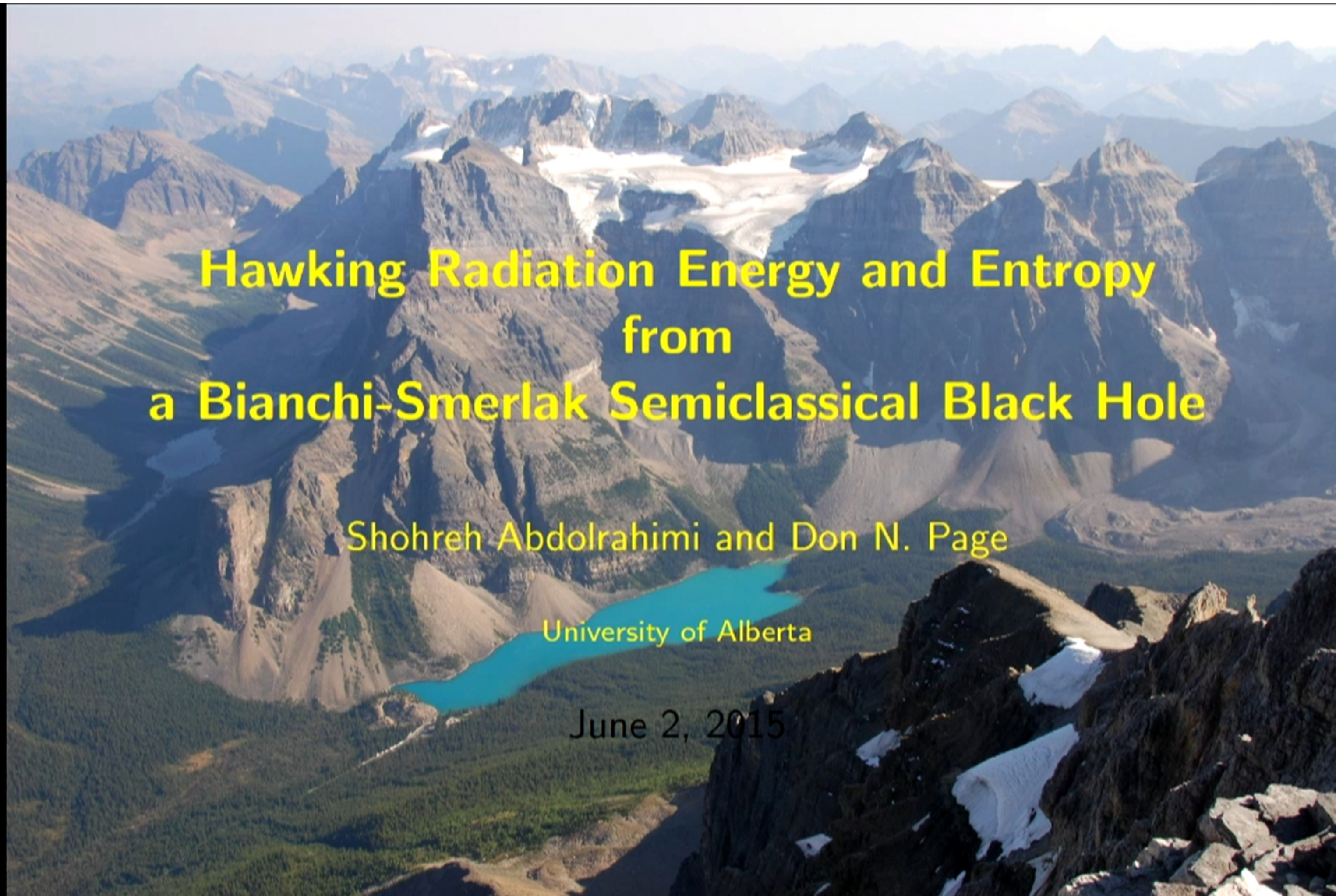

Hawking Radiation Energy and Entropy from a Bianchi-Smerlak Semiclassical Black Hole

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June 2, 2015

Introduction

Eugenio Bianchi and Matteo Smerlak have found a relationship between the Hawking radiation energy and von Neumann entropy in a conformal field emitted by a semiclassical two-dimensional black hole. We compare this relationship with what might be expected for unitary evolution of a quantum black hole in four and higher dimensions. If one neglects the expected increase in the radiation entropy over the decrease in the black hole Bekenstein-Hawking $A/4$ entropy that arises from the scattering of the radiation by the barrier near the black hole, the relation works very well, except near the peak of the radiation von Neumann entropy and near the final evaporation. These discrepancies are calculated and discussed as tiny differences between a semiclassical treatment and a quantum gravity treatment.

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The Bianchi-Smerlak Relation

Eugenio Bianchi and Matteo Smerlak have found a beautiful formula relating the Hawking radiation energy flux F and the retarded time derivatives of the von Neumann entanglement entropy S at future null infinity for a two-dimensional conformal field theory in a fixed two-dimensional classical or semiclassical spacetime:

$$F = \frac{1}{2\pi}\left(\frac{6}{c}\dot{S}^2 + \ddot{S}\right).$$

Here F is the energy flux at future null infinity ($\mathcal{I}^+$) as a function of the retarded time u, S is the renormalized entanglement entropy of the radiation at $\mathcal{I}^+$ up to the time u, c is a constant that depends on the conformal field, and an overdot represents a derivative with respect to the time u. This formula has been applied to many solvable models of gravitational collapse by Bianchi, De Lorenzo, and Smerlak.

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Comparison with 4-Dimensional Spherical Black Holes

Here we wish to compare the predictions of this formula with what is expected to be the case for the energy flux and the von Neumann entropy of Hawking radiation of massless fields from a four-dimensional spherically symmetric black hole. For the emission of a conformally invariant scalar field, one might expect the Hawking radiation to be dominated by the scalar field modes that have zero angular momentum (S-waves) and hence are spherically symmetric, effectively reducing the problem to a two-dimensional one for which one might expect the formula of Bianchi and Smerlak to apply, at least to some level of approximation.

Point of Agreement

We find that during most of the Hawking emission by a black hole of initial mass M_0 that is large in Planck units, the first term (the $\dot{S}^2$ term) on the right hand side dominates over the second term (the $\ddot{S}$ term) of

$$F = \frac{1}{2\pi}\left(\frac{6}{c}\dot{S}^2 + \ddot{S}\right).$$

The power-law dependence of this first term on the mass M is the same as that of the flux F on the left hand side in the case of 4 and higher dimensions. Therefore, by choosing the constant c appropriately, one can get a good match between the left and right hand sides.

First Difference

If $S_{\rm BH}(u) = A/(4G)$ (not the von Neumann entropy S but the logarithm of the dimension of the black hole Hilbert space), then the logarithm of the effective dimension of the radiation Hilbert space is $S_{\rm rad}(u) = bS_{\rm BH}(0) - bS_{\rm BH}(u)$, where $b > 1$ because the emission of radiation is not adiabatic. (In $d = 4$, the emission of photons and gravitons gives $b \approx 1.48472$.) S is approximately the smaller of these two dimensions, so during the first stage $\dot{S} \approx -b\dot{S}_{\rm BH}$, but during the second stage $\dot{S} \approx \dot{S}_{\rm BH}$.

Therefore, the ratio $(\dot{S})^2/F$ drops by a factor of $1/b^2$ from the first stage to the second stage, which would require c to drop by this same factor, contrary to the Bianchi-Smerlak formula in which c is supposed to be a constant.

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Second Difference

When $S(u)$ goes from increasing to decreasing, there is a brief period during which the negative $\ddot{S}$ term dominates over the positive $(6/c)(\dot{S})^2$ term, leading to a negative expression for the energy flux F in the Bianchi-Smerlak formula

$$F = \frac{1}{2\pi}\left(\frac{6}{c}\dot{S}^2 + \ddot{S}\right).$$

If one trusted this formula during this stage of the evaporation, it would seem that the black hole would have to gain a bit of energy near the peak in the von Neumann entropy S, rather than continuously losing energy.

Third Difference

Third, if one extrapolates the semiclassical approximation for $\dot{S} \approx +\dot{S}_{\rm BH}$ as a function of the black hole mass M down to Planck and sub-Planck values (where the semiclassical approximation is not believed to be valid), one again gets a regime in which the second term on the right hand side of

$$F = \frac{1}{2\pi}\left(\frac{6}{c}\dot{S}^2 + \ddot{S}\right).$$

dominates over the first term and again gives a negative expression for the energy flux.

Schwarzschild-Tangherlini Black Hole

Consider a d-dimensional, non-rotating, spherical, uncharged Schwarzschild-Tangherlini black hole of horizon radius R, with the metric

$$ds^2 = -(1 - \frac{\mu}{r^{d-3}})dt^2 + (1 - \frac{\mu}{r^{d-3}})^{-1} dr^2 + r^2 d\omega^2_{(d-2)},$$

where $\mu = R^{d-3}$ and $d\omega^2_{(d-2)}$ is the metric of a $(d-2)$-dimensional unit sphere. The ADM mass M of the black hole is given by

$$M = \frac{(d-2)\Omega_{(d-2)}\mu}{16\pi G} = \frac{(d-2)\Omega_{(d-2)}R^{d-3}}{16\pi G},$$

and the Bekenstein-Hawking thermodynamic entropy is

$$S_{\mathrm{BH}} = \frac{A}{4G} = \frac{\Omega_{(d-2)}R^{d-2}}{4G} = \frac{4\pi}{d-2}\left(\frac{16\pi G}{(d-2)\Omega_{(d-2)}}\right)^{\frac{1}{d-3}} M_0^{\frac{d-2}{d-3}}.$$

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Massless Hawking Radiation

The Hawking radiation energy flux into massless particles is

$$F = -\frac{dM}{du} = \frac{a}{R^2} = \frac{a}{\mu^{2/(d-3)}} = a\left(\frac{(d-2)\Omega_{(d-2)}}{16\pi GM}\right)^{\frac{2}{d-3}},$$

where photon and gravitons in $d = 4$ give $a \approx 0.00014990$.
The lifetime and evolution of mass and radiation entropy are

$$u_t = \frac{(d-2)(d-3)\Omega_{(d-2)}}{16\pi(d-1)Ga} R_0^{d-1} = \frac{d-3}{(d-1)a}\left(\frac{16\pi G}{(d-2)\Omega_{(d-2)}}\right)^{\frac{2}{d-3}} M_0^{\frac{d-1}{d-3}},$$

$$M(u) = M_0\left(1 - \frac{u}{u_t}\right)^{\frac{d-3}{d-1}},$$

$$S_{\rm rad}(u) = b[S_{\rm BH}(0) - S_{\rm BH}(u)] = bS_{\rm BH}(0)\left[1 - \left(1 - \frac{u}{u_t}\right)^{\frac{d-2}{d-1}}\right].$$

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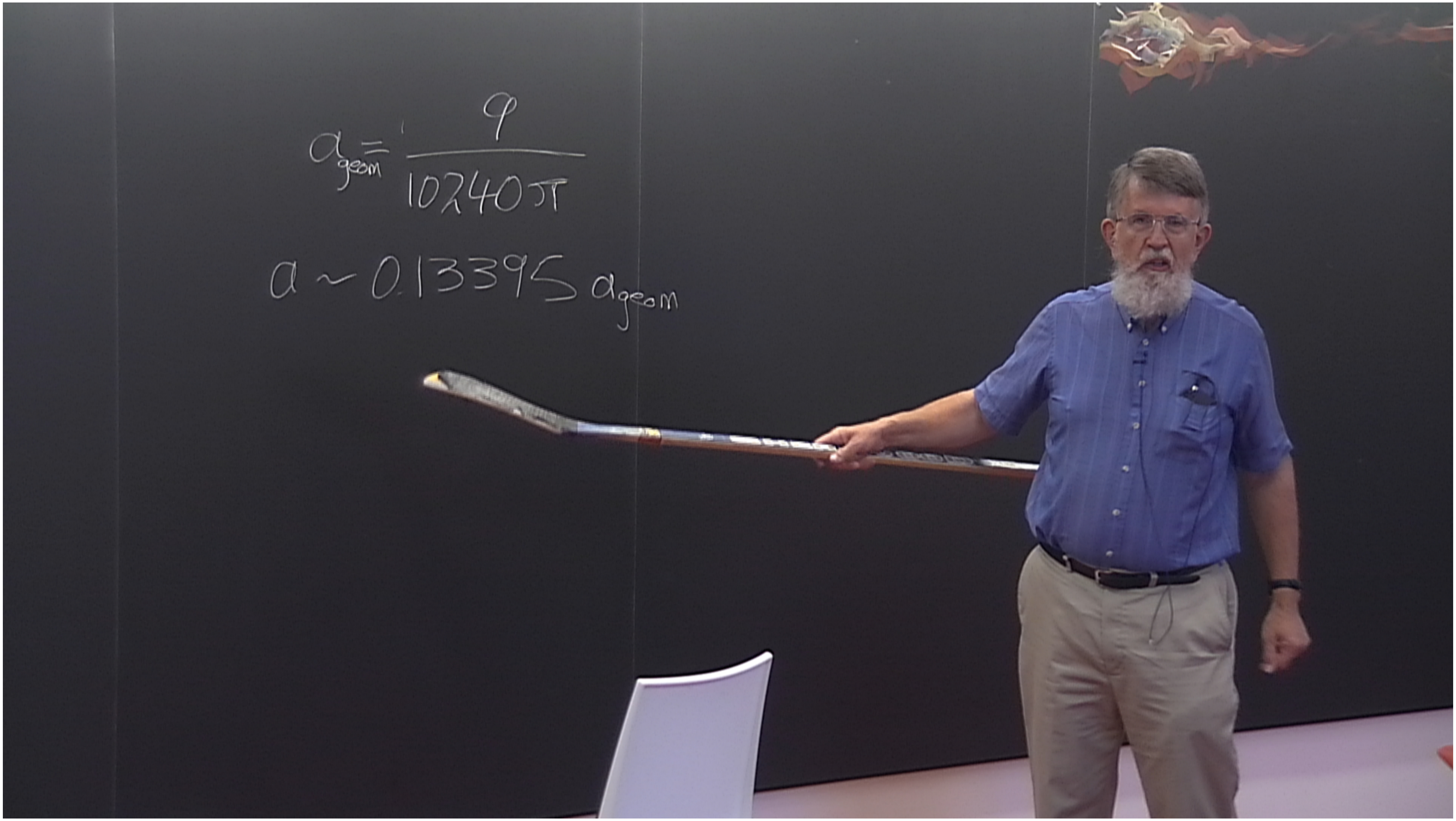

$a_{geom} = \frac{\varphi}{10240\pi}$
$a \sim 0.13395\, a_{geom}$

von Neumann Entropy of the Black Hole and Radiation

$S_{\rm BH}(u)$ and $S_{\rm rad}(u)$ should give approximate upper bounds on the von Neumann entropy S of the two subsystems, which should be near the minimum of the two upper bounds. These values cross at

$$u_p = u_t \left[1 - \left(\frac{b}{b+1} \right)^{\frac{d-1}{d-2}} \right].$$

Therefore, we expect

$$\begin{aligned} S &\approx b[S_{\rm BH}(0) - S_{\rm BH}(u)]\theta(u_p - u) + S_{\rm BH}(u)\theta(u - u_p) \\ &= bS_{\rm BH}(0) \left[1 - \left(1 - \frac{u}{u_t} \right)^{\frac{d-2}{d-1}} \right] \theta(u_p - u) \\ &\quad + S_{\rm BH}(0) \left(1 - \frac{u}{u_t} \right)^{\frac{d-2}{d-1}} \theta(u - u_p). \end{aligned}$$

Time Derivatives of the von Neumann Entropy

The Hawking temperature of the Schwarzschild-Tangherlini black hole is

$$T = \frac{\kappa}{2\pi} = \frac{d-3}{4\pi R},$$

so $\dot{M} = -F = -a/R^2$ gives

$$\dot{S}_{\rm BH} = \frac{\dot{M}}{T} = -\frac{F}{T} = -\frac{4\pi a}{(d-3)R},$$

$$\ddot{S}_{\rm BH} = -\frac{64\pi^2 G a^2}{(d-2)(d-3)\Omega_{(d-2)} R^d},$$

$$\dot{S} \approx +\frac{4\pi ab}{(d-3)R}\theta(u_p - u) - \frac{4\pi a}{(d-3)R}\theta(u - u_p),$$

$$\ddot{S} \approx +\frac{64\pi^2 G a^2 b}{(d-2)(d-3)\Omega_{(d-2)} R^d}\theta(u_p - u)$$

$$-\frac{64\pi^2 G a^2}{(d-2)(d-3)\Omega_{(d-2)} R^d}\theta(u - u_p) - \frac{4\pi a(b+1)}{(d-3)R}\delta(u - u_p).$$

Bianchi-Smerlak Flux for Most Times

Except for the retarded time u near either u_p or u_t, the first term dominates in

$$F_{\rm BS} = \frac{1}{2\pi}\left(\frac{6}{c}\dot{S}^2 + \ddot{S}\right).$$

Therefore, for most times the Bianchi-Smerlak flux is given by

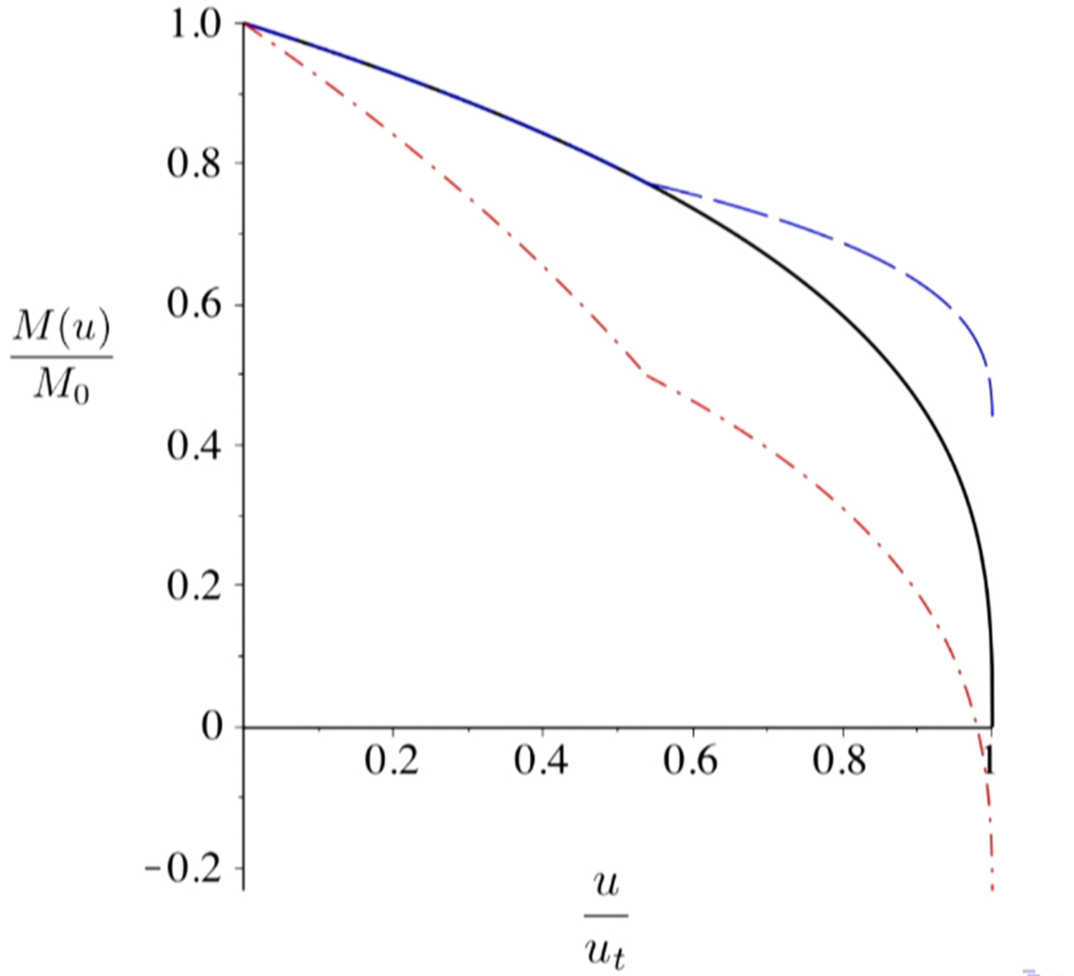
$$F_{\rm BS} \approx \frac{48\pi a^2}{c(d-3)R^2}[b^2\theta(u_p - u) + \theta(u - u_p)].$$

This would fit well with the expected $F = a/R^2$ if

$$c = \frac{48\pi a}{(d-3)^2}[b^2\theta(u_p - u) + \theta(u - u_p)].$$

However, c is supposed to be a constant, not changing between the periods $u < u_p$ and $u > u_p$.

Plot of Black Hole Mass vs. Time in 4-Dimensions



Behavior Near the Peak in S

For a random pure state in the tensor product of two Hilbert spaces of dimensions $1 \ll m \leq n$, the average von Neumann entropy of each subsystem is

$$S \approx \ln m - \frac{m}{2n}..$$

Then if

$$x \equiv bS_{\mathrm{BH}}(0) - (b+1)S_{\mathrm{BH}}(u),$$

$$S \approx \left(s_p + \frac{b}{b+1}x - \frac{1}{2}e^{x}\right)\theta(-x) + \left(s_p - \frac{1}{b+1}x - \frac{1}{2}e^{-x}\right)\theta(x),$$

$$F \approx \frac{\dot{x}^2}{4\pi}\left[\frac{3}{c}\left(\frac{2b}{b+1} - e^{x}\right)^2 - e^{x}\right]\theta(-x)$$
$$+\frac{\dot{x}^2}{4\pi}\left[\frac{3}{c}\left(\frac{2}{b+1} - e^{-x}\right)^2 - e^{-x}\right]\theta(x).$$

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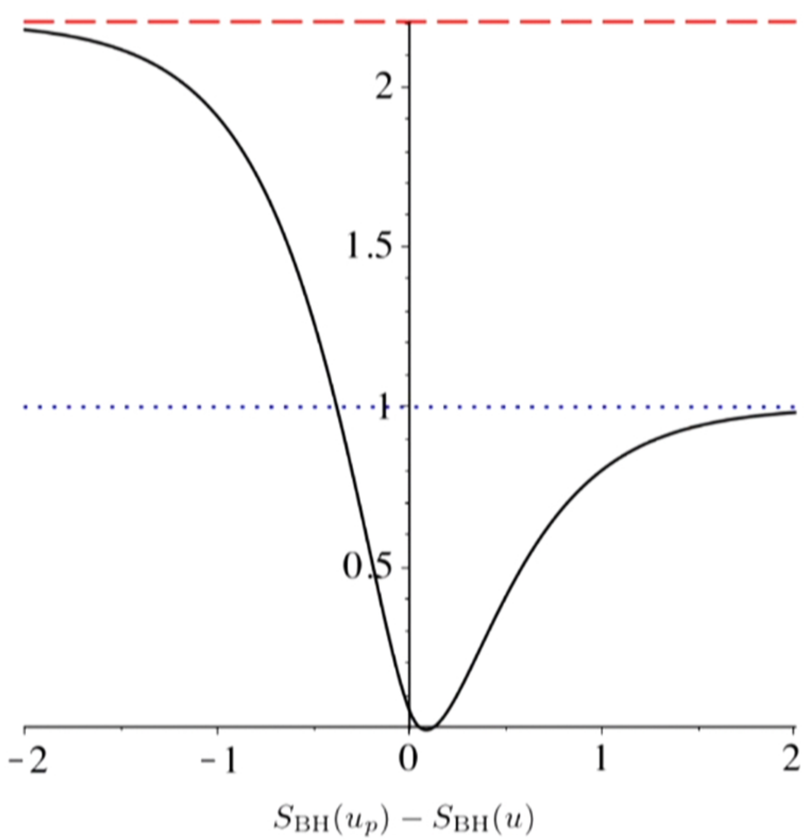
Plot of Bianchi-Smerlak Flux Formula Near the Peak of S

$$\frac{R^2}{a}F = -\frac{4M^2}{a}\frac{dM}{du}$$



Mass Increase When the Flux F Is Negative

For

$$\delta = \frac{(b+1)c}{24} = \frac{2\pi a(b+1)}{(d-3)^2} \approx 0.0018836,$$

$$y \equiv 4\ln(\sqrt{1+\delta} + \sqrt{\delta}) = 4\sinh^{-1}\sqrt{\delta} \approx 0.17355,$$

over a time period $[(d-3)yR_p/[\pi a(b+1)] \approx 37R_p$ the flux F is negative and gives a mass increase of

$$\Delta M = T_p(\sinh y - y) \approx 0.0008725\,T_p,$$

where $T_p = (d-3)/(4\pi R_p)$ is the Hawking temperature of the black hole at the time very near $u = u_p$ when the von Neumann entropy S is maximized and when the flux $F = -dM/du$ is negative, leading to the increase in the mass M.

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Uncertainty in the Time of the Negative Flux

Because of random root-N fluctuations in the number of particles emitted, which in 4-dimensions is $\sqrt{N} \sim R/\sqrt{G}$, with each particle taking a time $\sim R$ to be emitted, one might consider the Hawking evaporation to give an ensemble of semiclassical spacetimes, each with its own time at which the von Neumann entropy of the radiation is maximized. The rms spread in these times would be of the order of $\sim R^2/\sqrt{G}$, which for a solar-mass black hole of about 10^{38} Planck masses would be of the order of 10^{76} Planck times or 10^{15} times the present age of the universe. If the Bianchi-Smerlak formula applied to each spacetime in the ensemble, then there would be a negative mass flux of energy $\sim -0.00087\ T_p$ over a time less than a millisecond but uncertain by a time of the order of 10^{38} times the duration of the negative flux, a time roughly fifteen orders of magnitude larger than the present age of the universe. Therefore, the effect of this negative energy flux, if it really occurs, seems to be quite negligible and virtually impossible to detect. Surely it would be washed out by any averaging over the uncertain time at which the von Neumann entropy S is maximized.

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Avoiding Negative Flux Near the End of the Emission

If one takes $S \approx S_{\rm BH}(u) \approx S_{\rm BH}(0)(1 - u/u_t)^{(d-2)/(d-1)}$ at late times, one finds that the Bianchi-Smerlak flux becomes negative for $u_t - u$ small in Planck units. However, this can be avoided by taking, for example, in $d = 4$ spacetime dimensions

$$S = \frac{8\pi a X}{1 + 1/X} = \frac{S_{\rm BH}^2}{S_{\rm BH} + 8\pi a}$$

for $X \equiv R^2/(8Ga) = S_{\rm BH}/(8\pi a)$, giving

$$F = \frac{a}{R^2}\left[1 - \frac{X^3 + 8X^2 + 9X}{2(X+1)^4}\right].$$

This never goes negative but instead increases monotonically with the time u as R decreases, always remaining within a factor of 3 of the expression $F = a/R^2$. Therefore, there is no problem constructing $S(u)$ that is close to the Bekenstein-Hawking black hole entropy $S_{\rm BH} = A/(4G)$ for a large hole but deviates when the hole gets small so that the Bianchi-Smerlak flux stays positive.

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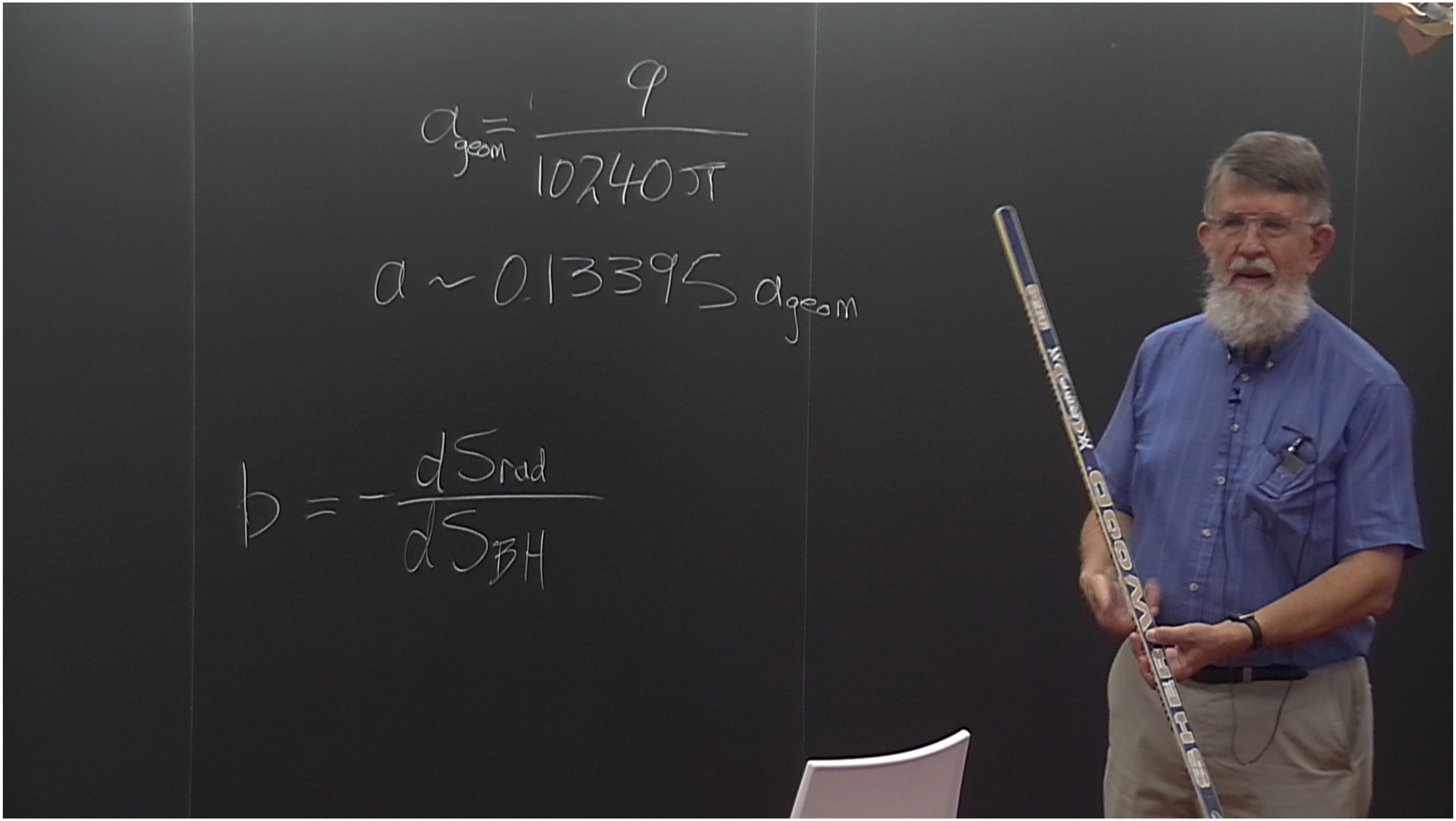
Conclusions 1

In conclusion, the Bianchi-Smerlak formula

$$F = \frac{1}{2\pi}\left(\frac{6}{c}\dot{S}^2 + \ddot{S}\right),$$

which was derived for a 2-dimensional model black hole with quantum field theory on a definite spacetime metric (say with boundary conditions given by a moving mirror), seems to work fairly well for quantum black holes in higher dimensions, but there are some inadequacies in the model. Perhaps the most serious is the fact that one cannot match the entropy and energy flux both before and after the peak in the von Neumann entropy with a fixed value of the constant c in the Bianchi-Smerlak formula if the black hole evaporation process is not adiabatic but generates extra entropy in the radiation in excess of the decrease in the Bekenstein-Hawking entropy $A/(4G)$ of the black hole.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