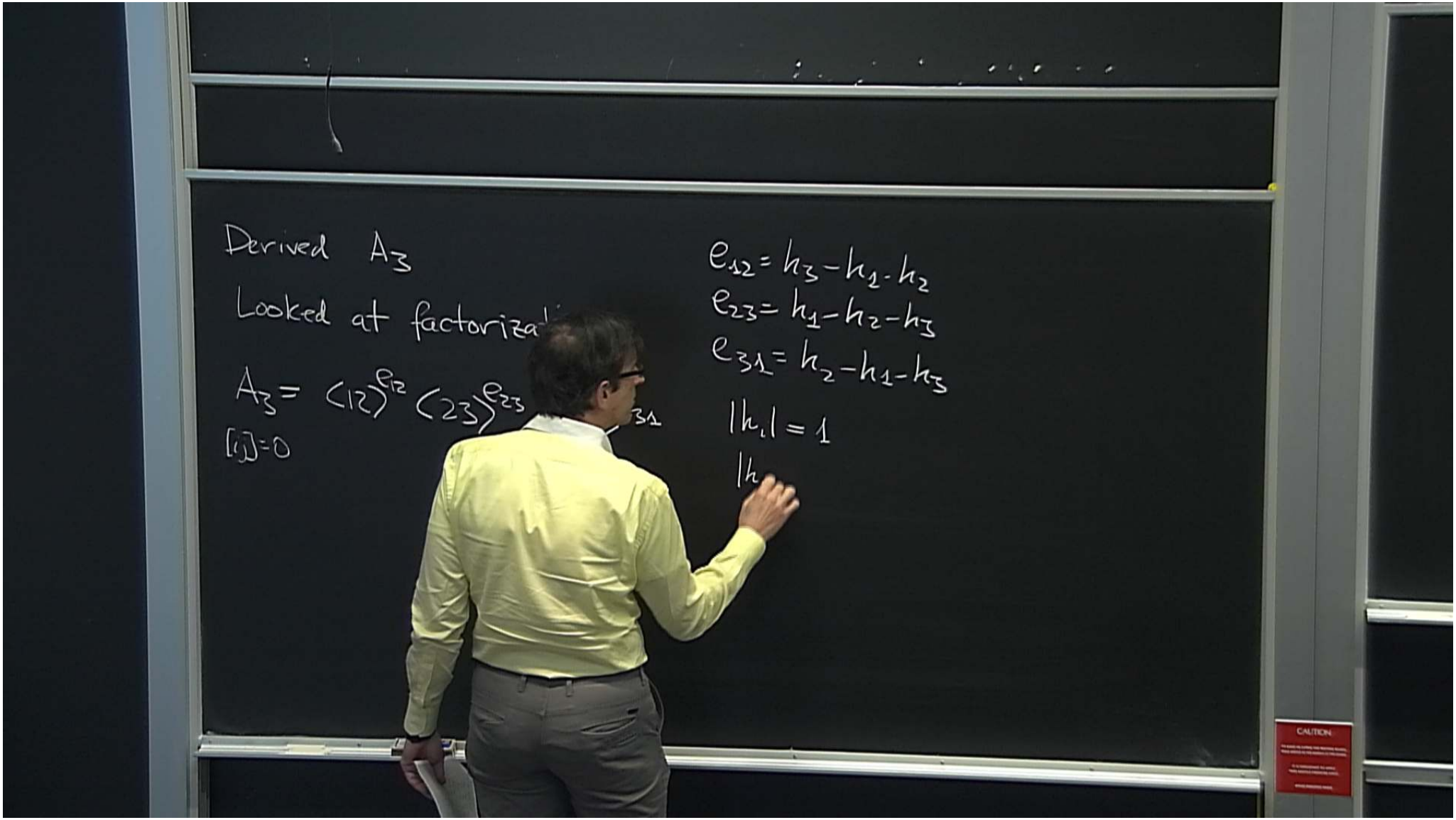
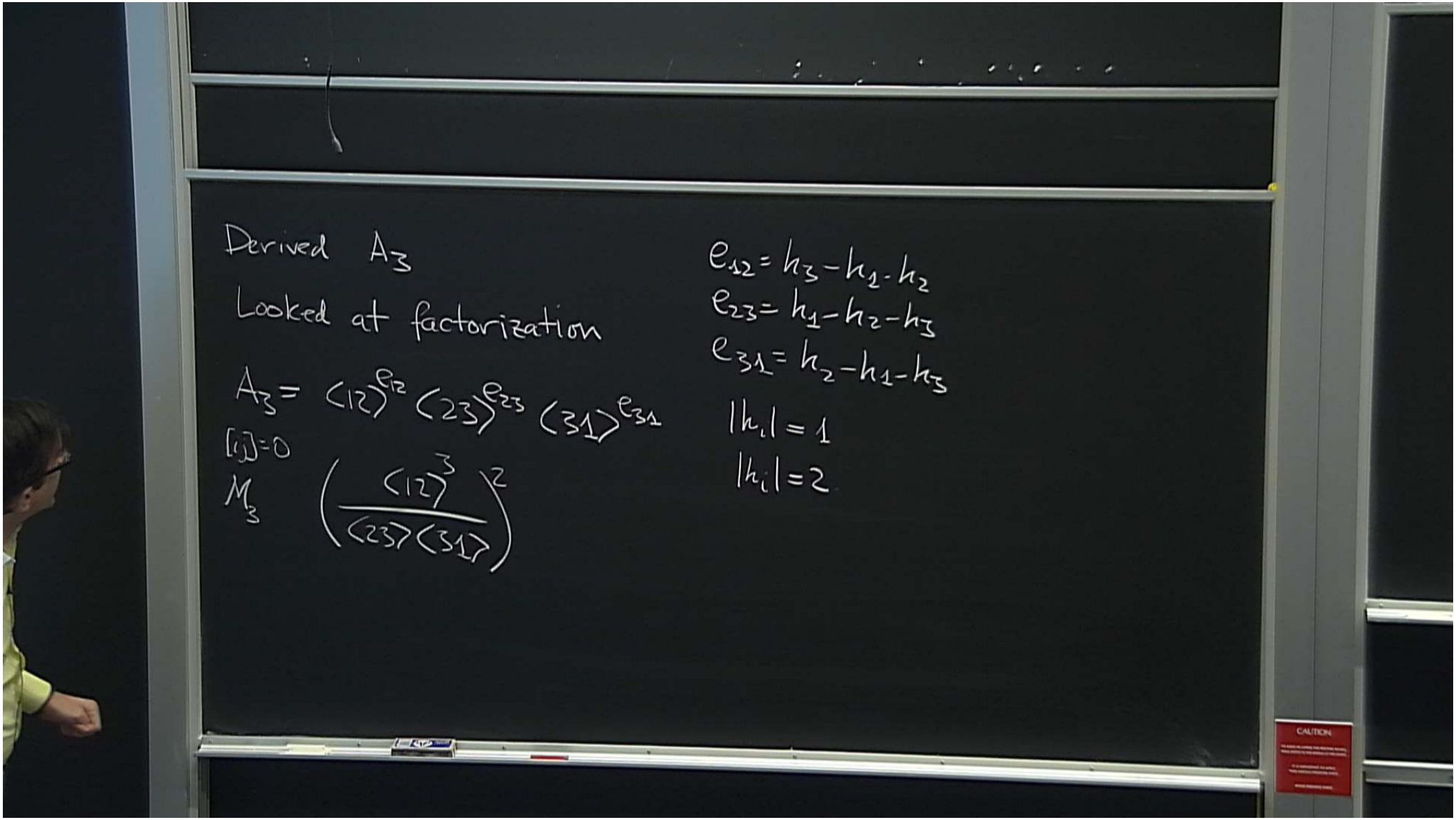
Title: Amplitudes: BCFW Recursion

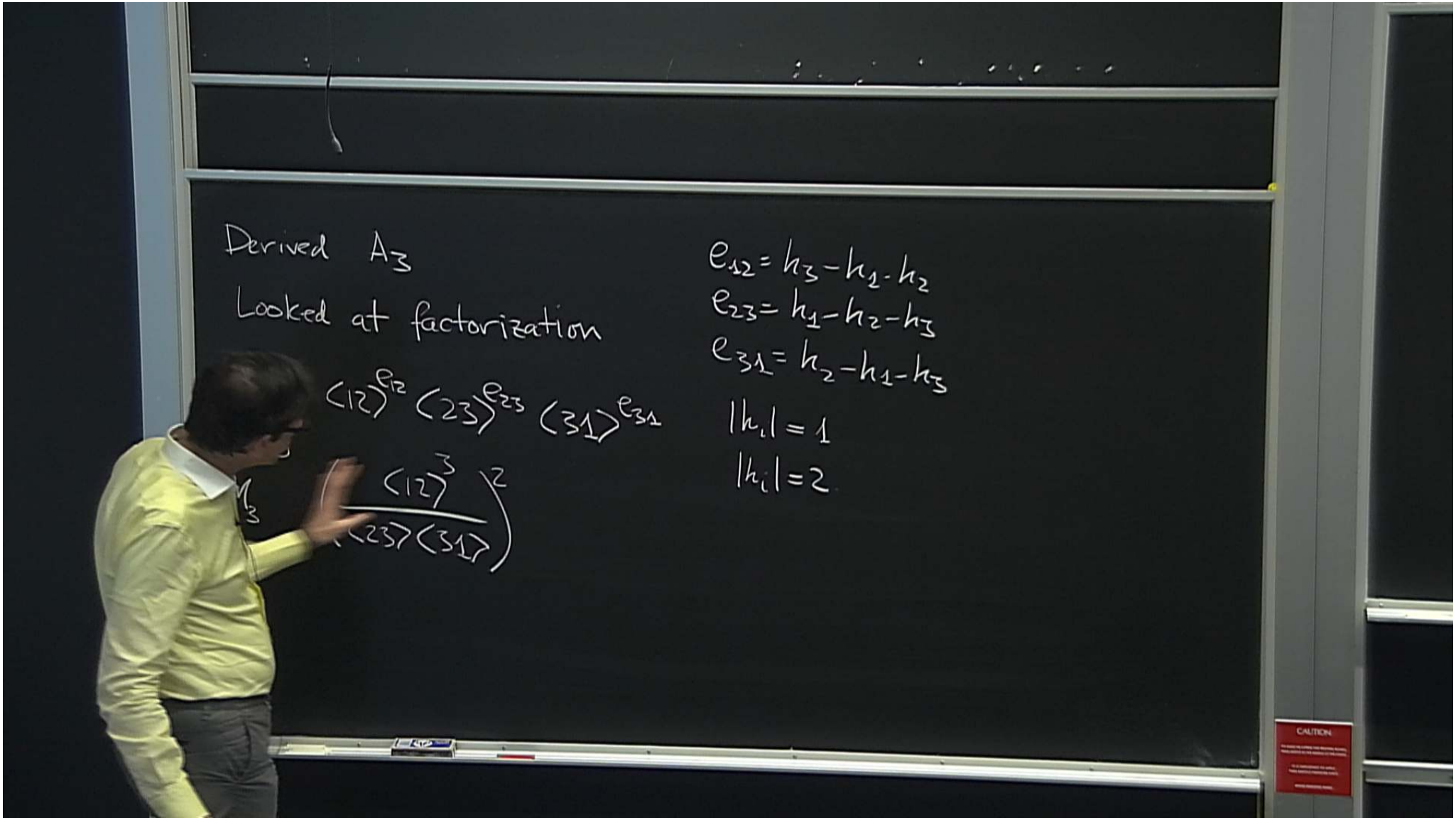
Date: Jul 15, 2015 02:30 PM

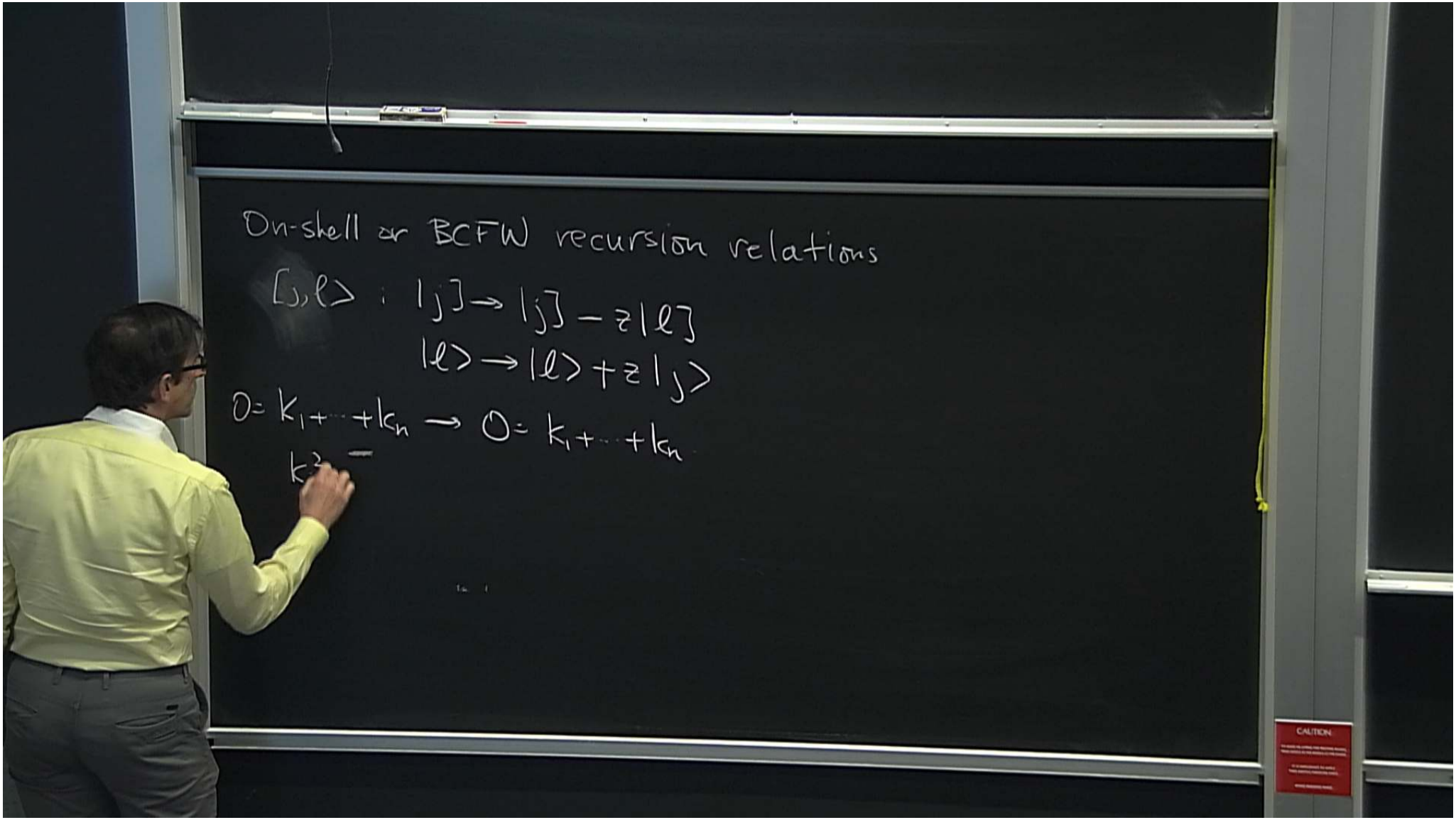
URL: http://pirsa.org/15070050

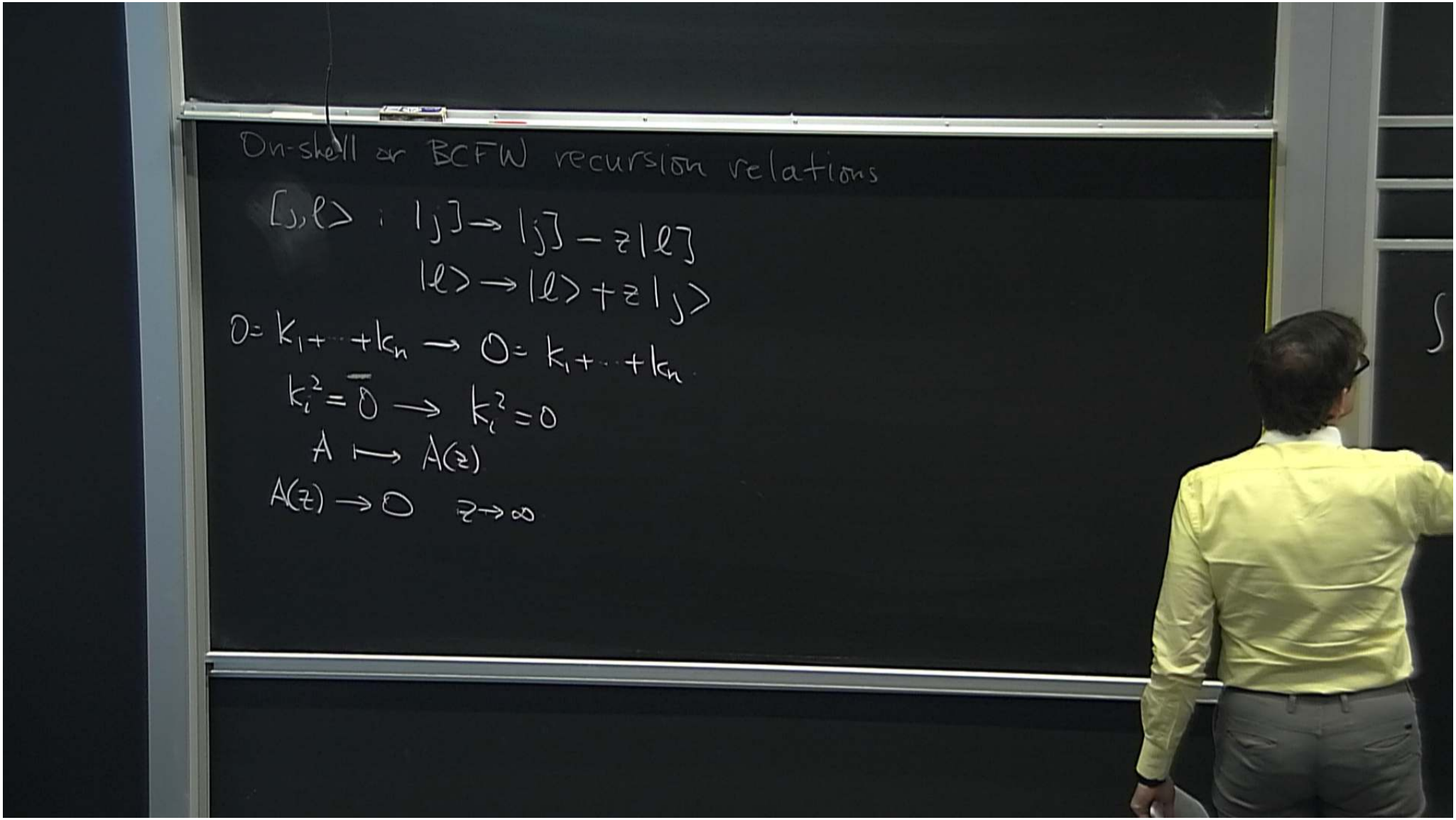
Abstract:

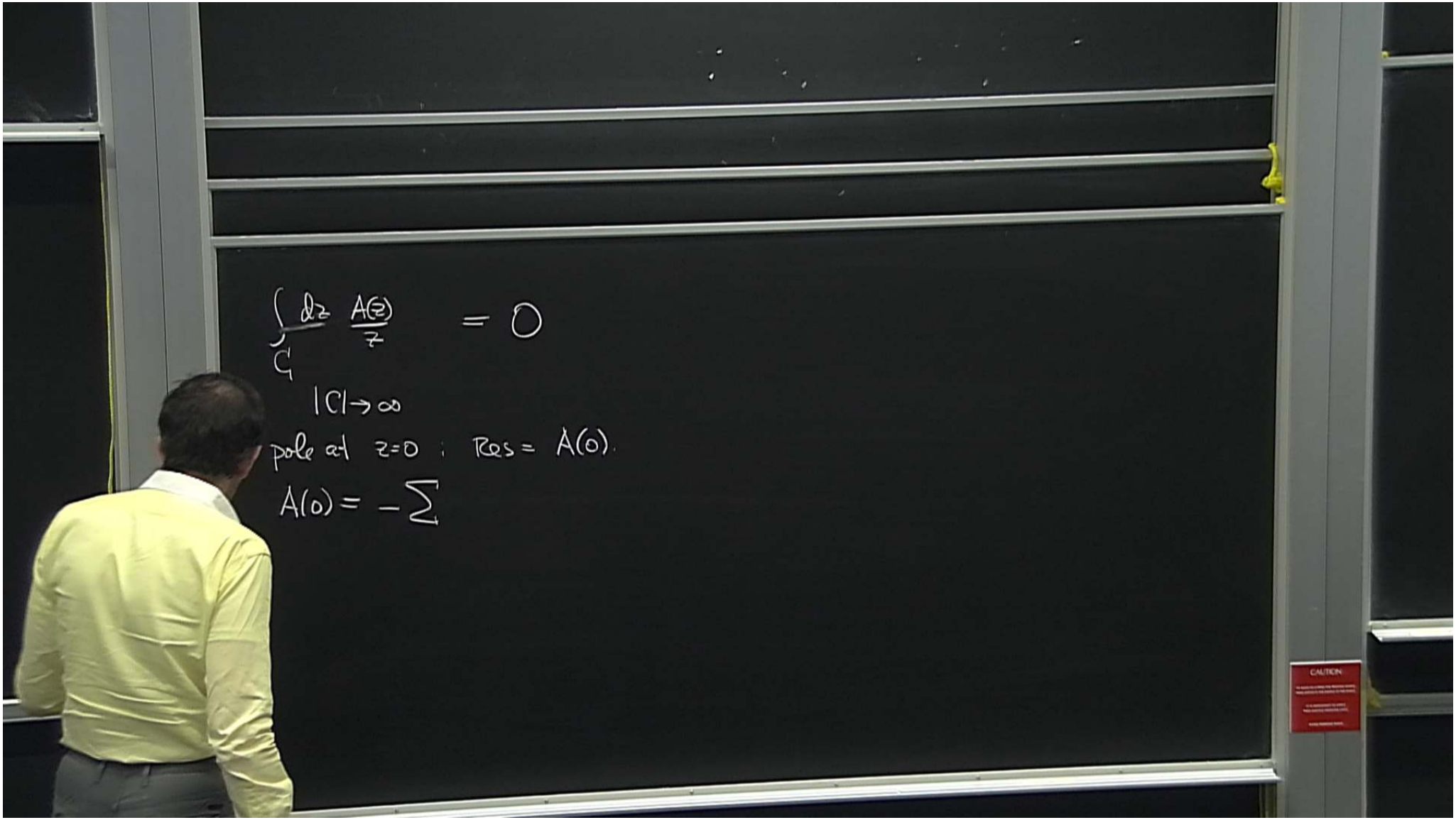

Derived A_3
Looked at factoriza
$A_3 = (12)^{e_{12}} (23)^{e_{23}}$
$[,] = 0$
$e_{12} = h_3 - h_1 - h_2$
$e_{23} = h_1 - h_2 - h_3$
$e_{31} = h_2 - h_1 - h_3$
$|h_1| = 1$
$|h$
CAUTION

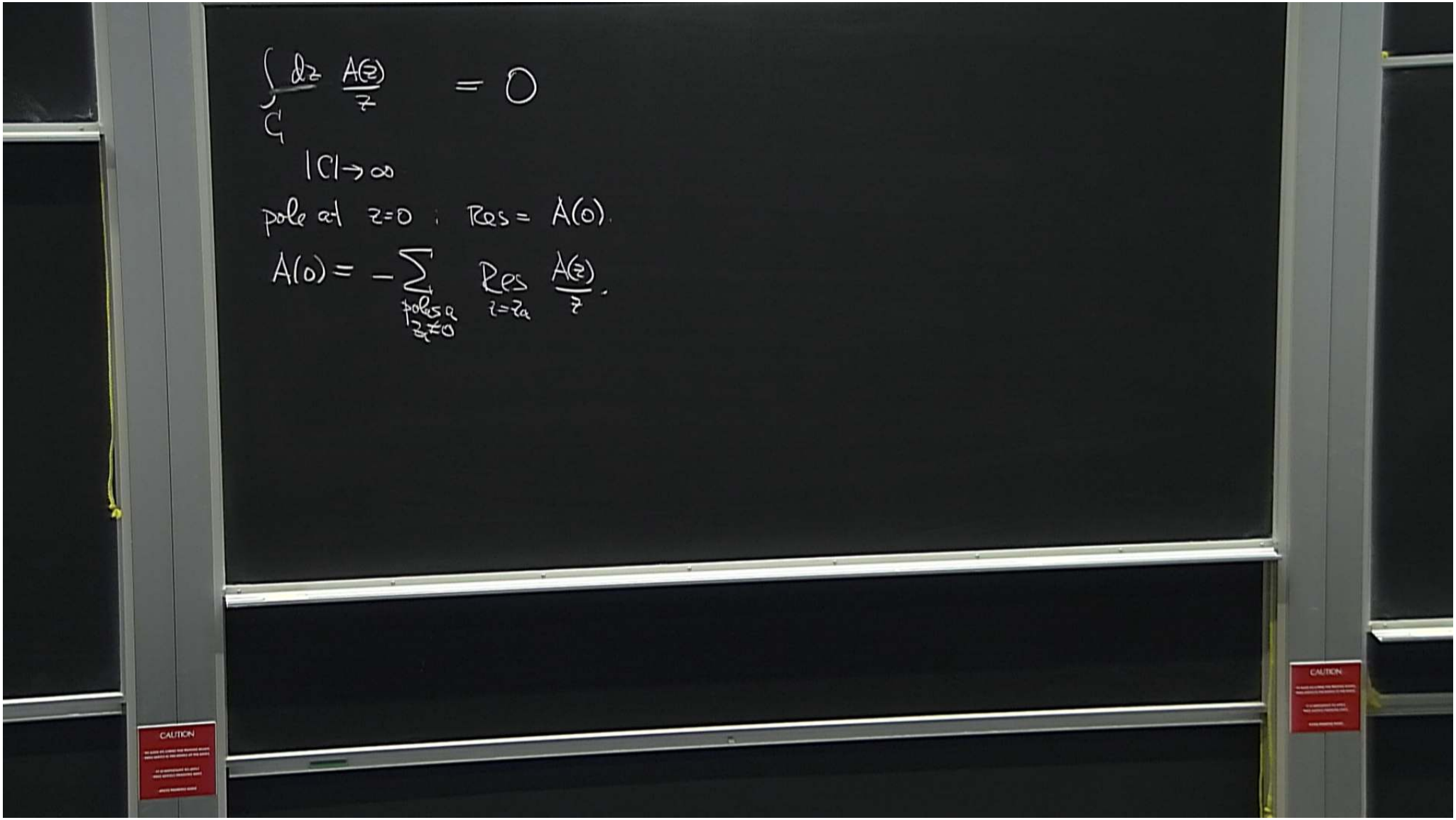

Derived A_3
Looked at factorization
$A_3 = \langle 12\rangle^{e_{12}} \langle 23\rangle^{e_{23}} \langle 31\rangle^{e_{31}}$
$[j]=0$
M_3
$\left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^2$
$e_{12} = h_3 - h_1 - h_2$
$e_{23} = h_1 - h_2 - h_3$
$e_{31} = h_2 - h_1 - h_3$
$|h_i| = 1$
$|h_i| = 2$
CAUTION

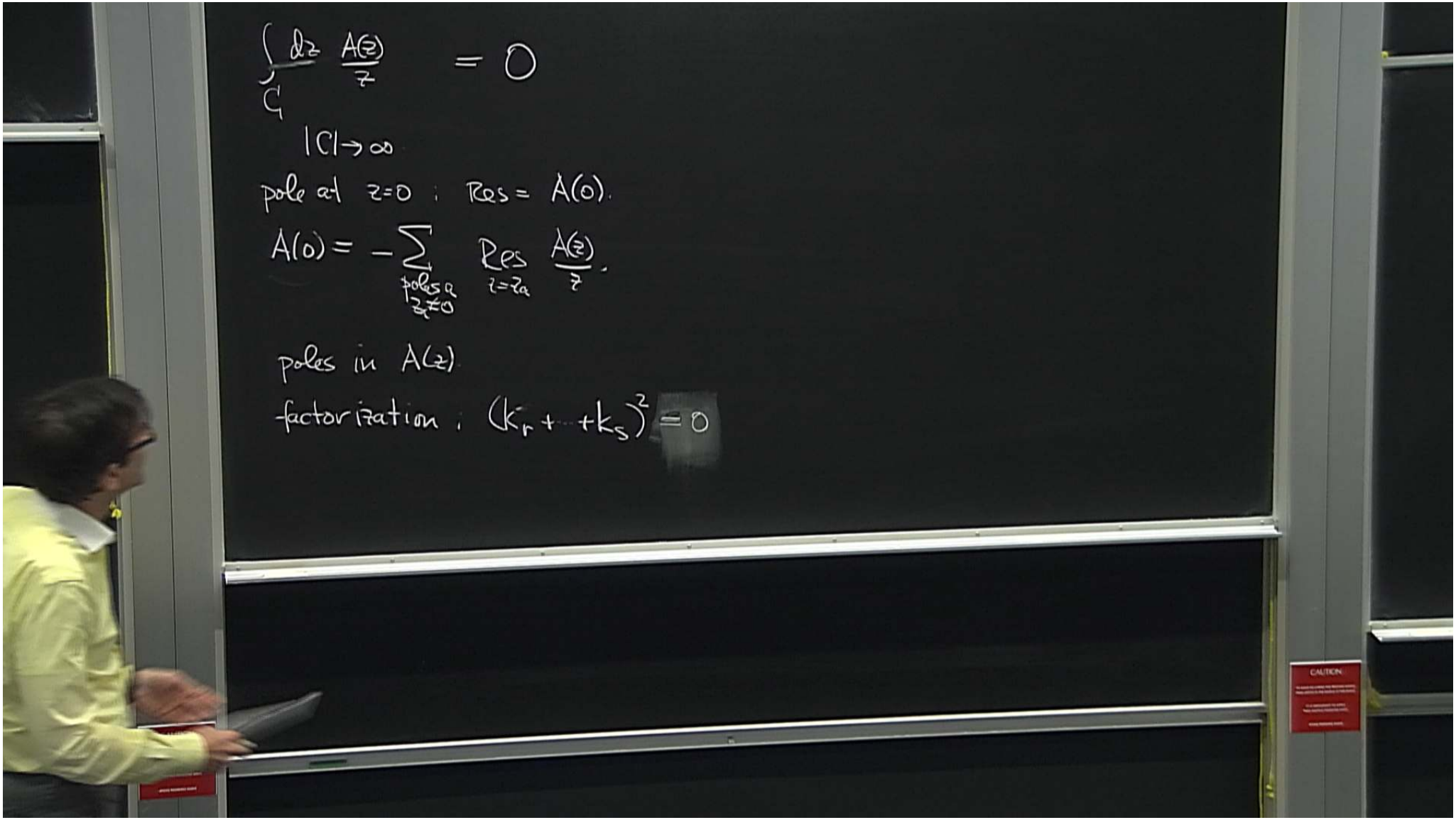

Derived A_3
Looked at factorization
$\langle 12\rangle^{e_{12}} \langle 23\rangle^{e_{23}} \langle 31\rangle^{e_{31}}$
$\left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^2$
$e_{12} = h_3 - h_1 - h_2$
$e_{23} = h_1 - h_2 - h_3$
$e_{31} = h_2 - h_1 - h_3$
$|h_i| = 1$
$|h_i| = 2$

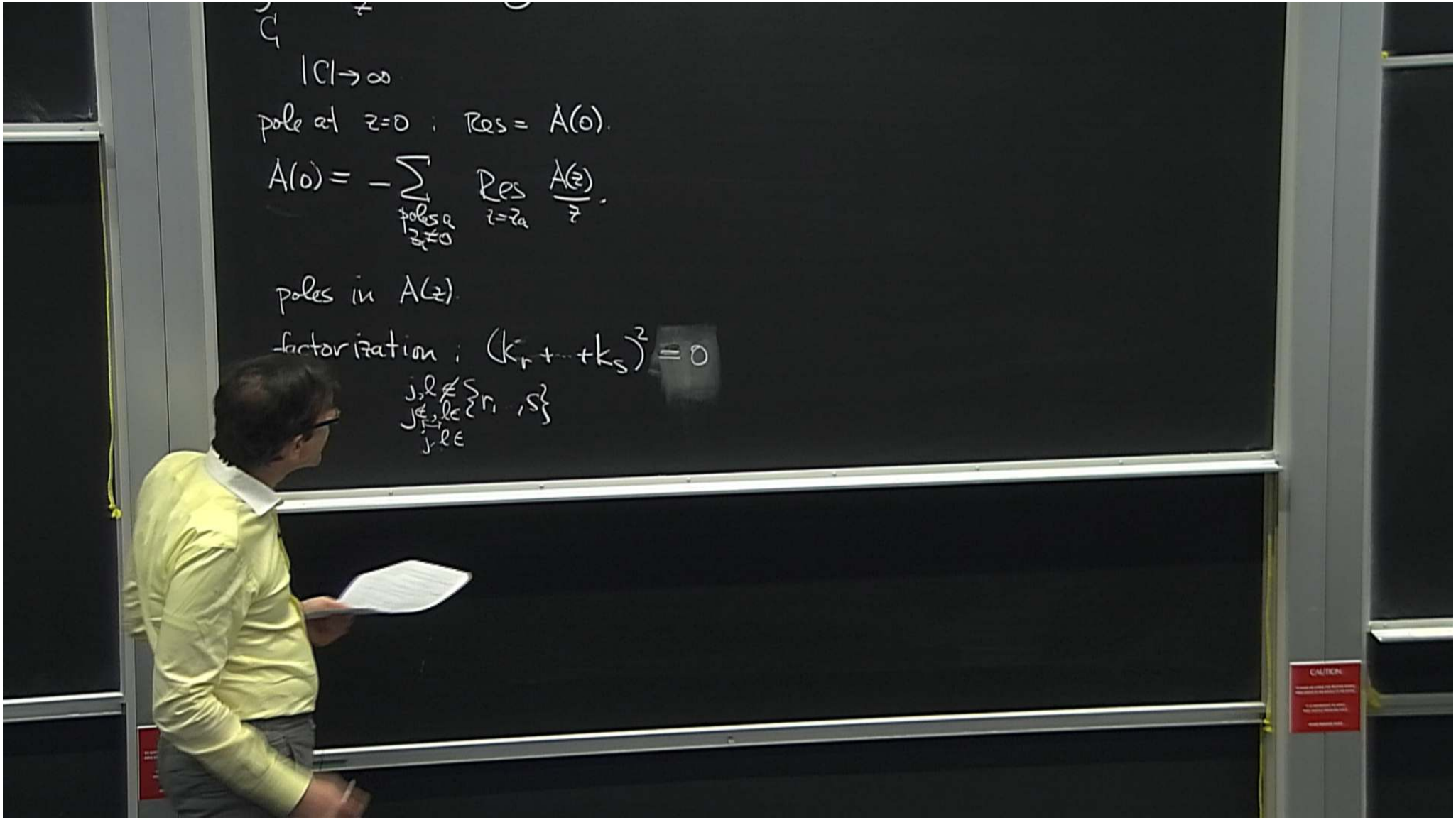

On-shell or BCFW recursion relations
$[j,\ell\rangle : |j] \to |j] - z|\ell]$
$|\ell\rangle \to |\ell\rangle + z|j\rangle$
$0 = k_1 + \cdots + k_n \to 0 = k_1 + \cdots + k_n$
$k^2 =$
CAUTION

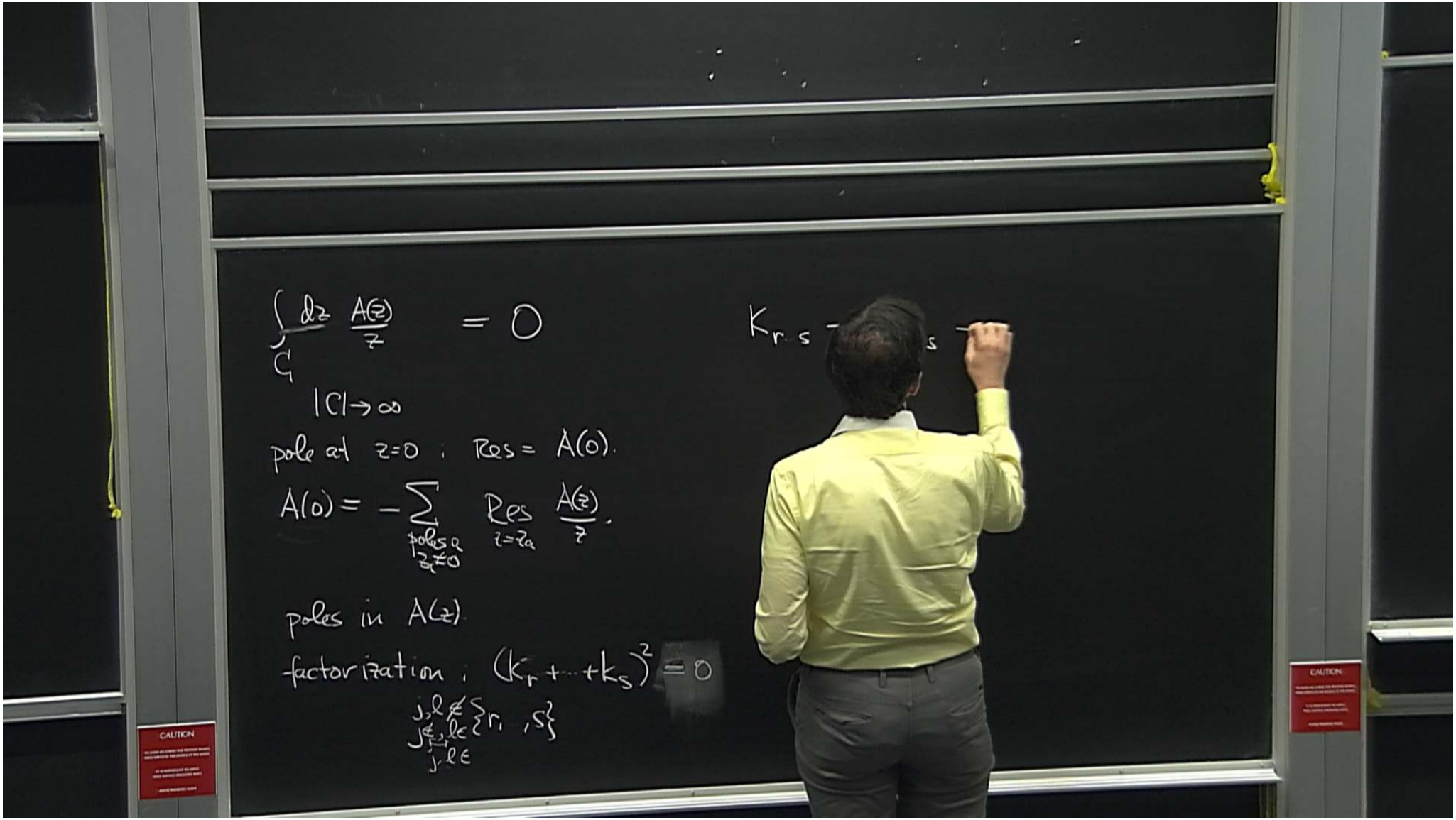

On-shell or BCFW recursion relations
$[j,\ell\rangle : |j] \to |j] - z|\ell]$
$|\ell\rangle \to |\ell\rangle + z|j\rangle$
$0 = k_1 + \cdots + k_n \to 0 = k_1 + \cdots + k_n$
$k_i^2 = 0 \to k_i^2 = 0$
$A \longmapsto A(z)$
$A(z) \to 0 \quad z \to \infty$

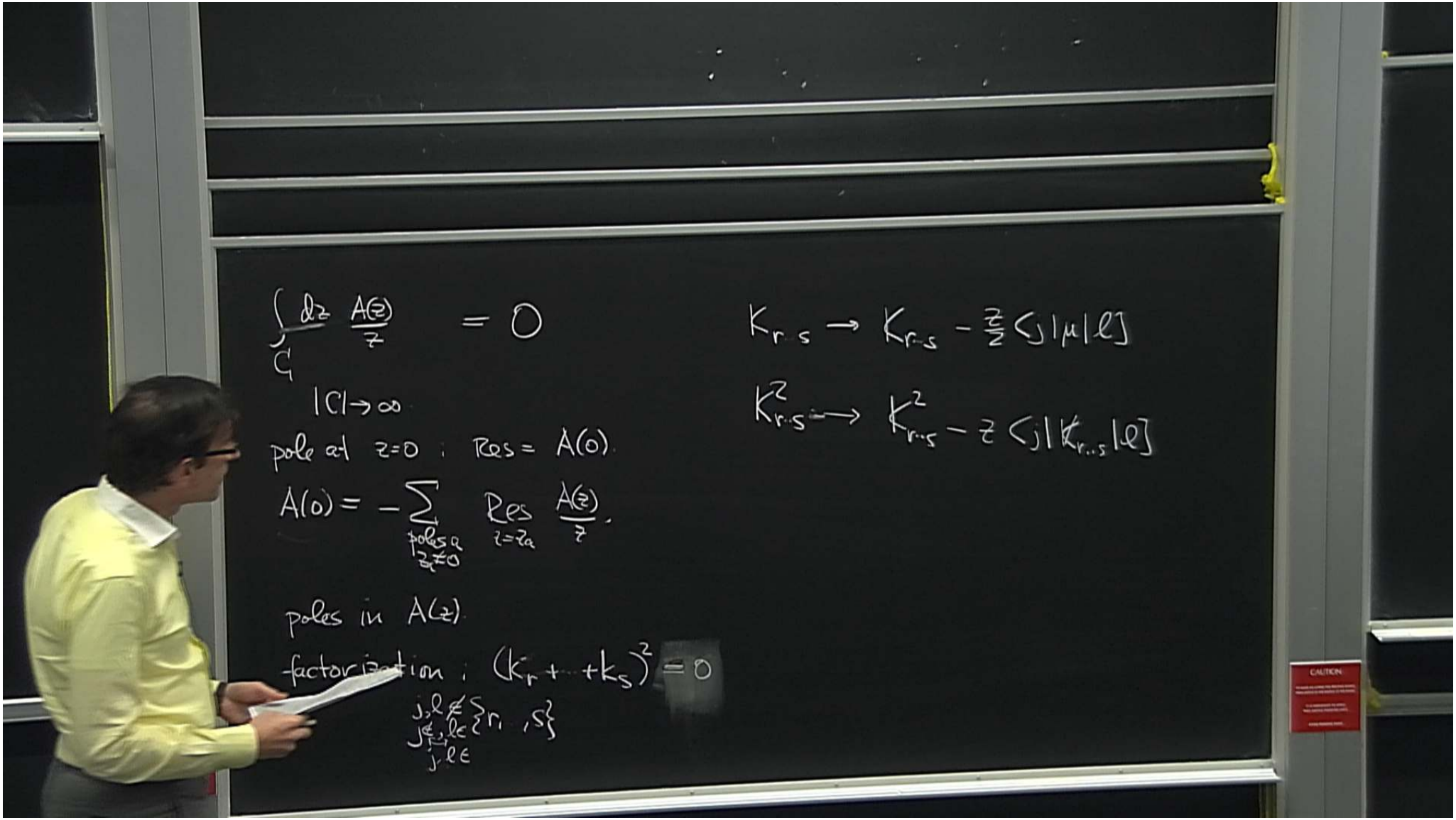

$\oint_C dz \frac{A(z)}{z} = 0$
$|C| \to \infty$
pole at $z=0$; Res $= A(0)$.
$A(0) = -\sum$


$\oint_C dz \frac{A(z)}{z} = 0$
$|C| \to \infty$
pole at $z=0$: Res $= A(0)$
$A(0) = -\sum_{\substack{\text{poles } a \\ z_a \neq 0}} \underset{z=z_a}{\text{Res}} \frac{A(z)}{z}$.
CAUTION
CAUTION


$\oint_C dz \frac{A(z)}{z} = 0$
$|C| \to \infty$
pole at $z=0$: $\text{Res} = A(0)$.
$A(0) = -\sum_{\substack{\text{poles } a \\ z_a \neq 0}} \underset{z=z_a}{\text{Res}} \frac{A(z)}{z}$.
poles in $A(z)$
factorization : $(k_r + \cdots + k_s)^2 = 0$


C
$|C| \to \infty$
pole at $z=0$: $\text{Res} = A(0)$
$A(0) = -\sum_{\substack{\text{poles } a \\ z_a \neq 0}} \underset{z=z_a}{\text{Res}} \frac{A(z)}{z}$
poles in $A(z)$
factorization : $(k_r + \cdots + k_s)^2 = 0$
$j, l \notin \{r, \ldots, s\}$
$j \notin \{r, \ldots, s\}, \; l \in \{r, \ldots, s\}$
$j, l \in \{r, \ldots, s\}$


$\oint_C dz \frac{A(z)}{z} = 0$
$|C| \to \infty$
pole at $z=0$; Res $= A(0)$.
$A(0) = -\sum_{\substack{\text{poles } a \\ z_a \neq 0}} \underset{z=z_a}{\text{Res}} \frac{A(z)}{z}$.
poles in $A(z)$
factorization ; $(k_r + \cdots + k_s)^2 = 0$
$j, \ell \notin \{r, \ldots, s\}$
$j \in, \ell \in$
$j \cdot \ell \in$
$K_{r \cdots s}$
CAUTION
CAUTION


$\oint_C dz \frac{A(z)}{z} = 0$
$|C| \to \infty$
pole at $z=0$; Res $= A(0)$
$A(0) = -\sum_{\substack{\text{poles of} \\ z \neq 0}} \operatorname{Res}_{z=z_a} \frac{A(z)}{z}$
poles in $A(z)$
factorization ; $(k_r + \ldots + k_s)^2 = 0$
$j, l \in \{r, \ldots, s\}$
$j \in, l \in$
$j, l \in$
$K_{r\,s} \to K_{r\,s} - \frac{z}{2} \zeta_j |\mu| l]$
$K^2_{r\,s} \longrightarrow K^2_{r\,s} - z \zeta_j |K_{r\,s}| l]$
CAUTION

$$\oint_C dz\, \frac{A(z)}{z} = 0$$

$|C| \to \infty$

pole at $z=0$: $\text{Res} = A(0)$

$$A(0) = -\sum_{\substack{\text{poles } a \\ z_a \neq 0}} \operatorname*{Res}_{z=z_a} \frac{A(z)}{z}$$

poles in $A(z)$

factorization : $(k_r + \cdots + k_s)^2 = 0$

$j, l \in \{r, \ldots, s\}$

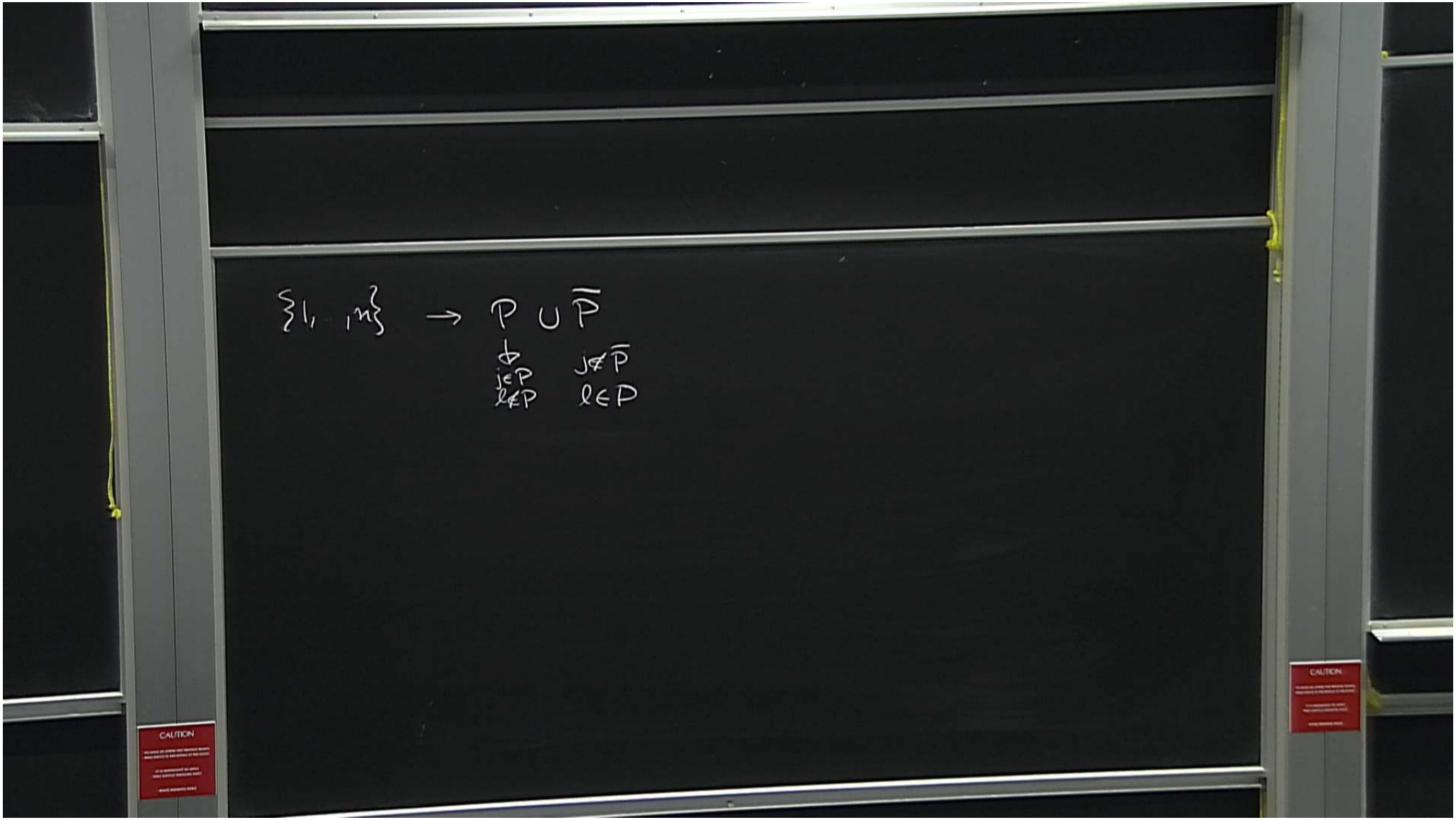
$j \in, l \in$

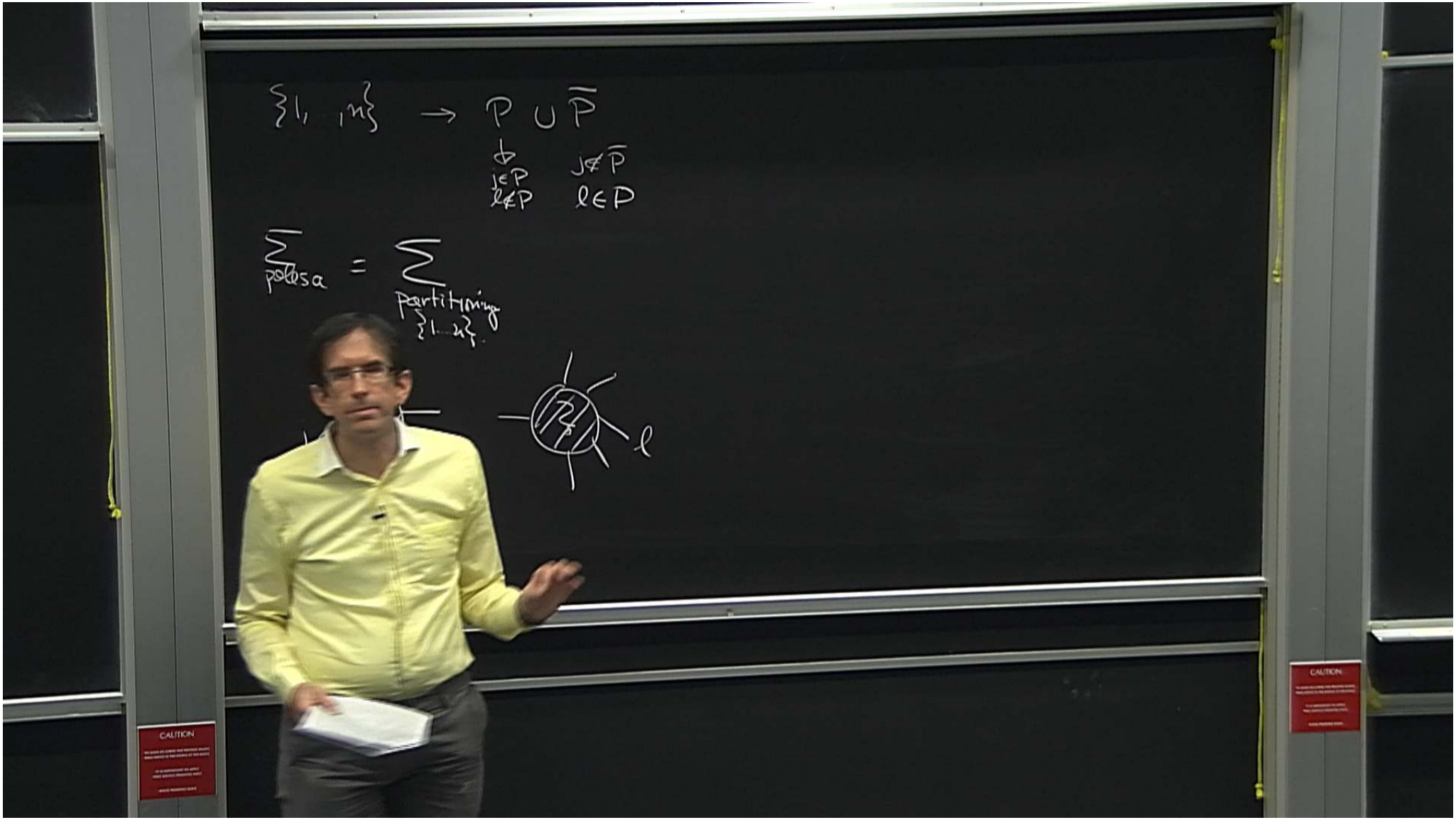
$j \cdot l \in$

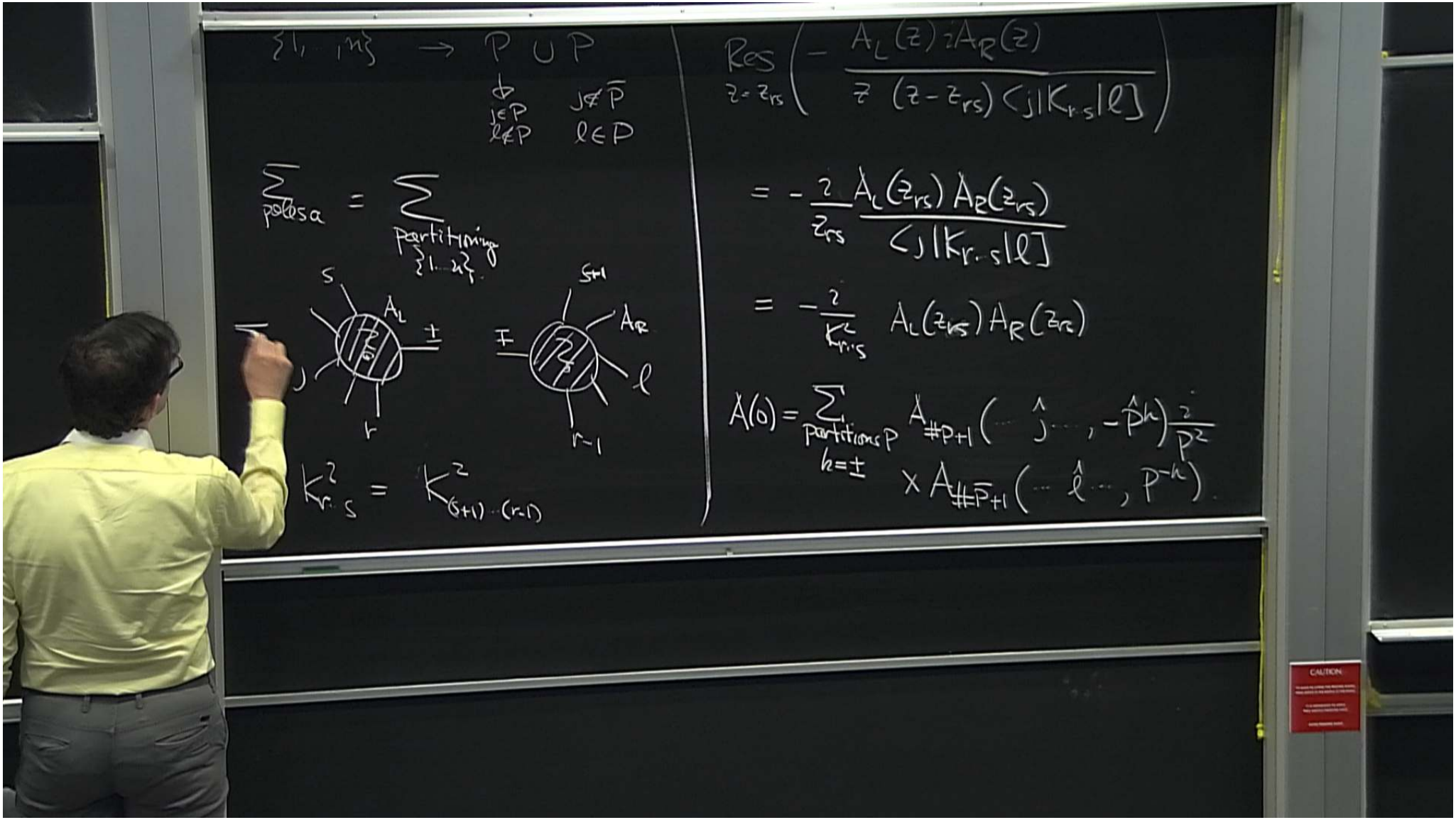
$$K_{r\cdots s} \to K_{r\cdots s} - \frac{z}{2} [j|\mu|l]$$

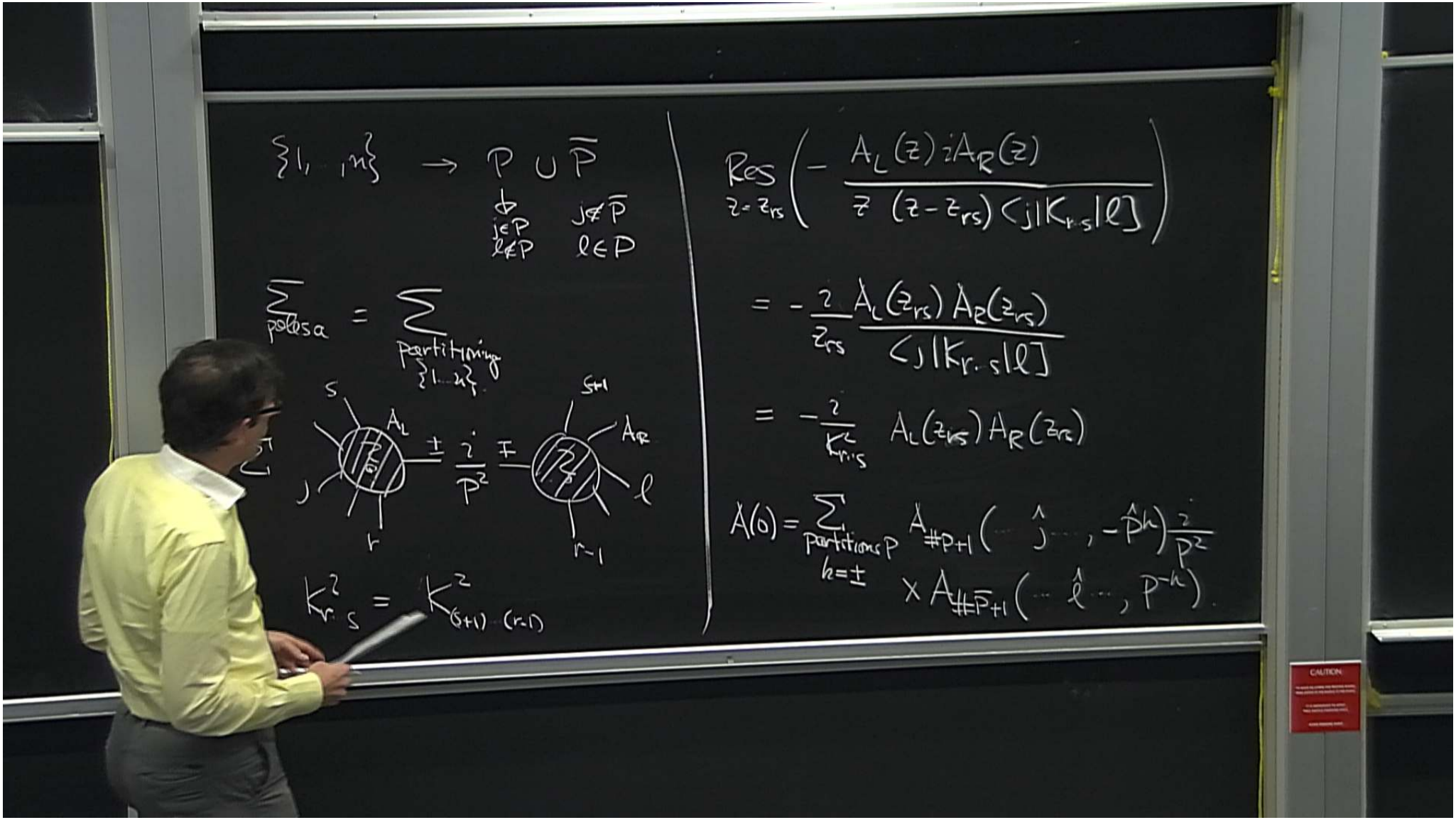
$$K_{r\cdots s}^2 \longrightarrow K_{r\cdots s}^2 - z\,[j|K_{r\cdots s}|l] = 0.$$

$$z = z_{rs} \equiv \frac{K_{r\cdots s}^2}{[j|K_{r\cdots s}|l]}$$


$\{1, \ldots, n\} \to P \cup \bar{P}$
$\emptyset$
$j \in P$ $j \notin \bar{P}$
$l \notin P$ $l \in P$
CAUTION
CAUTION


$\{1, \ldots, n\} \to P \cup \bar{P}$
$\emptyset$
$j \in P \quad j \notin \bar{P}$
$l \notin P \quad l \in P$
$\sum_{\text{polosa}} = \sum_{\text{partitioning } \{1 \ldots n\}}$
l
CAUTION
CAUTION


$\{1,\dots,n\} \to P \cup \bar{P}$
$j \in P$ $j \notin \bar{P}$
$l \notin P$ $l \in P$
$\sum_{\text{poles } a} = \sum_{\text{partitionings } \{1\dots n\}}$
A_L
A_R
$K^2_{r\dots s} = K^2_{(s+1)\dots(r-1)}$
$\operatorname{Res}_{z=z_{rs}} \left(- \frac{A_L(z) A_R(z)}{z\,(z-z_{rs})\langle j|K_{r\dots s}|l]} \right)$
$= - \frac{2}{z_{rs}} \frac{A_L(z_{rs})\,A_R(z_{rs})}{\langle j|K_{r\dots s}|l]}$
$= - \frac{2}{K^2_{r\dots s}} A_L(z_{rs})\,A_R(z_{rs})$
$A(0) = \sum_{\substack{\text{partitions } P \\ h=\pm}} A_{\#P+1}(\dots \hat{j} \dots, -\hat{P}^h) \frac{2}{P^2} \times A_{\#\bar{P}+1}(\dots \hat{l} \dots, P^{-h})$


$\{1, \dots, n\} \to P \cup \bar{P}$
$j \in P \quad j \notin \bar{P}$
$l \notin P \quad l \in P$
$\sum_{\text{poles } a} = \sum_{\text{partitioning } \{1 \dots n\}}$
A_L
A_R
$\pm \frac{i}{P^2} \mp$
$K^2_{r,s} = K^2_{(s+1)\cdots(r-1)}$
$\operatorname{Res}_{z=z_{rs}} \left(- \frac{A_L(z)\, i A_R(z)}{z\,(z-z_{rs})\langle j | K_{r,s} | l]} \right)$
$= - \frac{2}{z_{rs}} \frac{A_L(z_{rs})\, A_R(z_{rs})}{\langle j | K_{r,s} | l]}$
$= - \frac{2}{K^2_{r,s}} A_L(z_{rs})\, A_R(z_{rs})$
$A(0) = \sum_{\substack{\text{partitions } P \\ h = \pm}} A_{\#P+1}(\cdots \hat{j} \cdots, -\hat{P}^h) \frac{2}{P^2} \times A_{\#\bar{P}+1}(\cdots \hat{l} \cdots, P^{-h})$
CAUTION

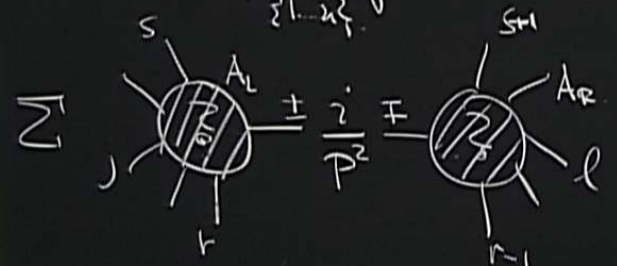
$\{1, \ldots, n\} \to \mathbb{P} \cup \bar{\mathbb{P}}$

$j \in \mathbb{P}$, $j \notin \bar{\mathbb{P}}$

$l \notin \mathbb{P}$, $l \in \bar{\mathbb{P}}$

$$\sum_{\text{poles } a} = \sum_{\text{partitioning } \{1 \ldots n\}}$$

$$\sum$$



$$K^2_{r\,s} = K^2_{(s+1)\,(r-1)}$$

$$\operatorname{Res}_{z=z_{rs}} \left(- \frac{A_L(z)\, i A_R(z)}{z\,(z-z_{rs})\,\langle j | K_{r\,s} | l]} \right)$$

$$= - \frac{2}{z_{rs}} \frac{A_L(z_{rs})\, A_R(z_{rs})}{\langle j | K_{r\,s} | l]}$$

$\hat{\lambda} = \lambda(z_{rs})$

$$= - \frac{2}{K^2_{r\,s}} A_L(z_{rs})\, A_R(z_{rs})$$

$$A(0) = \sum_{\substack{\text{partitions } \mathbb{P} \\ h = \pm}} A_{\#\mathbb{P}+1}(\ldots \hat{j} \ldots, -\hat{P}^h) \frac{2}{P^2} \times A_{\#\bar{\mathbb{P}}+1}(\ldots \hat{l} \ldots, \hat{P}^{-h})$$

CAUTION

CAUTION

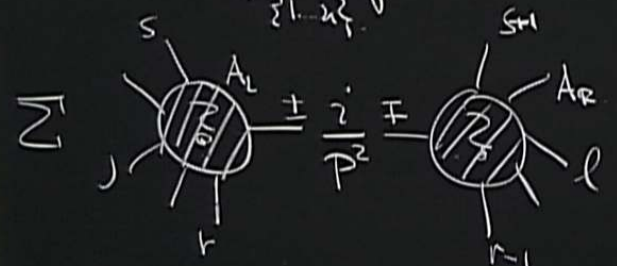
$\{1, \ldots, n\} \to \mathbb{P} \cup \bar{\mathbb{P}}$

$j \in \mathbb{P}$, $j \notin \bar{\mathbb{P}}$

$\ell \notin \mathbb{P}$, $\ell \in \bar{\mathbb{P}}$

$$\sum_{\text{poles } a} = \sum_{\text{partitioning } \{1 \ldots n\}}$$

$$\sum$$



$$K^2_{r\,s} = K^2_{(s+1)\cdots(r-1)}$$

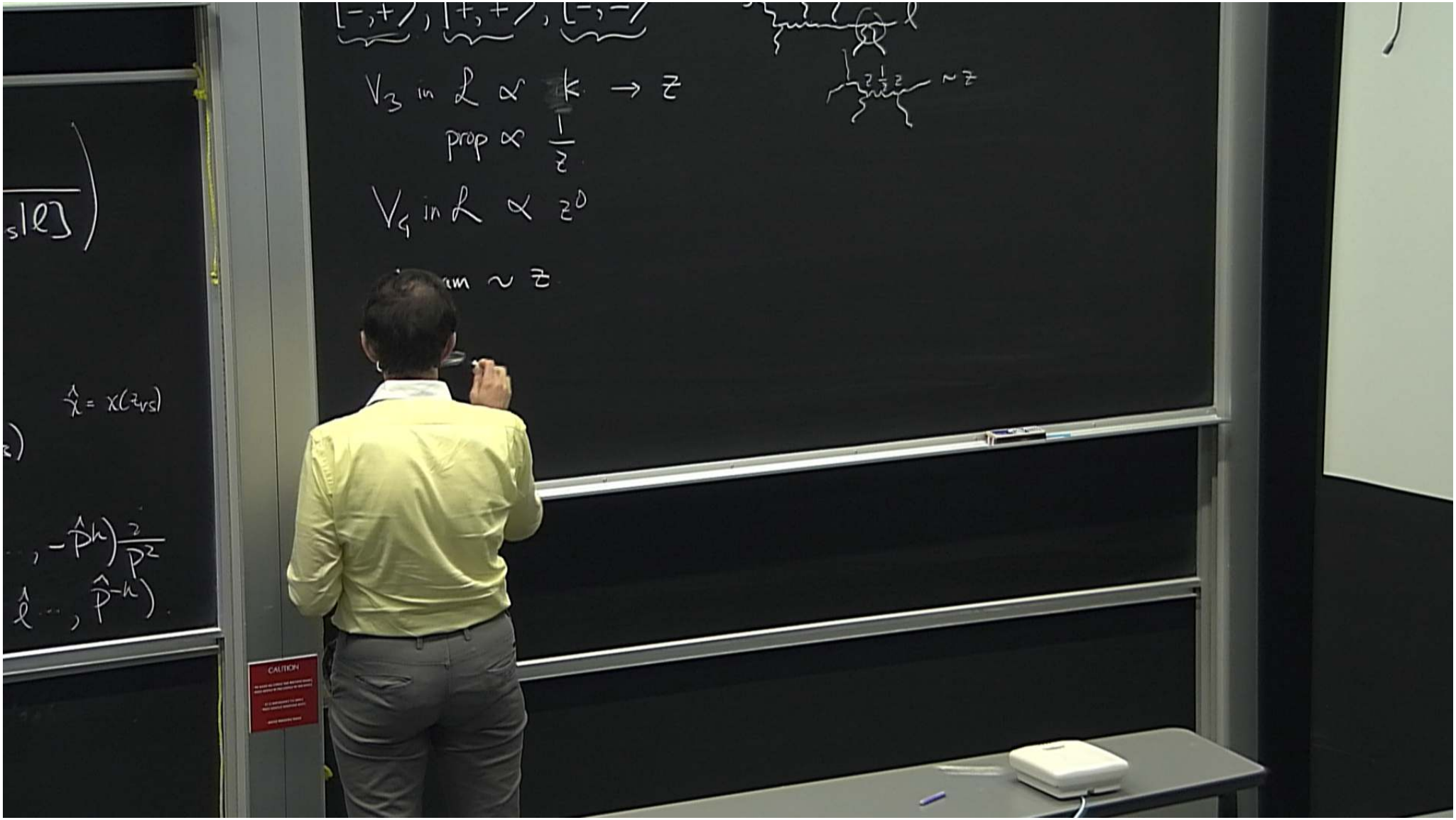
$$\operatorname*{Res}_{z=z_{rs}} \left(- \frac{A_L(z)\, i A_R(z)}{z\,(z - z_{rs})\,\langle j | K_{r\cdots s} | \ell]} \right)$$

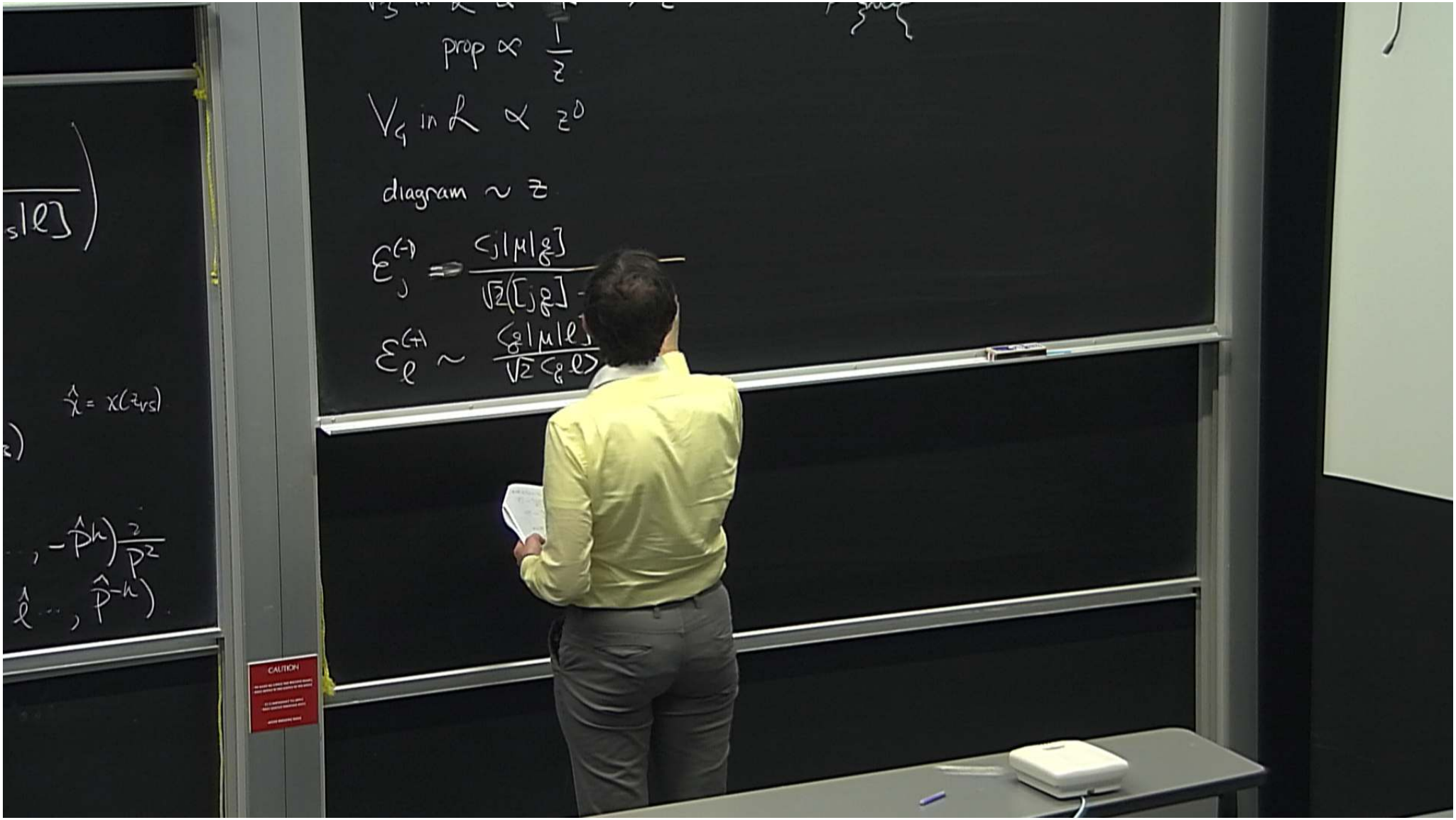
$$= - \frac{2}{z_{rs}} \frac{A_L(z_{rs})\, A_R(z_{rs})}{\langle j | K_{r\cdots s} | \ell]}$$

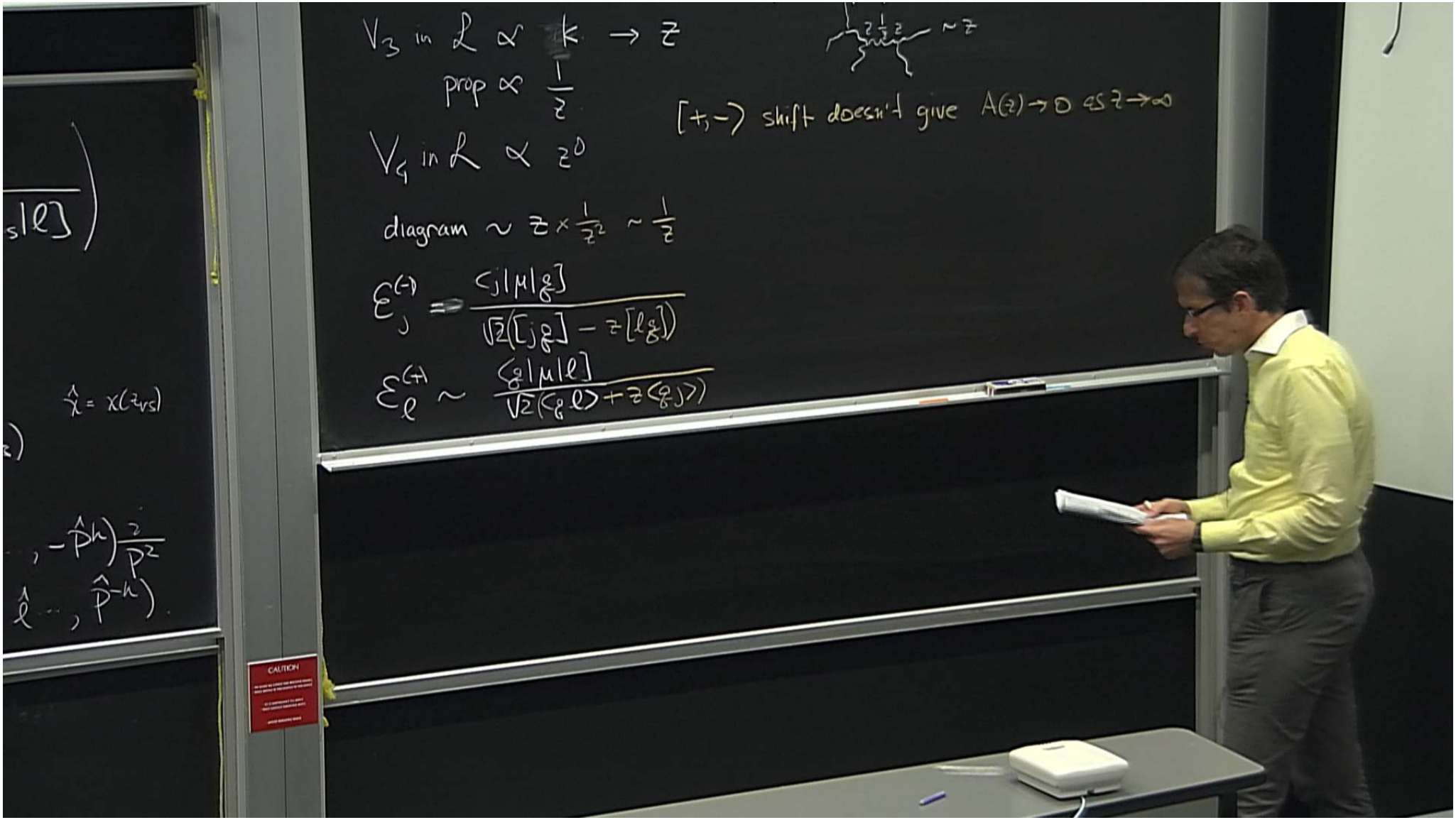
$\hat{\lambda} = \lambda(z_{rs})$

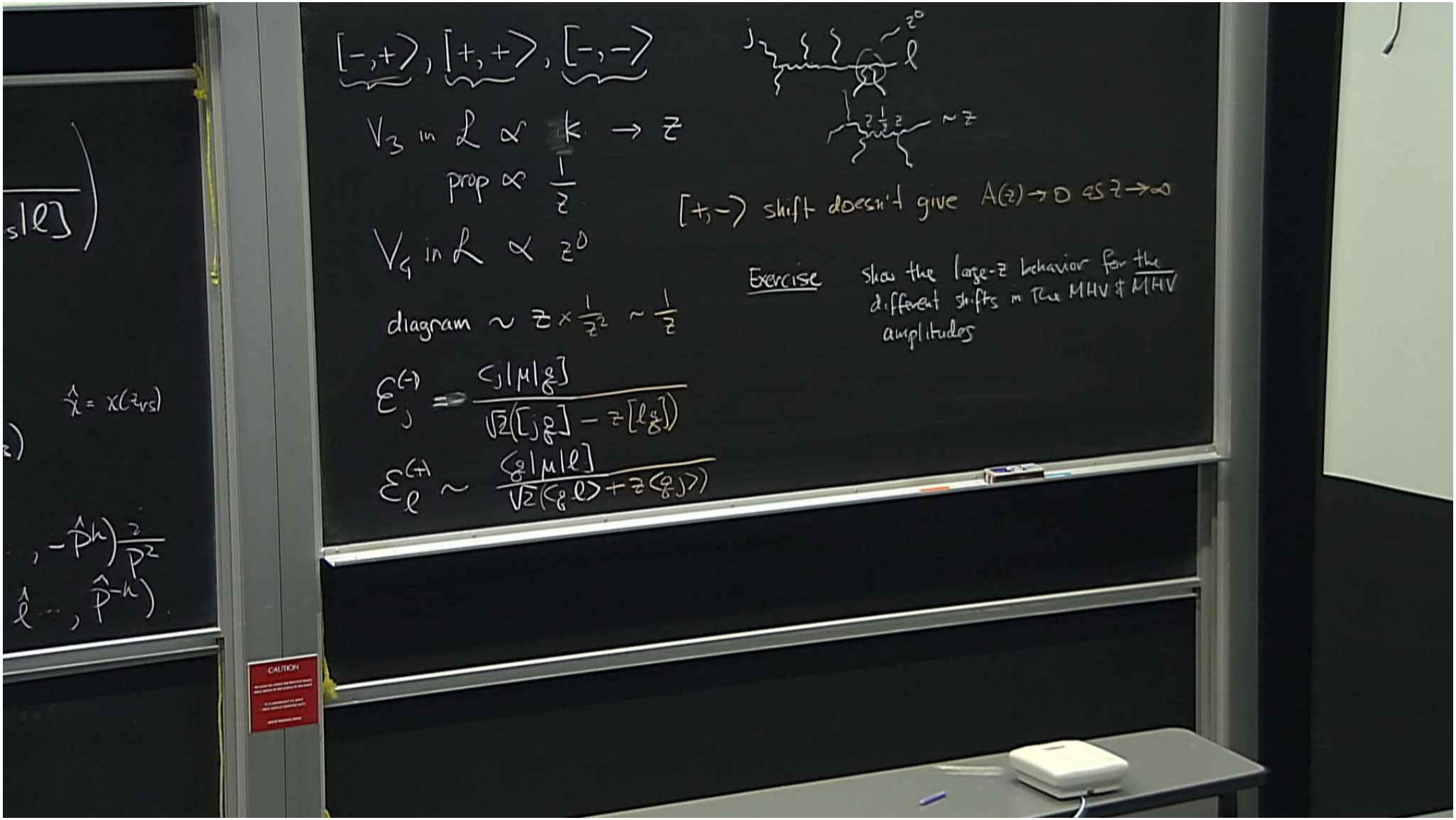
$$= - \frac{2}{K^2_{r\cdots s}} A_L(z_{rs})\, A_R(z_{rs})$$

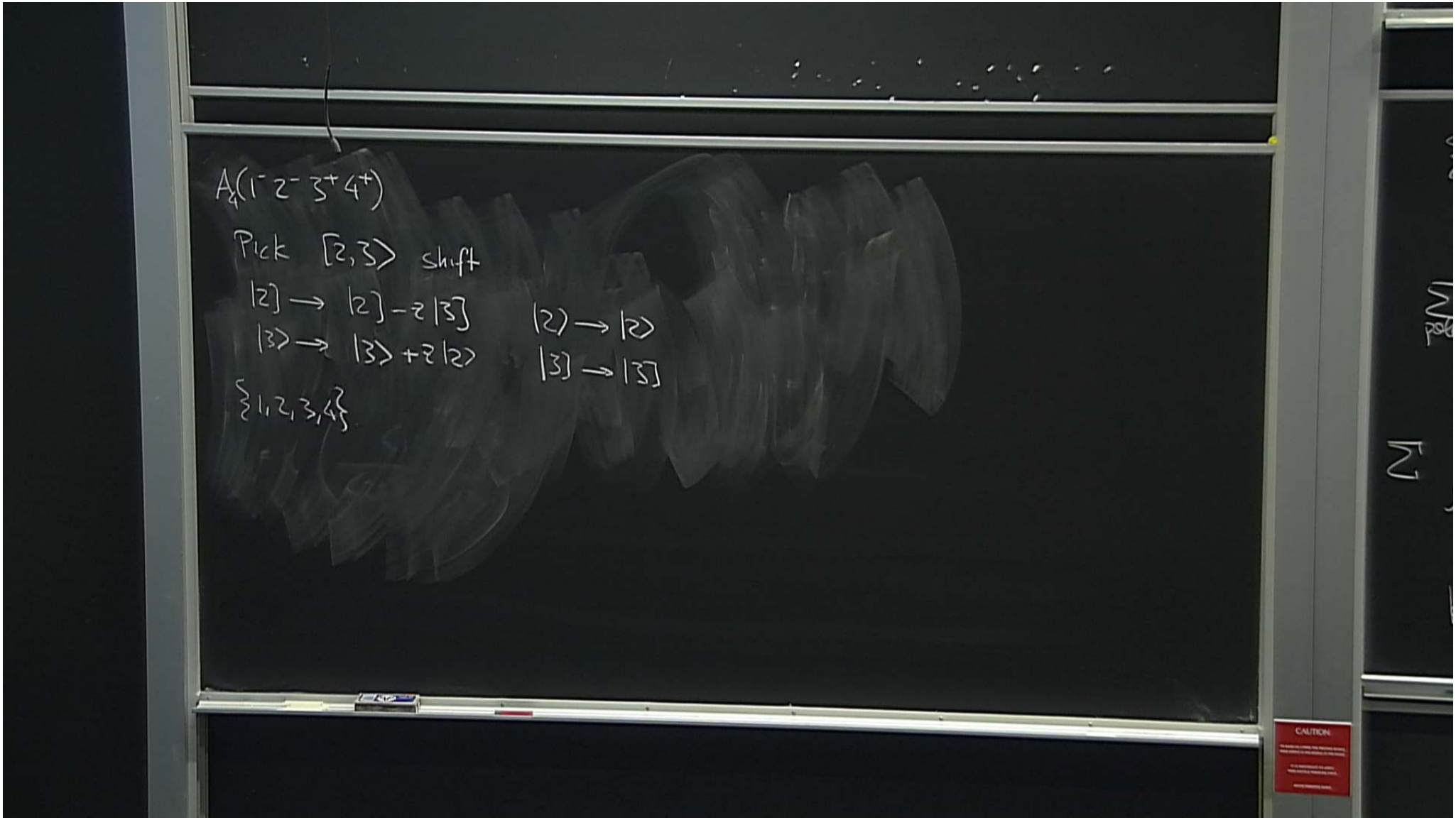
$$A(0) = \sum_{\substack{\text{partitions } \mathbb{P} \\ h = \pm}} A_{\#\mathbb{P}+1}(\cdots \hat{j} \cdots, -\hat{P}^h) \frac{2}{P^2} \times A_{\#\bar{\mathbb{P}}+1}(\cdots \hat{\ell} \cdots, \hat{P}^{-h})$$


V_3 in $\mathcal{L} \propto k \to z$
prop $\propto \frac{1}{z}$
V_4 in $\mathcal{L} \propto z^0$
$\sim z$
CAUTION


prop ∝ 1/z
V_G in L ∝ z^0
diagram ~ z
CAUTION


V_3 in $\mathcal{L} \propto k \to z$
prop $\propto \frac{1}{z}$
V_4 in $\mathcal{L} \propto z^0$
diagram $\sim z \times \frac{1}{z^2} \sim \frac{1}{z}$
$\varepsilon_j^{(-1)} = \frac{\langle j|\mu|\xi]}{\sqrt{2}([j\xi] - z[l\xi])}$
$\varepsilon_l^{(+1)} \sim \frac{\langle \xi|\mu|l]}{\sqrt{2}(\langle \xi l\rangle + z\langle \xi j\rangle)}$
$\sim z$
[+,−⟩ shift doesn't give $A(z) \to 0$ as $z \to \infty$
CAUTION


[−,+⟩, [+,+⟩, [−,−⟩
V_3 in $\mathcal{L} \propto k \to z$
prop $\propto \frac{1}{z}$
V_4 in $\mathcal{L} \propto z^0$
diagram $\sim z \times \frac{1}{z^2} \sim \frac{1}{z}$
$\varepsilon_j^{(-)} = \frac{\langle j|M|\xi]}{\sqrt{2}([j\xi] - z[l\xi])}$
$\varepsilon_l^{(+)} \sim \frac{\langle \xi|M|l]}{\sqrt{2}(\langle \xi l\rangle + z\langle \xi j\rangle)}$
[+,−⟩ shift doesn't give $A(z) \to 0$ as $z \to \infty$
Exercise Show the large-z behavior for the different shifts in the MHV & $\overline{\text{MHV}}$ amplitudes
CAUTION


$A_4(1^- 2^- 3^+ 4^+)$
Pick $[2,3\rangle$ shift
$|2] \to |2] - z|3]$
$|3\rangle \to |3\rangle + z|2\rangle$
$|2\rangle \to |2\rangle$
$|3] \to |3]$
$\{1,2,3,4\}$
CAUTION

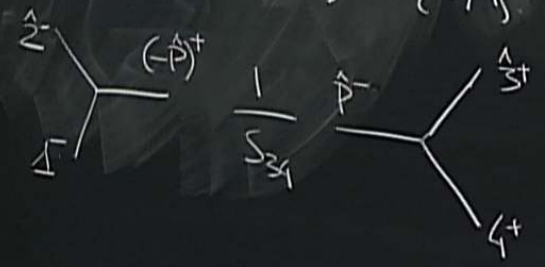
$A_4(1^- 2^- 3^+ 4^+)$

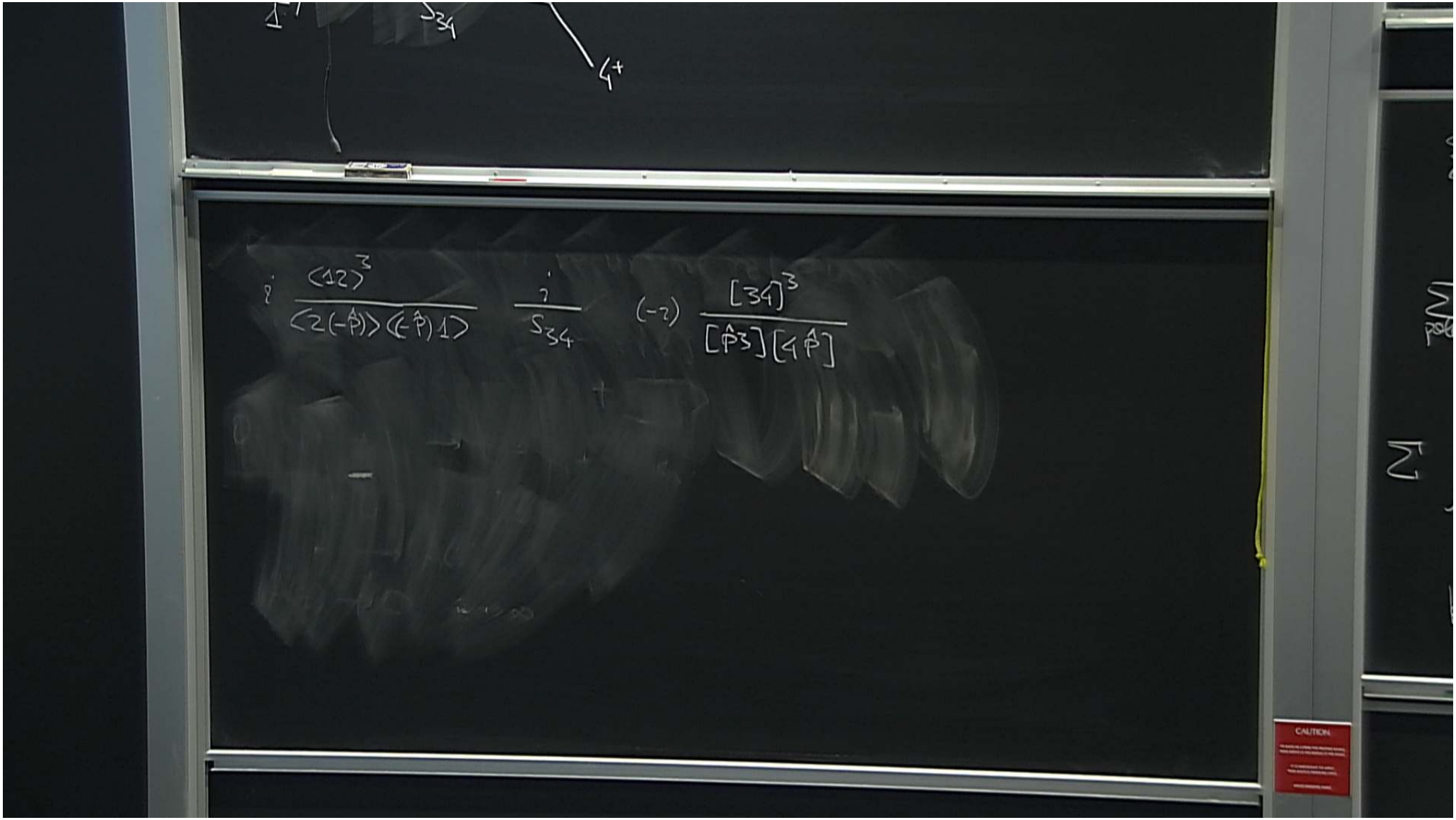
Pick $[2,3\rangle$ shift

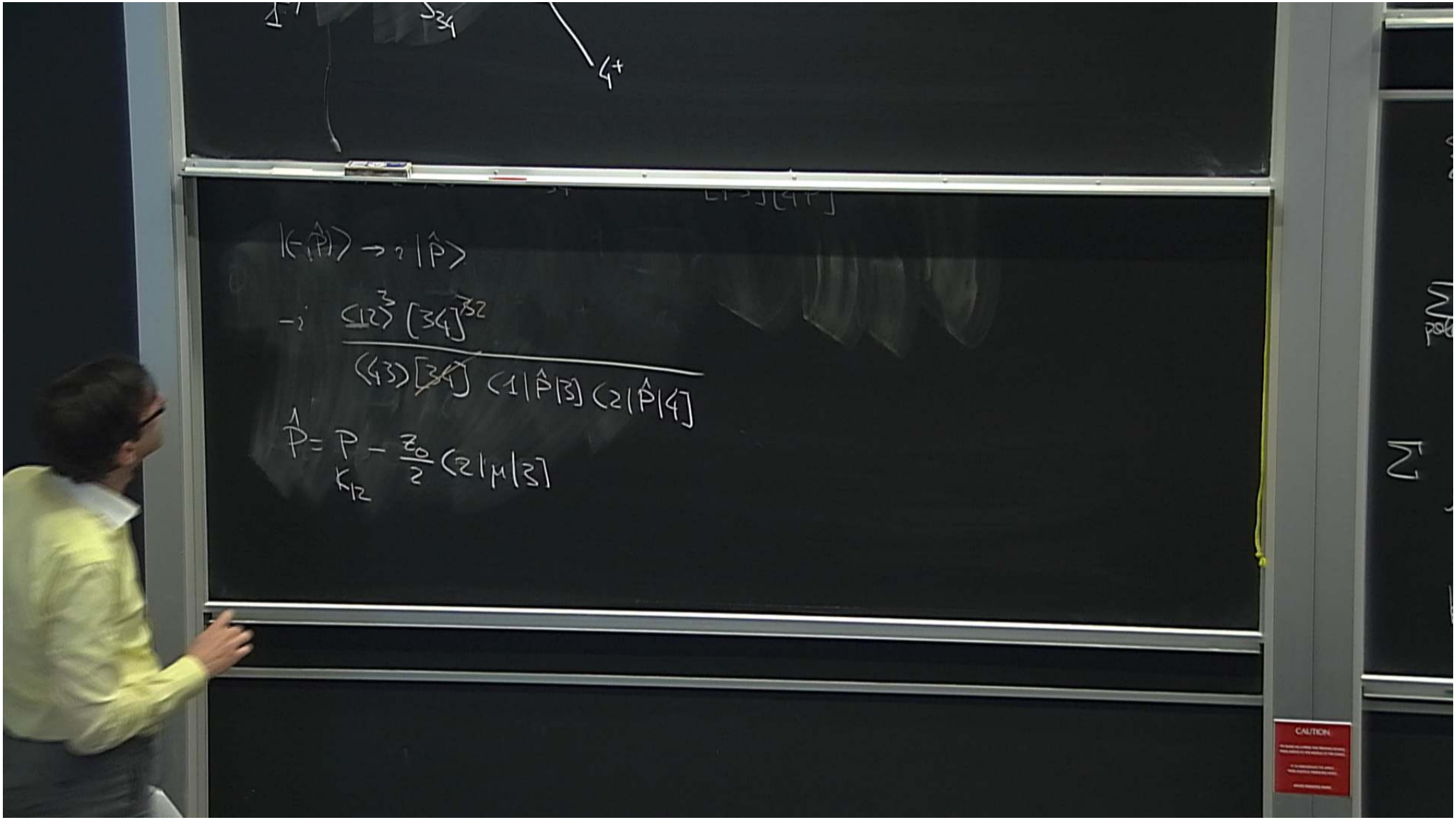
$|2] \to |2] - z|3]$ $\quad |2\rangle \to |2\rangle$

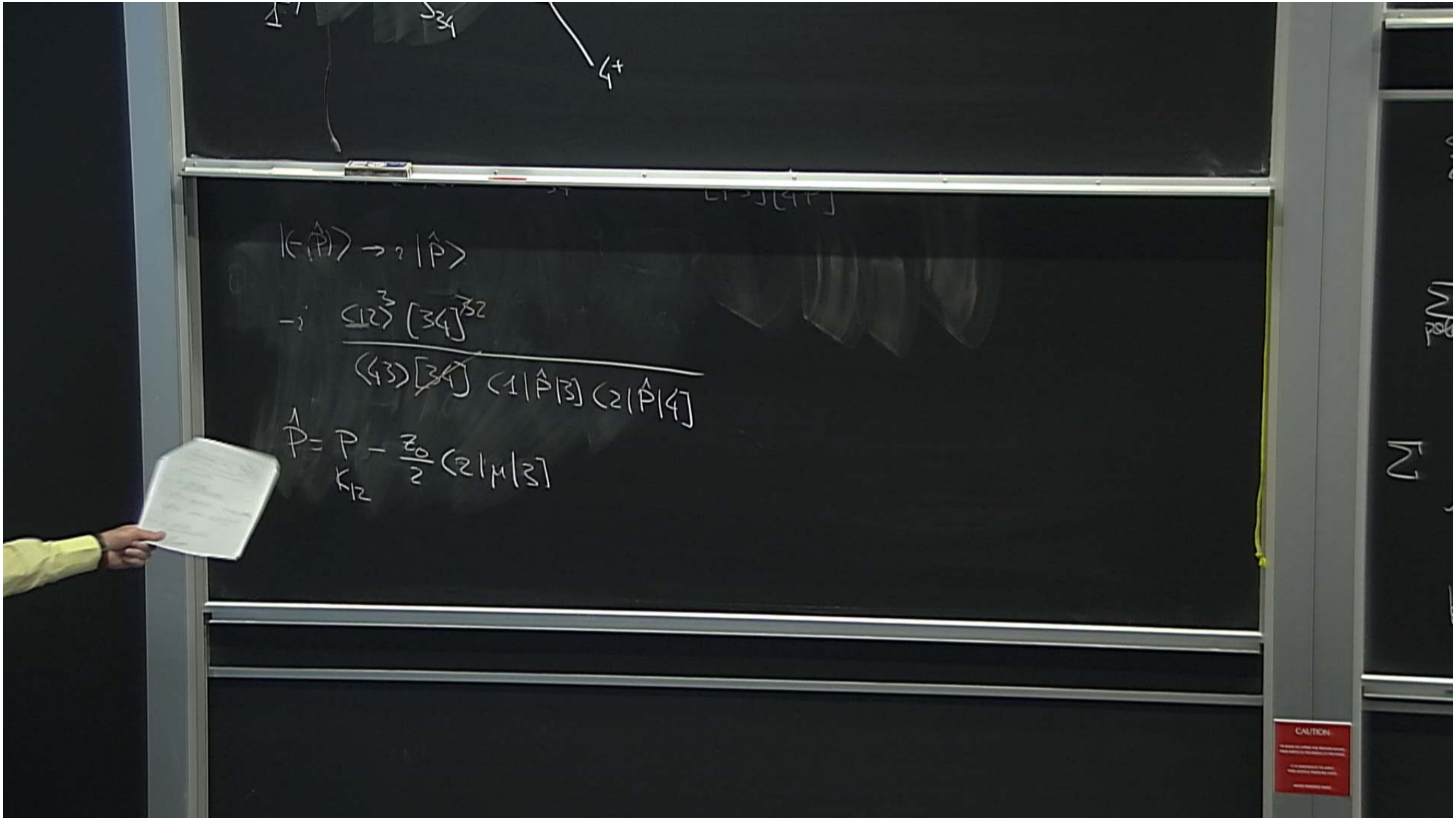
$|3\rangle \to |3\rangle + z|2\rangle$ $\quad |3] \to |3]$

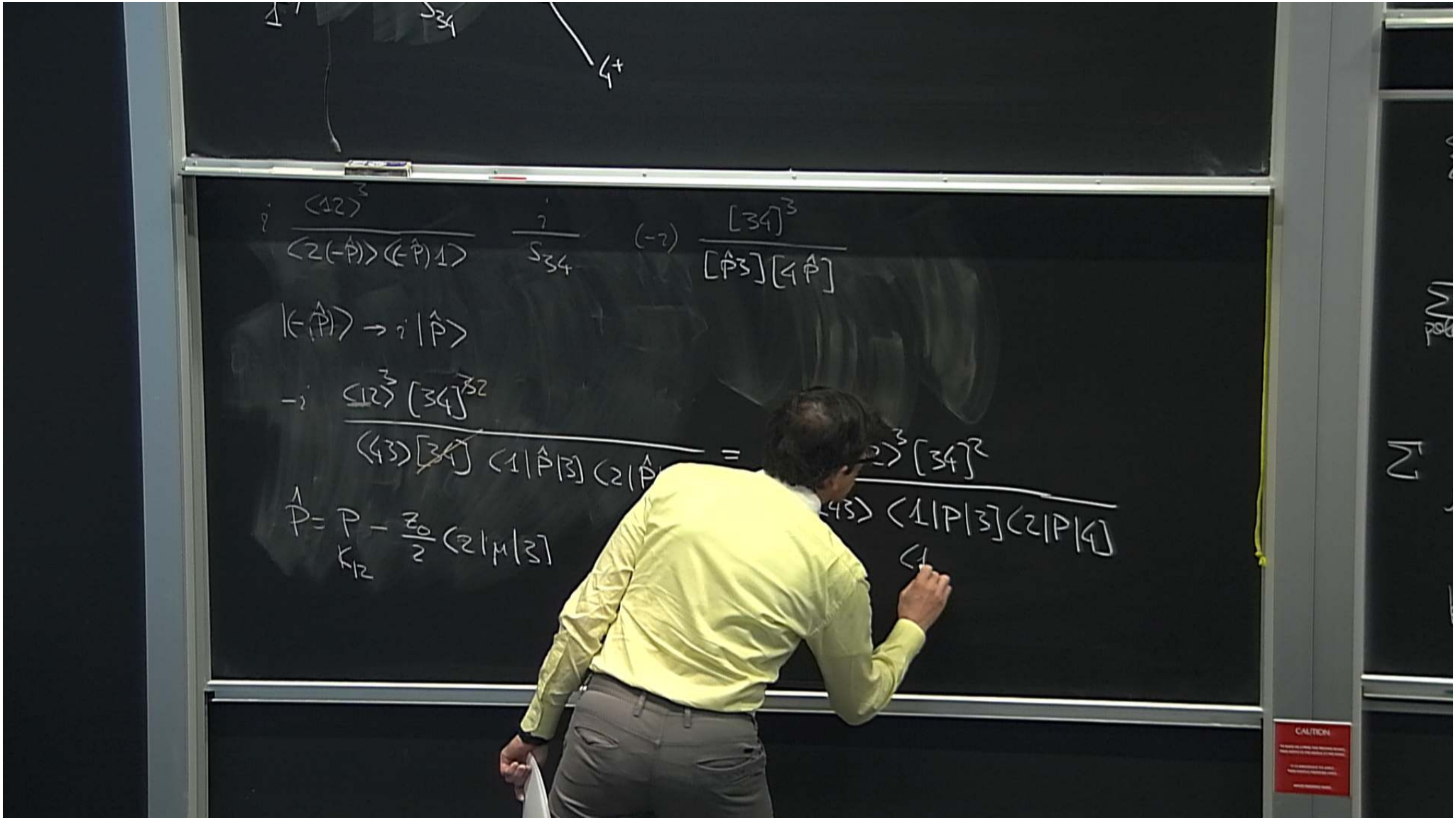
$\{1,2,3,4\}$: $\{1,2\} \cup \{3,4\}$

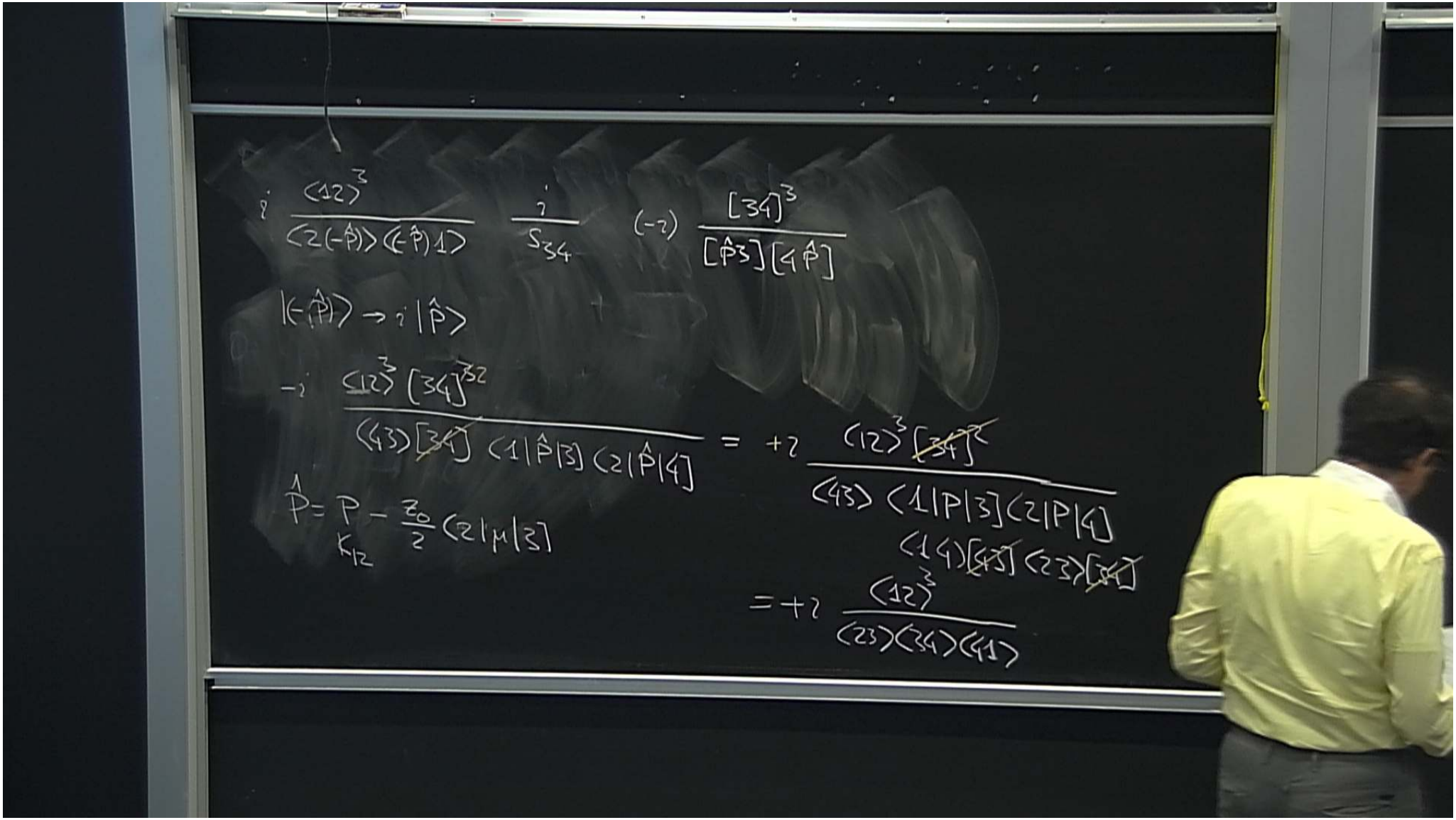


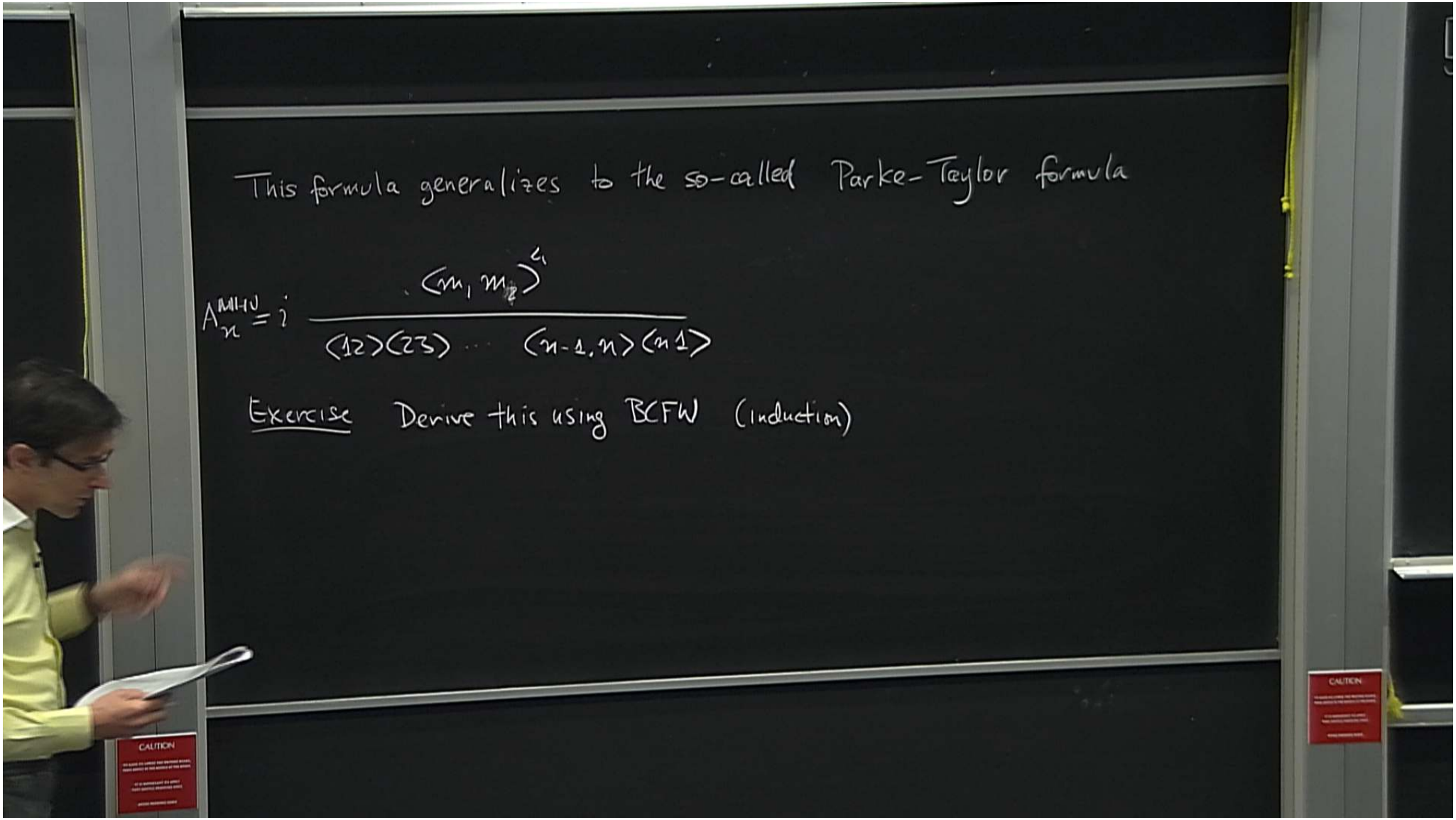

$\frac{\langle 12\rangle^3}{\langle 2(-\hat{P})\rangle\langle(-\hat{P})1\rangle}\ \frac{i}{s_{34}}\ (-i)\ \frac{[34]^3}{[\hat{P}3][4\hat{P}]}$

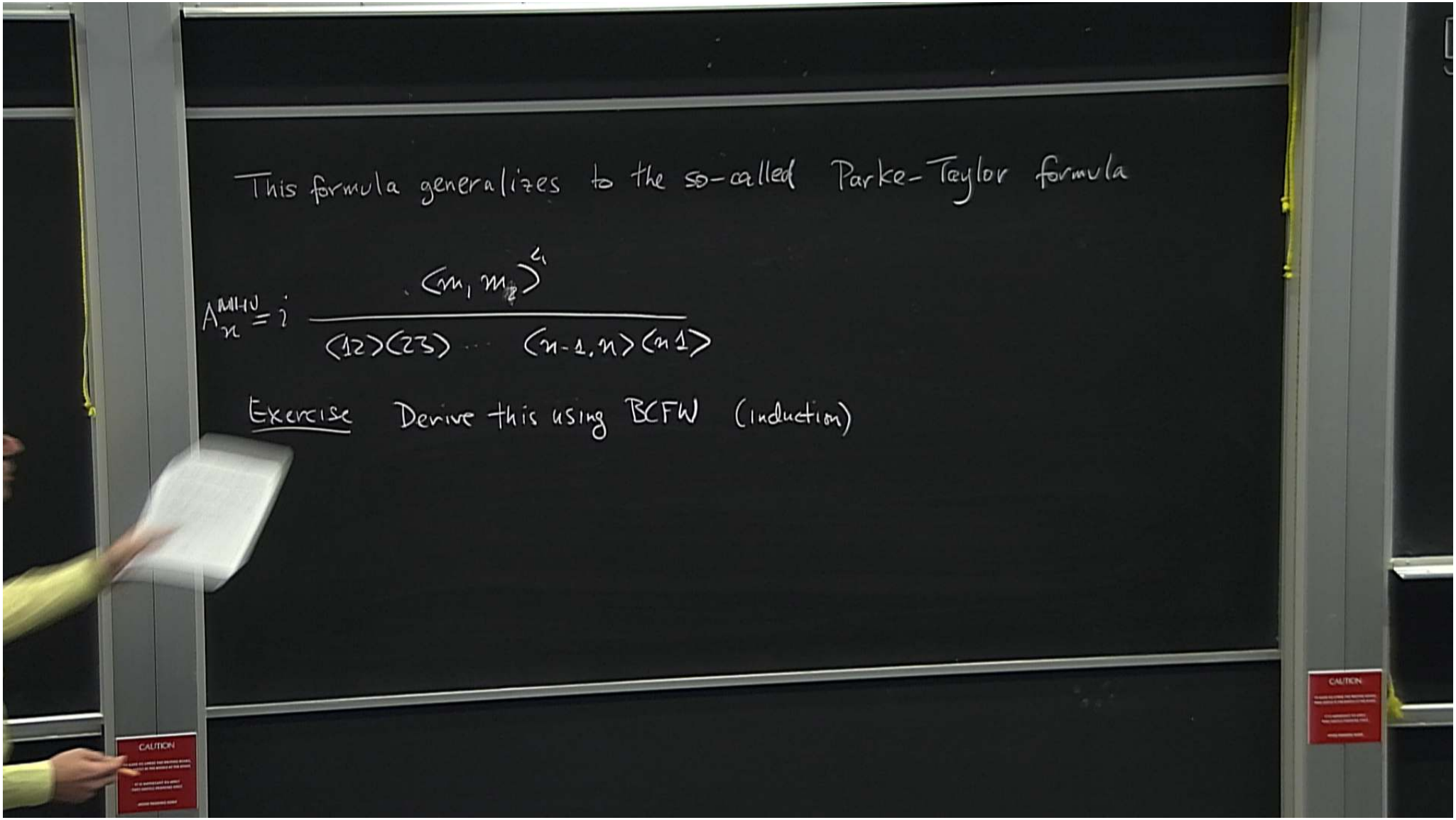

$|(-,\hat{P})\rangle \to i|\hat{P}\rangle$
$-i \frac{\langle 12\rangle^3 [34]^{32}}{\langle 43\rangle \cancel{[34]} \langle 1|\hat{P}|3] \langle 2|\hat{P}|4]}$
$\hat{P} = P - \frac{z_0}{2} \langle 2|\mu|3]$
K_{12}
4^+
CAUTION


$|(-,\hat{P})\rangle \to i|\hat{P}\rangle$
$-i \dfrac{\langle 12\rangle^3 [34]^{32}}{\langle 43\rangle \cancel{[34]} \langle 1|\hat{P}|3] \langle 2|\hat{P}|4]}$
$\hat{P} = P - \frac{z_0}{2} \langle 2|\mu|3]$
K_{12}
4^+


$\frac{\langle 12\rangle^3}{\langle 2(-\hat{P})\rangle\langle(-\hat{P})1\rangle}$ $\frac{i}{s_{34}}$ $(-i)\frac{[34]^3}{[\hat{P}3][4\hat{P}]}$
$|(-\hat{P})\rangle \to i|\hat{P}\rangle$
$-i\frac{\langle 12\rangle^3[34]^{32}}{\langle 43\rangle[34]\langle 1|\hat{P}|3]\langle 2|\hat{P}}$ =
$\hat{P} = P - \frac{z_0}{2}\langle 2|\mu|3]$
K_{12}
$\langle 43\rangle\langle 1|P|3]\langle 2|P|4]$
CAUTION


$i \frac{\langle 12\rangle^3}{\langle 2(-\hat{P})\rangle\langle(-\hat{P})1\rangle} \frac{i}{s_{34}} (-i) \frac{[34]^3}{[\hat{P}3][4\hat{P}]}$
$|(-\hat{P})\rangle \to i|\hat{P}\rangle$
$-i \frac{\langle 12\rangle^3 [34]^{32}}{\langle 43\rangle [34] \langle 1|\hat{P}|3\rangle \langle 2|\hat{P}|4\rangle} = +i \frac{\langle 12\rangle^3 [34]^2}{\langle 43\rangle \langle 1|P|3\rangle \langle 2|P|4\rangle}$
$\hat{P} = P - \frac{z_0}{2}\langle 2|\mu|3]$
$\langle 14\rangle[43]\langle 23\rangle[34]$
$= +i \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}$

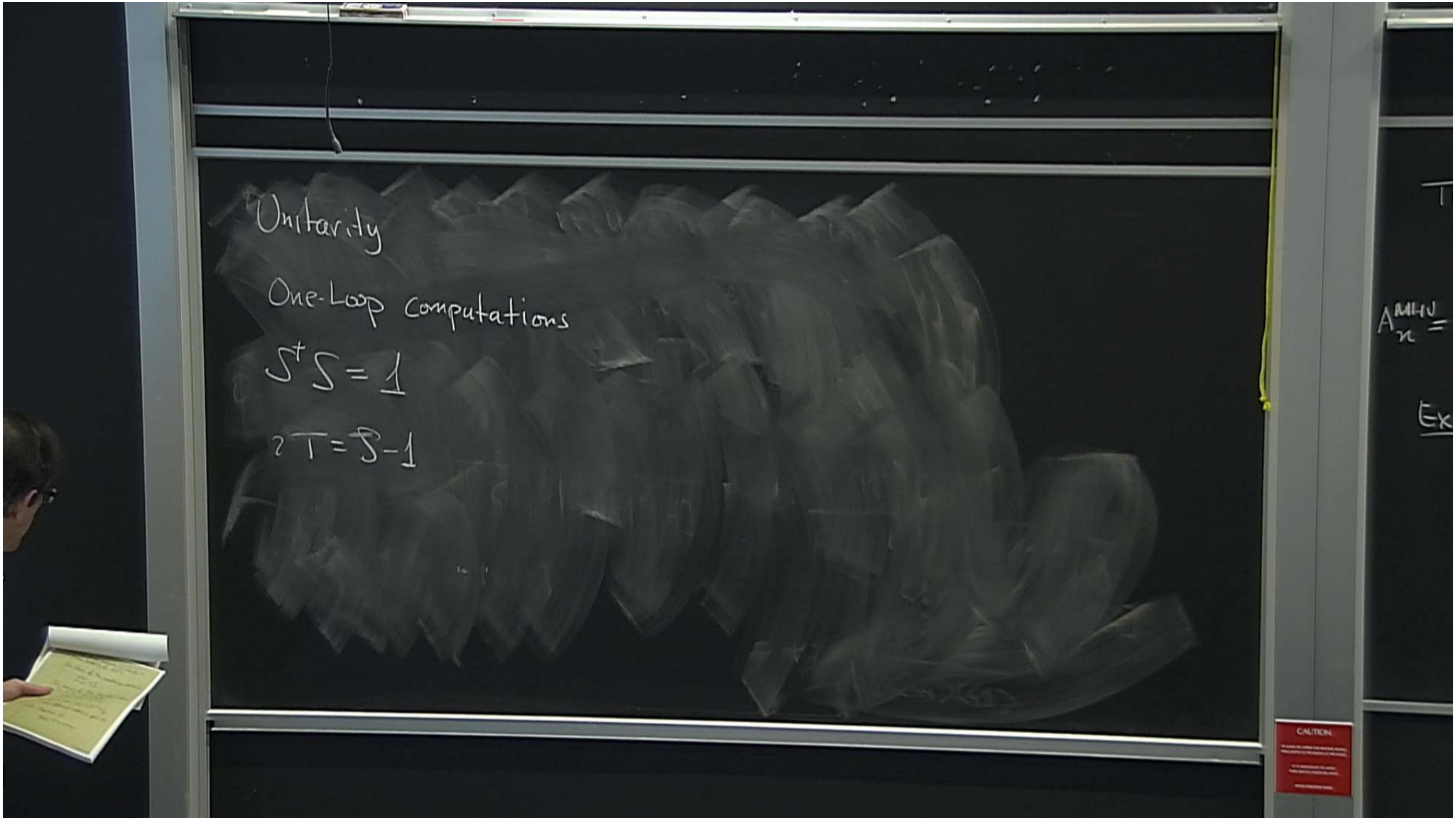

This formula generalizes to the so-called Parke-Taylor formula
$A_n^{MHV} = i \dfrac{\langle n_1 n_2 \rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n-1, n\rangle\langle n1\rangle}$
Exercise Derive this using BCFW (induction)
CAUTION
CAUTION

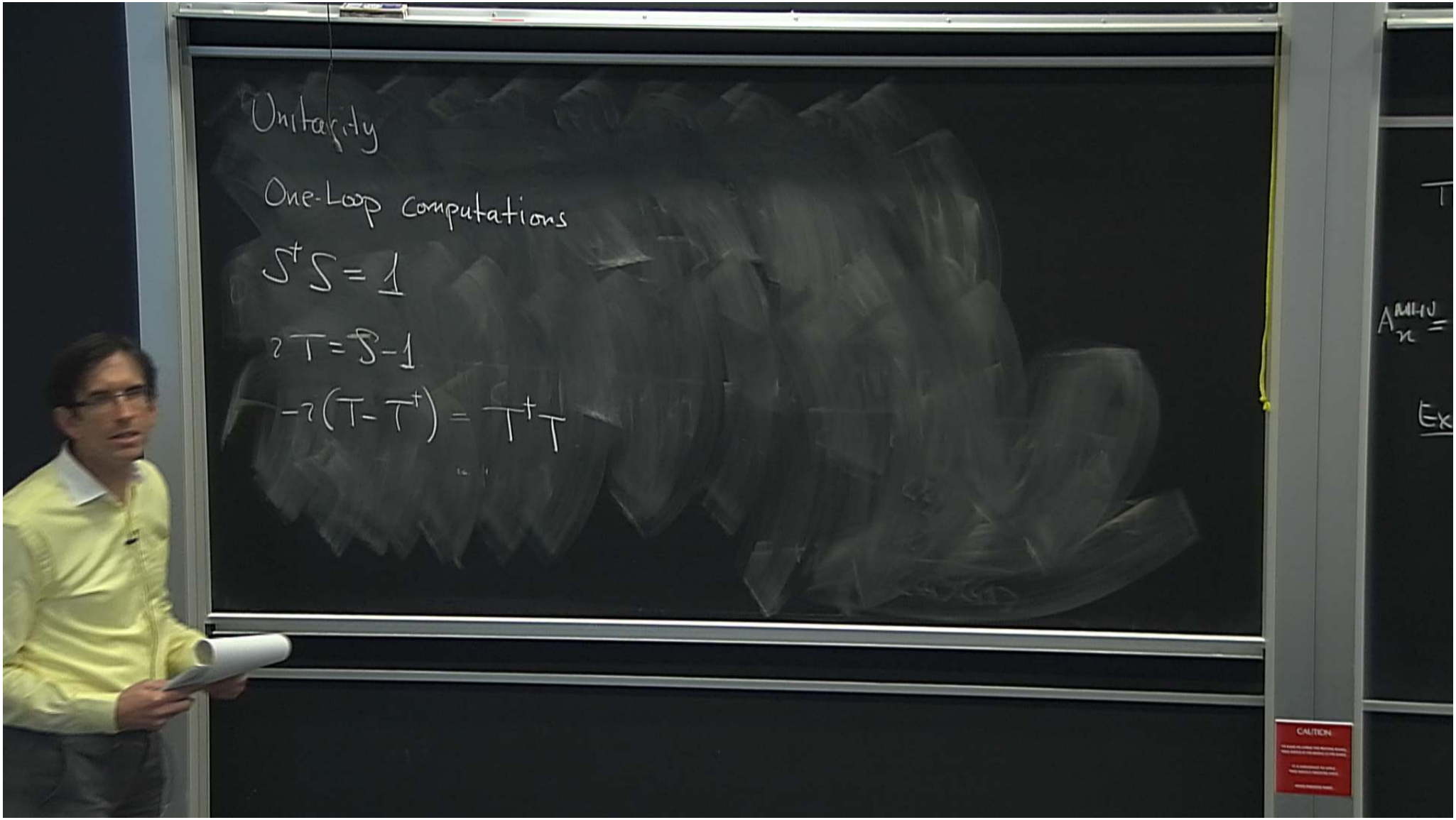

This formula generalizes to the so-called Parke-Taylor formula
$A_n^{MHV} = i \dfrac{\langle n_1 n_2 \rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n-1, n\rangle\langle n1\rangle}$
Exercise Derive this using BCFW (induction)
CAUTION
CAUTION

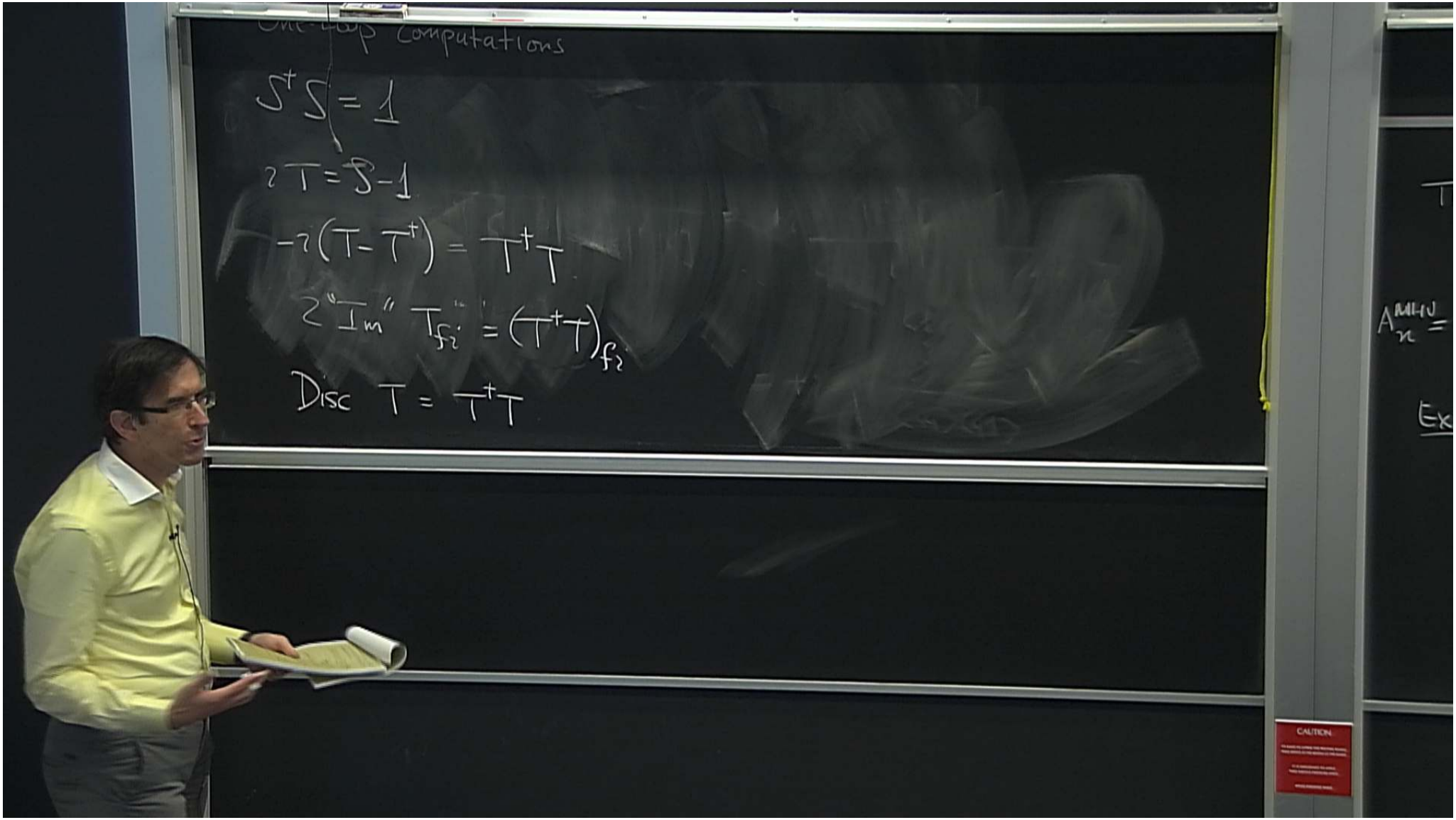
This formula generalizes to the so-called Parke-Taylor formula

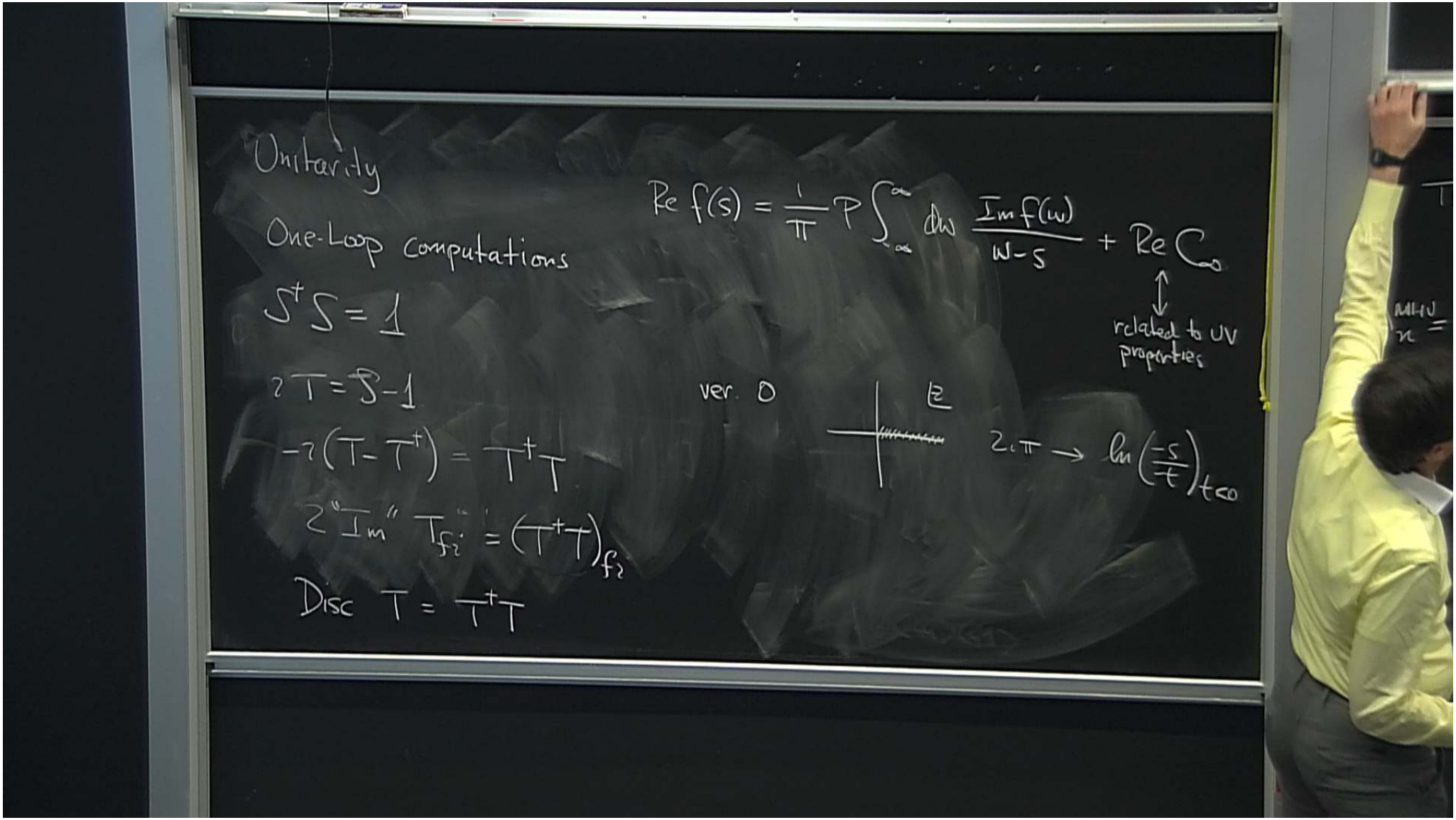
$$A_n^{MHV} = i \frac{\langle m_1 m_2 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n-1, n \rangle \langle n 1 \rangle} \qquad (1^+ \cdots m_1^- \cdots m_2^- \cdots n^+)$$

Exercise Derive this using BCFW (induction)


Unitarity
One-Loop Computations
$S^\dagger S = 1$
$? \; T = S - 1$
CAUTION


Unitarity
One-Loop Computations
$S^\dagger S = 1$
$2T = S - 1$
$-2(T - T^\dagger) = T^\dagger T$
CAUTION

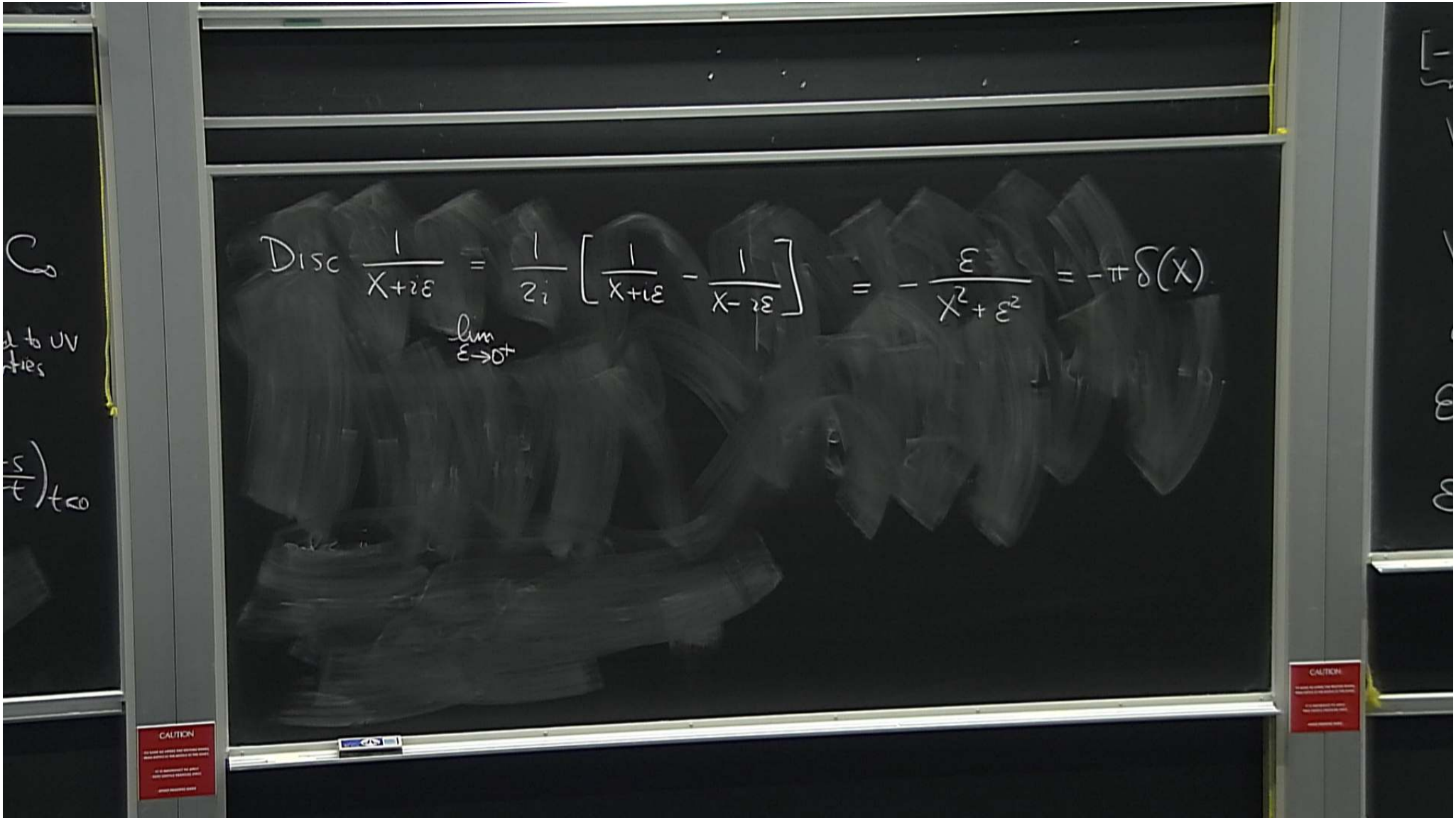

Computations
$S^\dagger S = 1$
$2T = S - 1$
$-2(T - T^\dagger) = T^\dagger T$
$2\,\text{"Im"}\,T_{fi} = (T^\dagger T)_{fi}$
Disc $T = T^\dagger T$
CAUTION


Unitarity
One-Loop computations
$S^\dagger S = 1$
$2T = S - 1$
$-2(T - T^\dagger) = T^\dagger T$
$2\,\text{"Im"}\,T_{fi} = (T^\dagger T)_{fi}$
Disc $T = T^\dagger T$
$\text{Re}\,f(s) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dw \frac{\text{Im}\,f(w)}{w - s} + \text{Re}\,C_\infty$
related to UV properties
ver. 0
$2i\pi \to \ln\left(\frac{-s}{t}\right)_{t<0}$

This formula generalizes to the so-called Parke-Taylor formula

$$\mathrm{Disc}\ \frac{1}{X+i\varepsilon} = \frac{1}{2i}\left[\frac{1}{X+i\varepsilon} - \frac{1}{X-i\varepsilon}\right] = -\frac{\varepsilon}{X^2+\varepsilon^2}$$

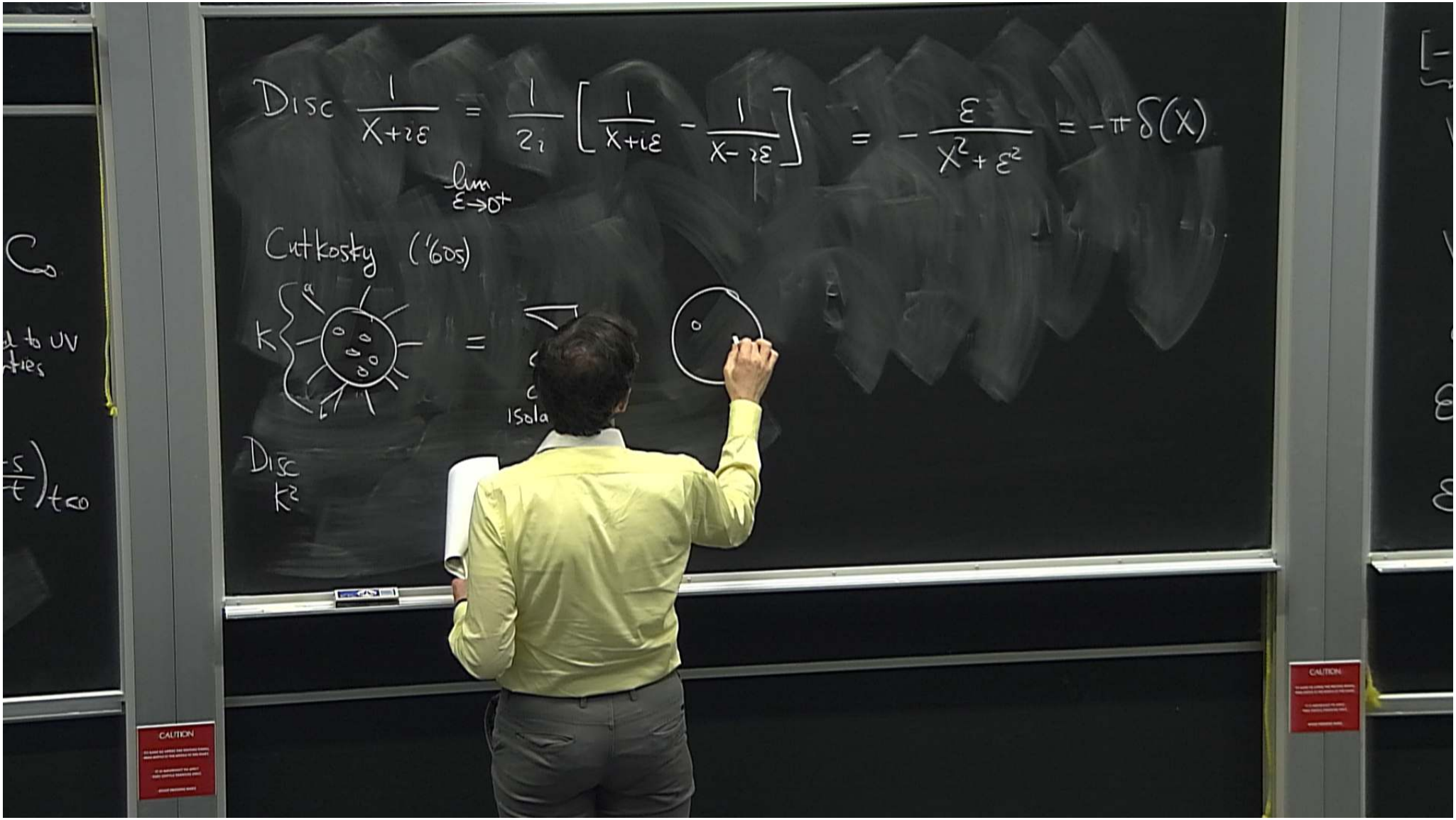
$\lim_{\varepsilon \to 0^+}$


$\mathrm{Disc}\ \frac{1}{x+i\varepsilon} = \frac{1}{2i}\left[\frac{1}{x+i\varepsilon} - \frac{1}{x-i\varepsilon}\right] = -\frac{\varepsilon}{x^2+\varepsilon^2} = -\pi\delta(x)$
$\lim_{\varepsilon\to 0^+}$

$$\text{Disc}\ \frac{1}{x+i\varepsilon} = \frac{1}{2i}\left[\frac{1}{x+i\varepsilon} - \frac{1}{x-i\varepsilon}\right] = -\frac{\varepsilon}{x^2+\varepsilon^2} = -\pi\delta(x)$$

$\lim_{\varepsilon \to 0^+}$

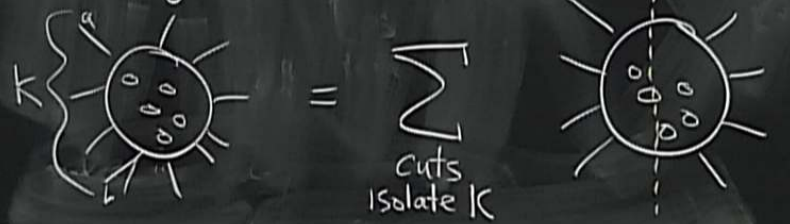
Cutkosky ('60s)


Disc $\frac{1}{x+i\varepsilon} = \frac{1}{2i}\left[\frac{1}{x+i\varepsilon} - \frac{1}{x-i\varepsilon}\right] = -\frac{\varepsilon}{x^2+\varepsilon^2} = -\pi\delta(x)$
$\lim_{\varepsilon\to 0^+}$
Cutkosky ('60s)
Disc
k^2

$$\text{Disc}\ \frac{1}{X+i\varepsilon} = \frac{1}{2i}\left[\frac{1}{X+i\varepsilon} - \frac{1}{X-i\varepsilon}\right] = -\frac{\varepsilon}{X^2+\varepsilon^2} = -\pi\delta(X)$$

$\lim_{\varepsilon\to 0^+}$

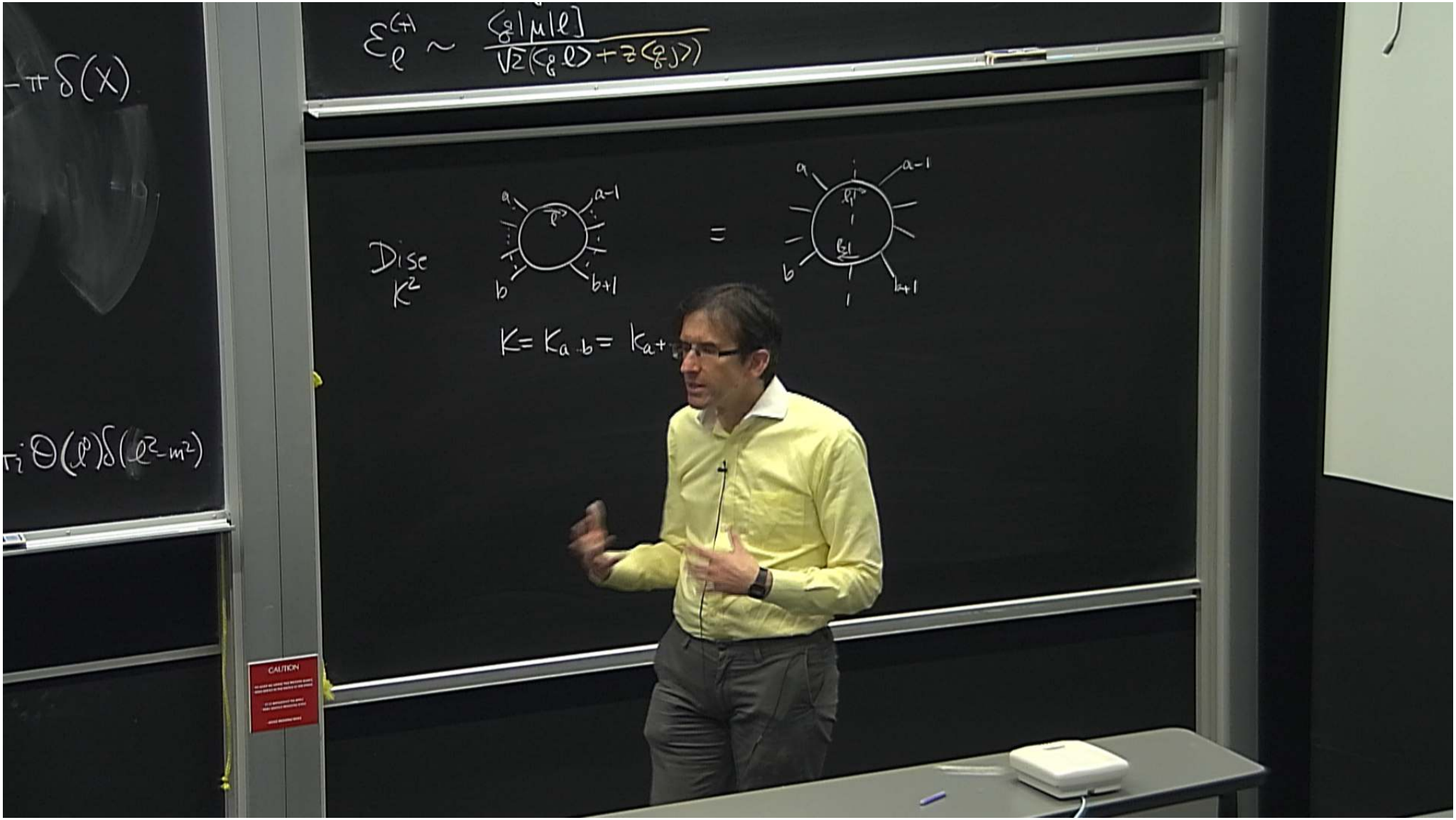
Cutkosky ('60s)

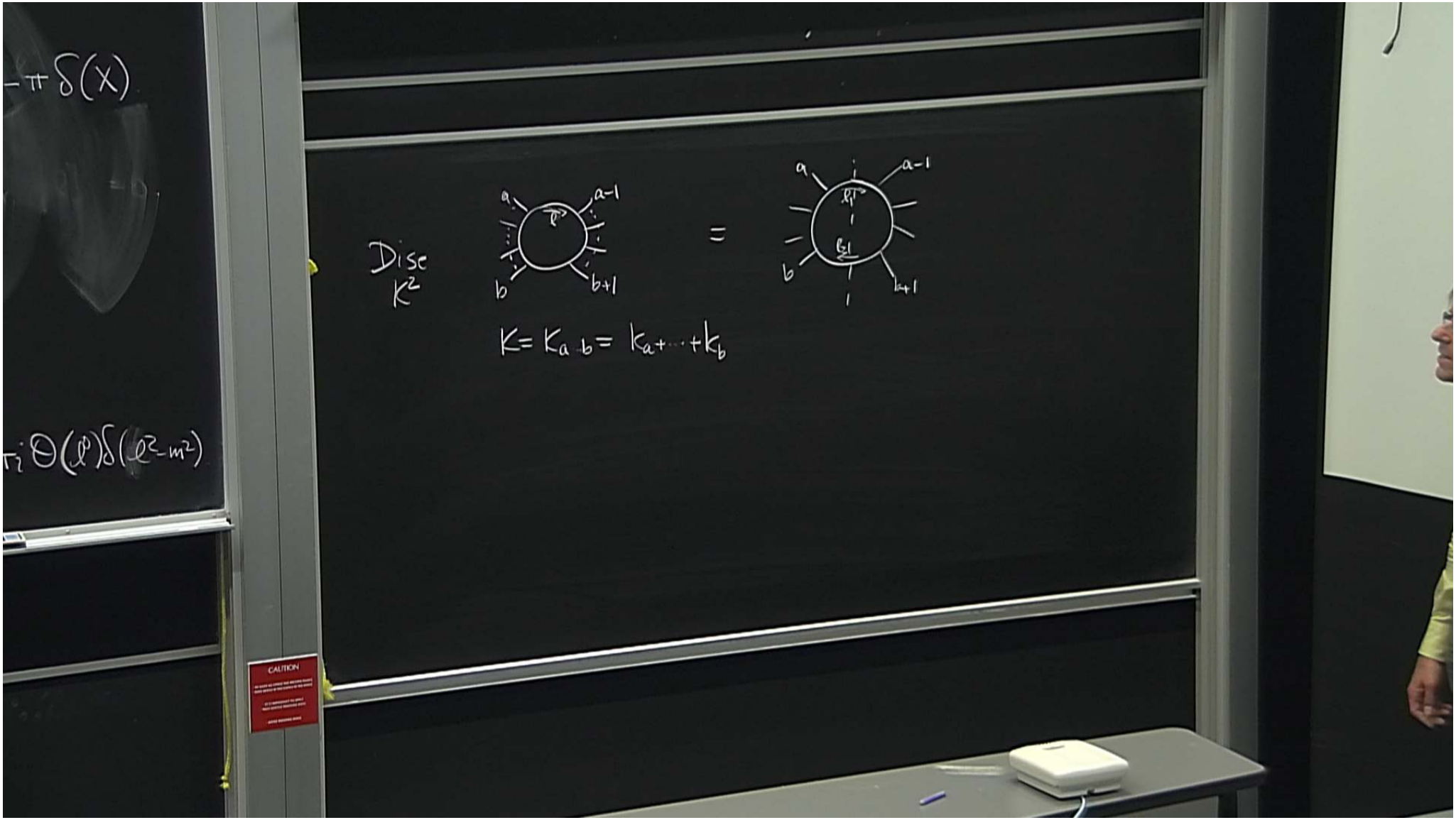


RHS $+i\varepsilon \to -i\varepsilon$, $2 \to -2$

Disc $\frac{1}{k^2}$

$$\frac{1}{\ell^2-m^2+i\varepsilon} = -2\pi i\,\delta^{(+)}(\ell^2-m^2) = -2\pi i\,\theta(\ell^0)\,\delta(\ell^2-m^2)$$


Dise
K²
a
a-1
b
b+1
=
K= Ka b= Ka+
CAUTION


Dise
K^2
a
a-1
b
b+1
=
$K = K_{a\ldots b} = k_a + \cdots + k_b$