Title: Amplitudes: Unitarity

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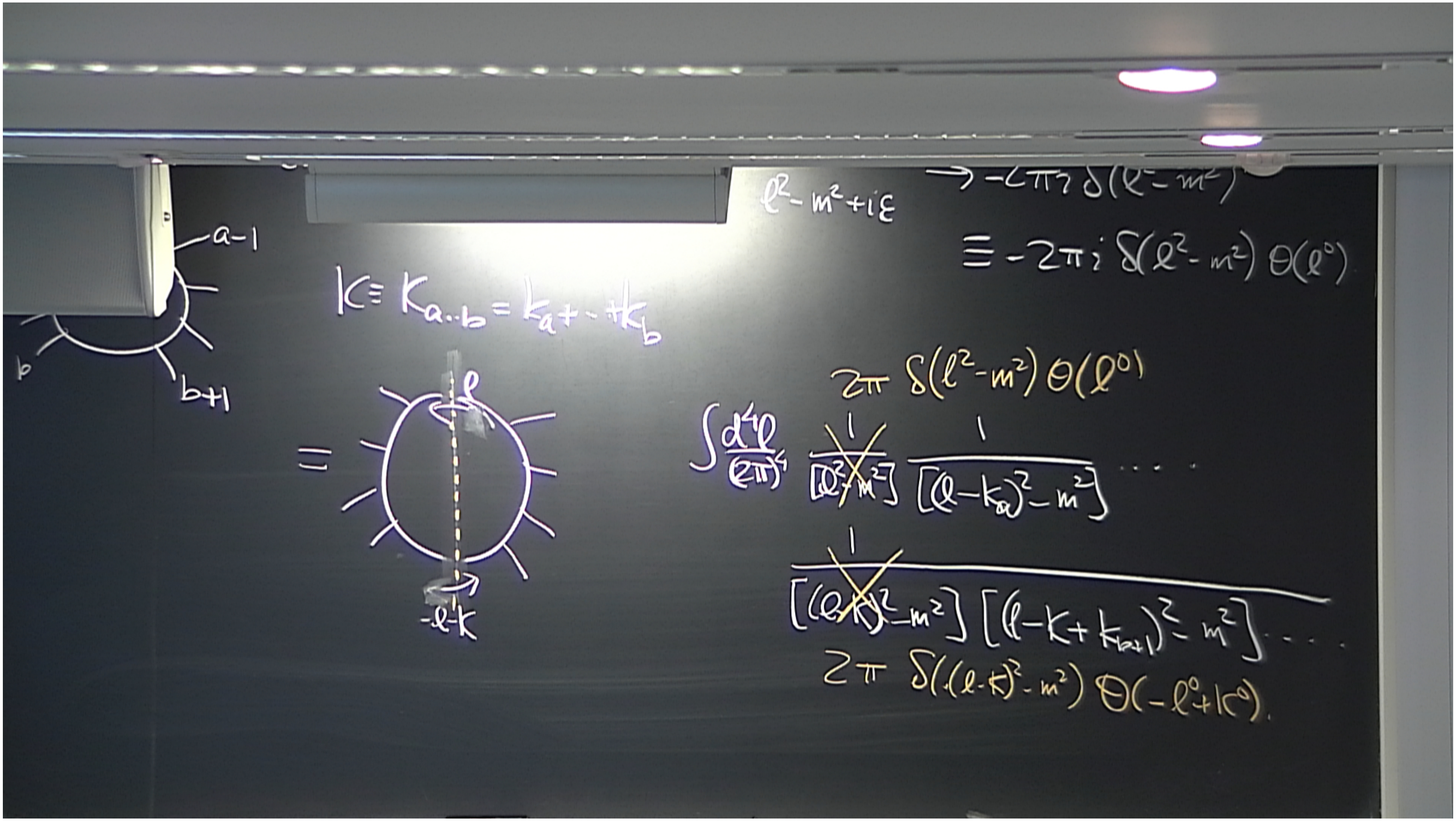
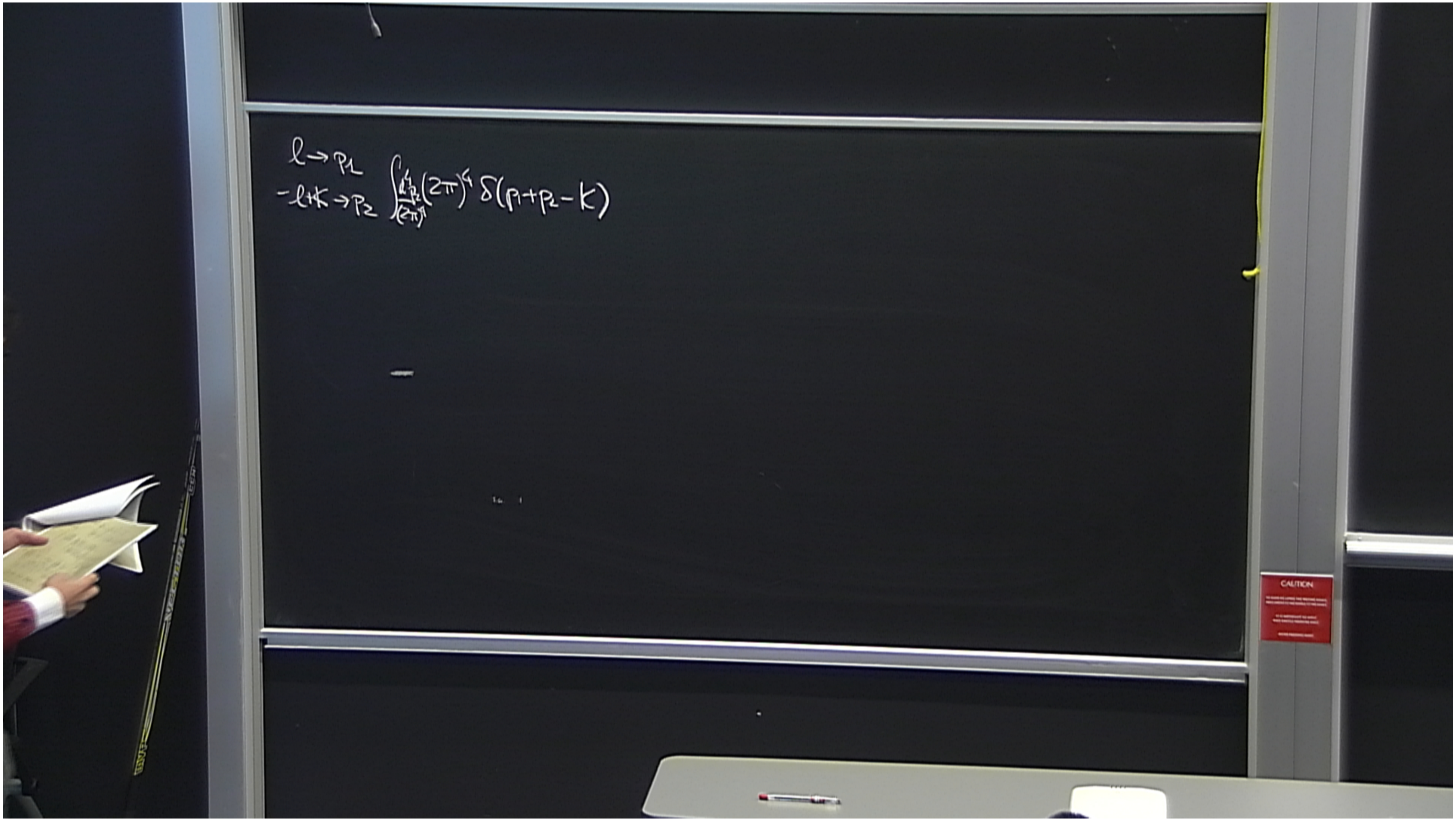
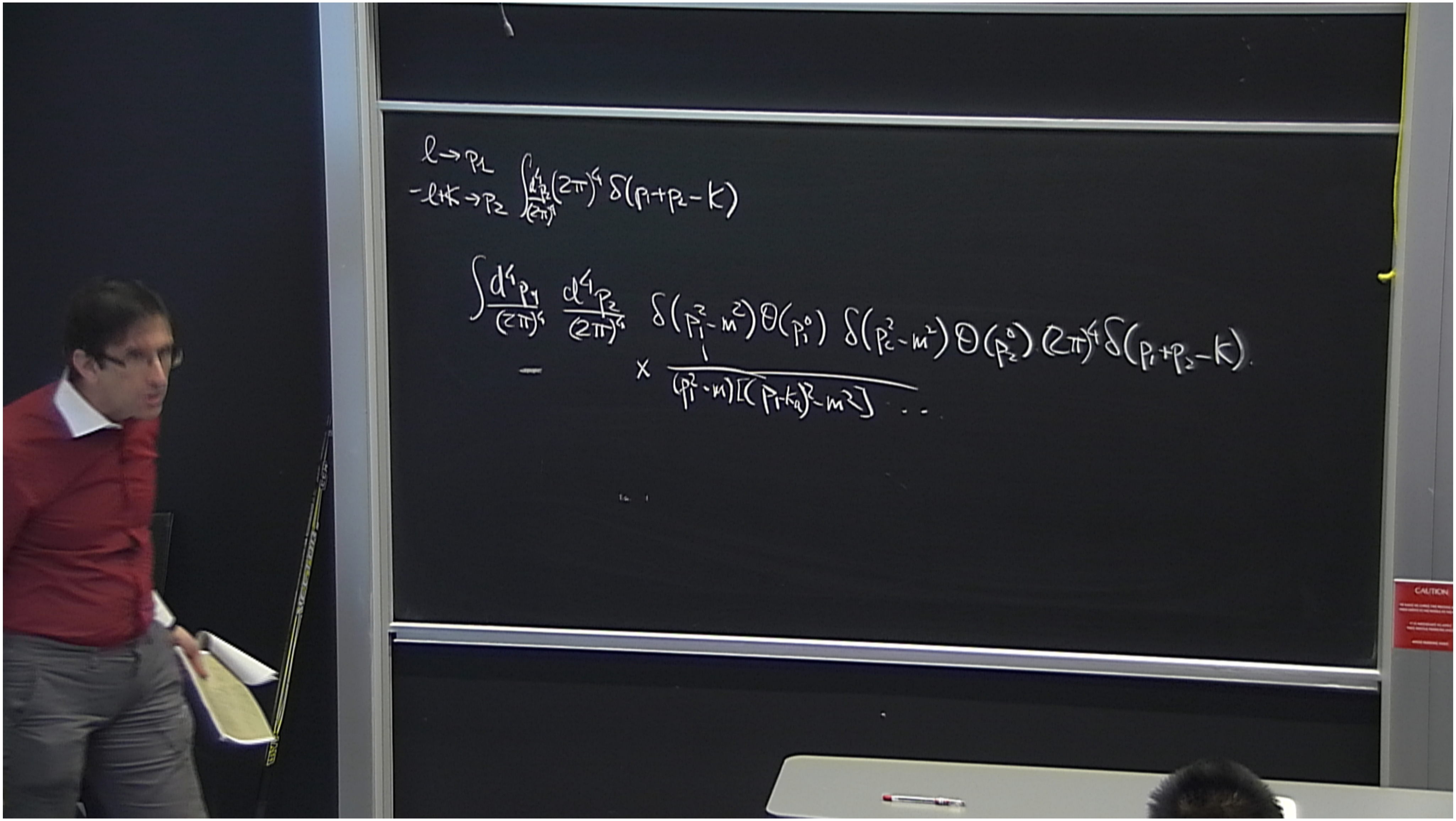
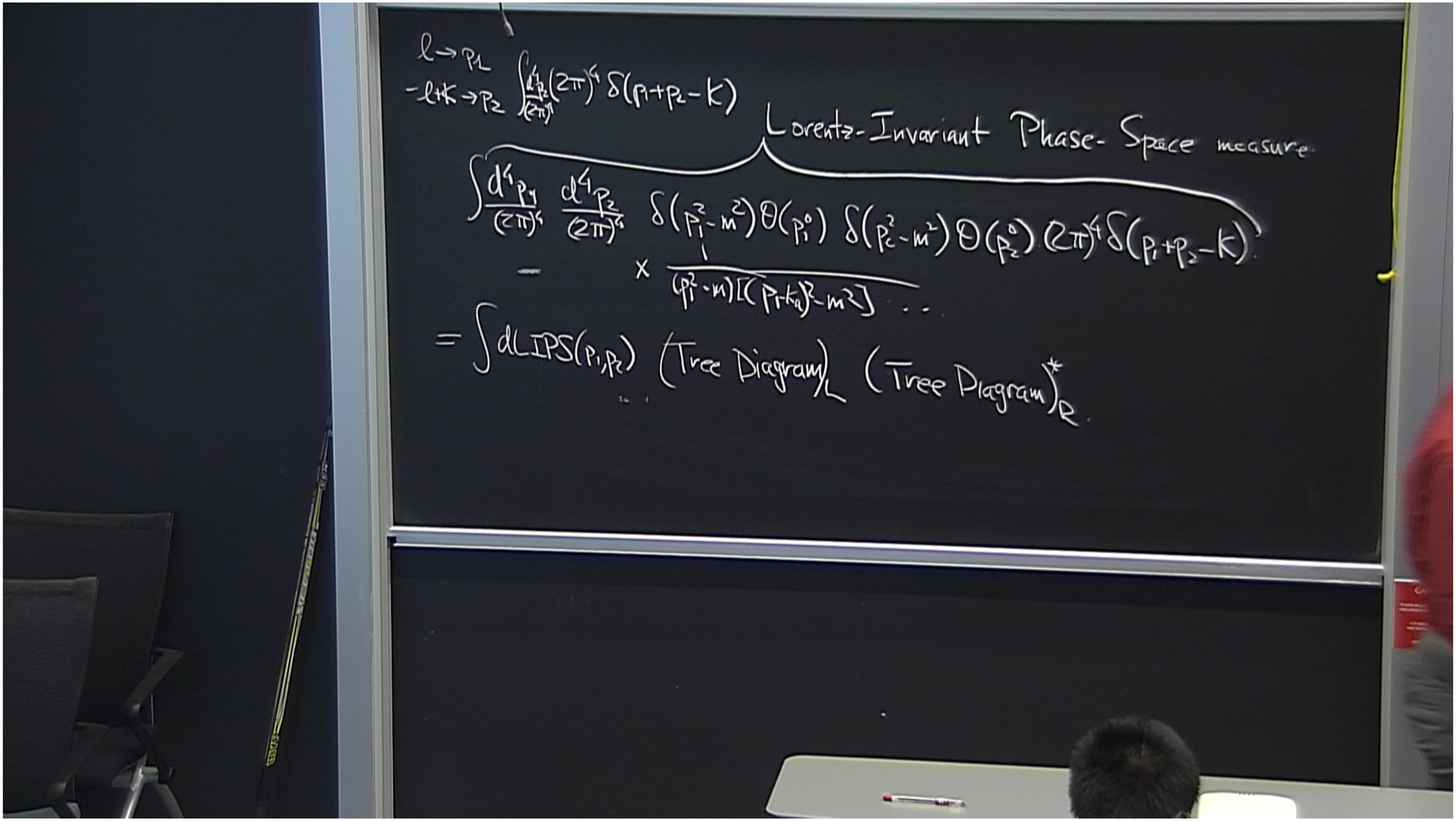
URL: http://pirsa.org/15070054

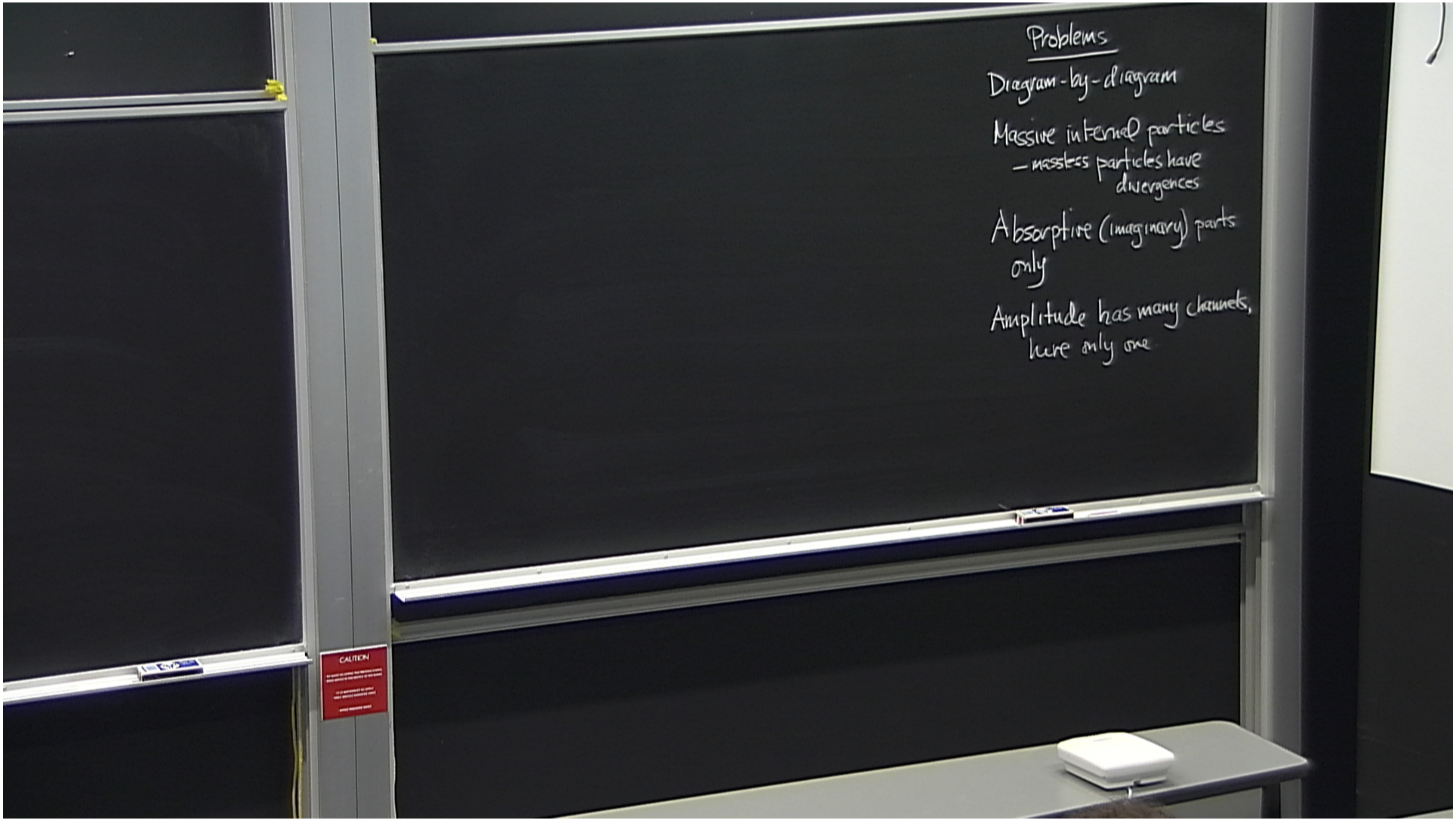
Abstract:

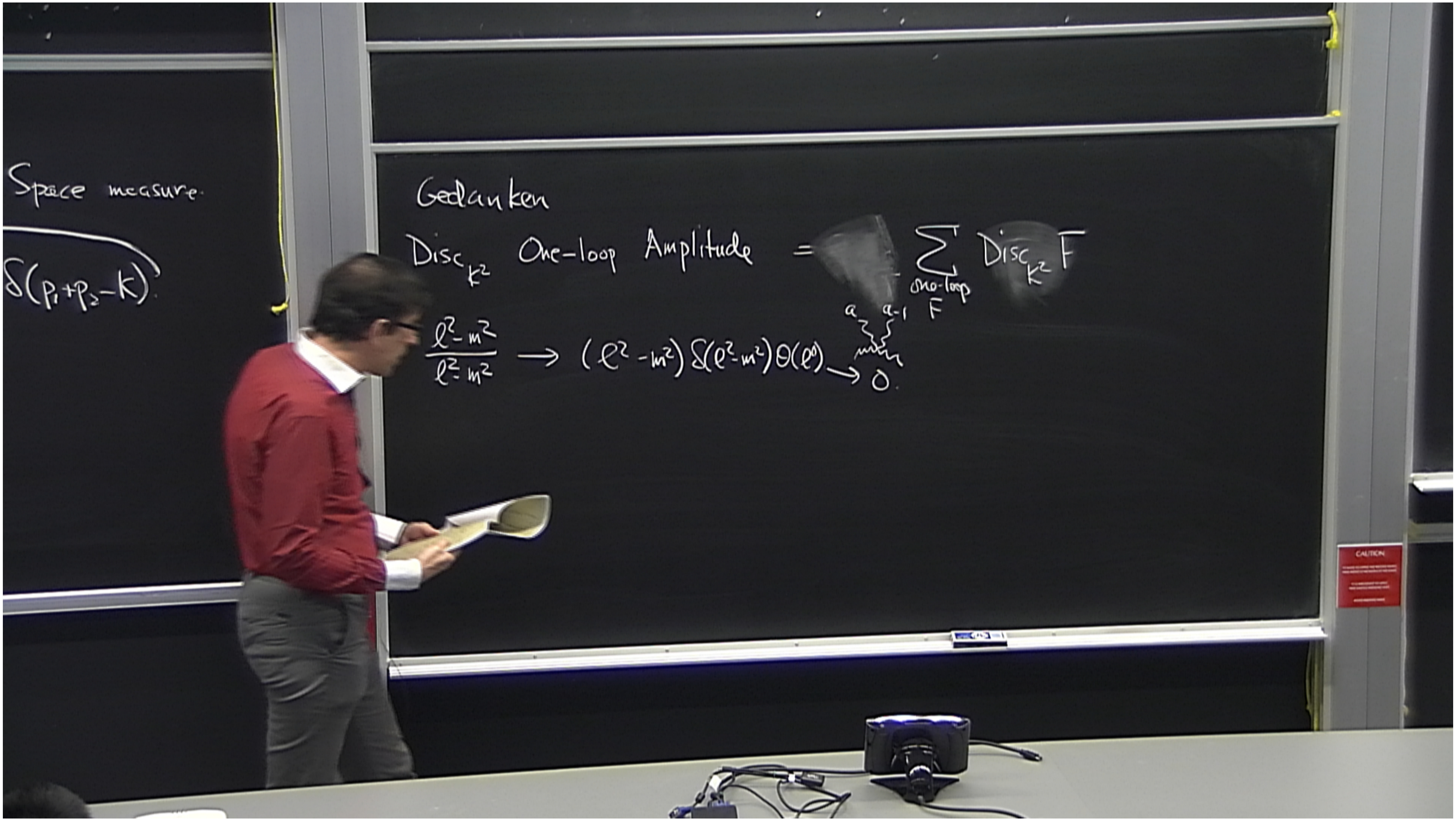

a-1
b
b+1
$K \equiv K_{a..b} = k_a + \cdots + k_b$
ℓ
$=$
$-\ell - K$
$\ell^2 - m^2 + i\varepsilon$
$\equiv -2\pi i\, \delta(\ell^2 - m^2)\, \theta(\ell^0)$
$2\pi\, \delta(\ell^2 - m^2)\, \theta(\ell^0)$
$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - m^2]} \frac{1}{[(\ell - k_a)^2 - m^2]} \cdots$
$\frac{1}{[(\ell - K)^2 - m^2]\,[(\ell - K + k_{b+1})^2 - m^2]} \cdots$
$2\pi\, \delta((\ell - K)^2 - m^2)\, \theta(-\ell^0 + K^0)$

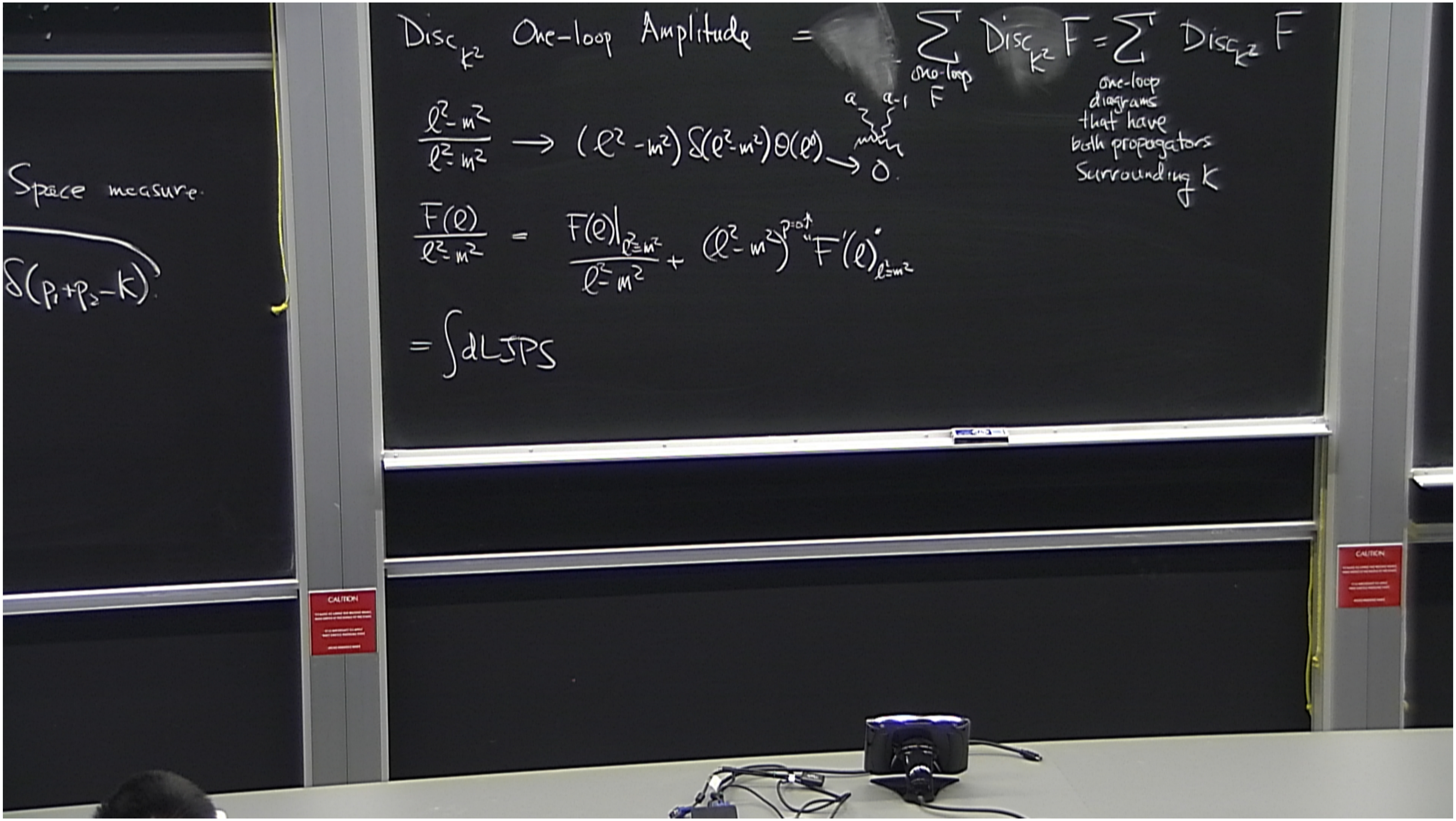

$\ell \to p_1$
$-\ell + k \to p_2 \quad \int \frac{d^4 p_2}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 - k)$

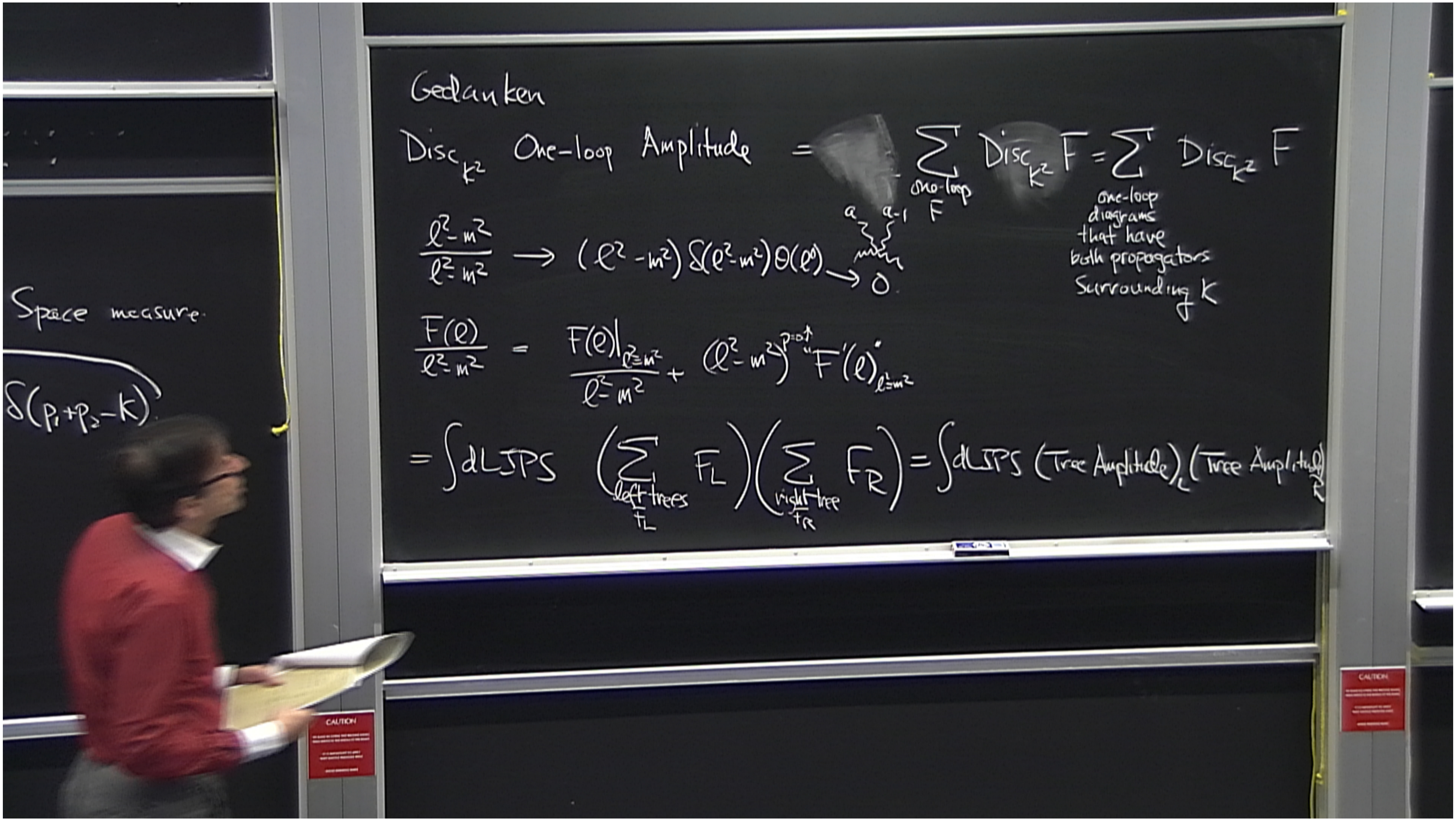

$\ell \to p_1$
$-\ell + k \to p_2$ $\int \frac{d^4 p_2}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 - k)$
$\int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \delta(p_1^2 - m^2)\Theta(p_1^0)\, \delta(p_2^2 - m^2)\Theta(p_2^0)\,(2\pi)^4 \delta(p_1 + p_2 - k)$
$\times \frac{}{(p_1^2 - m)[(p_1 - k_a)^2 - m^2]} \ldots$

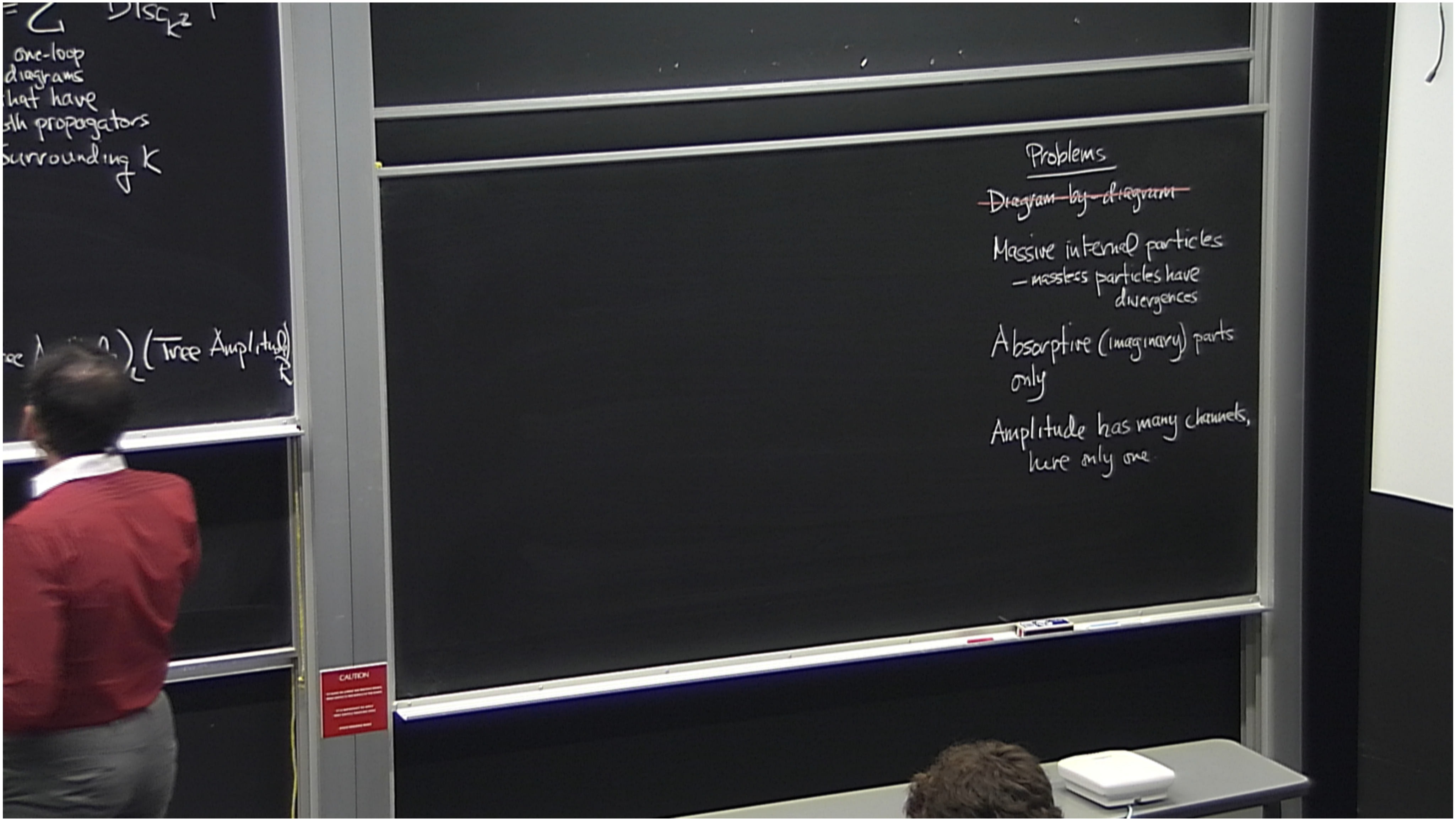

$\ell \to p_1$
$-\ell + K \to p_2$ $\int \frac{d^4 p_2}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 - K)$
Lorentz-Invariant Phase-Space measure
$\int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \delta(p_1^2 - m^2) \Theta(p_1^0) \delta(p_2^2 - m^2) \Theta(p_2^0) (2\pi)^4 \delta(p_1 + p_2 - K)$
$\times \frac{1}{(p_1^2 - m)[(p_1 - k_1)^2 - m^2]} \ldots$
$= \int dLIPS(p_1, p_2)$ (Tree Diagram)$_L$ (Tree Diagram)$^*_R$

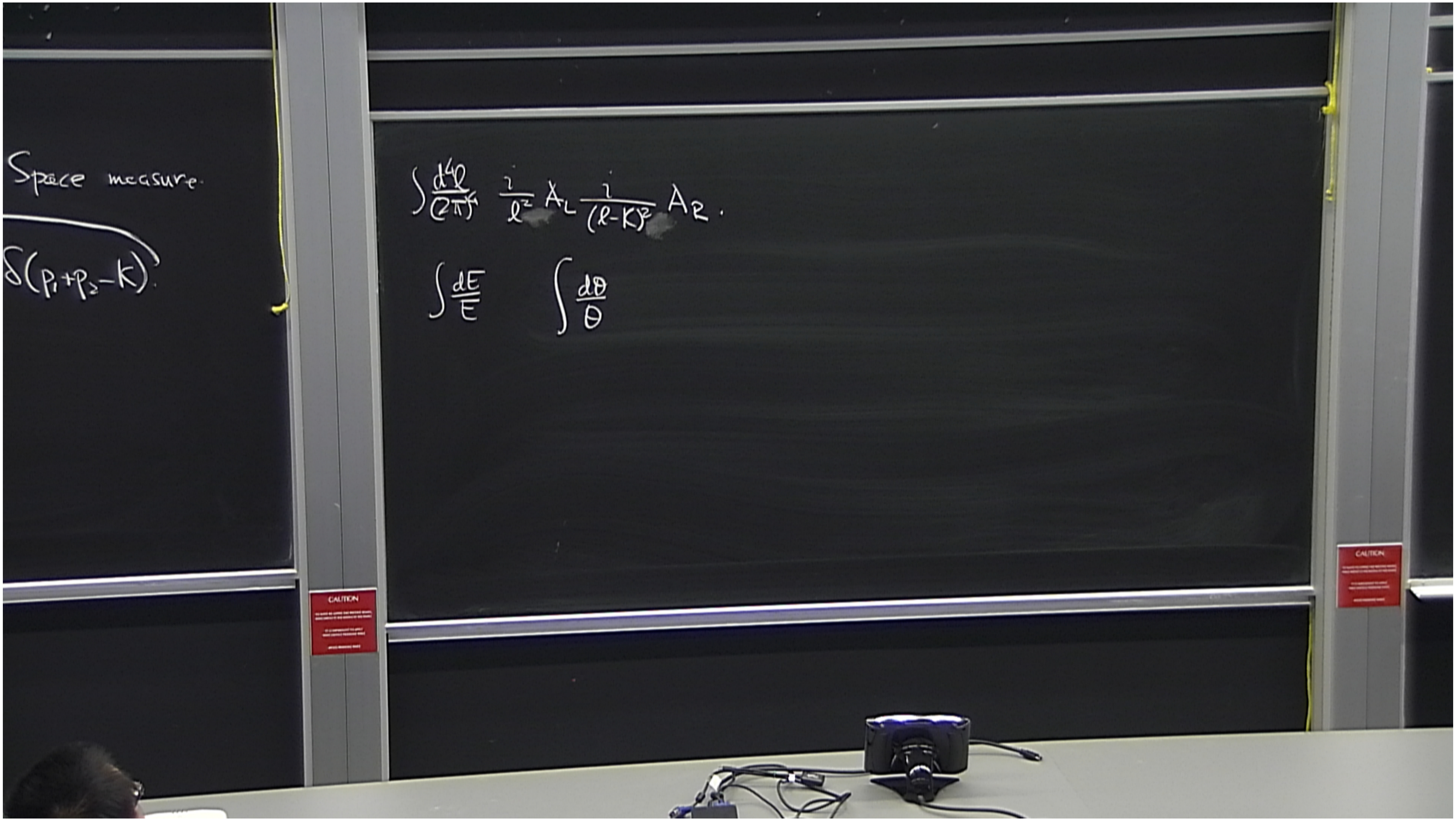

Problems
Diagram-by-diagram
Massive internal particles
— massless particles have divergences
Absorptive (imaginary) parts only
Amplitude has many channels, here only one
CAUTION

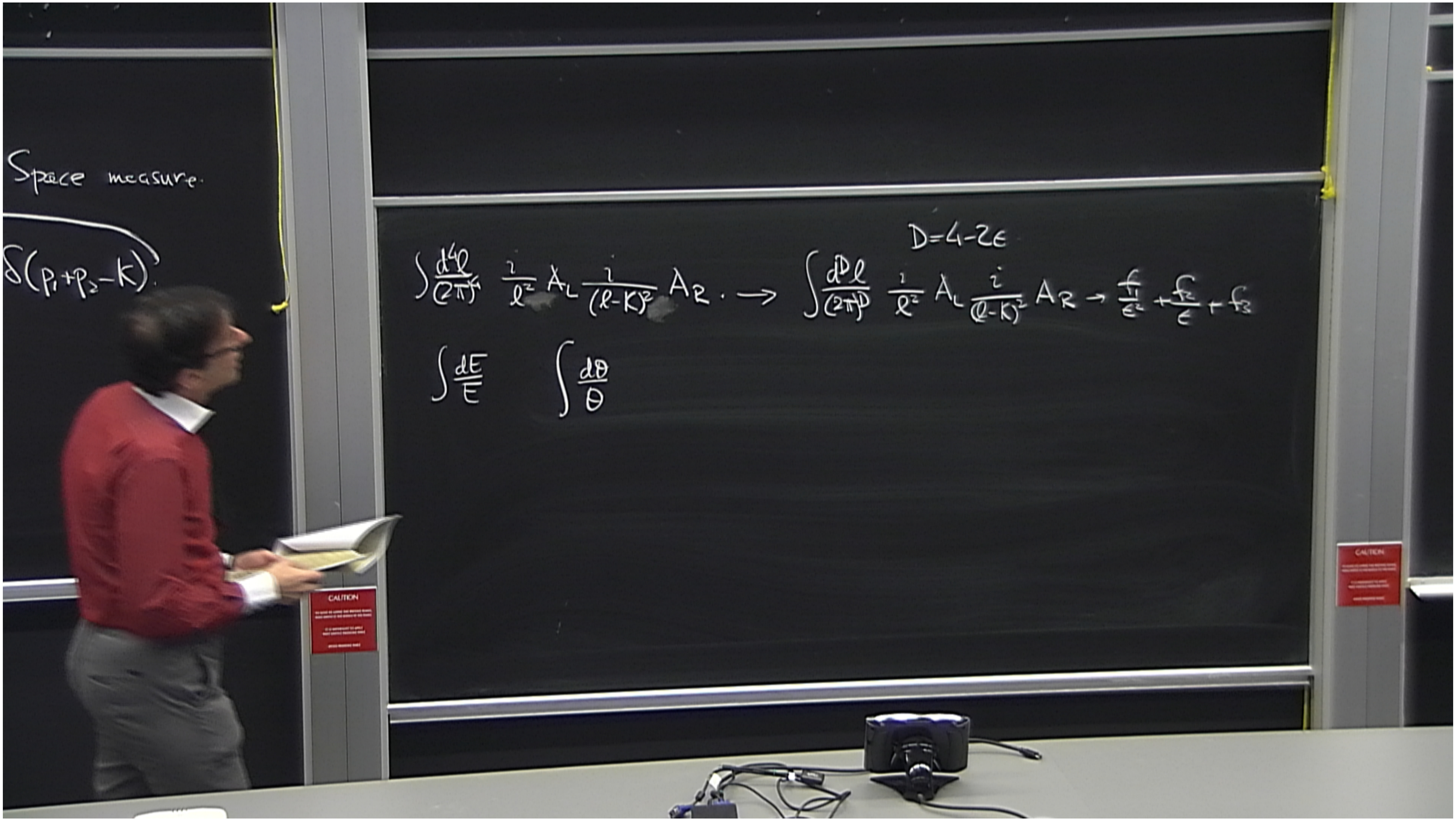

Space measure
δ(p₁+p₂−k)
Gedanken
Disc_{k²} One-loop Amplitude = Σ_{one-loop F} Disc_{k²} F
a a-1
(ℓ²−m²)/(ℓ²−m²) → (ℓ²−m²) δ(ℓ²−m²) θ(ℓ⁰) → 0.

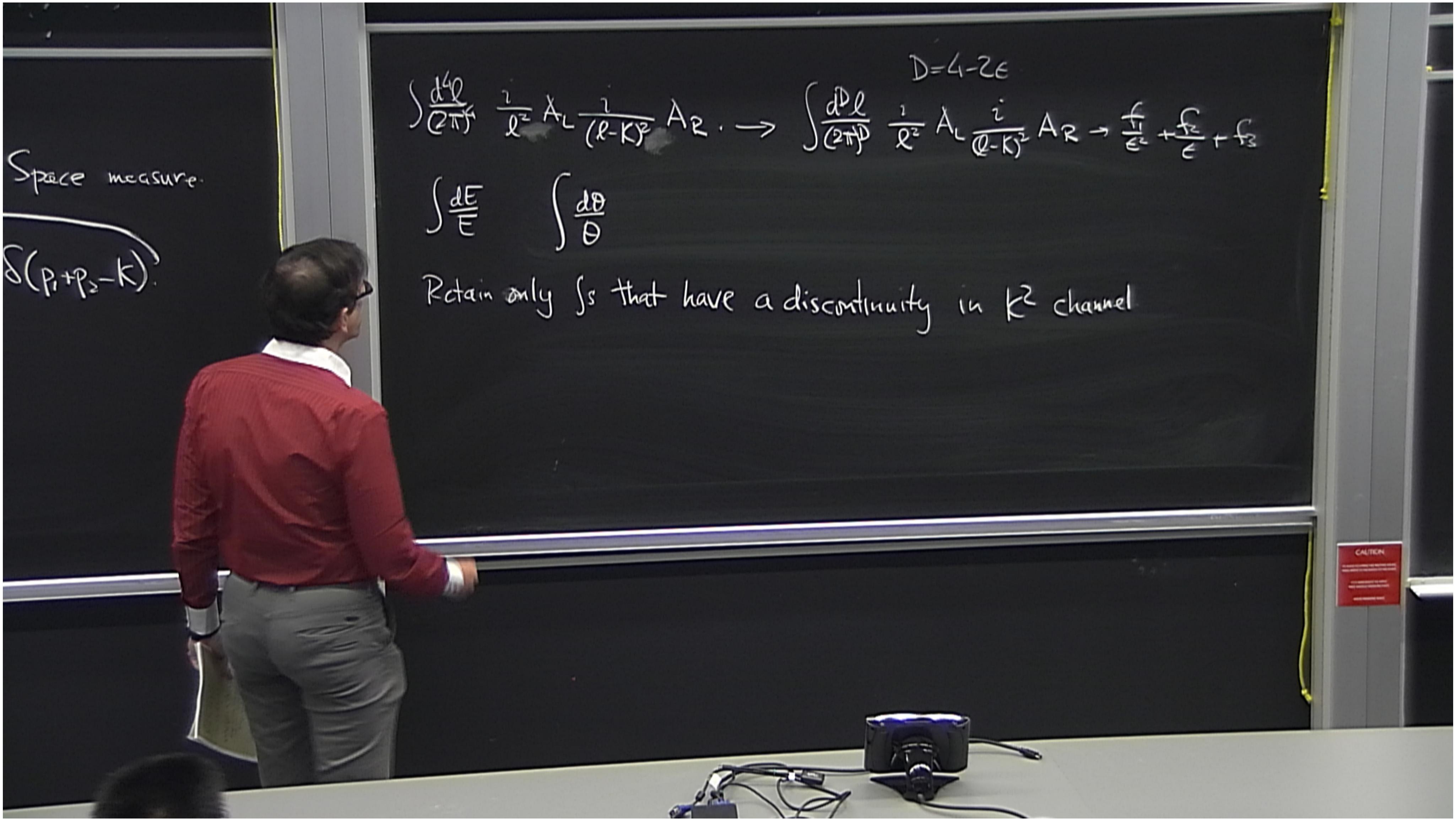

Space measure
$\delta(p_1+p_2-K)$
Disc_{K^2} One-loop Amplitude $= \sum_{\text{one-loop } F} \mathrm{Disc}_{K^2} F = \sum \mathrm{Disc}_{K^2} F$
one-loop diagrams that have both propagators surrounding K
$\frac{\ell^2-m^2}{\ell^2-m^2} \to (\ell^2-m^2)\,\delta(\ell^2-m^2)\,\theta(\ell^0) \to 0$
$\frac{F(\ell)}{\ell^2-m^2} = \frac{F(\ell)|_{\ell^2=m^2}}{\ell^2-m^2} + (\ell^2-m^2)\, F'(\ell)|_{\ell^2=m^2}$
$= \int d\mathrm{LIPS}$

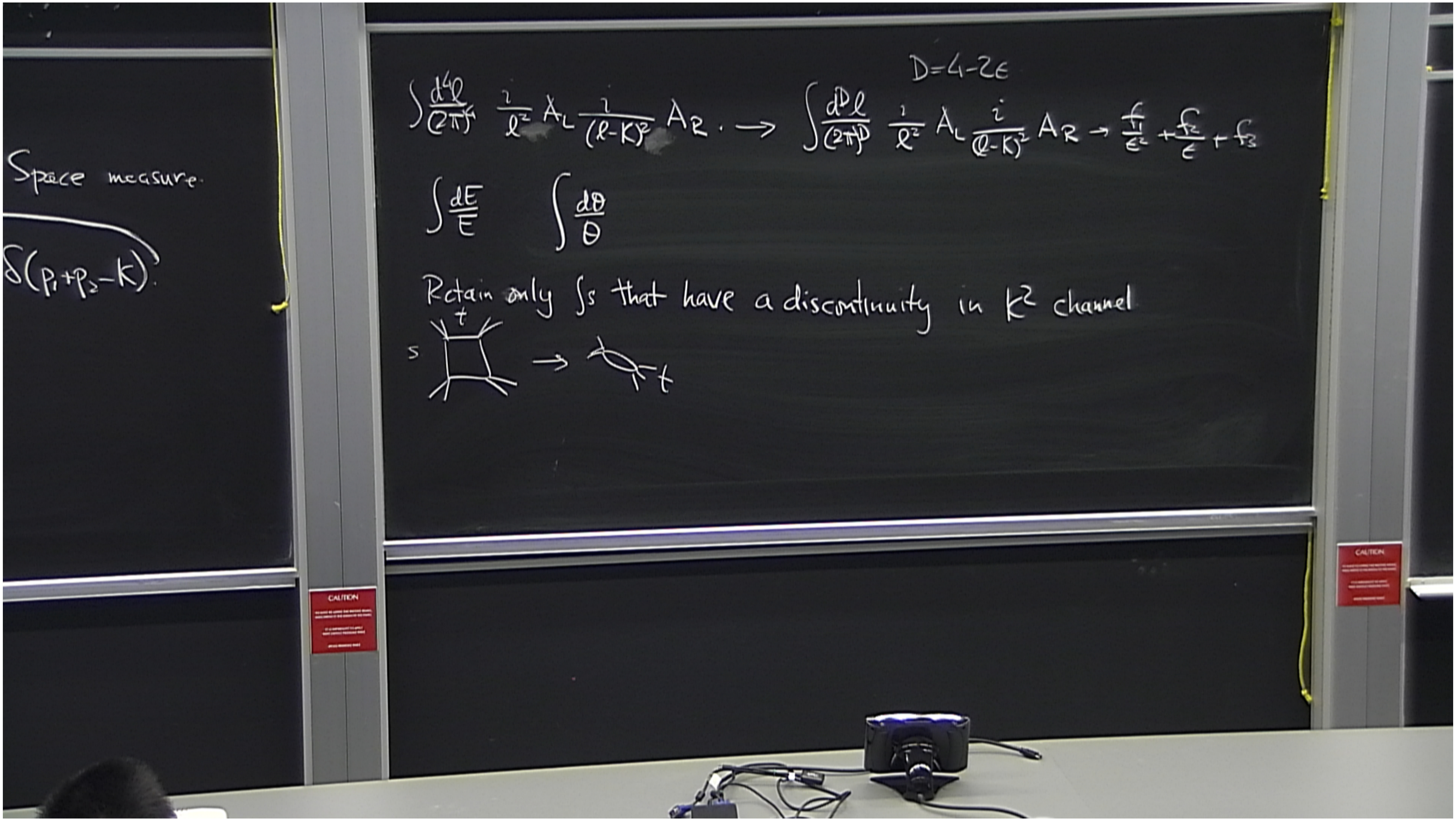

Gedanken
Disc_{K^2} One-loop Amplitude $= \sum_{\text{one-loop } F} \mathrm{Disc}_{K^2} F = \sum' \mathrm{Disc}_{K^2} F$
a a-1
one-loop diagrams that have both propagators surrounding K
$\frac{\ell^2-m^2}{\ell^2-m^2} \longrightarrow (\ell^2-m^2)\,\delta(\ell^2-m^2)\,\theta(\ell^0) \longrightarrow 0$
$\frac{F(\ell)}{\ell^2-m^2} = \frac{F(\ell)|_{\ell^2=m^2}}{\ell^2-m^2} + (\ell^2-m^2)^{p=0}\,\text{"}F'(\ell)\text{"}_{\ell^2=m^2}$
$= \int dLIPS \left(\sum_{\text{left trees } F_L} F_L\right)\left(\sum_{\text{right tree } F_R} F_R\right) = \int dLIPS\,(\text{Tree Amplitude})_L\,(\text{Tree Amplitude})_R$
Space measure
$\delta(p_1+p_2-K)$


one-loop
diagrams
propagators
Problems
~~Diagram-by-diagram~~
Massive internal particles
– massless particles have divergences
Absorptive (imaginary) parts only
Amplitude has many channels, here only one


Space measure.
$\delta(p_1+p_2-K)$.
$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2} A_L \frac{1}{(l-K)^2} A_R$.
$\int \frac{dE}{E}$ $\int \frac{d\theta}{\theta}$


Space measure
$\delta(p_1+p_2-K)$
$D=4-2\epsilon$
$\int\frac{d^4\ell}{(2\pi)^4}\frac{i}{\ell^2}A_L\frac{i}{(\ell-K)^2}A_R \rightarrow \int\frac{d^D\ell}{(2\pi)^D}\frac{i}{\ell^2}A_L\frac{i}{(\ell-K)^2}A_R \rightarrow \frac{f_1}{\epsilon^2}+\frac{f_2}{\epsilon}+f_3$
$\int\frac{dE}{E}$ $\int\frac{d\theta}{\theta}$


Space measure
$\delta(p_1+p_2-K)$
$D=4-2\epsilon$
$\int \frac{d^4 \ell}{(2\pi)^4} \frac{i}{\ell^2} A_L \frac{i}{(\ell-K)^2} A_R \rightarrow \int \frac{d^D \ell}{(2\pi)^D} \frac{i}{\ell^2} A_L \frac{i}{(\ell-K)^2} A_R \rightarrow \frac{f_1}{\epsilon^2} + \frac{f_2}{\epsilon} + f_3$
$\int \frac{dE}{E} \quad \int \frac{d\theta}{\theta}$
Retain only ∫s that have a discontinuity in K^2 channel


$D=4-2\epsilon$
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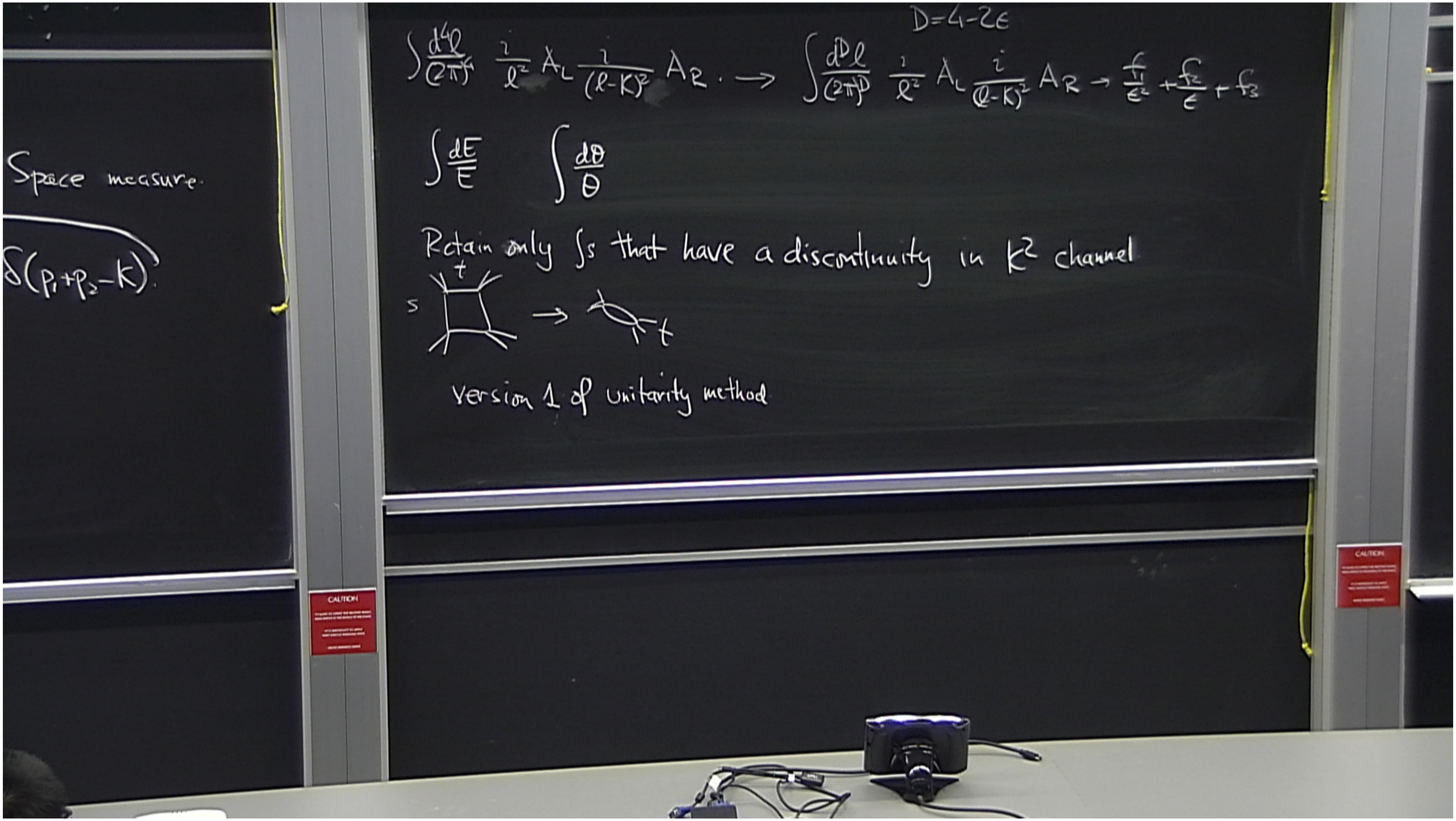
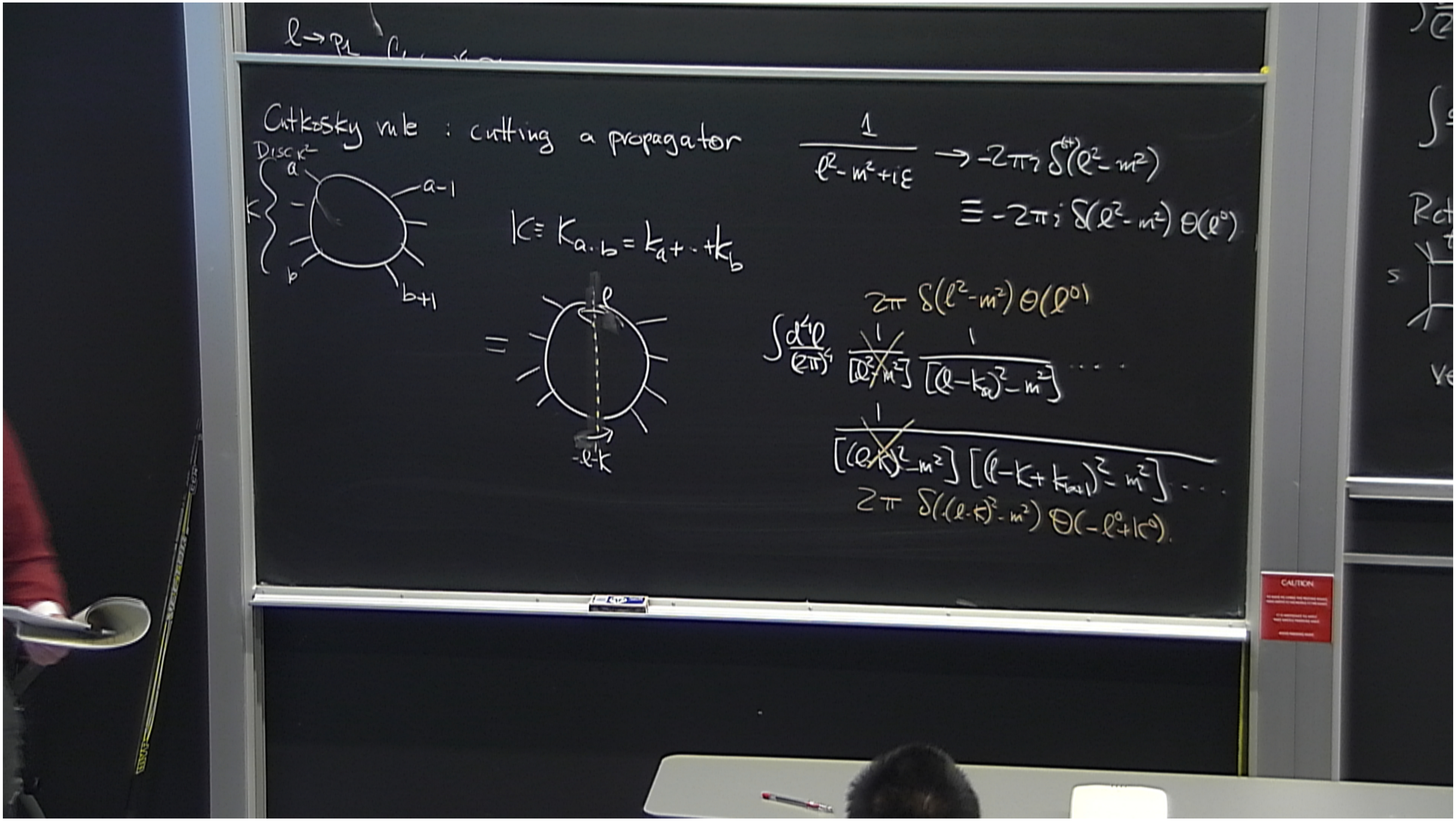
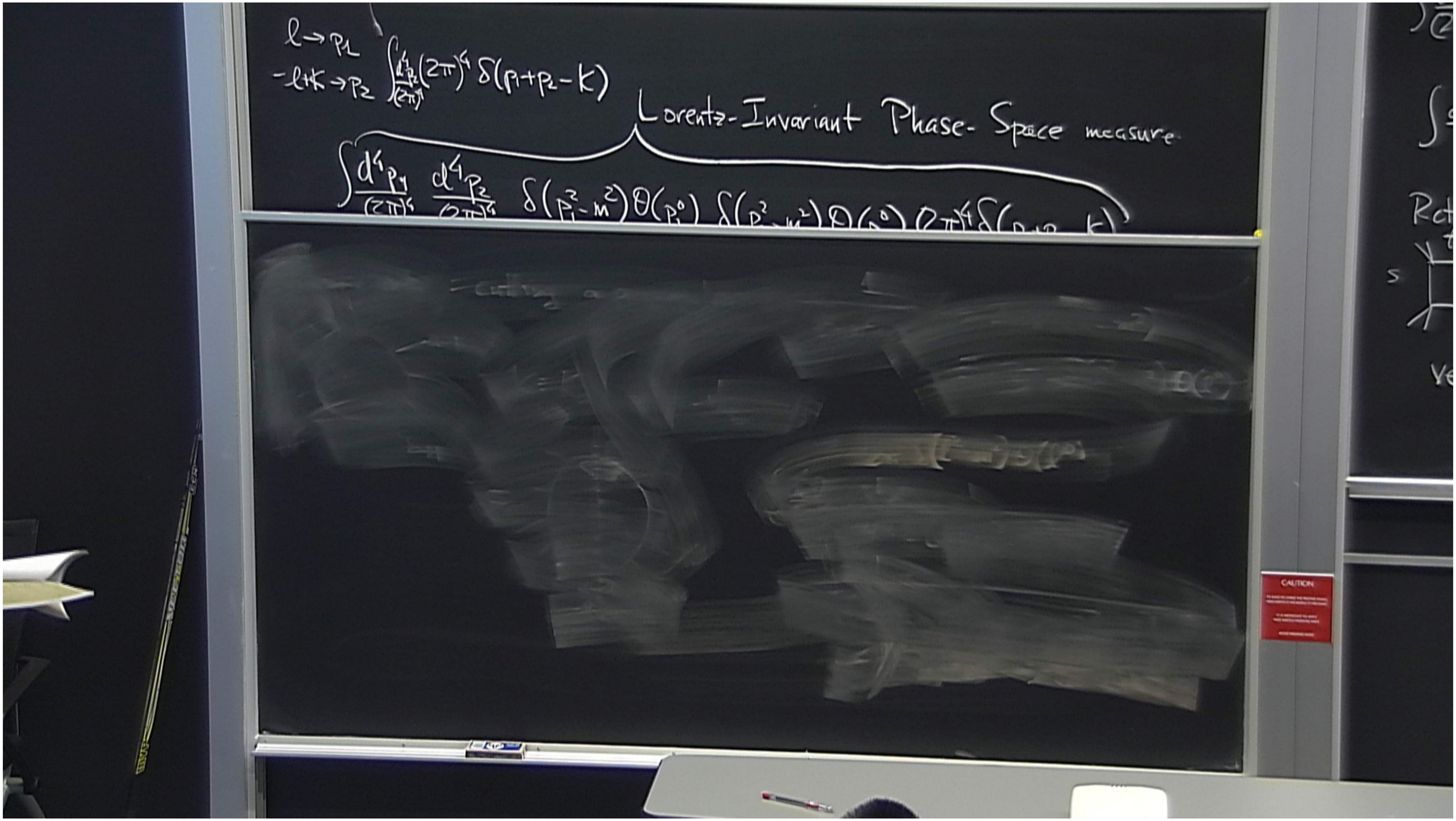
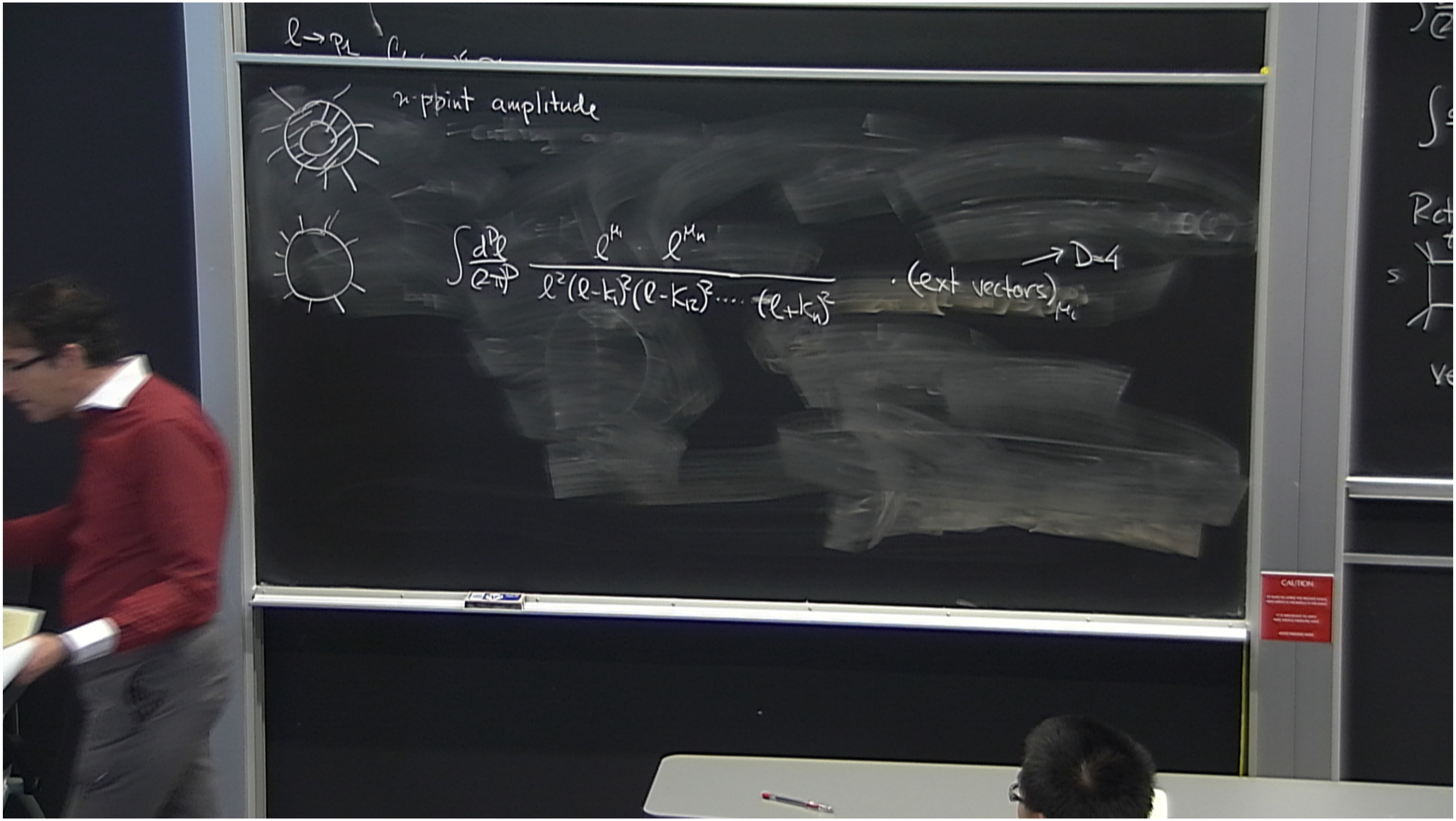
Space measure
$\delta(p_1+p_2-K)$
$\int \frac{dE}{E} \quad \int \frac{d\theta}{\theta}$
Retain only $\int$s that have a discontinuity in K^2 channel
t
s
$\rightarrow$
t

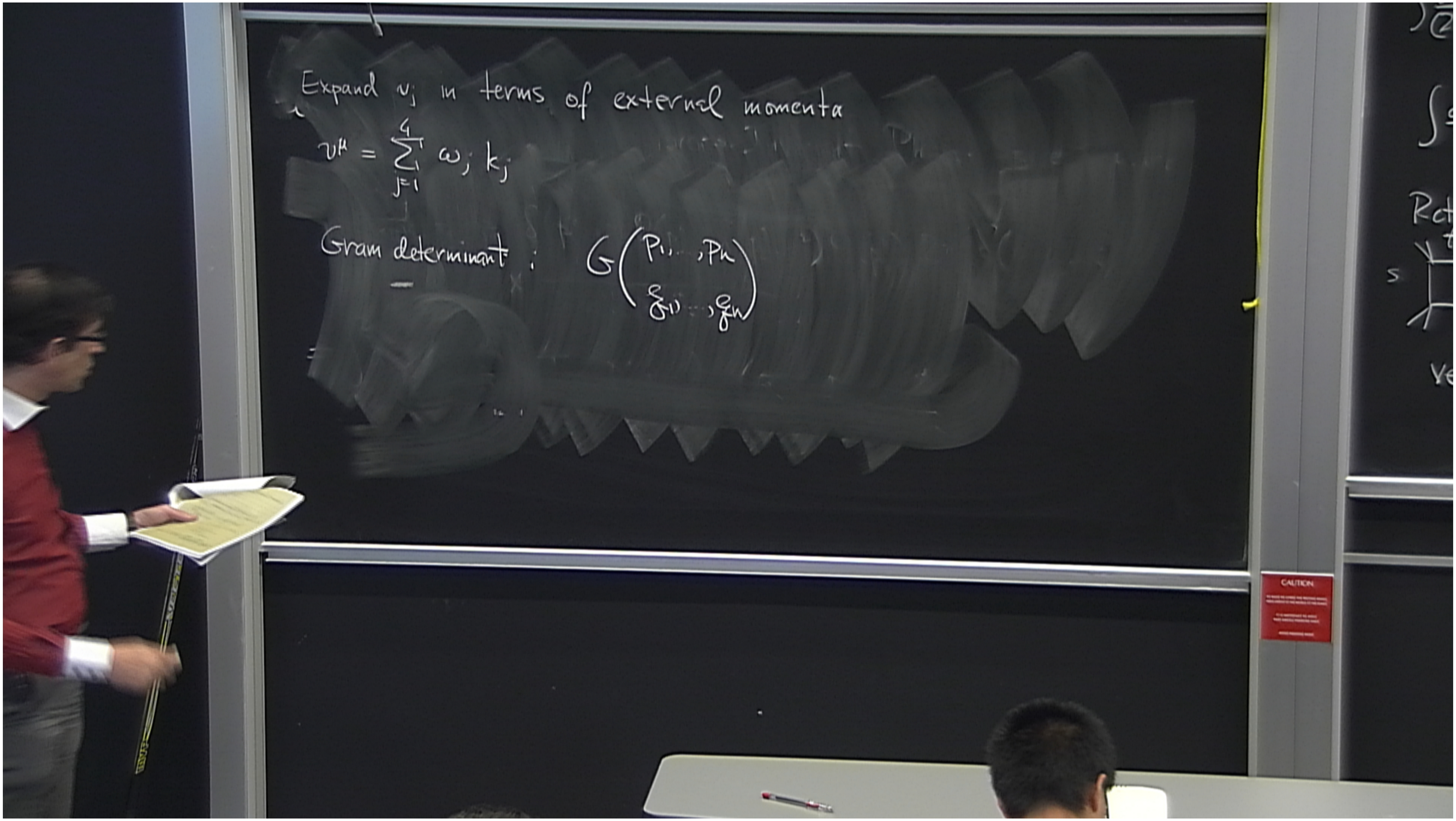

Space measure
$\delta(p_1+p_2-K)$
$D=4-2\epsilon$
$\int \frac{d^4\ell}{(2\pi)^4} \frac{i}{\ell^2} A_L \frac{i}{(\ell-K)^2} A_R \to \int \frac{d^D\ell}{(2\pi)^D} \frac{i}{\ell^2} A_L \frac{i}{(\ell-K)^2} A_R \to \frac{f_1}{\epsilon^2} + \frac{f_2}{\epsilon} + f_3$
$\int \frac{dE}{E}$ $\int \frac{d\theta}{\theta}$
Retain only ∫s that have a discontinuity in K^2 channel
t
s
t
version 1 of unitarity method
CAUTION

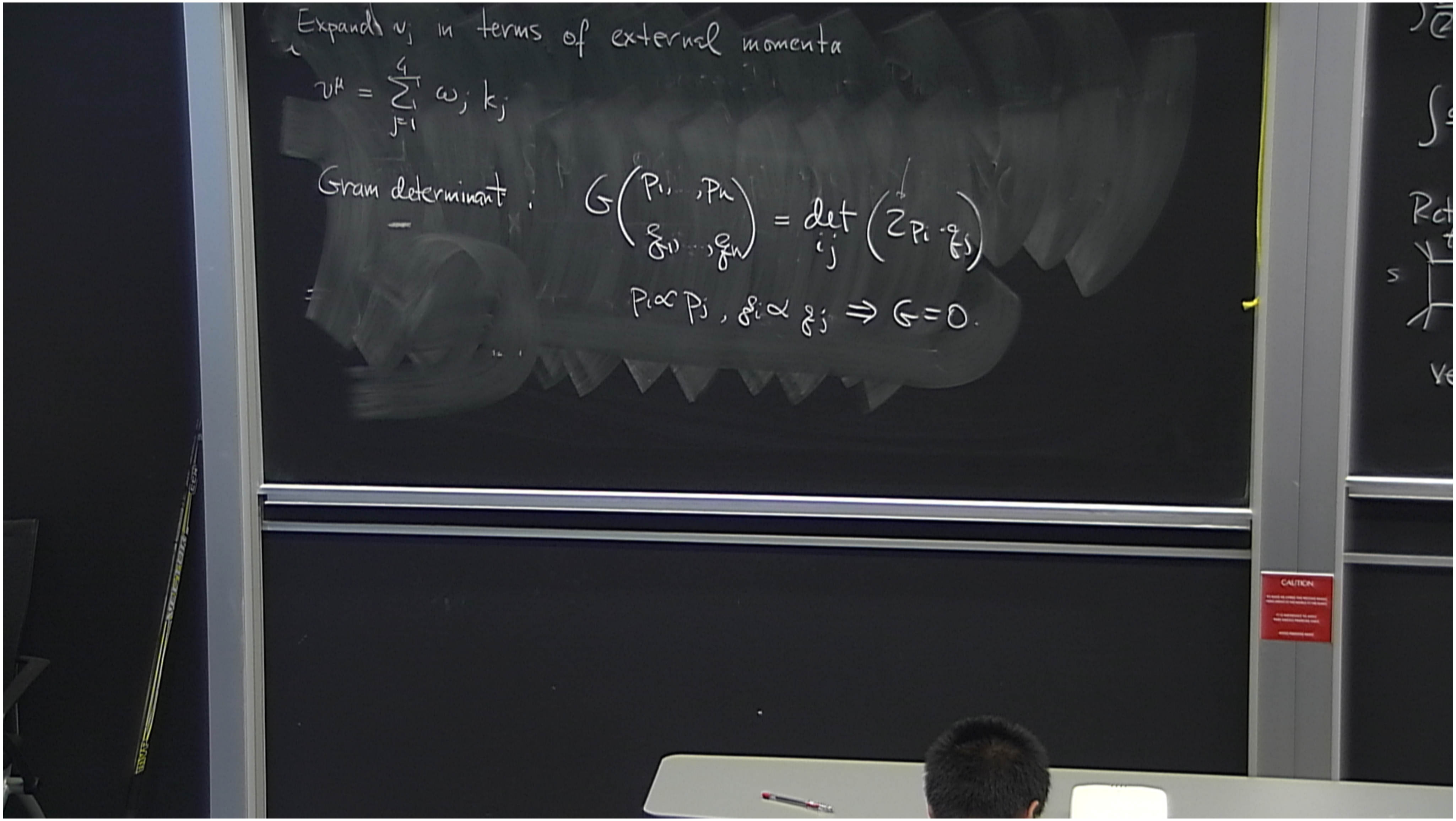

Cutkosky rule : cutting a propagator
Disc K²
a
a-1
K
b
b+1
K ≡ K_{a..b} = k_a + .. + k_b
ℓ
=
-ℓ-K
1 / (ℓ² - m² + iε) → -2πi δ^{(+)}(ℓ² - m²)
≡ -2πi δ(ℓ² - m²) Θ(ℓ⁰)
2π δ(ℓ² - m²) Θ(ℓ⁰)
∫ d⁴ℓ/(2π)⁴ 1/[ℓ² - m²] 1/[(ℓ - k_a)² - m²] ....
1/[(ℓ-K)² - m²] [(ℓ - K + k_{b+1})² - m²] ....
2π δ((ℓ-K)² - m²) Θ(-ℓ⁰ + K⁰).

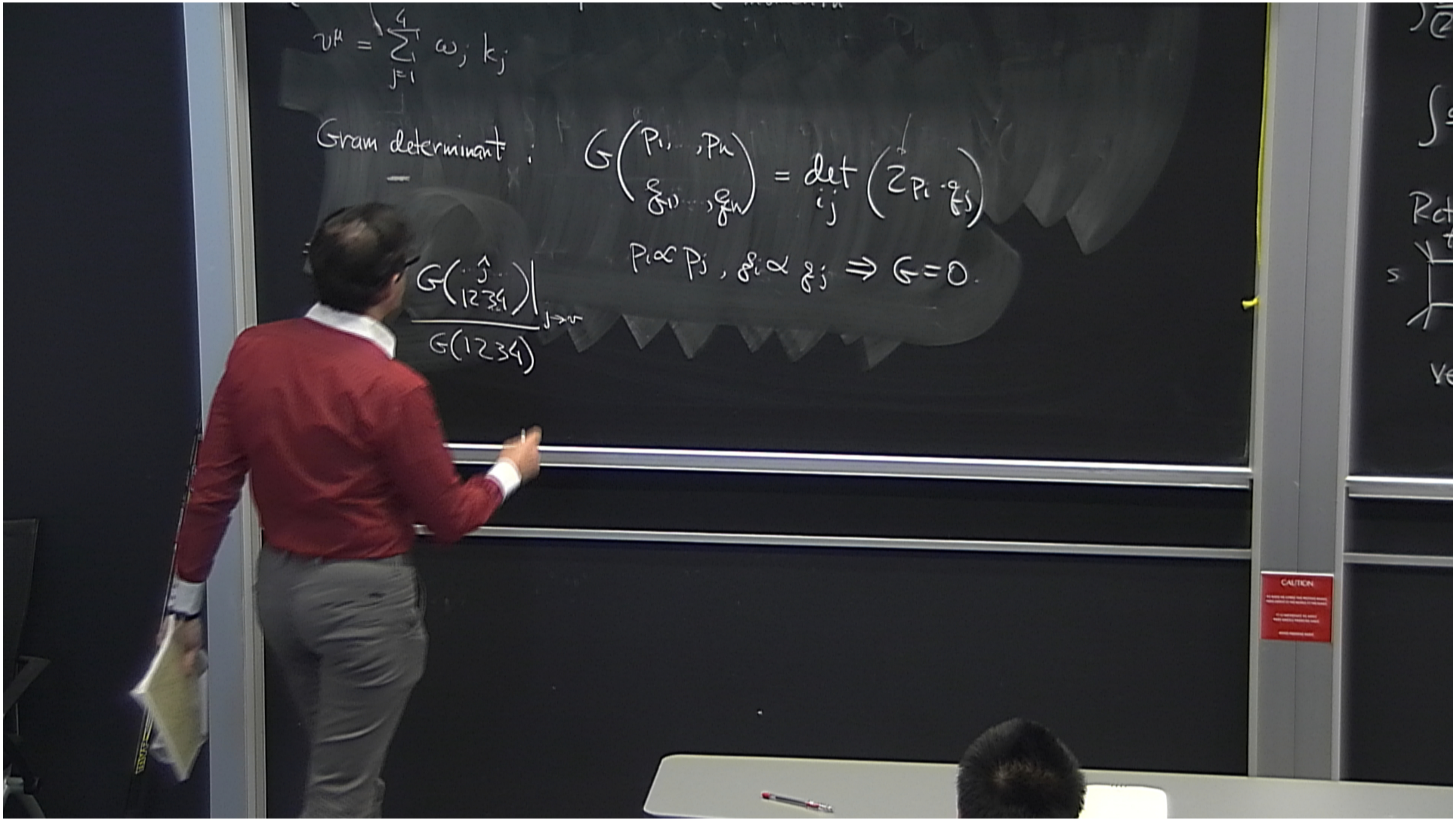


$l \to p_1$
$-l+K \to p_2$ $\int \frac{d^4p_2}{(2\pi)^4}(2\pi)^4\,\delta(p_1+p_2-K)$
Lorentz-Invariant Phase-Space measure
$\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\,\delta(p_1^2-m^2)\,\theta(p_1^0)\,\delta(p_2^2-m^2)\,\theta(p_2^0)\,(2\pi)^4\,\delta(p_1+p_2-K)$

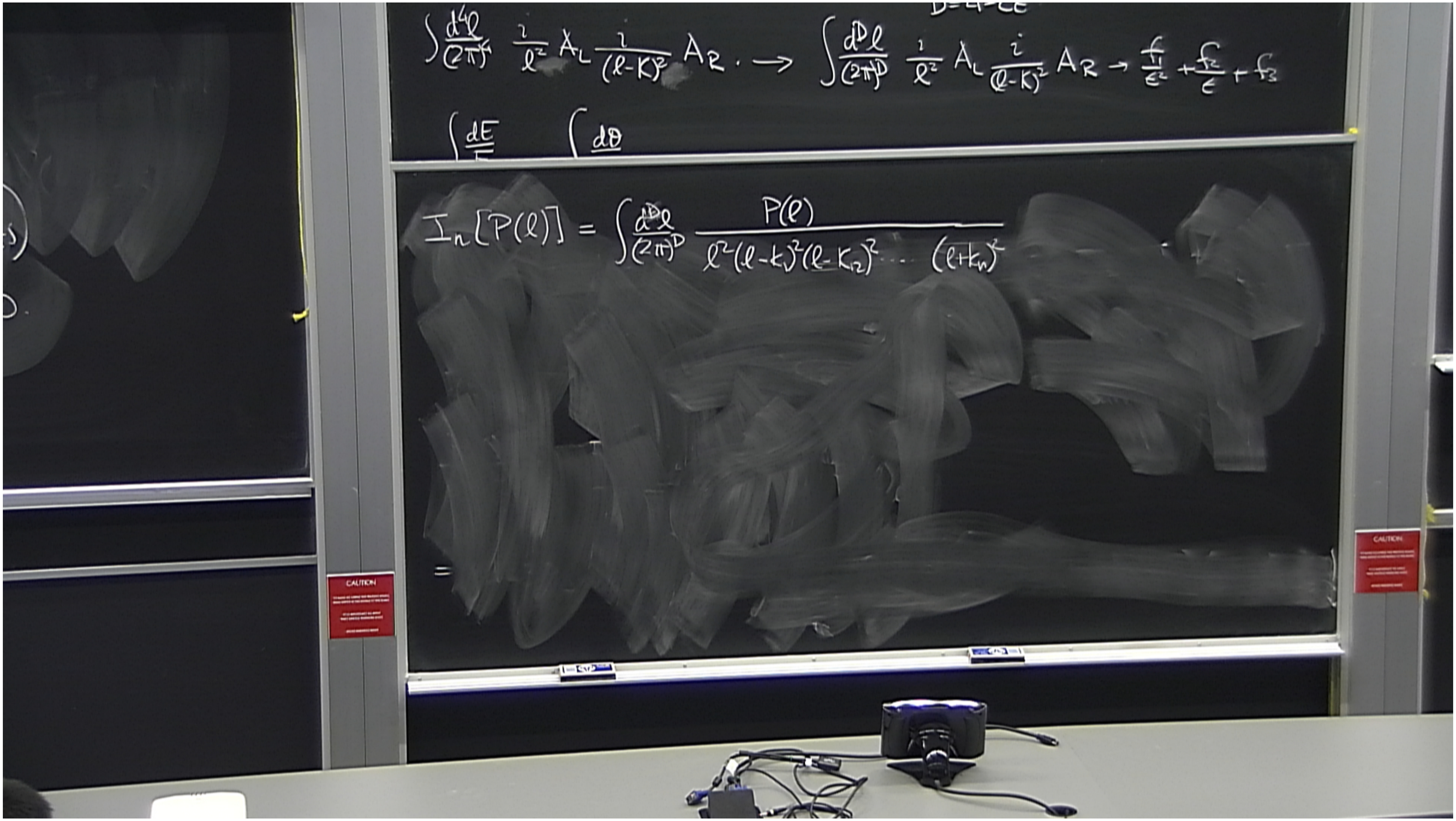

n-point amplitude
∫ d^D l/(2π)^D l^μ1 … l^μn / l²(l−k1)²(l−k12)²….(l+kn)² · (ext vectors)_μi
→ D=4


Expand v_j in terms of external momenta
$v^\mu = \sum_{j=1}^{4} \omega_j k_j$
Gram determinant : $G\begin{pmatrix} p_1, \dots, p_n \\ q_1, \dots, q_n \end{pmatrix}$

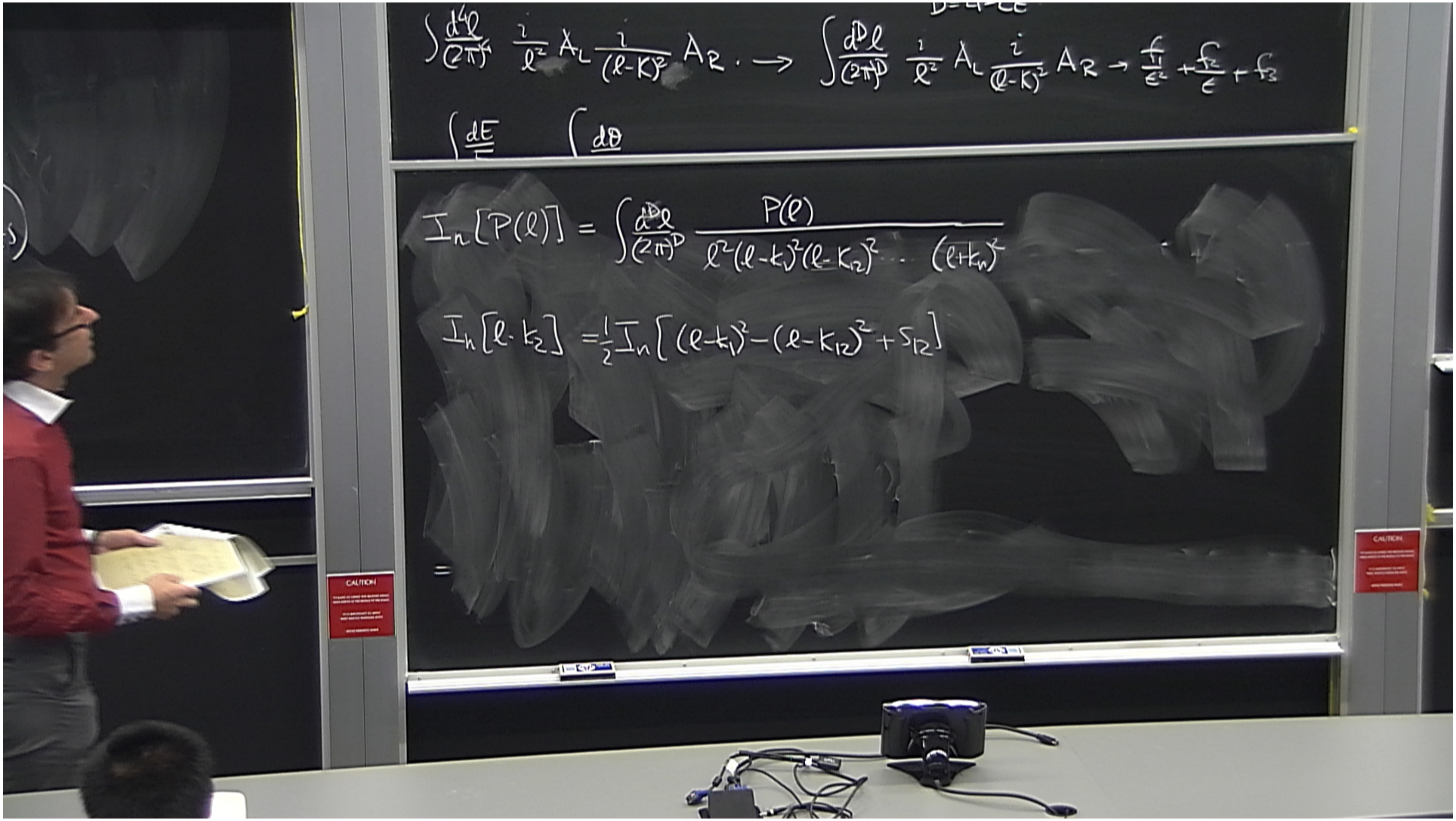

Expand v_j in terms of external momenta
$v^\mu = \sum_{j=1}^{4} \omega_j k_j$
Gram determinant: $G\begin{pmatrix} p_1, \dots, p_n \\ q_1, \dots, q_n \end{pmatrix} = \det_{ij}\left(2 p_i \cdot q_j\right)$
$p_i \propto p_j,\ q_i \propto q_j \Rightarrow G = 0.$

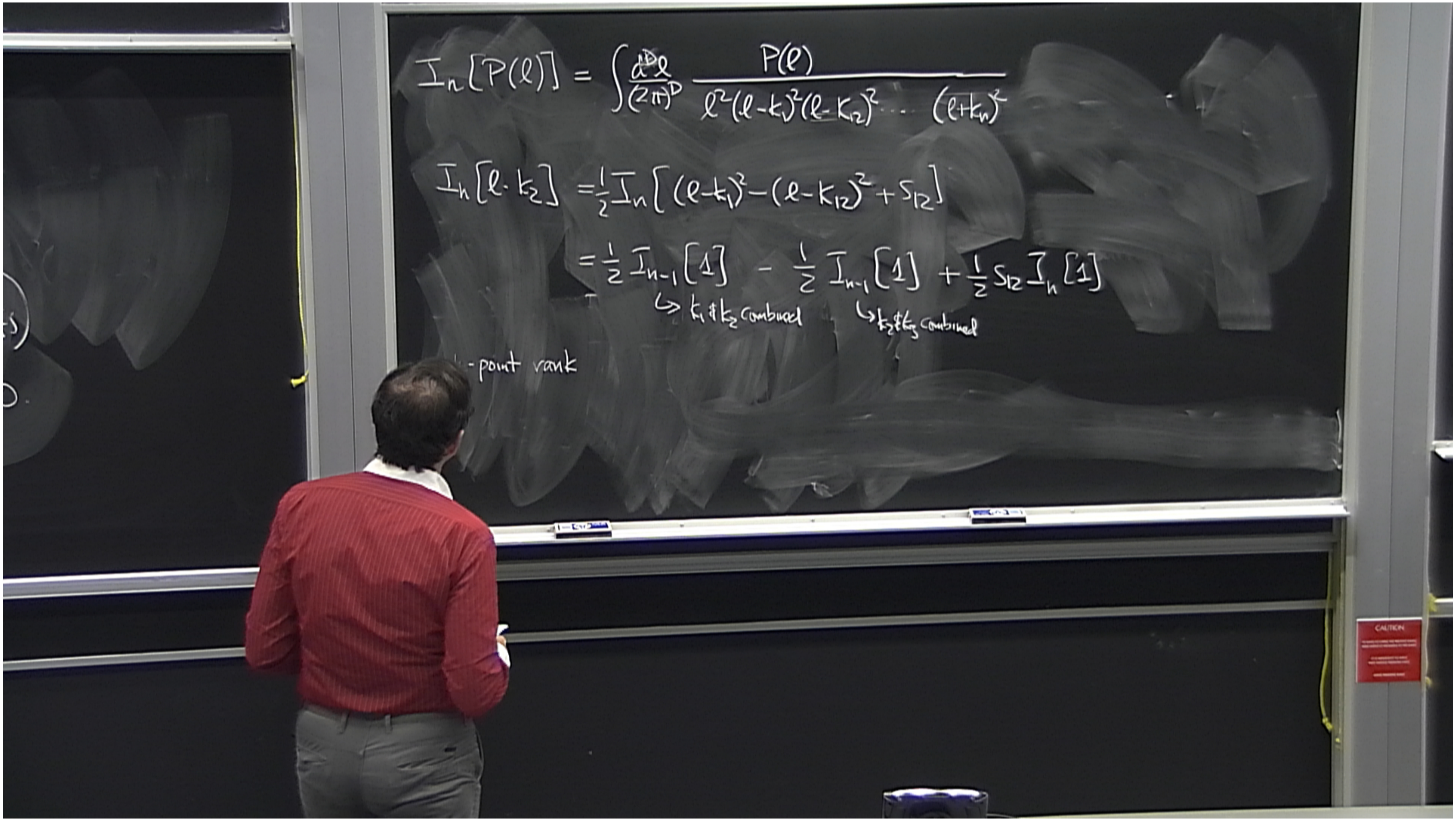


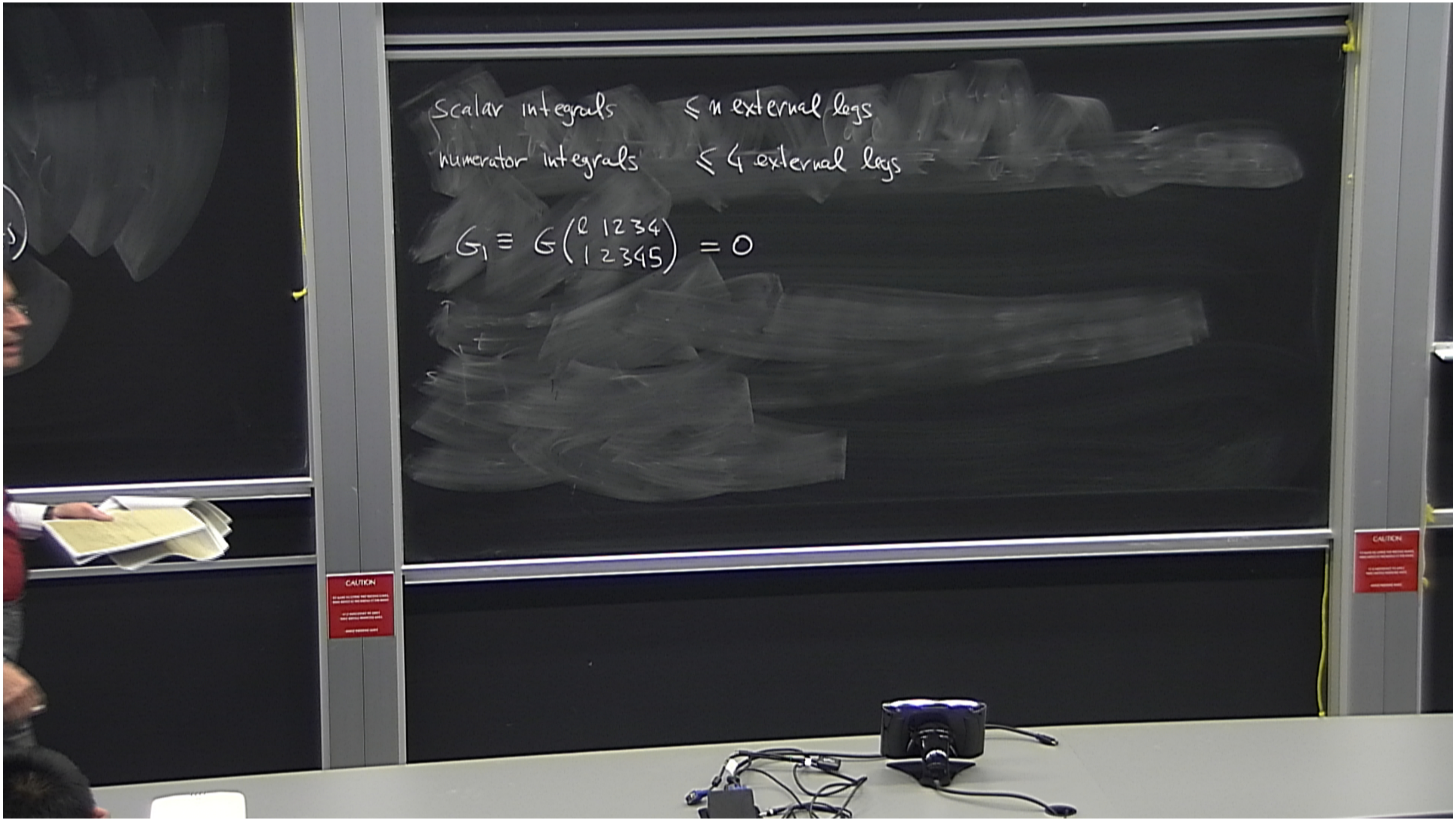
v^μ = Σ_{j=1}^{4} ω_j k_j
Gram determinant :
G(p1 … pn ; q1 … qn) = det_ij (2 pi · qj)
pi ∝ pj , qi ∝ qj ⇒ G = 0
G(ĵ ; 1234) / G(1234)
CAUTION

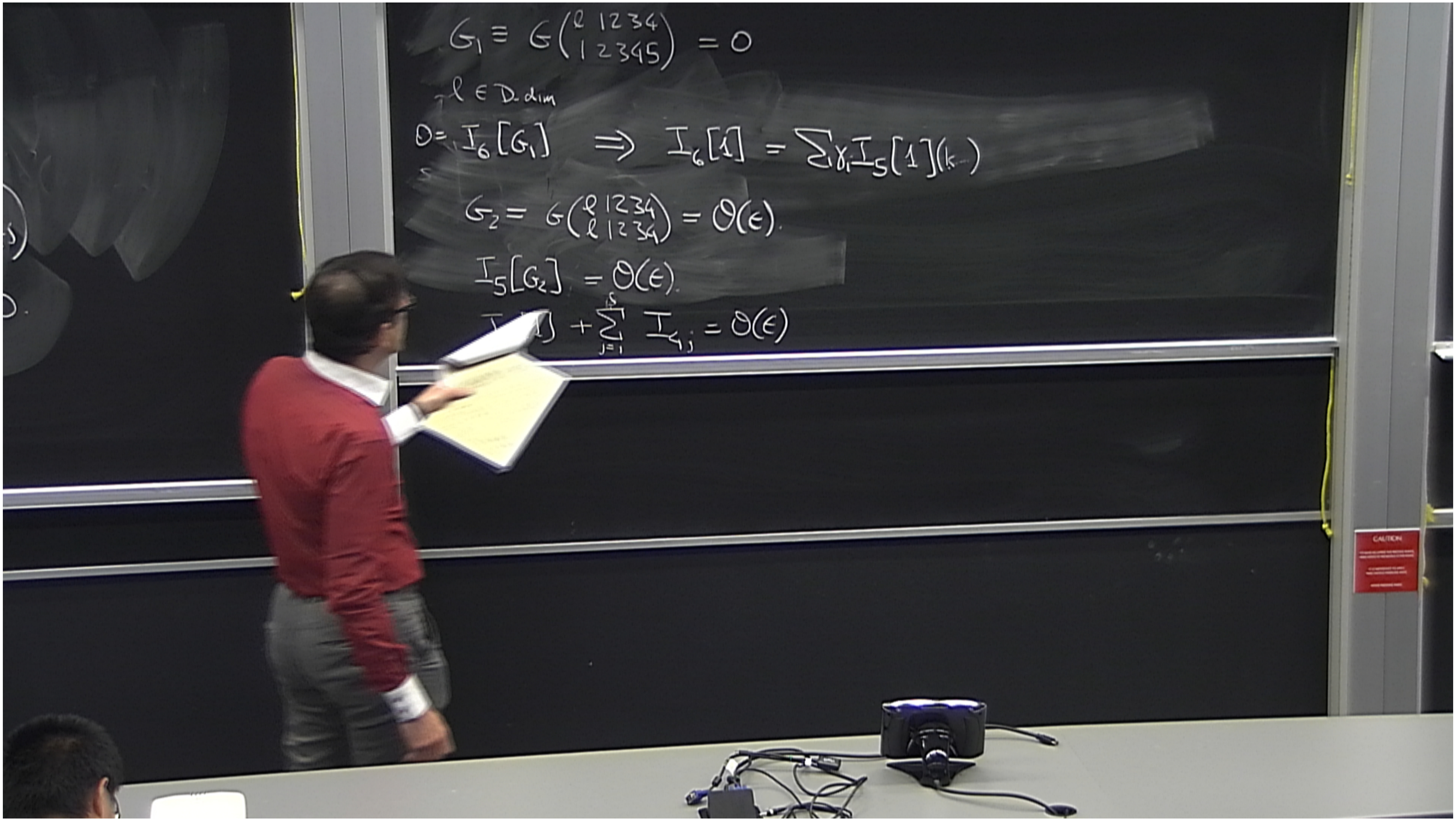


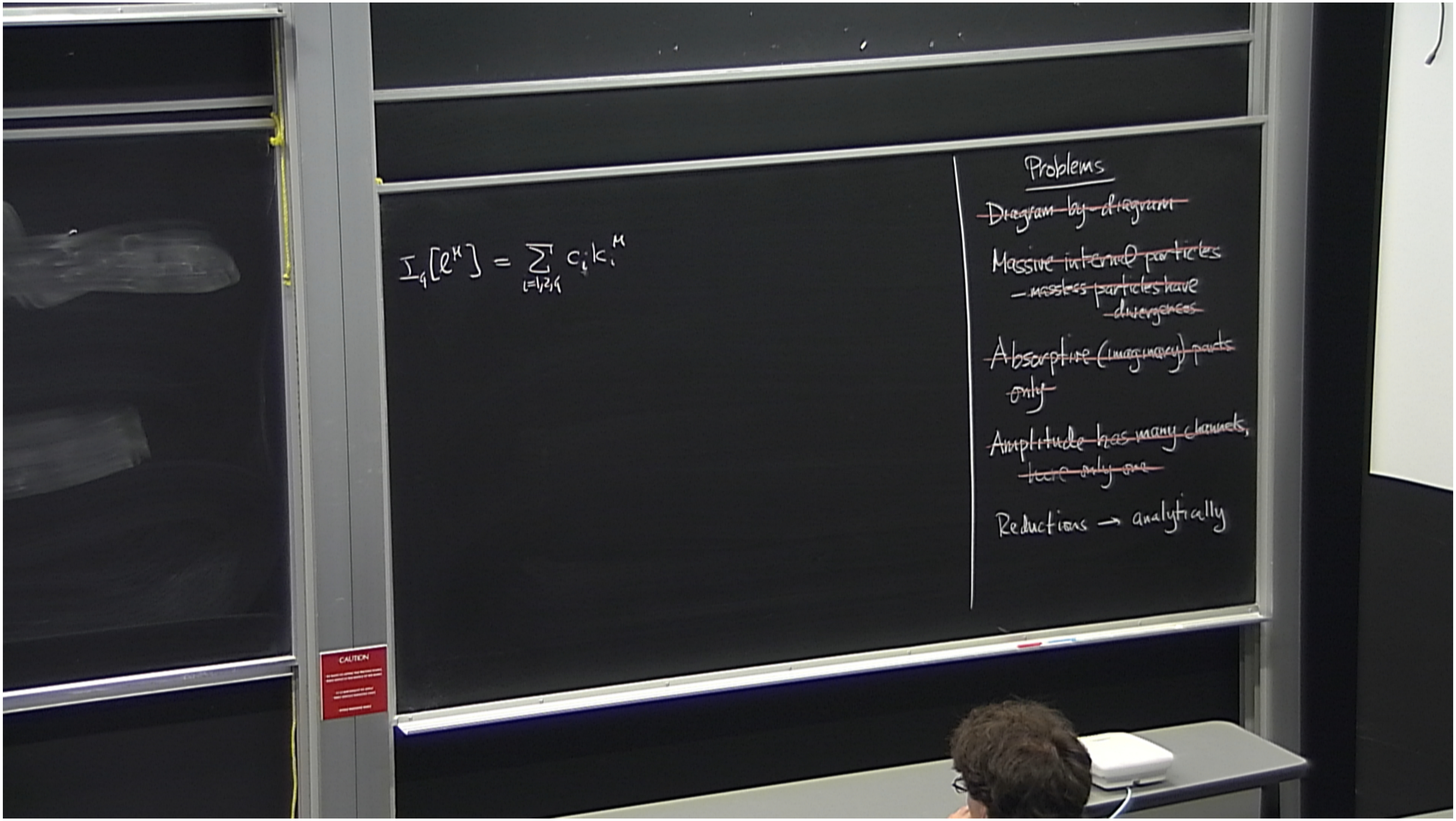
$D=4-2\epsilon$
$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2} A_L \frac{1}{(\ell-K)^2} A_R . \rightarrow \int \frac{d^D\ell}{(2\pi)^D} \frac{i}{\ell^2} A_L \frac{i}{(\ell-K)^2} A_R \rightarrow \frac{f_1}{\epsilon^2} + \frac{f_2}{\epsilon} + f_3$
$\int \frac{dE}{E} \quad \int d\theta$
$I_n[P(\ell)] = \int \frac{d^D\ell}{(2\pi)^D} \frac{P(\ell)}{\ell^2(\ell-k_1)^2(\ell-K_2)^2 \cdots (\ell+k_n)^2}$
$=$

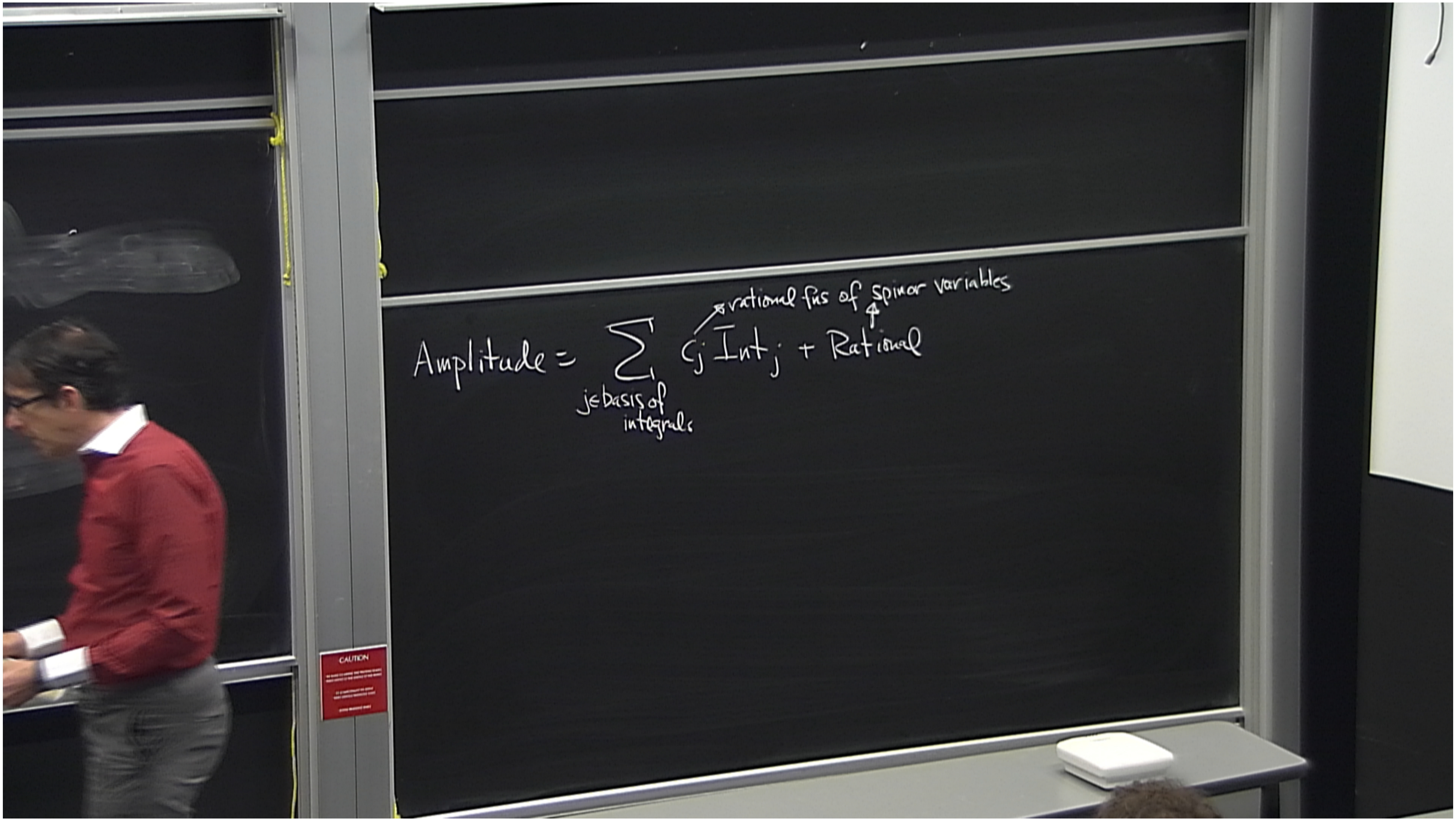

$\int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2} A_L \frac{i}{(l-K)^2} A_R \rightarrow \int \frac{d^D l}{(2\pi)^D} \frac{i}{l^2} A_L \frac{i}{(l-K)^2} A_R \rightarrow \frac{f_1}{\epsilon^2} + \frac{f_2}{\epsilon} + f_3$
$\int dE \quad \int d\theta$
$I_n[P(l)] = \int \frac{d^D l}{(2\pi)^D} \frac{P(l)}{l^2 (l-k_1)^2 (l-K_{12})^2 \cdots (l+k_n)^2}$
$I_n[l \cdot k_2] = \frac{1}{2} I_n[(l-k_1)^2 - (l-K_{12})^2 + s_{12}]$
$=$


$I_n[P(\ell)] = \int \frac{d^D\ell}{(2\pi)^D} \frac{P(\ell)}{\ell^2(\ell-k_1)^2(\ell-K_{12})^2 \cdots (\ell+k_n)^2}$
$I_n[\ell\cdot k_2] = \frac{1}{2} I_n[(\ell-k_1)^2-(\ell-K_{12})^2+s_{12}]$
$= \frac{1}{2} I_{n-1}[1] - \frac{1}{2} I_{n-1}[1] + \frac{1}{2} s_{12} I_n[1]$
↳ k_1 & k_2 combined
↳ k_2 & k_3 combined
-point rank

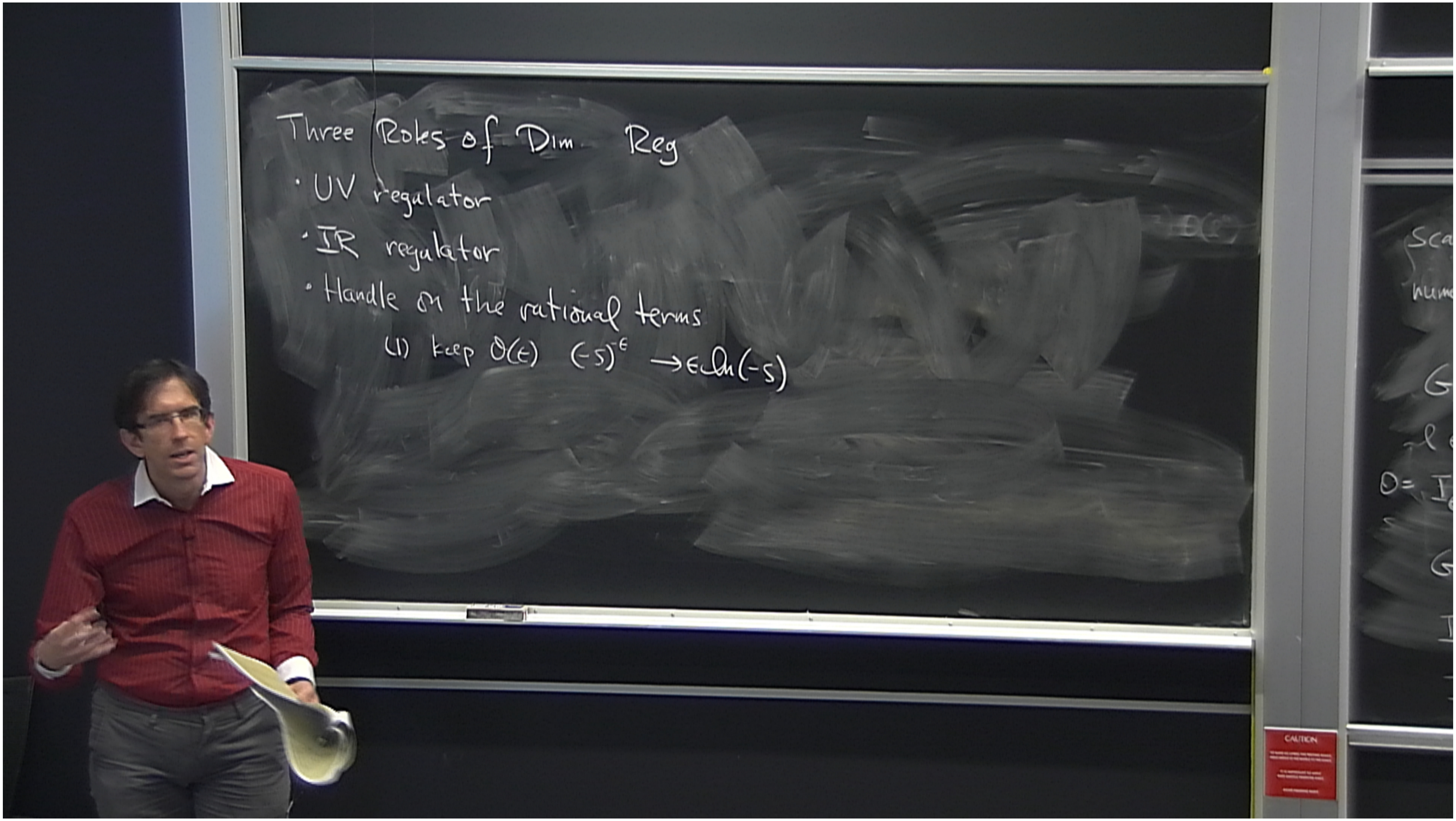

Scalar integrals $\leq n$ external legs
numerator integrals ≤ 4 external legs
$G_1 \equiv G\begin{pmatrix} \ell\ 1\,2\,3\,4 \\ 1\,2\,3\,4\,5 \end{pmatrix} = 0$
CAUTION


$G_1 \equiv G\begin{pmatrix} \ell\ 1\,2\,3\,4 \\ 1\,2\,3\,4\,5 \end{pmatrix} = 0$
$\ell \in D\text{-dim}$
$0 = I_6[G_1] \Rightarrow I_6[1] = \sum_i \gamma_i I_5[1](k\ldots)$
$G_2 = G\begin{pmatrix} \ell\ 1\,2\,3\,4 \\ \ell\ 1\,2\,3\,4 \end{pmatrix} = \mathcal{O}(\epsilon)$
$I_5[G_2] = \mathcal{O}(\epsilon)$
$+ \sum_{j=1}^{5} I_{4,j} = \mathcal{O}(\epsilon)$


$I_4[\ell^\mu] = \sum_{i=1,2,4} c_i k_i^\mu$
Problems
~~Diagram-by-diagram~~
Massive ~~internal particles~~
~~– massless particles have divergences~~
~~Absorptive (imaginary) parts only~~
~~Amplitude has many channels, here only one~~
Reductions → analytically


rational fns of spinor variables
Amplitude = Σ_j c_j Int_j + Rational
j∈basis of integrals


Three Roles of Dim Reg
UV regulator
IR regulator
Handle on the rational terms
Scalar integrals
Amplitude


Three Roles of Dim. Reg
· UV regulator
· IR regulator
· Handle on the rational terms
(1) keep O(ε) (-s)^-ε → ε ln(-s)