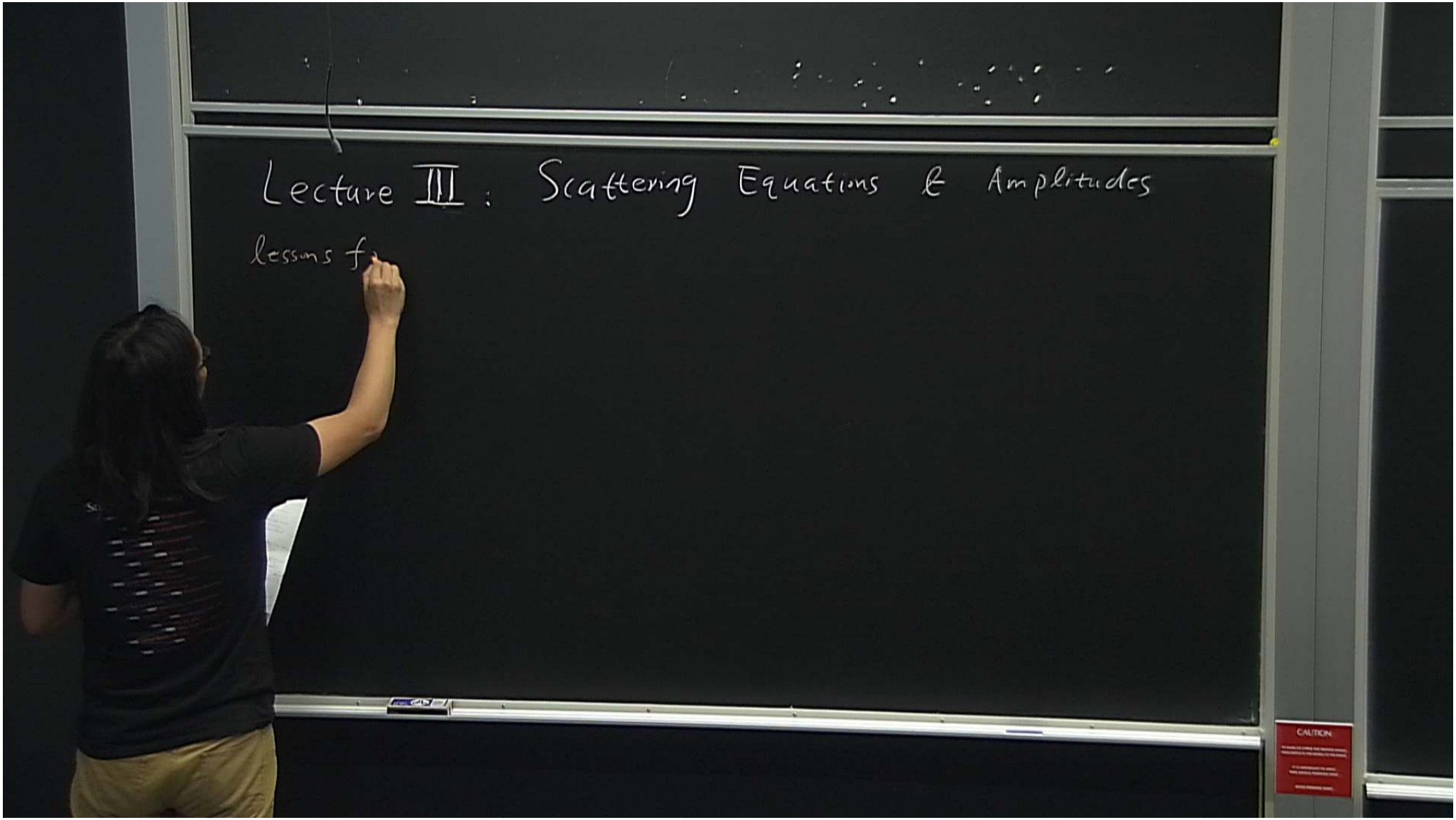
Title: Amplitudes: Reformulation of Feynman diagrams

Date: Jul 15, 2015 09:00 AM

URL: http://pirsa.org/15070047

Abstract:


Lecture III : Scattering Equations & Amplitudes
lessons f
CAUTION

Lecture III: Scattering Equations & Amplitudes

lessons from Lecture II: if $\{z_1, z_2, \cdots, z_n\}$ satisfy eqs.

$$E_a = \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{S_{ab}}{z_a - z_b} = 0 \quad (a = 1, 2 \cdots n), \qquad \text{for } b \text{ perm } w$$

$$\boxed{S_{ab} = (k_a + k_b)^2 = 2 k_a \cdot k_b}$$

4D: special sol $\lambda_a^{\alpha} \sim \begin{pmatrix} 1 \\ z_a \end{pmatrix}$

$$\frac{F(\{z\})}{(z_{w(1)} - z_{w(2)})(z_{w(2)} - z_{w(3)}) \cdots (z_{w(n)} - z_{w(1)})}$$

perm inv

with satisfy U(1)-decoupling, BCJ rels.

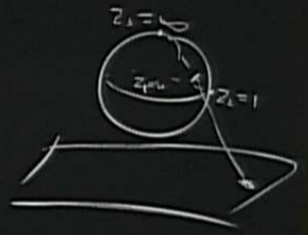
Properties ① data $\{S_{ab}\}$ ⟶ solutions $\{z_1, z_2 \cdots z_n\}$

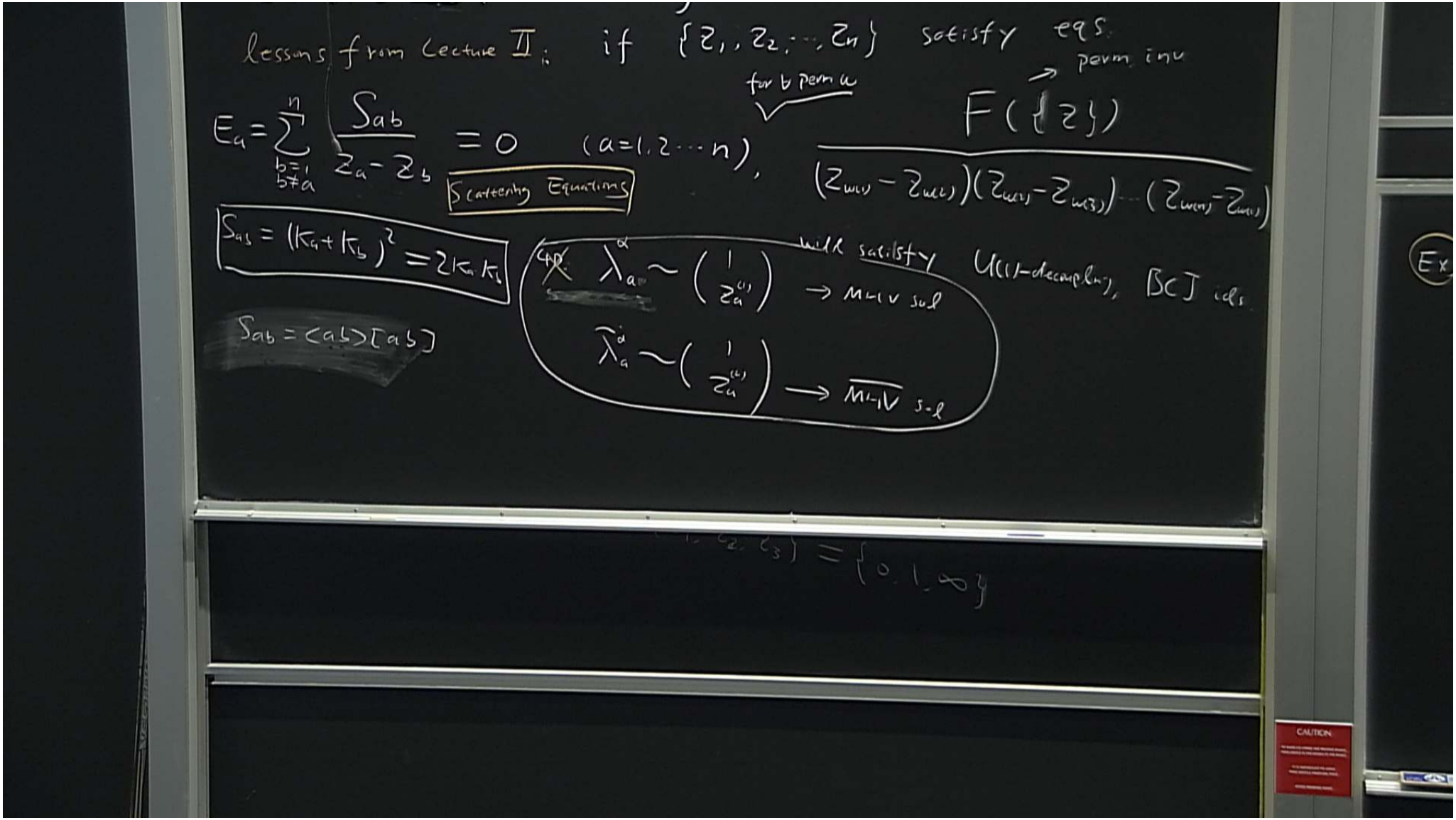
space of kin inv. K_n ⟶ moduli space of n-punctured sphere $\mathcal{M}_{n,0}$

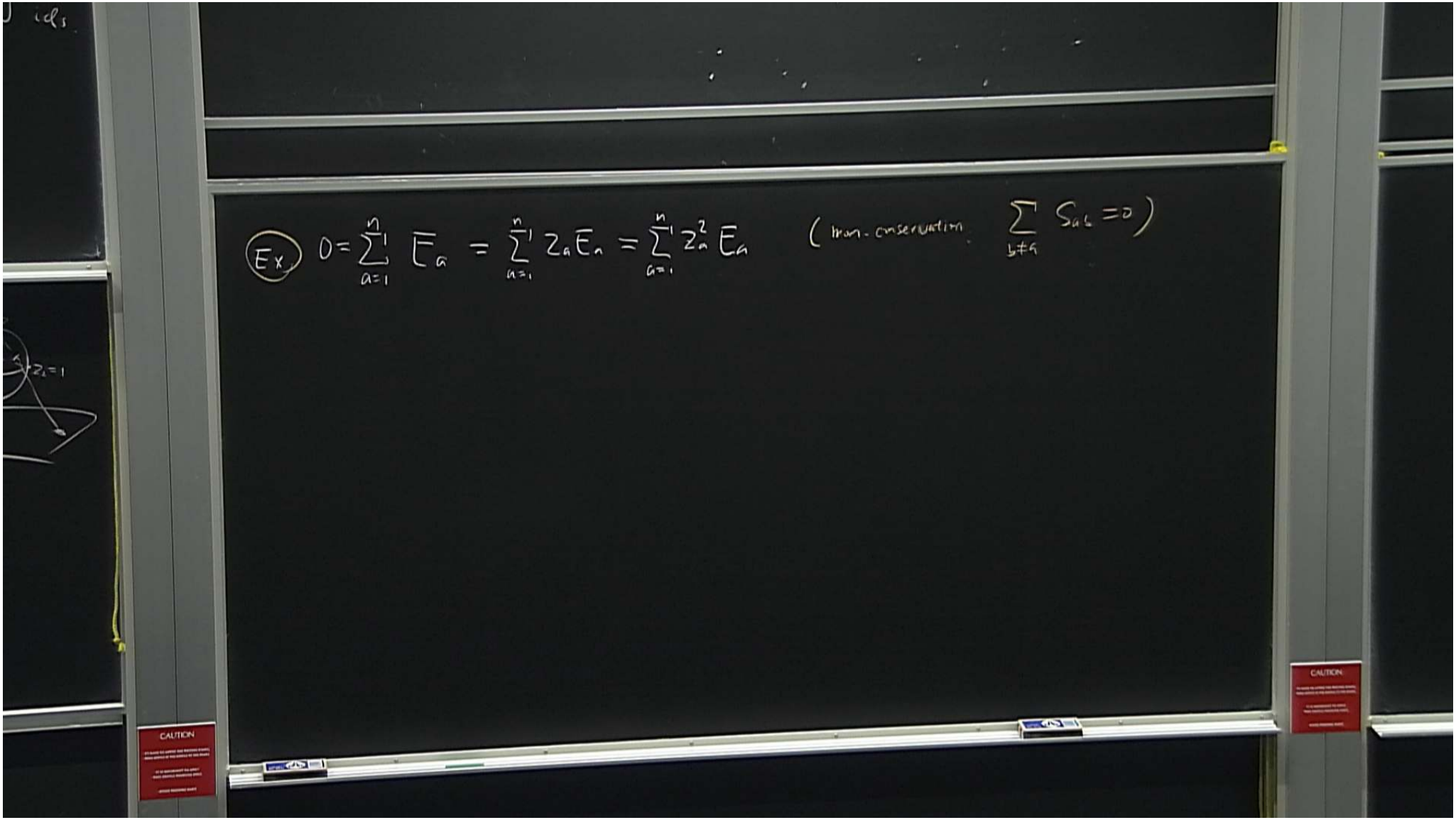
② redundancies: $SL(2,\mathbb{C})$

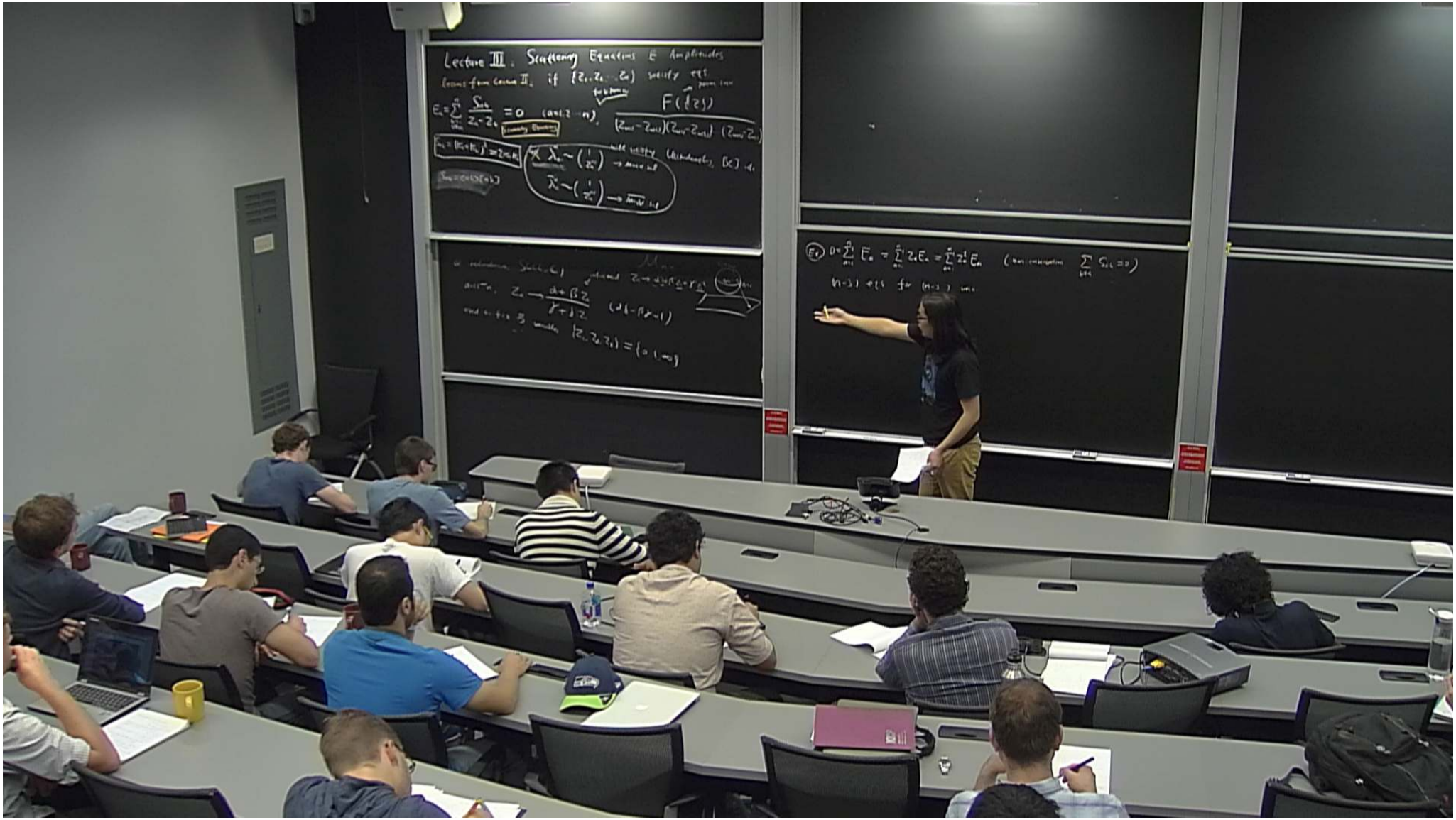
$a = 1 2 \cdots n$, $z_a \to \dfrac{\alpha + \beta z_a}{\gamma + \delta z_a}$ $(\alpha\delta - \beta\gamma = 1)$

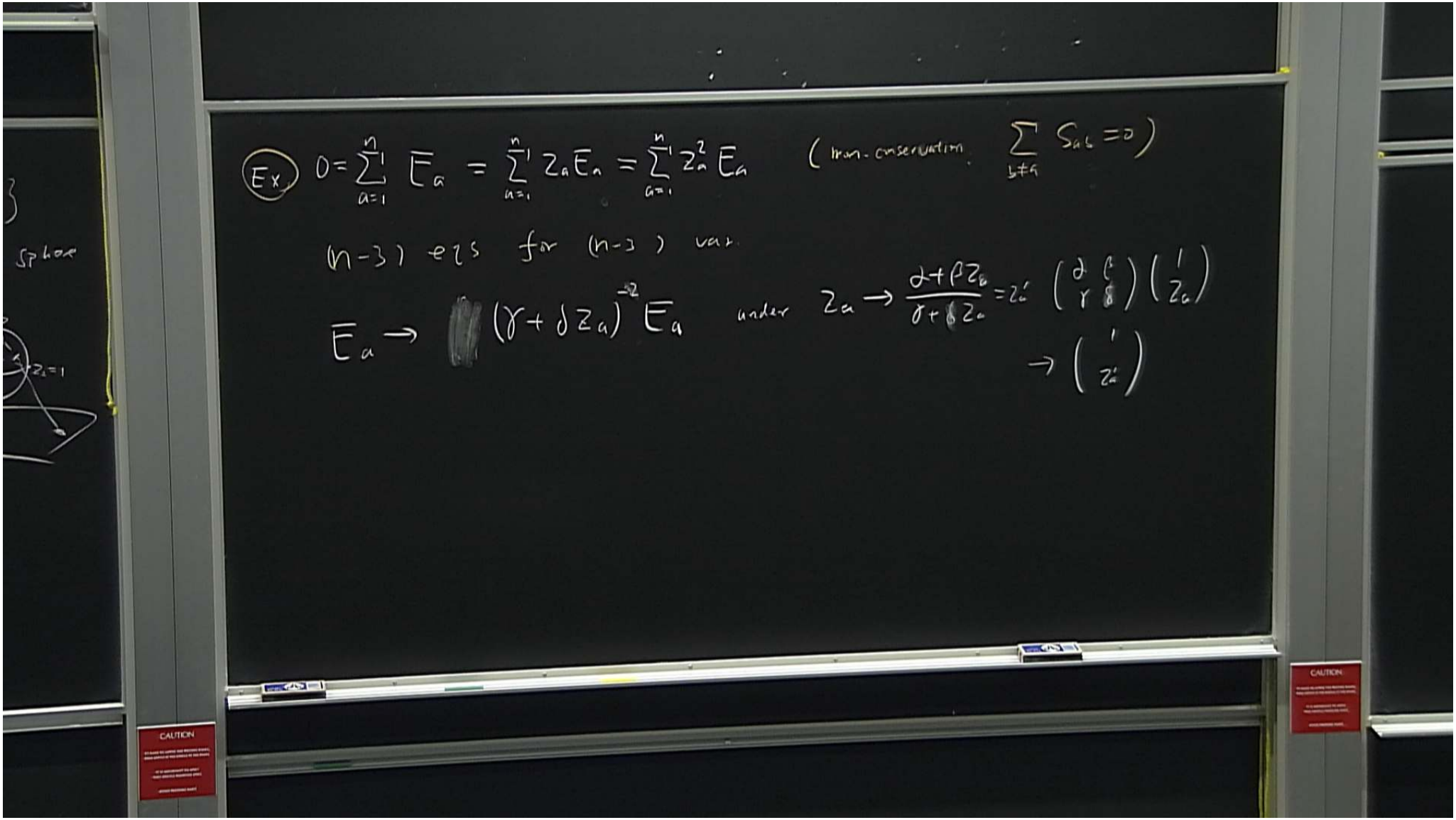
used to fix 3 variables $\{z_1, z_2, z_3\} = \{0, 1, \infty\}$

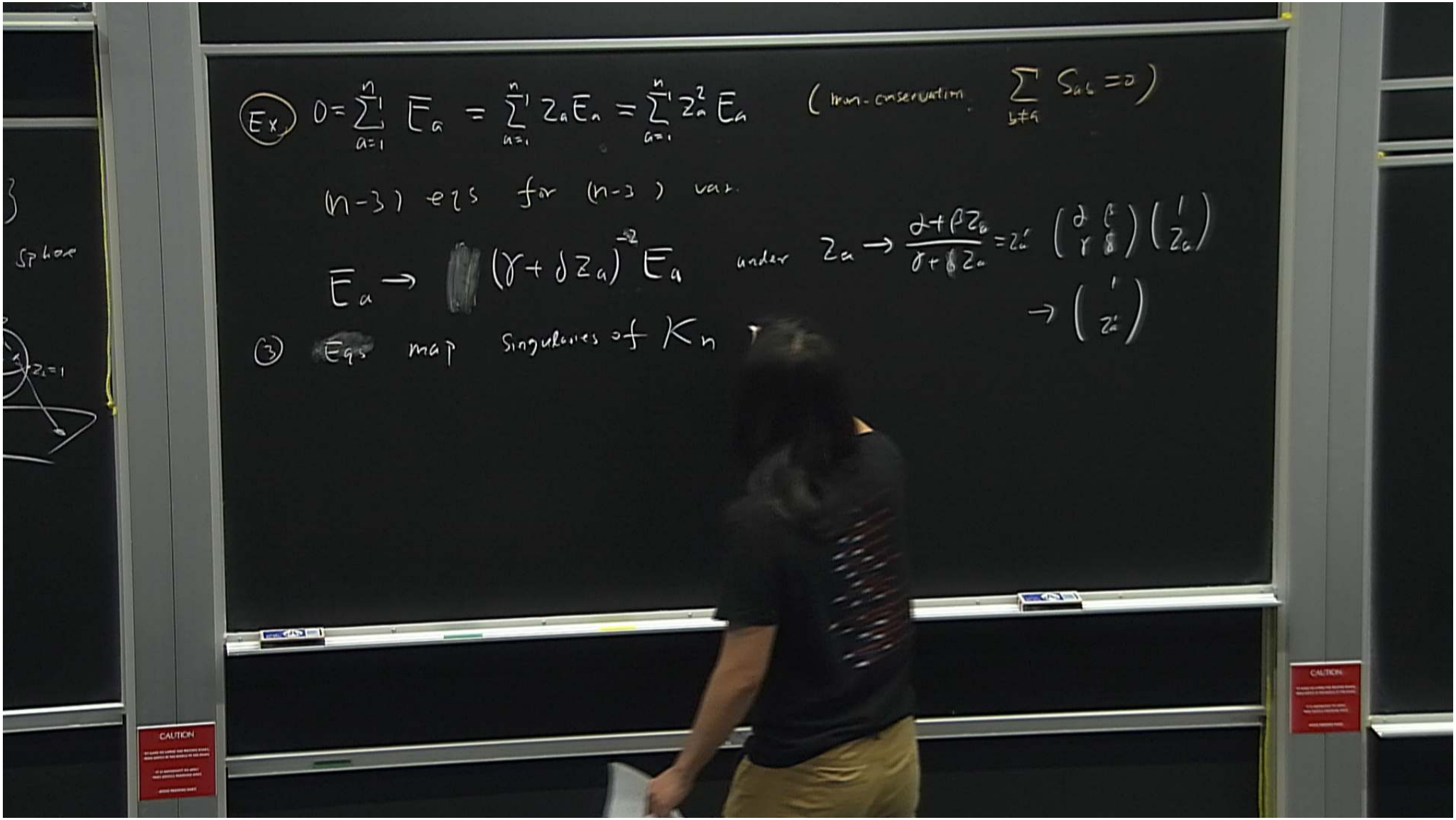


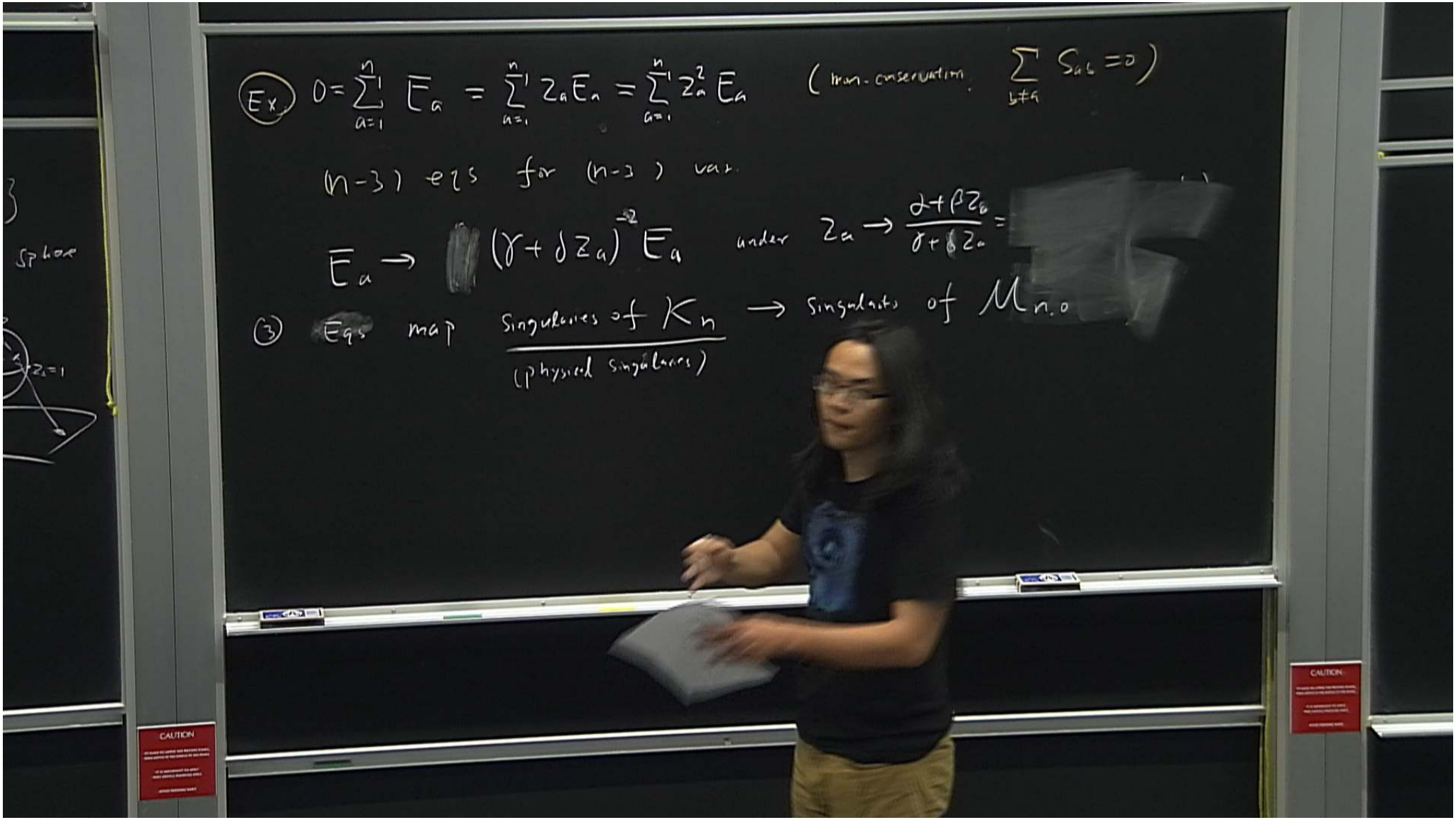

lessons from Lecture II: if {z1, z2, ..., zn} satisfy eqs
for b perm w
perm inv
$E_a = \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{S_{ab}}{z_a - z_b} = 0 \quad (a = 1, 2 \cdots n)$, Scattering Equations
$F(\{z\})$
$(z_{w(1)} - z_{w(2)})(z_{w(2)} - z_{w(3)}) \cdots (z_{w(n)} - z_{w(1)})$
$S_{ab} = (k_a + k_b)^2 = 2 k_a \cdot k_b$
$S_{ab} = \langle ab \rangle [ab]$
will satisfy U(1)-decoupln, BCJ rels
$\lambda_a^\alpha \sim \begin{pmatrix} 1 \\ z_a^{(1)} \end{pmatrix}$ → M-1 sol
$\tilde{\lambda}_a^{\dot{\alpha}} \sim \begin{pmatrix} 1 \\ z_a^{(1)} \end{pmatrix}$ → M-1 s-l
$\{0, 1, \infty\}$
CAUTION

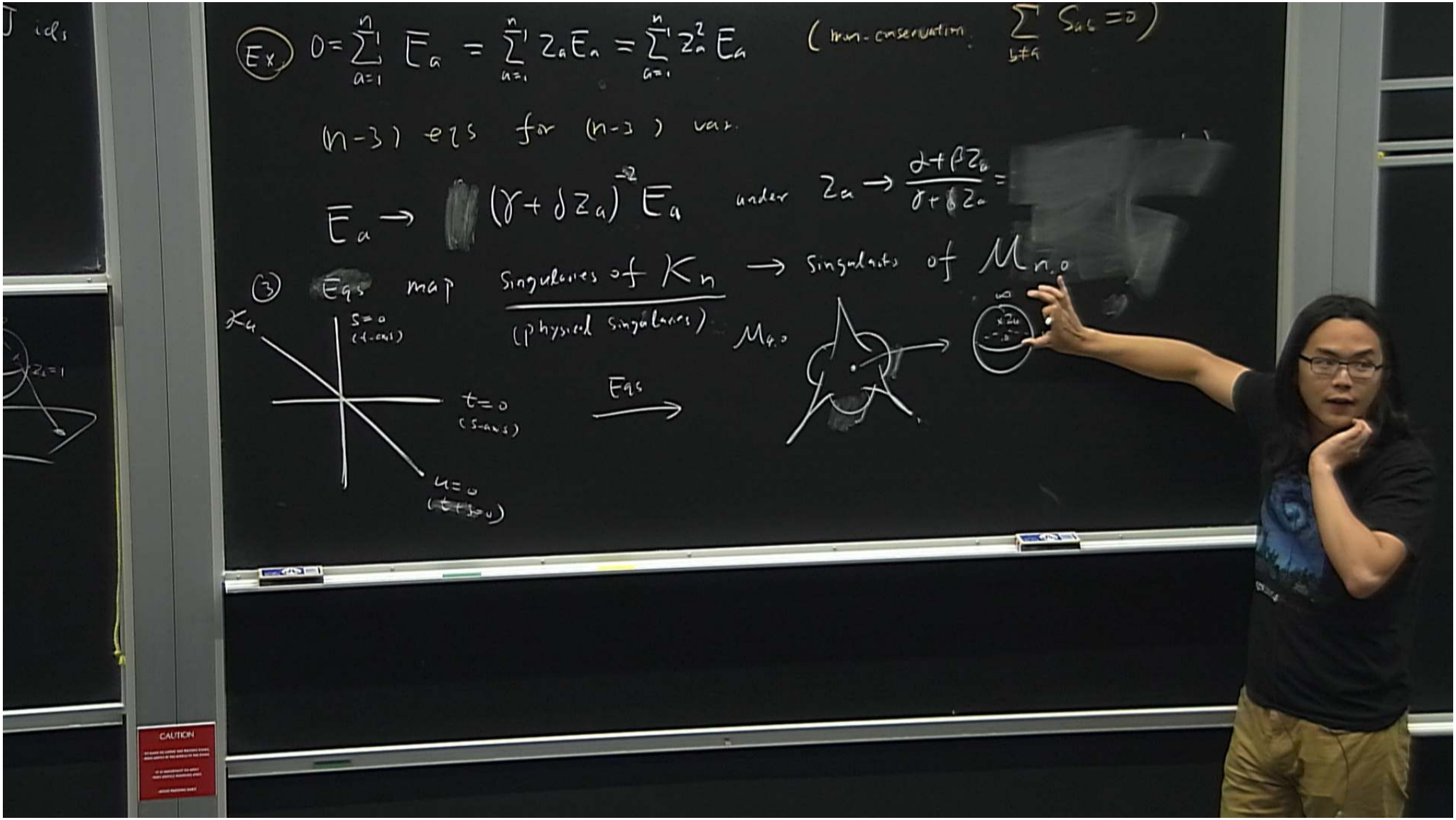

Ex.
$0 = \sum_{a=1}^{n} E_a = \sum_{a=1}^{n} z_a E_a = \sum_{a=1}^{n} z_a^2 E_a$
(trans-conservation $\sum_{b \neq a} S_{ab} = 0$)
$z_i = 1$
CAUTION
CAUTION

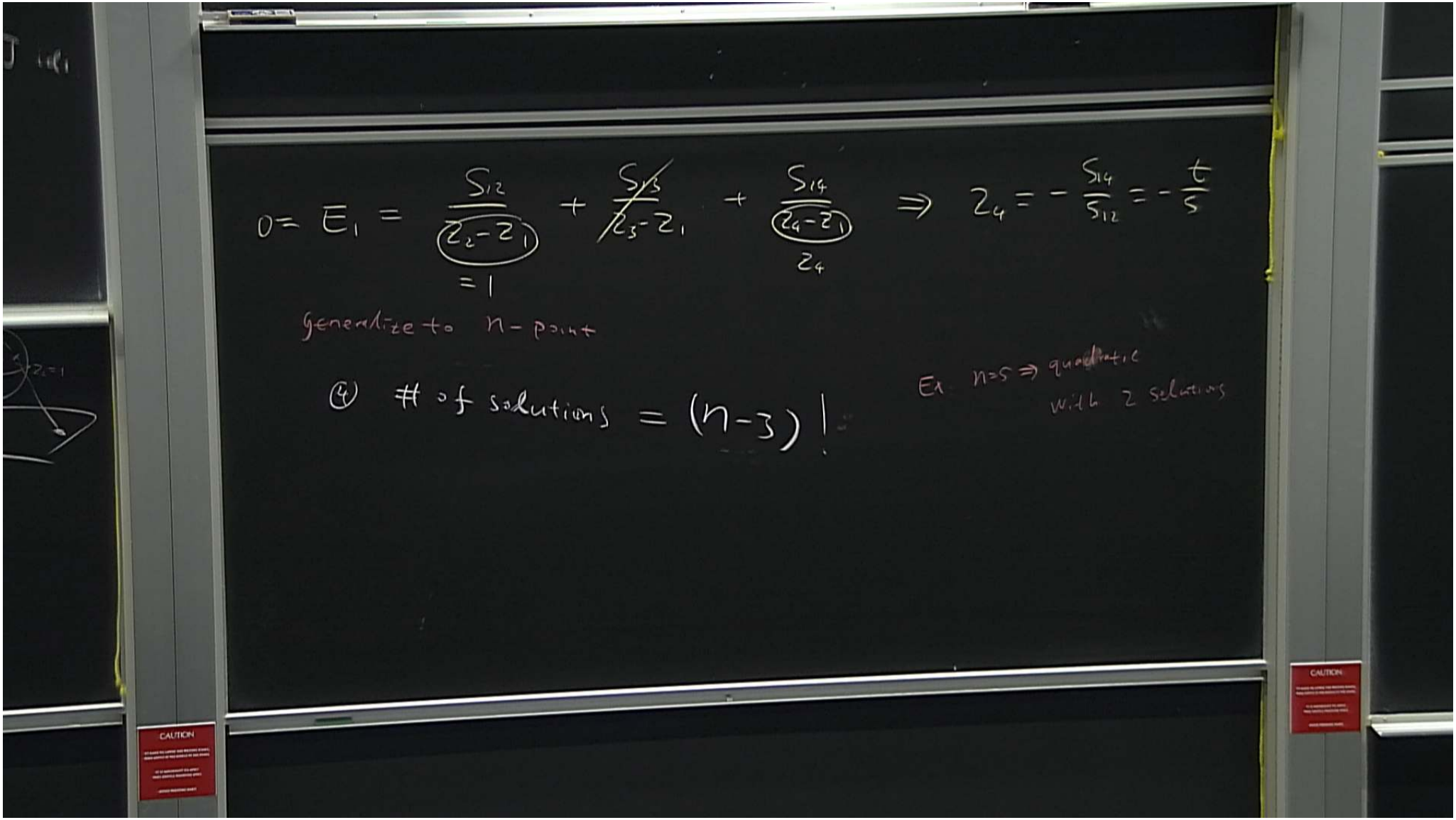

Lecture III. Scattering Equations & Amplitudes

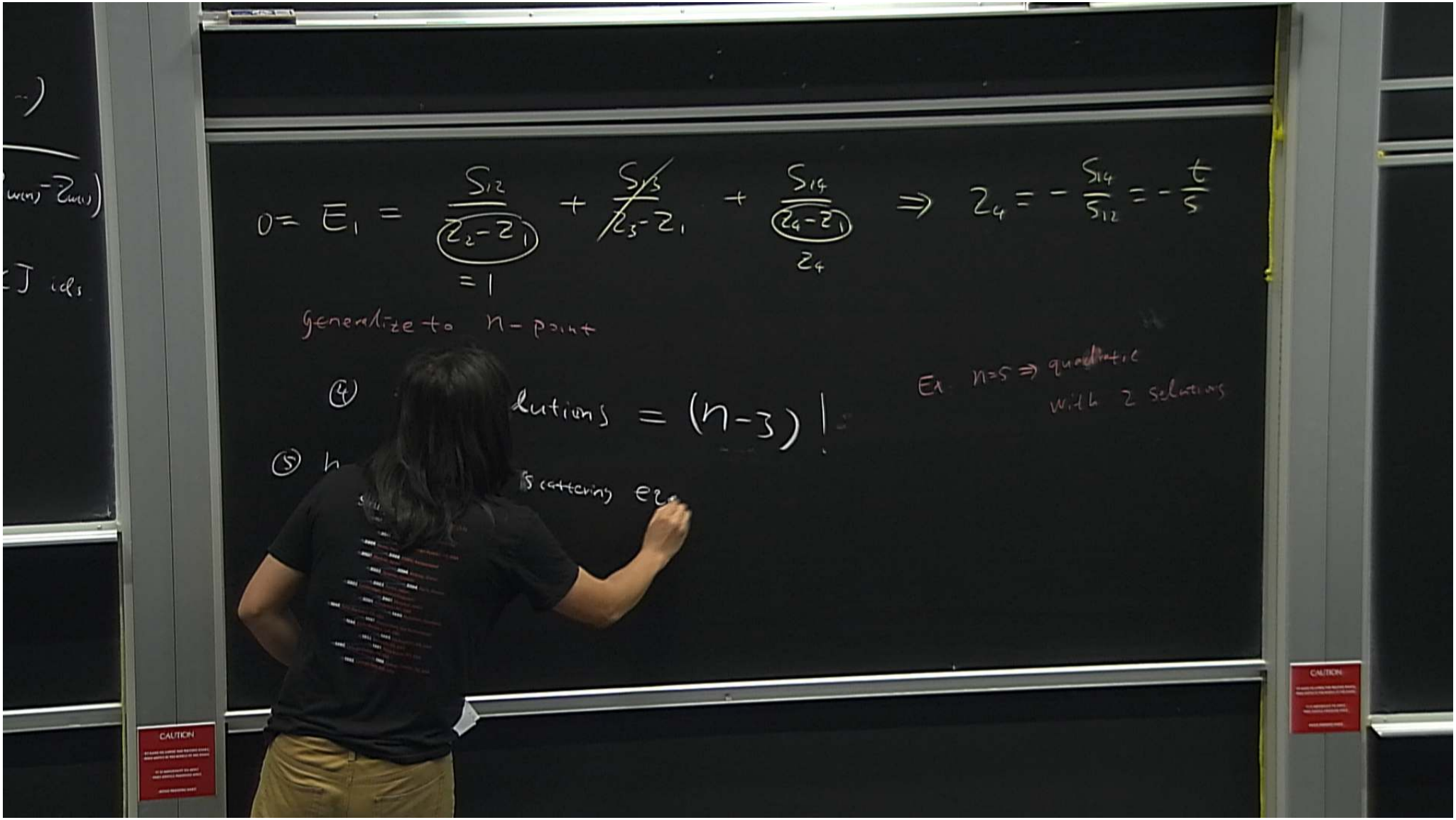

Ex.
$0 = \sum_{a=1}^{n} E_a = \sum_{a=1}^{n} z_a E_a = \sum_{a=1}^{n} z_a^2 E_a$
(mom. conservation $\sum_{b \neq a} S_{ab} = 0$)
$(n-3)$ eqs for $(n-3)$ vars
$E_a \to (\gamma + \delta z_a)^{-2} E_a$ under $z_a \to \frac{\alpha + \beta z_a}{\gamma + \delta z_a} = z'_a$
$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ z_a \end{pmatrix} \to \begin{pmatrix} 1 \\ z'_a \end{pmatrix}$
sphere
$z_1 = 1$
CAUTION
CAUTION


Ex
$0=\sum_{a=1}^{n}{}' E_a = \sum_{a=1}^{n}{}' z_a E_a = \sum_{a=1}^{n}{}' z_a^2 E_a$
(trans-conservation $\sum_{b\neq a} S_{ab}=0$)
(n-3) eqs for (n-3) vars.
$E_a \to (\gamma+\delta z_a)^{-2} E_a$ under $z_a \to \frac{\alpha+\beta z_a}{\gamma+\delta z_a} = z_a'$
$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 1 \\ z_a \end{pmatrix} \to \begin{pmatrix} 1 \\ z_a' \end{pmatrix}$
(3) Eqs map Singularities of K_n
sphere
$z_i=1$
CAUTION

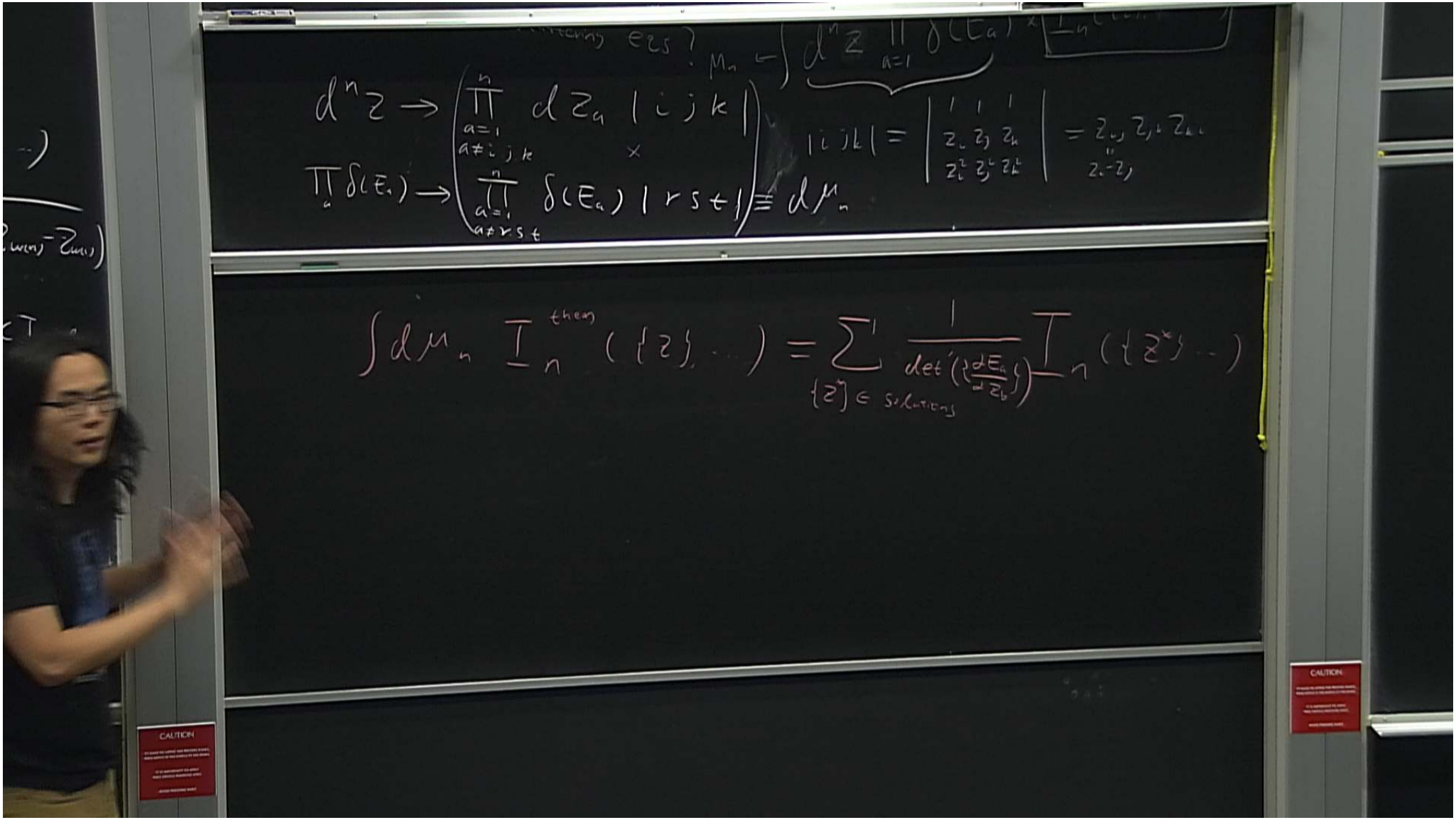

Ex. $0=\sum_{a=1}^{n} E_a = \sum_{a=1}^{n} z_a E_a = \sum_{a=1}^{n} z_a^2 E_a$ (trans-conservation $\sum_{b\neq a} S_{ab}=0$)
(n-3) eqs for (n-3) vars.
$E_a \to (\gamma+\delta z_a)^{-2} E_a$ under $z_a \to \frac{\alpha+\beta z_a}{\gamma+\delta z_a}$
(3) Eqs map $\frac{\text{Singularities of } K_n}{\text{(physical singularities)}} \to$ Singularities of $\mathcal{M}_{n,0}$
$z_1=1$
CAUTION
CAUTION

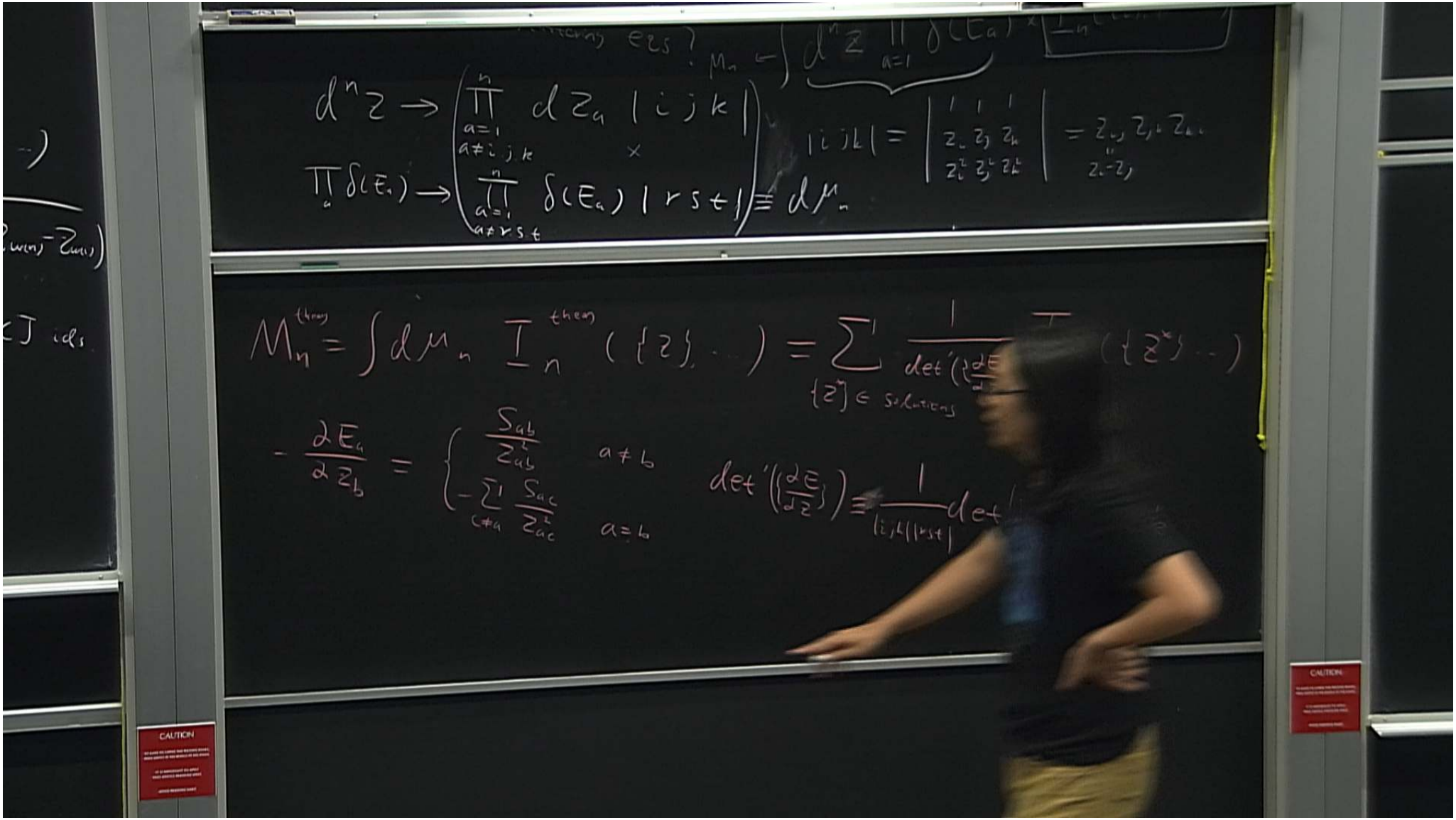

Ex
$0 = \sum_{a=1}^{n} E_a = \sum_{a=1}^{n} z_a E_a = \sum_{a=1}^{n} z_a^2 E_a$
(trans-conservation $\sum_{b \neq a} S_{ab} = 0$)
(n-3) eqs for (n-3) var.
$E_a \to (\gamma + \delta z_a)^{-2} E_a$ under $z_a \to \frac{\alpha + \beta z_a}{\gamma + \delta z_a}$
③ Eqs map Singularities of K_n / (physical singularities) $\to$ Singularities of $M_{n,0}$
$M_{4,0}$
Eqs
x_a
s=0 (t-axis)
t=0 (s-axis)
u=0
CAUTION

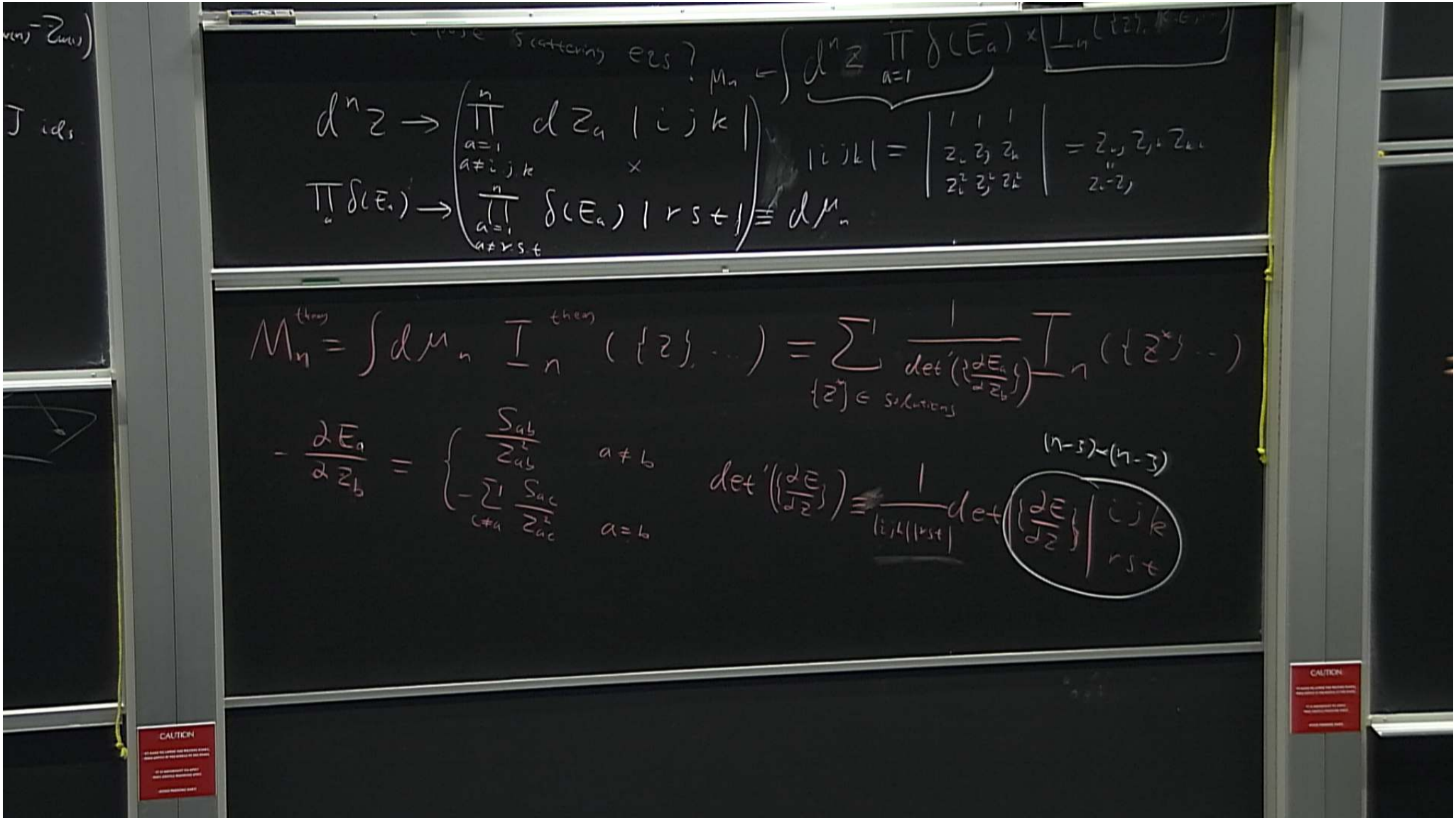

$0 = E_1 = \frac{S_{12}}{z_2 - z_1} + \frac{S_{13}}{z_3 - z_1} + \frac{S_{14}}{z_4 - z_1} \Rightarrow z_4 = -\frac{S_{14}}{S_{12}} = -\frac{t}{s}$
$= 1$
z_4
Generalize to n-point
④ # of solutions $= (n-3)!$
Ex. $n=5 \Rightarrow$ quadratic
with 2 solutions

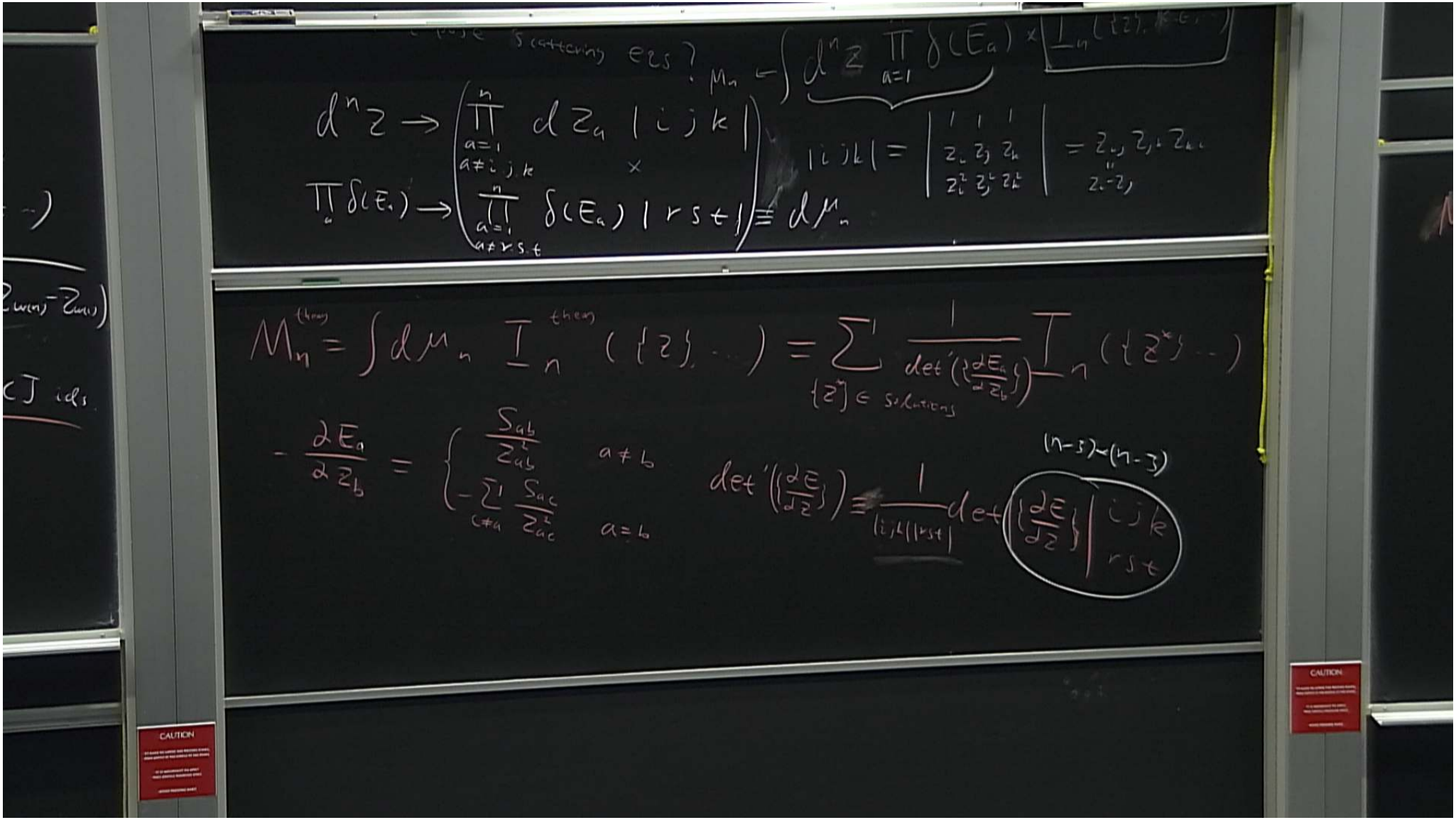


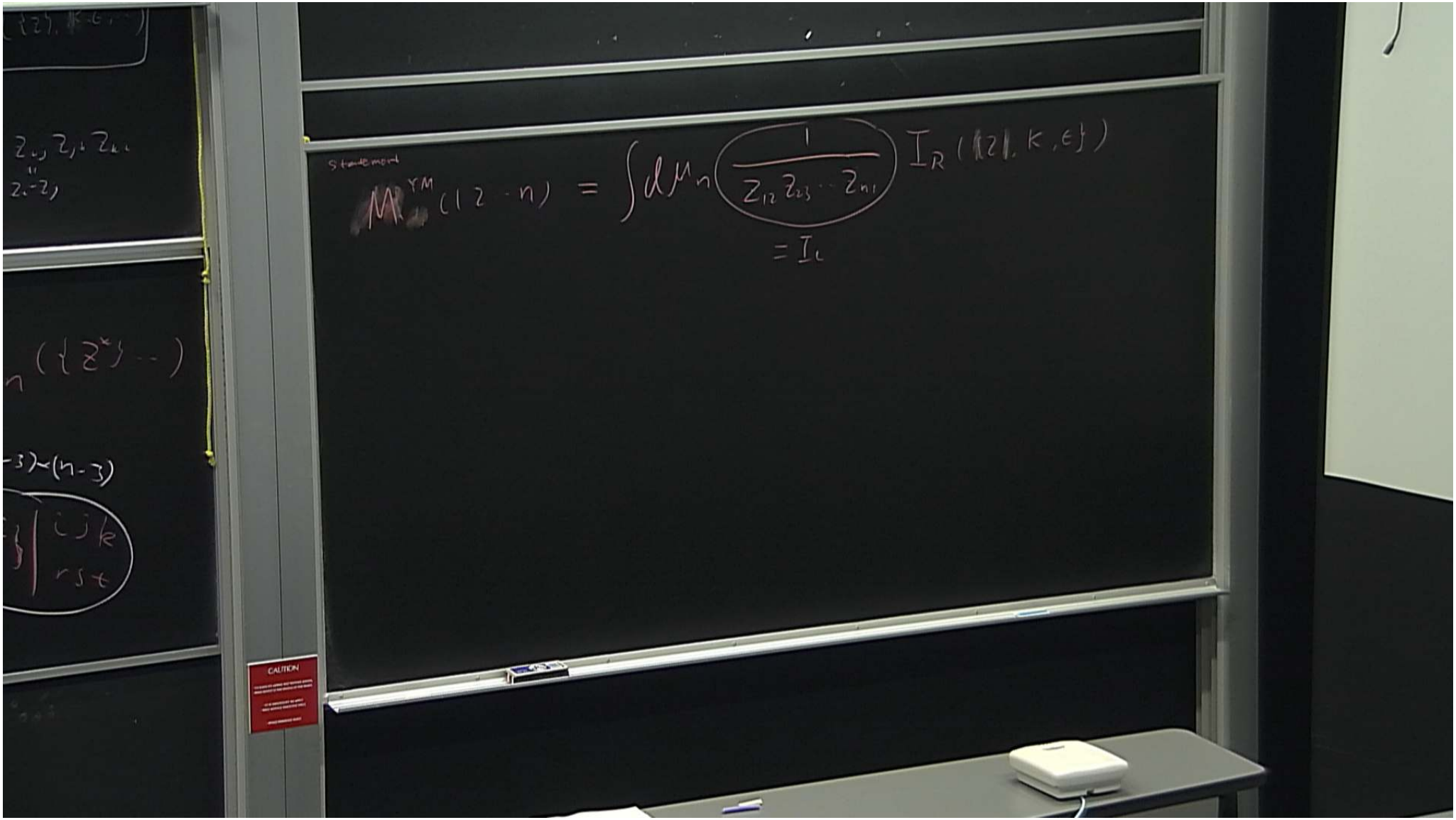
$0 = E_1 = \frac{S_{12}}{z_2 - z_1} + \frac{S_{13}}{z_3 - z_1} + \frac{S_{14}}{z_4 - z_1} \Rightarrow z_4 = -\frac{S_{14}}{S_{12}} = -\frac{t}{s}$
$= 1$
z_4
generalize to n-point
④ lutions $= (n-3)!$
Ex $n=5 \Rightarrow$ quadratic with 2 solutions
⑤ h
scattering eq

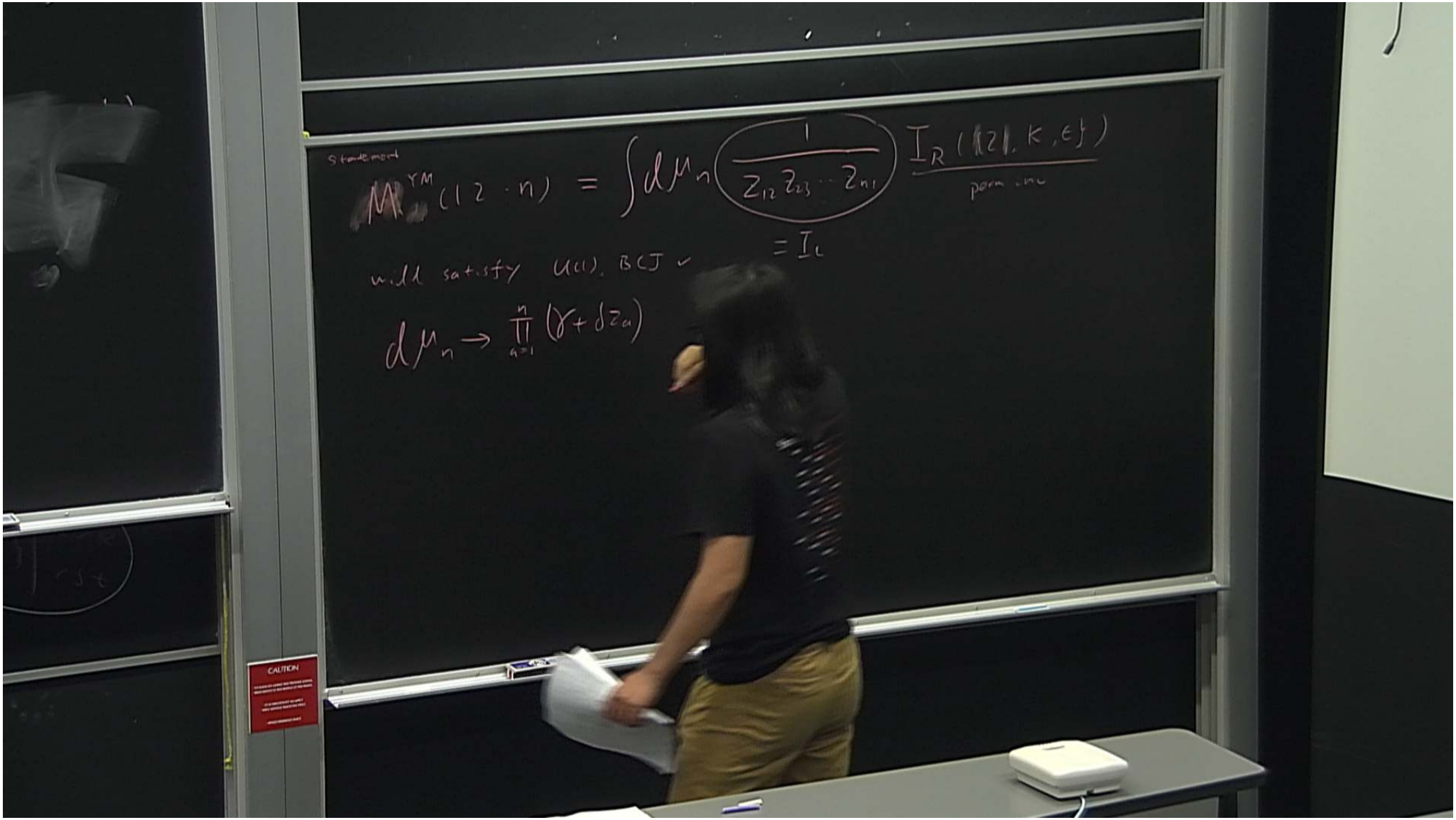

$d^n z \to \left(\prod_{\substack{a=1 \\ a\neq i,j,k}}^{n} dz_a \, |ijk|\right) \times$
$\prod_a \delta(E_a) \to \left(\prod_{\substack{a=1 \\ a\neq r,s,t}}^{n} \delta(E_a) \, |rst|\right) \equiv d\mu_n$
$|ijk| = \begin{vmatrix} 1 & 1 & 1 \\ z_i & z_j & z_k \\ z_i^2 & z_j^2 & z_k^2 \end{vmatrix} = z_{ij} z_{jk} z_{ki}$
$z_{ij} = z_i - z_j$
$\int d\mu_n \, I_n^{\text{theory}}(\{z\}, \ldots) = \sum'_{\{z^*\} \in \text{solutions}} \frac{1}{\det\left(\frac{\partial E_a}{\partial z_b}\right)} I_n(\{z^*\}, \ldots)$
CAUTION
CAUTION

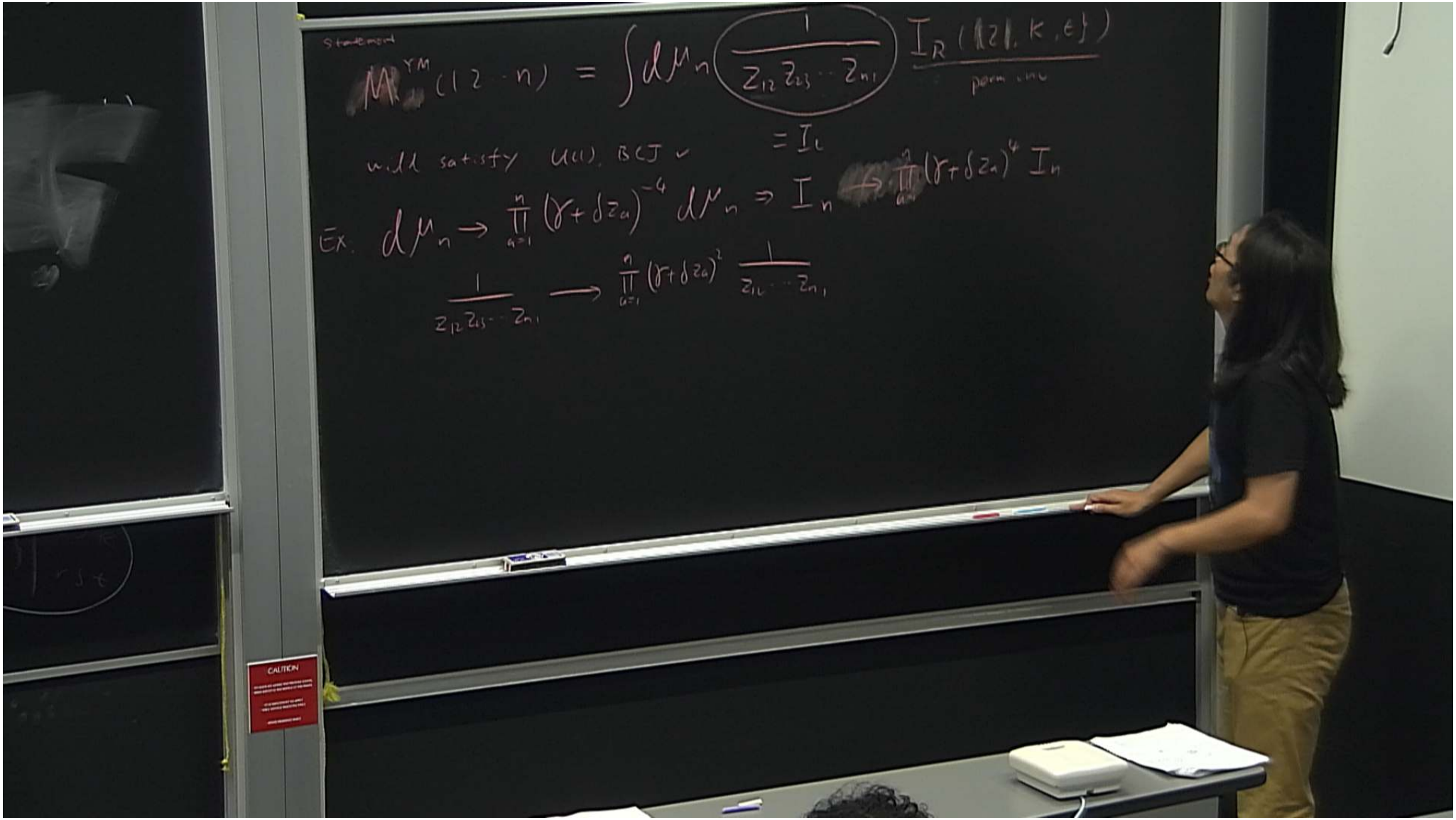

$d^n z \to \left(\prod_{\substack{a=1 \\ a \neq i,j,k}}^{n} dz_a \, |ijk| \right) \times$
$\prod_a \delta(E_a) \to \left(\prod_{\substack{a=1 \\ a \neq r,s,t}}^{n} \delta(E_a) \, |rst| \right) \equiv d\mu_n$
$|ijk| = \begin{vmatrix} 1 & 1 & 1 \\ z_i & z_j & z_k \\ z_i^2 & z_j^2 & z_k^2 \end{vmatrix} = z_i z_j z_k (z_i - z_j)$
$M_n^{(theory)} = \int d\mu_n \, I_n^{theory}(\{z\}, \ldots) = \sum_{\{z^*\} \in solutions} \frac{1}{\det'(\frac{\partial E}{\partial z})} \, T \, (\{z^*\}, \ldots)$
$-\frac{\partial E_a}{\partial z_b} = \begin{cases} \frac{S_{ab}}{z_{ab}^2} & a \neq b \\ -\sum_{c \neq a} \frac{S_{ac}}{z_{ac}^2} & a = b \end{cases}$
$\det'\left(\frac{\partial E_i}{\partial z}\right) \equiv \frac{1}{|i,j,k||r,s,t|} \det$
CAUTION
CAUTION

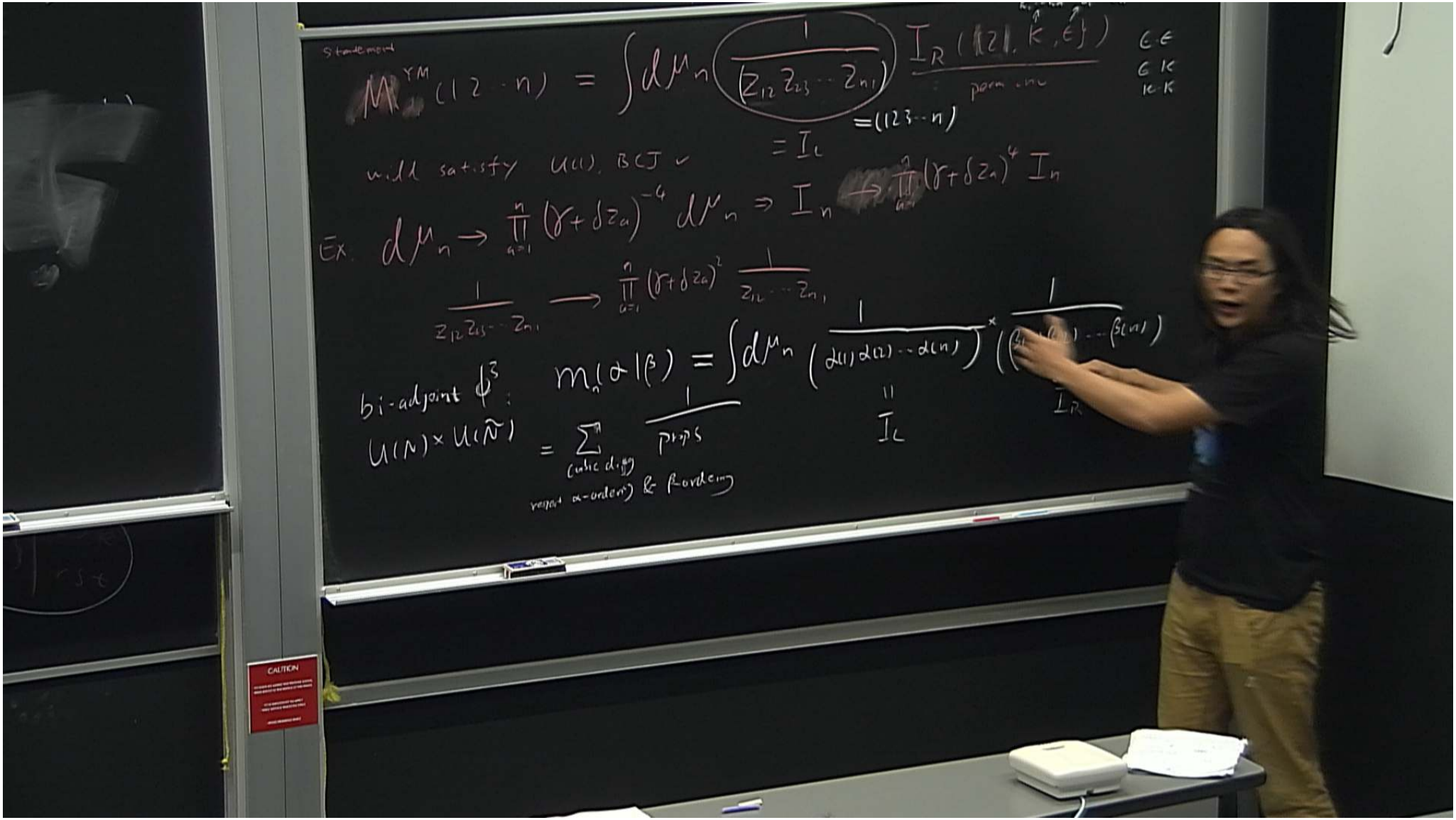

scattering eqs?
$\mu_n \leftarrow \int d^n z \prod_{a=1} \delta(E_a) \times I_n(\{z\}, \ldots)$
$d^n z \to \left(\prod_{\substack{a=1 \\ a \neq i,j,k}}^{n} dz_a \, |ijk| \right)$
$\prod_a \delta(E_a) \to \left(\prod_{\substack{a=1 \\ a \neq r,s,t}}^{n} \delta(E_a) \, |rst| \right) \equiv d\mu_n$
$|ijk| = \begin{vmatrix} 1 & 1 & 1 \\ z_i & z_j & z_k \\ z_i^2 & z_j^2 & z_k^2 \end{vmatrix} = z_{ij} z_{jk} z_{ki}$
$z_{ij} = z_i - z_j$
$M_n^{\text{theory}} = \int d\mu_n \, I_n^{\text{theory}}(\{z\}, \ldots) = \sum_{\{z^*\} \in \text{solutions}} \frac{1}{\det'\left(\left\{\frac{\partial E_a}{\partial z_b}\right\}\right)} I_n(\{z^*\} \ldots)$
$-\frac{\partial E_a}{\partial z_b} = \begin{cases} \frac{S_{ab}}{z_{ab}^2} & a \neq b \\ -\sum_{c \neq a} \frac{S_{ac}}{z_{ac}^2} & a = b \end{cases}$
$\det'\left(\left\{\frac{\partial E}{\partial z}\right\}\right) \equiv \frac{1}{|ijk||rst|} \det\left(\left\{\frac{\partial E}{\partial z}\right\} \Big|_{rst}^{ijk}\right)$
$(n-3) \times (n-3)$

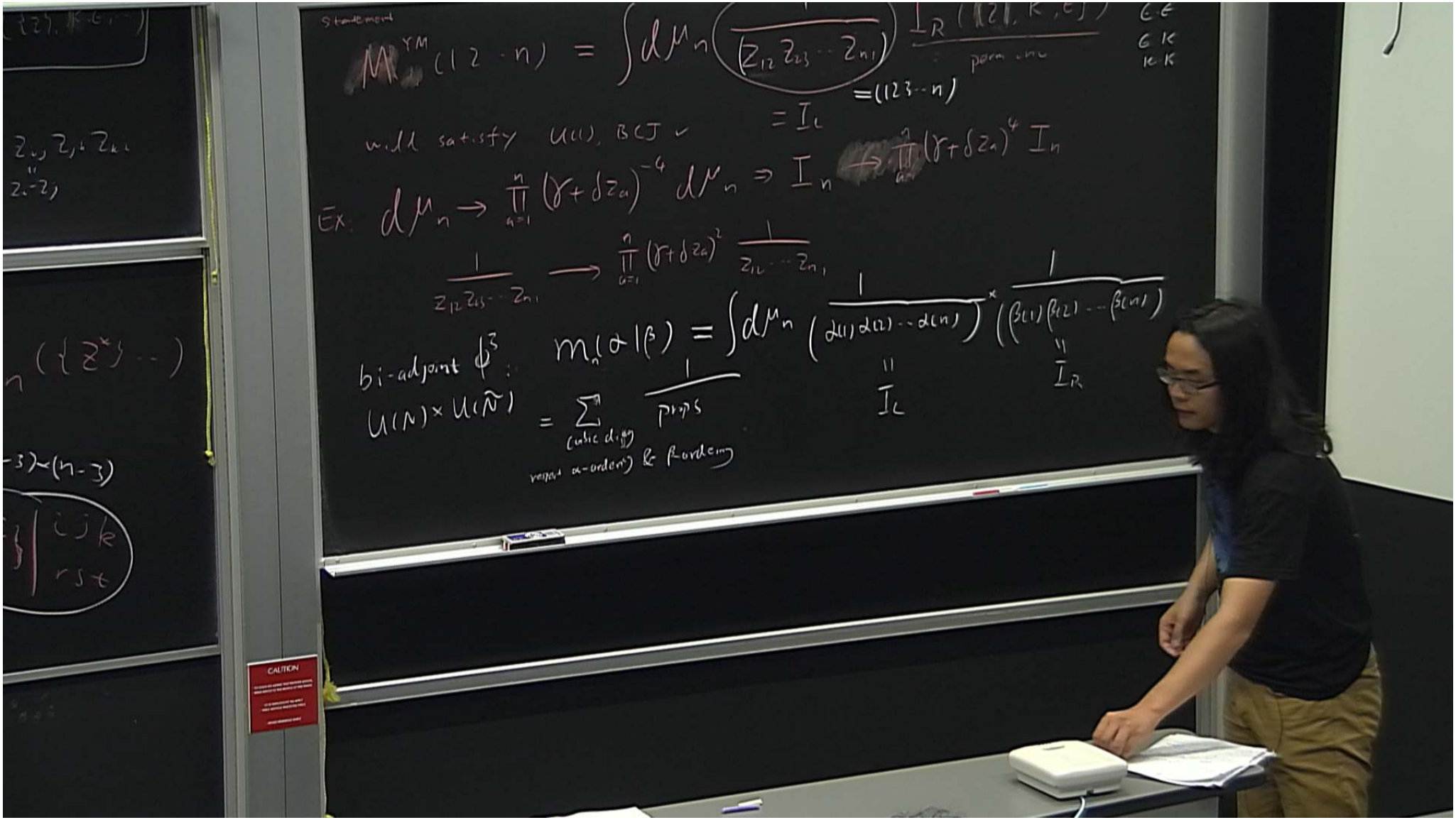

scattering eqs ?
$M_n = \int d^n z \prod_{a=1}^{n} \delta(E_a) \times$
$d^n z \to \prod_{a=1}^{n} dz_a \, |ijk| \times \prod_{\substack{a=1 \\ a\neq r,s,t}}^{n} \delta(E_a) \, |rst| \equiv d\mu_n$
$|ijk| = \begin{vmatrix} 1 & 1 & 1 \\ z_i & z_j & z_k \\ z_i^2 & z_j^2 & z_k^2 \end{vmatrix} = z_{ij} z_{jk} z_{ki}$
$z_{ij} = z_i - z_j$
$M_n^{(\text{theory})} = \int d\mu_n \, I_n^{\text{theory}}(\{z\}, \ldots) = \sum'_{\{z^*\} \in \text{solutions}} \frac{1}{\det'\left(\frac{\partial E_a}{\partial z_b}\right)} I_n(\{z^*\}, \ldots)$
$-\frac{\partial E_a}{\partial z_b} = \begin{cases} \frac{S_{ab}}{z_{ab}^2} & a \neq b \\ -\sum'_{c\neq a} \frac{S_{ac}}{z_{ac}^2} & a = b \end{cases}$
$\det'\left(\frac{\partial E}{\partial z}\right) \equiv \frac{1}{|ijk||rst|} \det \begin{pmatrix} \frac{\partial E}{\partial z} & ijk \\ & rst \end{pmatrix}$
$(n-3)\times(n-3)$

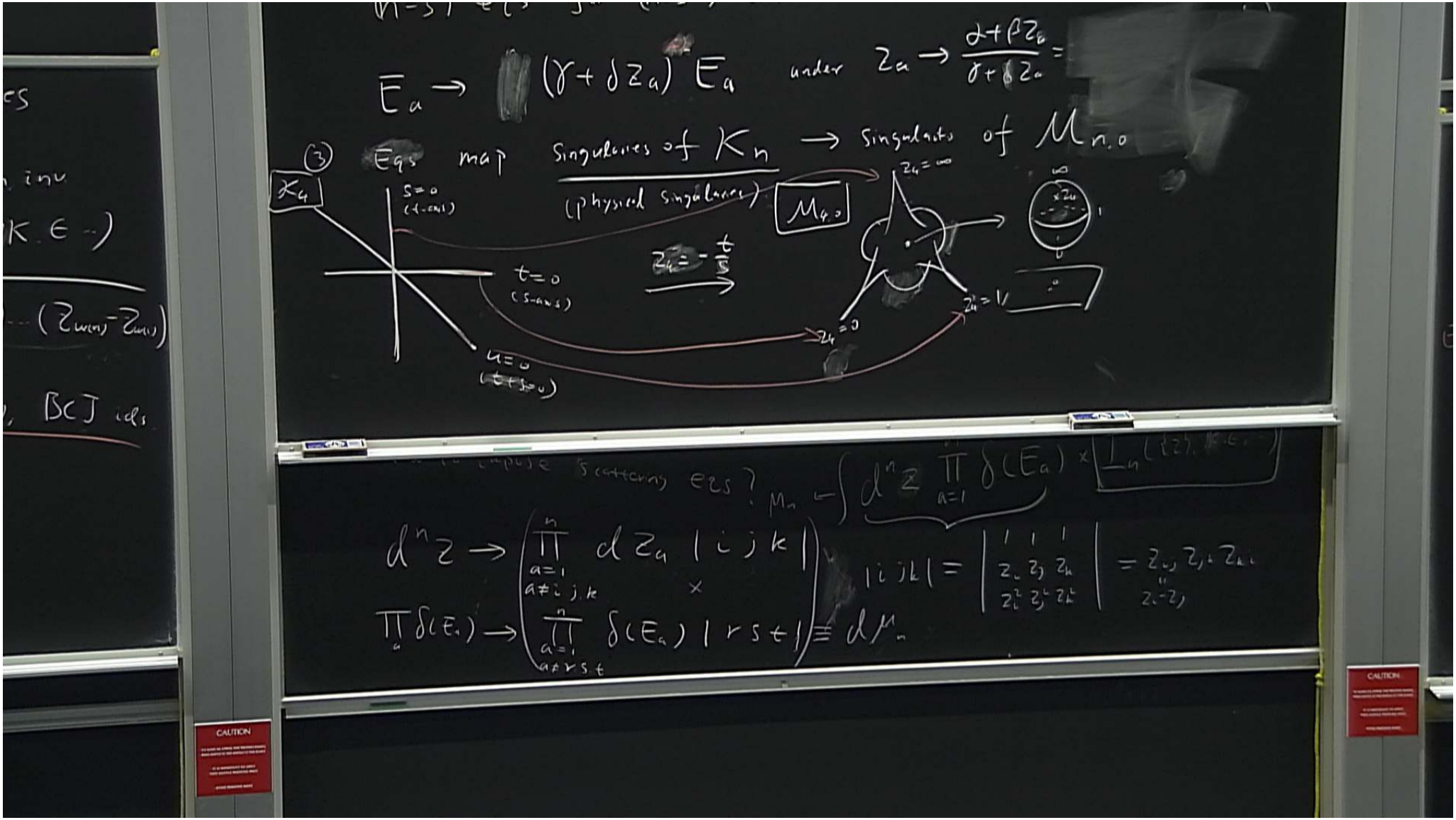

Statement
$M^{YM}(1\,2\cdots n) = \int d\mu_n \frac{1}{Z_{12}Z_{23}\cdots Z_{n1}} I_R(\{Z\}, K, \epsilon\})$
$= I_L$
CAUTION

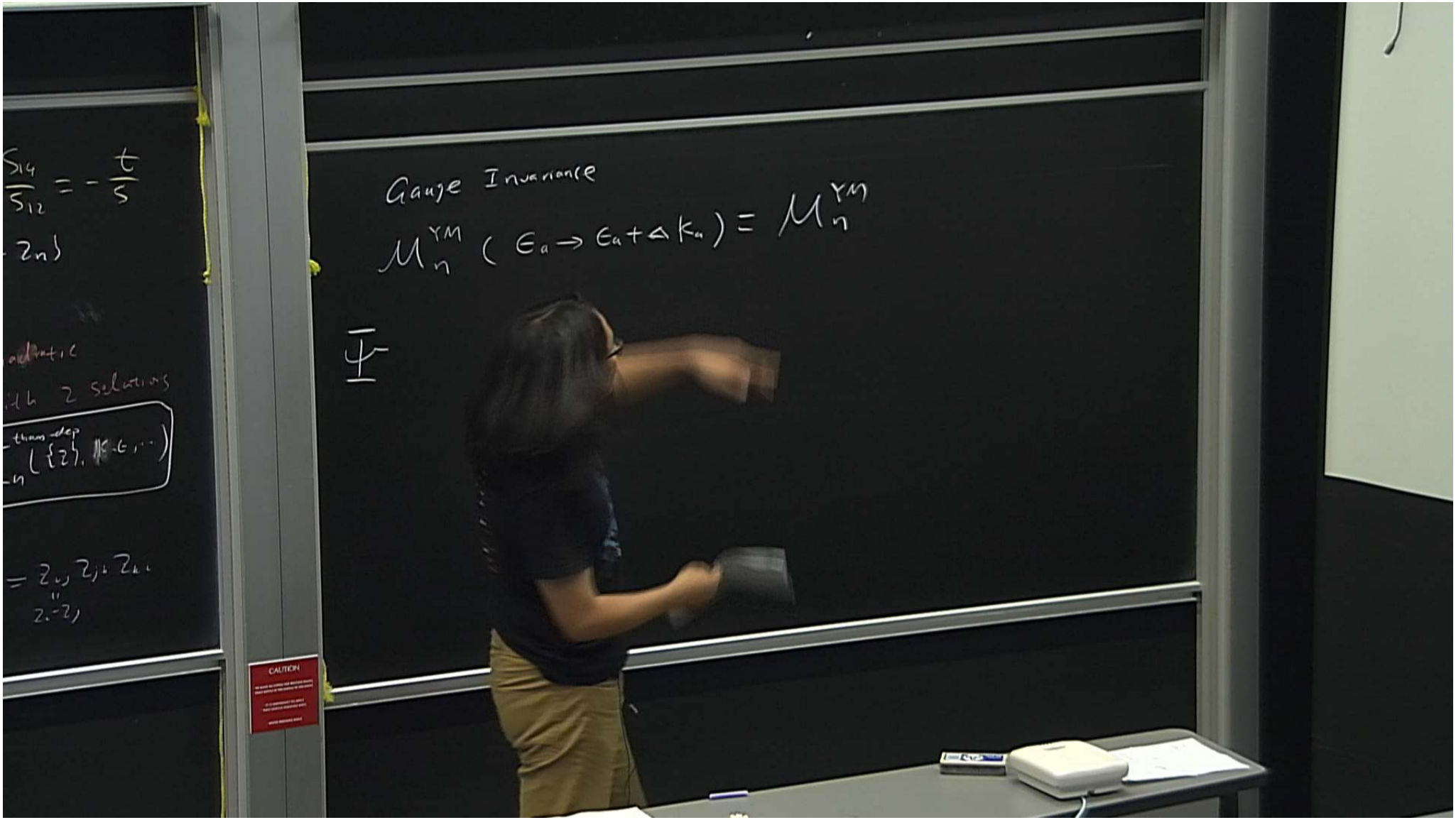

Statement
$M_n^{YM}(1\,2\cdots n) = \int d\mu_n \frac{1}{z_{12} z_{23} \cdots z_{n1}} \frac{I_R(\{12\}, K, \epsilon_j)}{\text{perm. inv}}$
$= I_L$
will satisfy U(1), BCJ ✓
$d\mu_n \to \prod_{a=1}^{n} (\gamma + \delta z_a)$
CAUTION

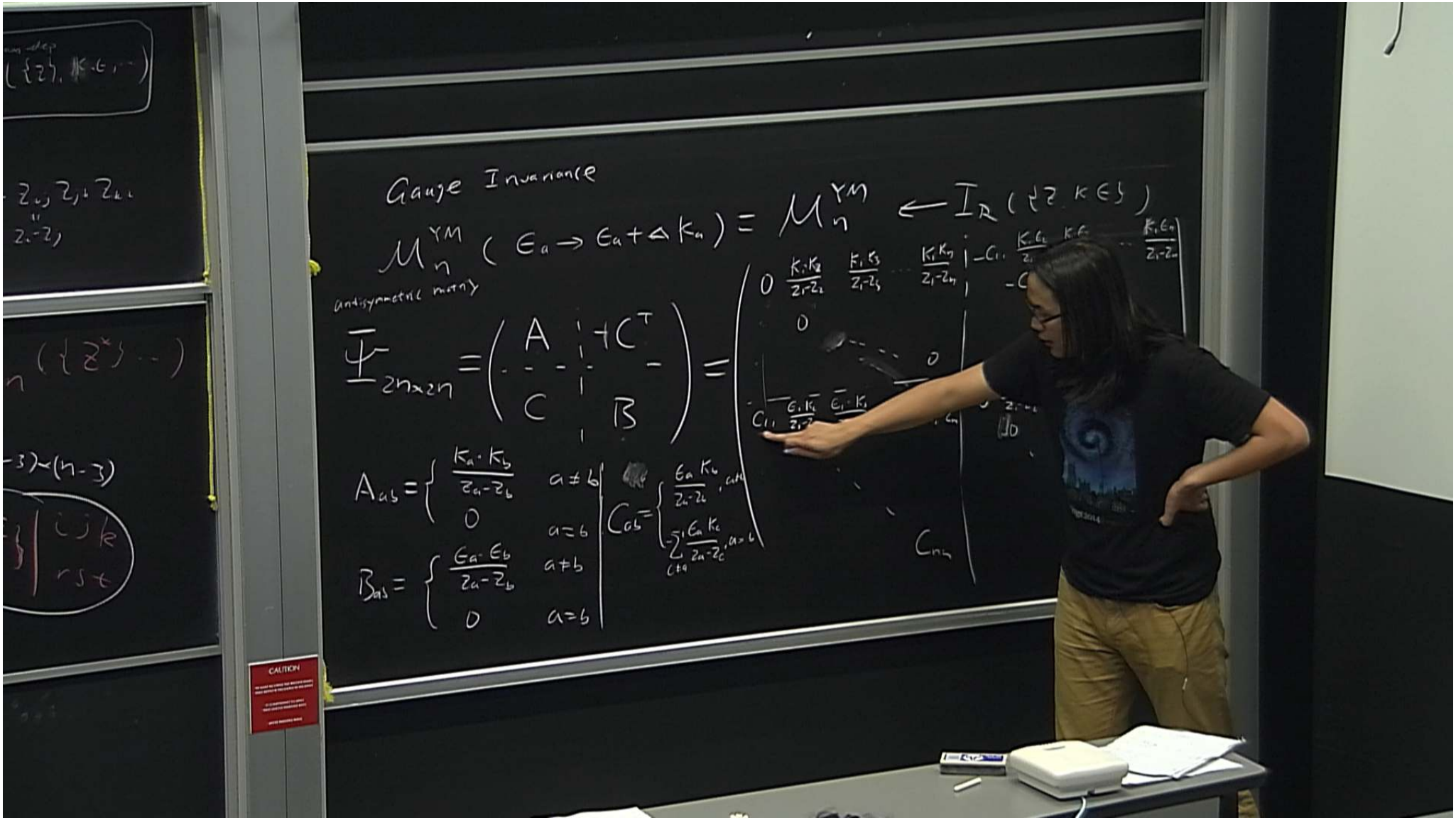

Statement
$M_n^{YM}(1\,2\cdots n) = \int d\mu_n \frac{1}{z_{12}z_{23}\cdots z_{n1}} \frac{I_R(\{z\},K,\epsilon)}{\text{perm. inv.}}$
$= I_L$
will satisfy U(1), BCJ ✓
Ex. $d\mu_n \to \prod_{a=1}^{n}(\gamma+\delta z_a)^{-4} d\mu_n \Rightarrow I_n \to \prod_{a=1}^{n}(\gamma+\delta z_a)^4 I_n$
$\frac{1}{z_{12}z_{23}\cdots z_{n1}} \longrightarrow \prod_{a=1}^{n}(\gamma+\delta z_a)^2 \frac{1}{z_{12}\cdots z_{n1}}$
CAUTION

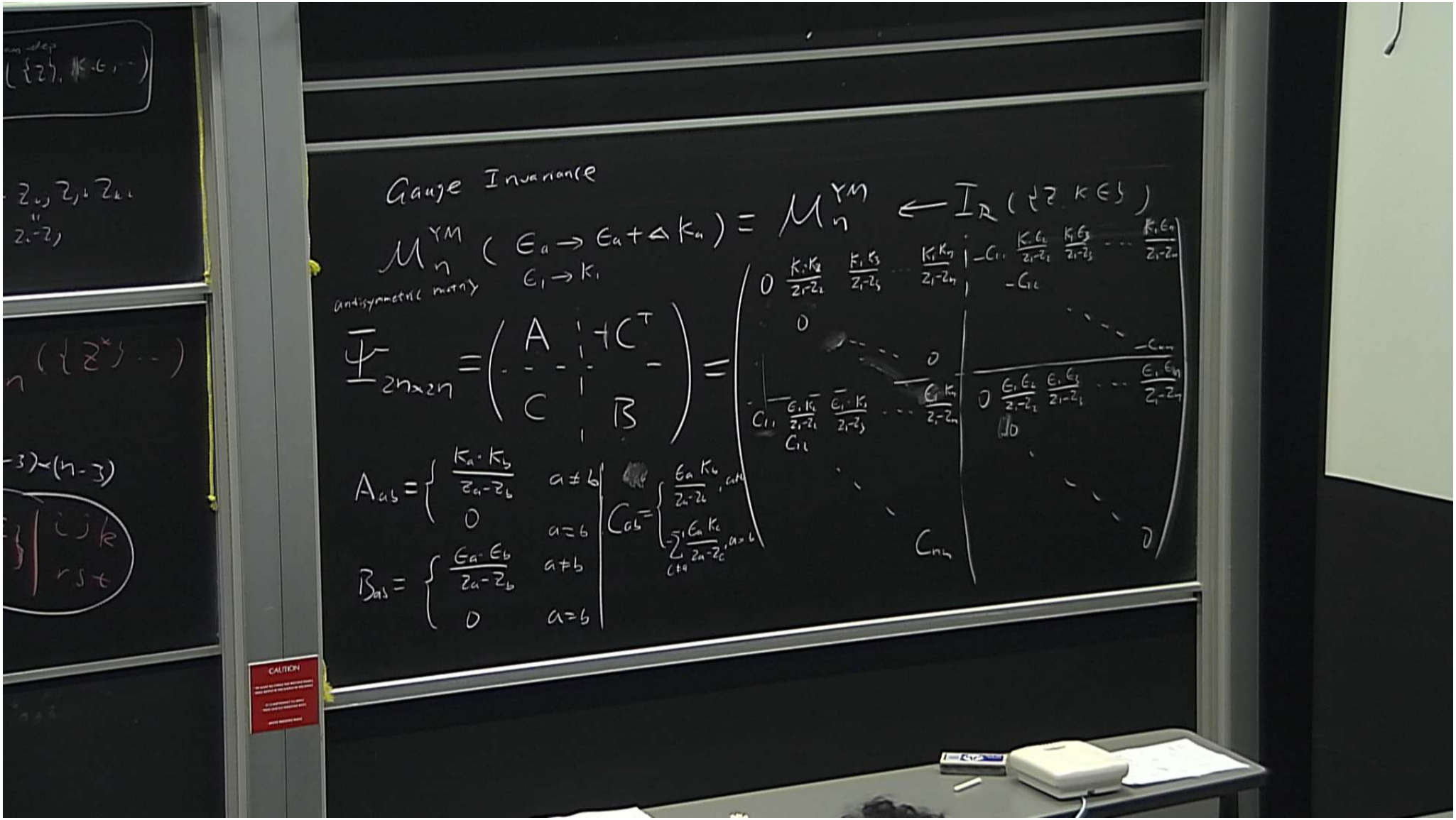

Statement
$M_n^{YM}(12\cdots n) = \int d\mu_n \frac{1}{(z_{12}z_{23}\cdots z_{n1})} I_R(\{12\cdots n\}, \{k\}, \{\epsilon\})$
$= I_L$
$= (123\cdots n)$
will satisfy U(1), BCJ
Ex. $d\mu_n \to \prod_{a=1}^{n} (\gamma + \delta z_a)^{-4} d\mu_n \Rightarrow I_n \to \prod_{a=1}^{n} (\gamma+\delta z_a)^4 I_n$
$\frac{1}{z_{12}z_{23}\cdots z_{n1}} \to \prod_{a=1}^{n} (\gamma+\delta z_a)^2 \frac{1}{z_{12}\cdots z_{n1}}$
bi-adjoint ϕ^3: $U(N)\times U(\tilde{N})$
$m_n(\alpha|\beta) = \int d\mu_n \frac{1}{(\alpha(1)\alpha(2)\cdots\alpha(n))} \times \frac{1}{(\beta(1)\cdots\beta(n))}$
$= I_L$
I_R
$= \sum_{\text{(cubic diag)}} \frac{1}{\text{props}}$
respect α-ordering & flavor-ordering
CAUTION

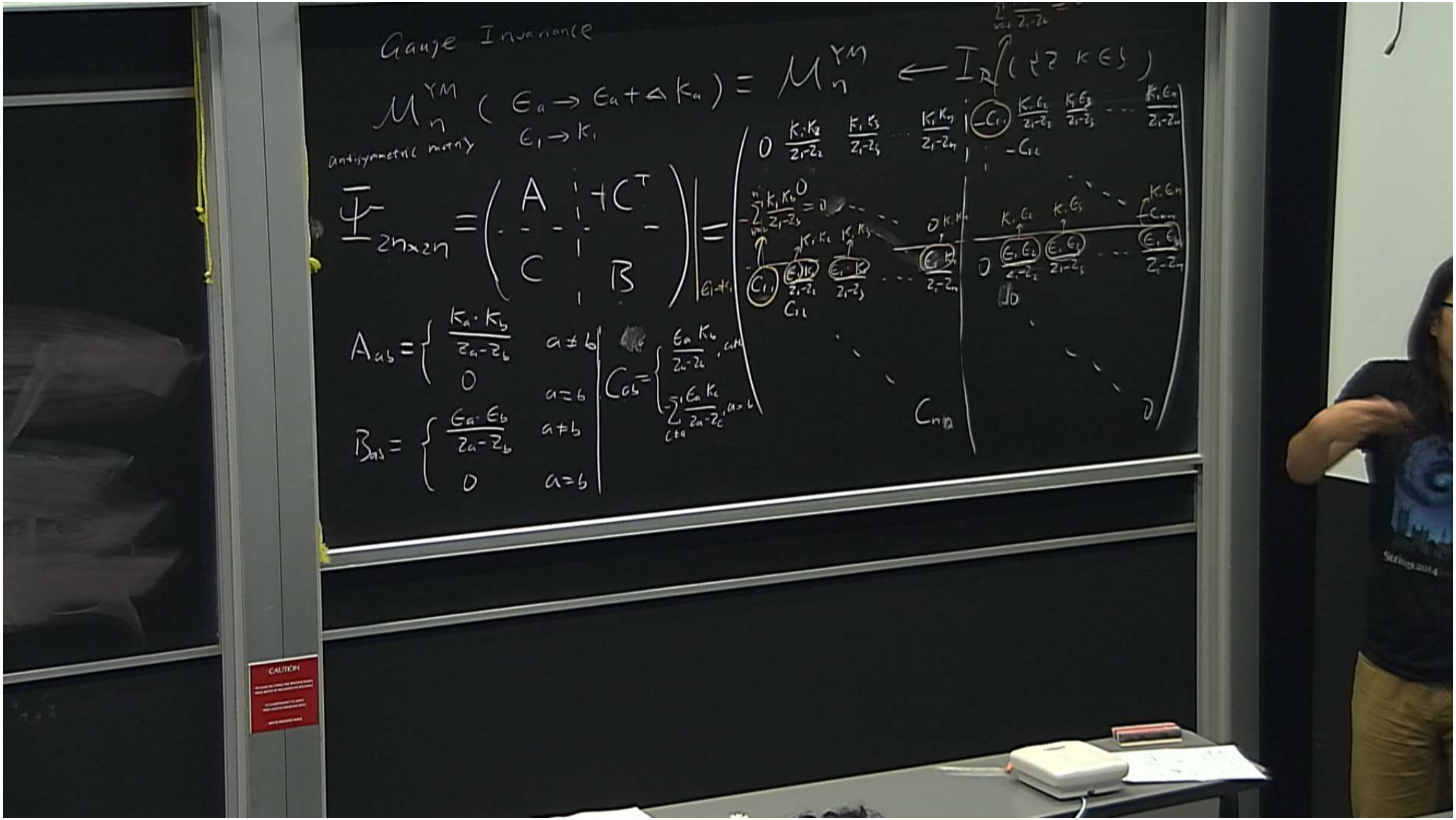

Statement
perm inv
= (123 ... n)
= I_L
will satisfy U(1), BCJ ✓
Ex:
bi-adjoint φ³:
U(N) × U(Ñ)
cubic diag
respect α-ordering & β-ordering
props
CAUTION

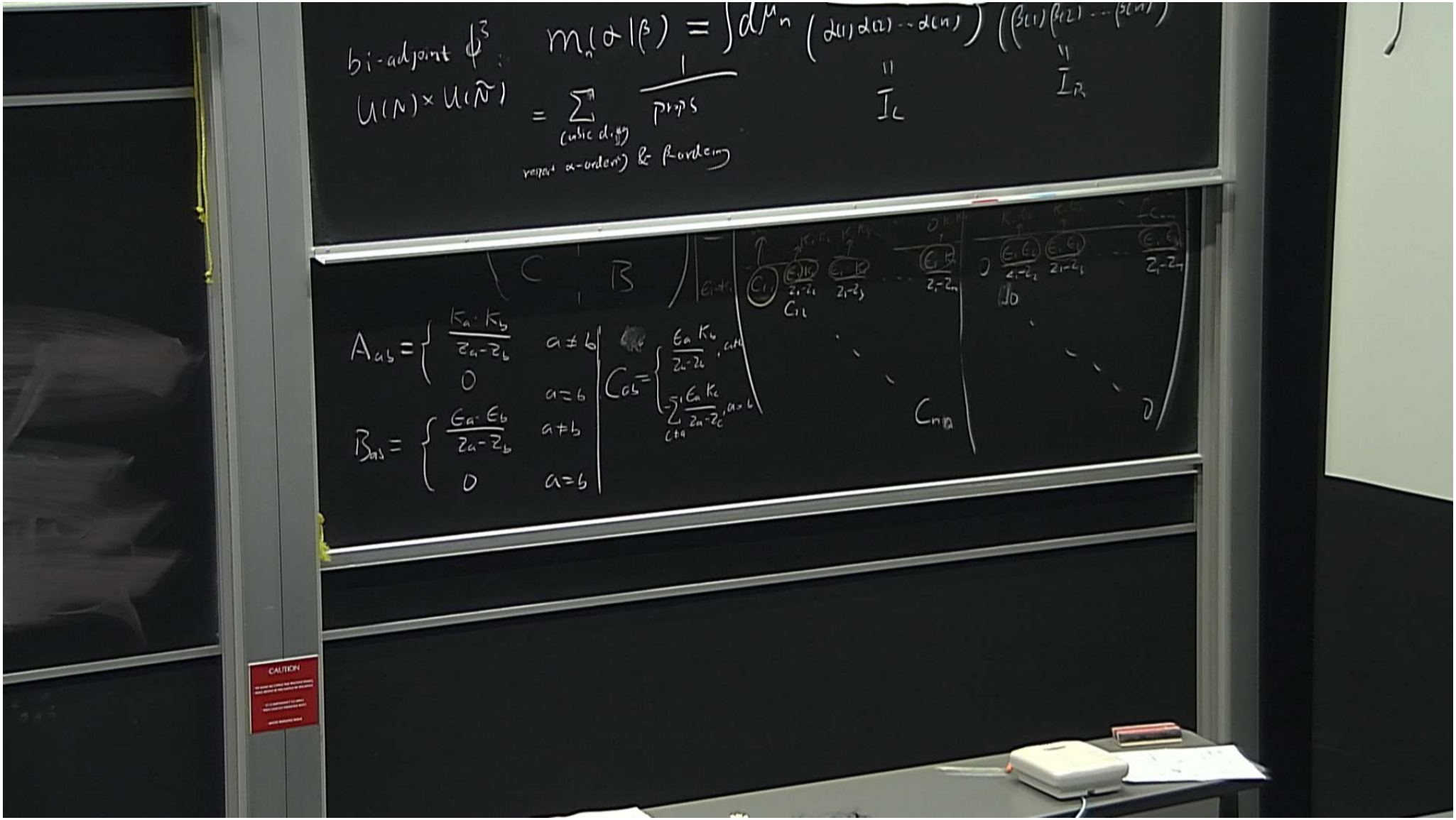

$E_a \to (\gamma+\delta z_a)^{-} E_a$ under $z_a \to \frac{\alpha+\beta z_a}{\gamma+\delta z_a}$
(3) Eqs map Singularities of K_n → Singularities of $\mathcal{M}_{n,0}$
(physical singularities)
$\mathcal{M}_{4,0}$
$z_4 = -\frac{t}{s}$
$z_4 = \infty$
$z_4 = 1$
$z_4 = 0$
$s=0$ (collinear)
$t=0$ (s-chan)
$u=0$ ($t+s=0$)
$d^n z \to \left(\prod_{\substack{a=1 \\ a\neq i,j,k}}^{n} dz_a\, |ijk|\right) \times \left(\prod_{\substack{a=1 \\ a\neq r,s,t}}^{n} \delta(E_a)\, |rst|\right) \equiv d\mu_n$
$|ijk| = \begin{vmatrix} 1 & 1 & 1 \\ z_i & z_j & z_k \\ z_i^2 & z_j^2 & z_k^2 \end{vmatrix} = z_{ij} z_{jk} z_{ki}$
$z_{ij} = z_i - z_j$
$M_n \sim \int d^n z \prod_{a=1}^{n} \delta(E_a) \times$
CAUTION


Gauge Invariance
$\mathcal{M}_n^{YM}(\epsilon_a \to \epsilon_a + \alpha k_a) = \mathcal{M}_n^{YM}$
$\frac{s_{14}}{s_{12}} = -\frac{t}{s}$
CAUTION


Gauge Invariance
antisymmetric matrix
CAUTION


Gauge Invariance
antisymmetric matrix
CAUTION


Gauge Invariance
antisymmetric many
CAUTION

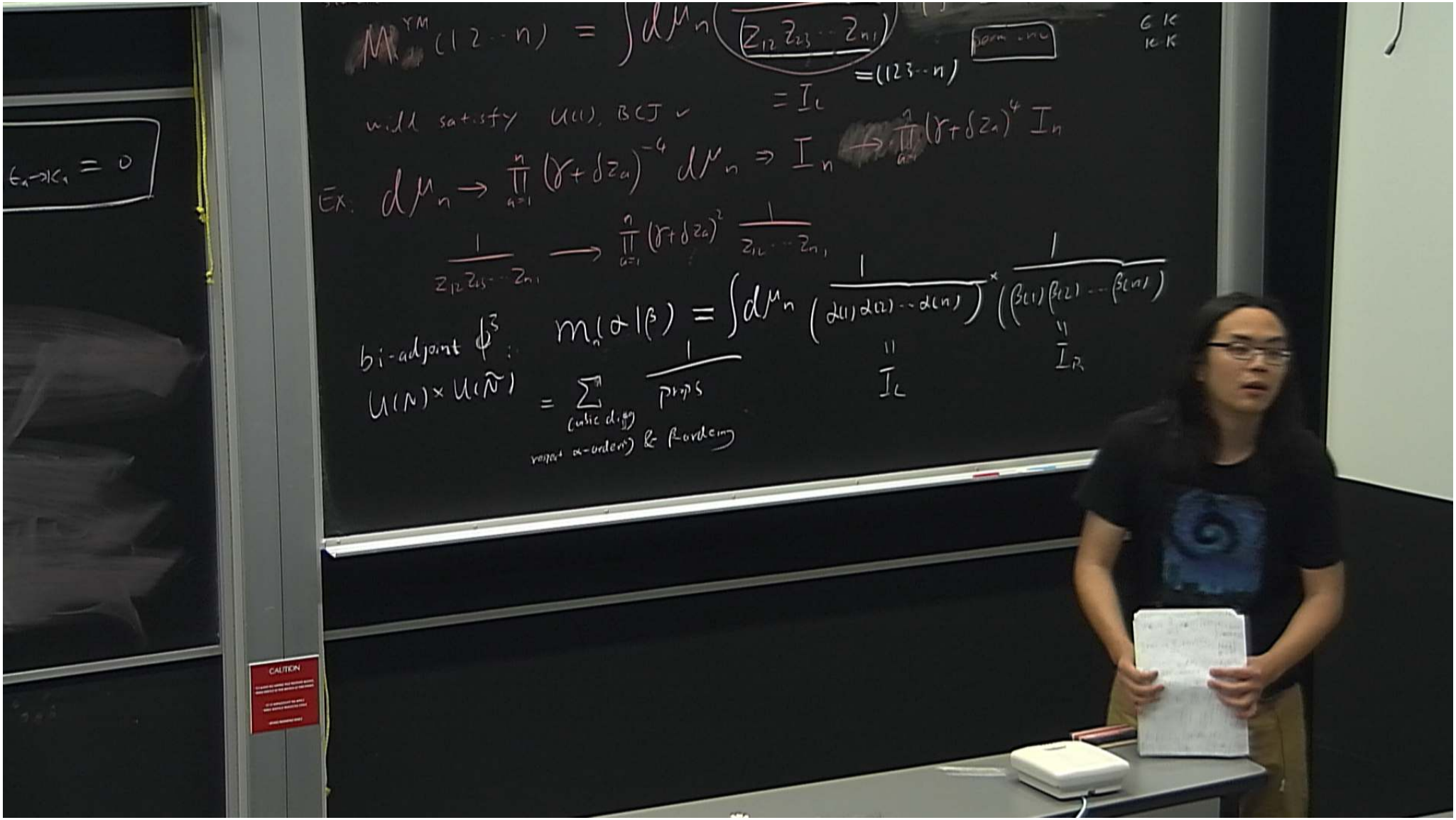

bi-adjoint ϕ^3:
$U(N) \times U(\tilde{N})$
$m_n(\alpha|\beta) = \int d\mu_n \left(\alpha(1)\alpha(2)\cdots\alpha(n)\right)\left(\beta(1)\beta(2)\cdots\beta(n)\right)$
$= \sum_{\text{cubic diag}} \frac{1}{\text{props}}$
I_L
I_R
respect α-ordering & flavor
$A_{ab} = \begin{cases} \frac{K_a \cdot K_b}{z_a - z_b} & a \neq b \\ 0 & a = b \end{cases}$
$B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{z_a - z_b} & a \neq b \\ 0 & a = b \end{cases}$
$C_{ab} = \begin{cases} \frac{\epsilon_a K_b}{z_a - z_b}, & \\ -\sum_{c \neq a} \frac{\epsilon_a K_c}{z_a - z_c}, & a = b \end{cases}$
C_{nn}
CAUTION

$$I_R^{YM}(z,\epsilon,k) = Pf'\,\bar{\Psi}(z,\epsilon,k)$$

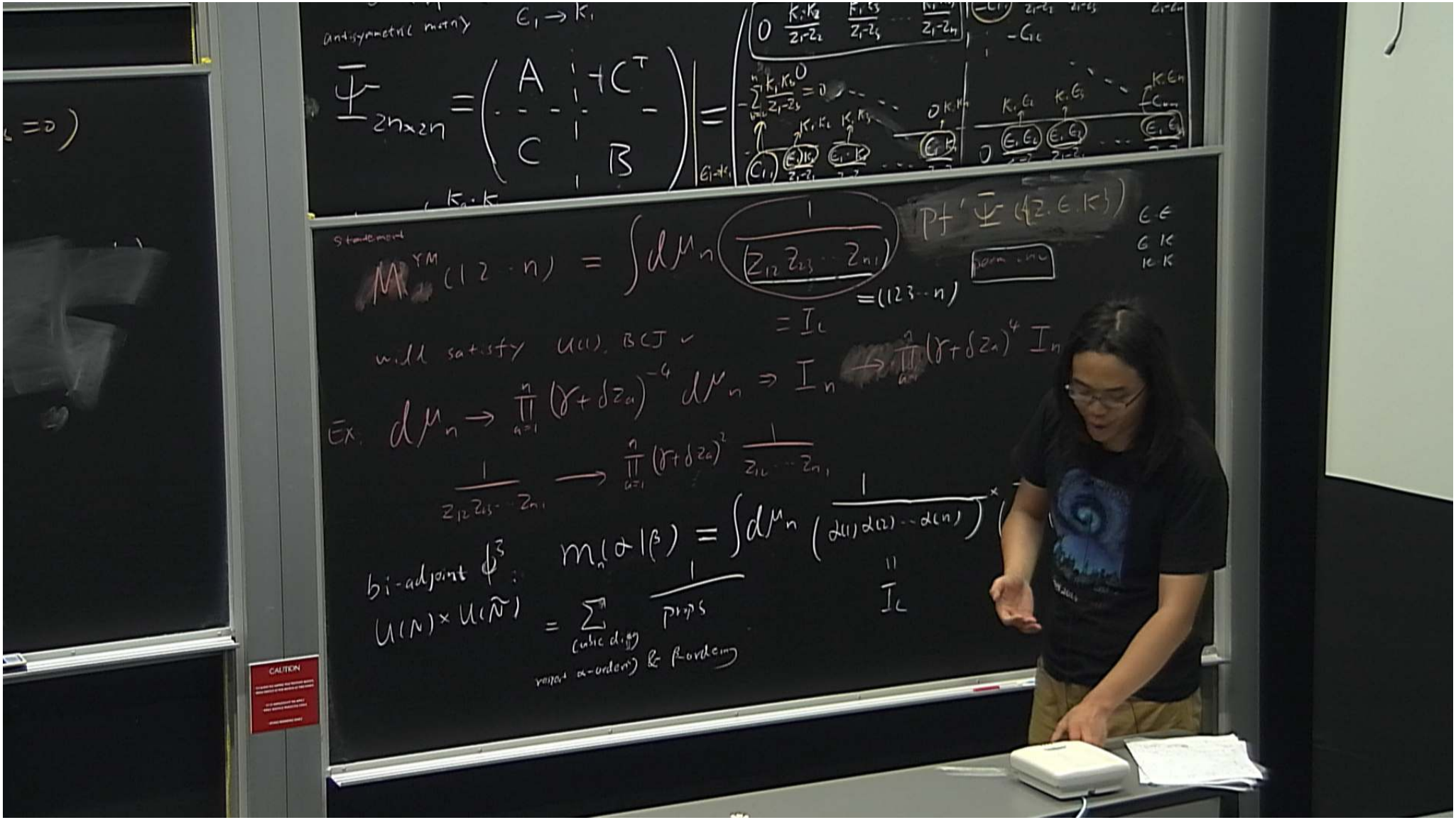
$$I_2\big|_{\epsilon_n \to k_n} = 0$$

$$\bar{\Psi}\begin{pmatrix} 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \bar{\Psi}\begin{pmatrix} z_1 \\ \vdots \\ z_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$$

$$\frac{(-1)^{i+j}}{z_i - z_j} Pf\,|\bar{\Psi}|^{i,j}_{i,j} \equiv Pf'\,\bar{\Psi}_n \qquad (1 \le i < j \le n)$$



$M^{YM}(12\cdots n) = \int d\mu_n \frac{1}{(z_{12}z_{23}\cdots z_{n1})}$
perm inv
$= (123\cdots n)$
$= I_L$
will satisfy U(1), BCJ ✓
$\epsilon_a \to k_a = 0$
Ex: $d\mu_n \to \prod_{a=1}^{n} (\gamma + \delta z_a)^{-4} d\mu_n \Rightarrow I_n \to \prod_{a=1}^{n} (\gamma + \delta z_a)^4 I_n$
$\frac{1}{z_{12}z_{23}\cdots z_{n1}} \longrightarrow \prod_{a=1}^{n} (\gamma + \delta z_a)^2 \frac{1}{z_{12}\cdots z_{n1}}$
bi-adjoint ϕ^3: $m_n(\alpha|\beta) = \int d\mu_n \frac{1}{(\alpha(1)\alpha(2)\cdots\alpha(n))} \times \frac{1}{(\beta(1)\beta(2)\cdots\beta(n))}$
$U(N) \times U(\tilde{N})$
$= \sum_{\text{cubic diag}} \frac{1}{\text{props}}$
respect α-ordering & β-ordering
$= I_L$
$= I_R$
CAUTION


antisymmetric matrix
Statement
Ex.
bi-adjoint ϕ^3:
$U(N) \times U(\tilde{N})$
cubic diag
CAUTION