Title: Topological Order Series

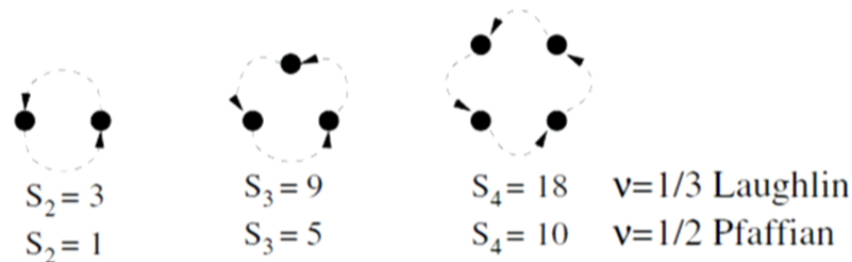
Date: Jul 02, 2015 03:00 PM

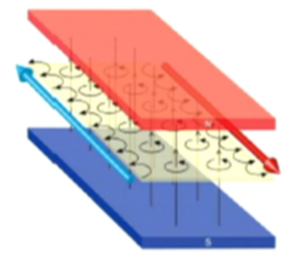
URL: http://pirsa.org/15070009

Abstract:

Local dancing rule → global dancing pattern

- Local dancing rules of a FQH liquid:
  (1) every electron dances around clock-wise
  (Φ_{FQH} only depends on $z = x + iy$)
  (2) takes exactly three steps to go around each other
  (Relative angular momentum $S_2 = 3$)   Wen-Wang arXiv:0803.1016
  → Global dancing pattern $\Phi_{\text{FQH}}(\{z_1, ..., z_N\}) = \prod(z_i - z_j)^3$
- A systematic theory of FQH state – **Pattern of zeros** S_a:
  a-electron cluster has a relative angular momentum S_a





- Local dancing rules are enforce by the Hamiltonian to lower energy.
- Only certain sequences S_a correspond to valid FQH states. Which?
- Different POZ S_a give rise to different topological properties

GPE rule and edge spectrum for the Pfaffian state

First kind of edge:

$M_0 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots :$ 2020202020202|00000000 $\cdots$

$M_0 + 1 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots :$ 20202020201|10000000 $\cdots$

$M_0 + 1 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots :$ 20202020112|00000000 $\cdots$ not allowed

$M_0 + 2 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots :$ 20202020201|01000000 $\cdots$

$M_0 + 2 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots :$ 20202020200|20000000 $\cdots$

$M_0 + 2 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots :$ 20202020111|10000000 $\cdots$

$M_0 + 3 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots :$ 20202020201|00100000 $\cdots$

$M_0 + 3 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots :$ 20202020200|11000000 $\cdots$

$M_0 + 3 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots :$ 20202020111|01000000 $\cdots$

$M_0 + 3 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots :$ 20202020110|20000000 $\cdots$

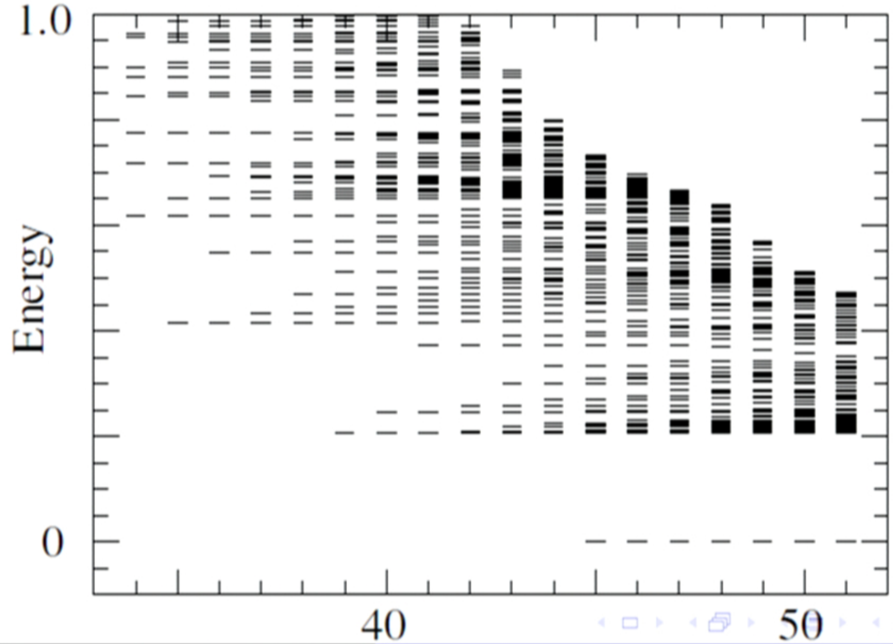
$M_0 + 3 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots :$ 20202011111|10000000 $\cdots$

M	M_0	M_0+1	M_0+2	M_0+3	M_0+4
# of states D_n	1	1	3	5	10

Edge states = zero-energy states of ideal Hamiltonian

- For ideal Hamiltonian $V_{1/3}(z_1, z_2) = -\partial_{z_1}\delta(z_1 - z_2)\partial_{z_1}$, the N electron state $P_{1/3} = \prod_{i<j}(z_i - z_j)^3$, is the zero-energy state with minimal angular momentum (the order of z_i's) $M_0 = N(N-1)$.
- Other zero-energy state has higher anglar momenta. Those zero energy states are the so called **edge states**:

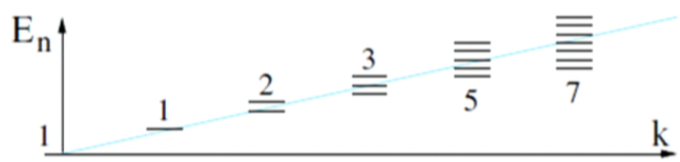
The energy spectrum of 100 lowest levels of the ideal Hamiltonian with 6 electrons



Edge spectrum of Laughlin state

For $\nu = 1/2$ **Laughlin state**, the edge states are obtained by deforming the Laughlin wave function without reducing the order of zeros:
$\Psi_{\text{edge}} = P_{\text{sym}}(\{z_i\})\Psi_{1/2}$ where P_{sym} is a symmetric polynomial, such as $\sum z_i$, $(\sum z_i)^2$, $\sum z_i^2, \dots$

M	M_0	M_0+1	M_0+2	M_0+3	M_0+4
# of states	1	1	2	3	5
P_{sym}	1	$\sum z_i$	$(\sum z_i)^2$	...	...
			$\sum z_i^2$	...	...



Pattern of zeros and generalized Pauli exclusion rule

- $l_{\gamma;a+1} - l_{\gamma;a} \geq S_2$ → the spacing between any two occupied orbitals by different electrons is no less than S_2
  → a generalized Pauli exclusion rule

  More general Pauli exclusion (GPE) rule

  In terms of $l_{\gamma;a} = S_{\gamma;a} - S_{\gamma;a-1}$, the concave condition for quasiparticles becomes

$$\sum_{k=1}^{b} l_{\gamma;a+k} \geq S_b = \sum_{k=1}^{b} l_{a+k}, \quad \to \quad l_{\gamma;a} \geq l_a,$$

$$\sum_{k=1}^{c} (l_{\gamma;a+b+k} - l_{\gamma;a+k}) \geq S_{b+c} - S_b - S_c = \sum_{k=1}^{c} (l_{b+k} - l_k)$$

  for any $a, b, c \in Z_+$. Note that $l_{\gamma;a+b} - l_{\gamma;a}$ is the spread of $b+1$ electrons. Setting $c = 1$ → **The spread of b electrons $\geq l_b$.**
  → **The number of obitals occupied by b electrons $\geq l_b + 1$.**
  (l_b = the spread of the first b electrons in the ground state.)

Edge CFT → bulk CFT

- The above theory $\frac{m}{4\pi}\partial_x\phi(\partial_t + v\partial_x)\phi$ for edge states is a chiral conformal field theory – a Gaussian model or a $U(1)$ current algebra, with electron operator $c(z) \propto \mathrm{e}^{\mathrm{i}m\phi(z)}$
- The correlation of the N electron operators in the CFT produce the N-electron bulk wave function:

$$P(\{z_i\}) = \prod (z_i - z_j)^m \propto \langle [c^\dagger(z_\infty)]^N_{\text{point-split}} \prod c(z_i)\rangle$$

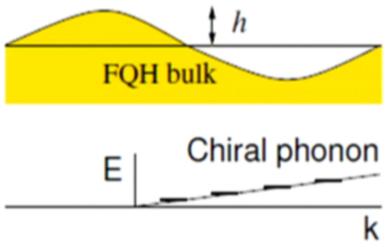
The above can be shown by using

$$\langle \mathrm{e}^{-\mathrm{i}mN\phi(z_\infty)} \prod \mathrm{e}^{\mathrm{i}m\phi(z_i)}\rangle = \langle \mathrm{e}^{-\mathrm{i}mN\phi(z_\infty)+\sum \mathrm{i}m\phi(z_i)}\rangle$$

$$= \mathrm{e}^{\frac{1}{2}\langle[-\mathrm{i}mN\phi(z_\infty)+\sum \mathrm{i}m\phi(z_i)]^2\rangle} \propto \mathrm{e}^{-m^2\sum_{i<j}\langle\phi(z_i)\phi(z_j)\rangle} = \prod_{i<j}(z_i - z_j)^m$$

If the bulk wave function can be written as a correlation of "electron operators in a "bulk CFT", then the same CFT will also "describe" the gapless edge states.

Low energy effective theory of edge excitations



- The $\nu = 1/m$ state $\Psi_{1/m} = \prod(z_i - z_j)^m$.
  $\Rightarrow$ edge excitations = edge waves
  $\Rightarrow$ edge phonons after quantization.
  Displacement $h \propto$ 1D edge density $\rho = h\rho_{2D}$.
- Confining electric field induces edge current

$$\mathbf{j} = \sigma_{xy}\hat{z} \times \mathbf{E}, \qquad \sigma_{xy} = \nu \frac{e^2}{2\pi\hbar}$$

- Electron drift velocity at the edge

$$v = \frac{E}{B}c$$

- The wave equation for the propagating edge wave:

$$\partial_t \rho + v\partial_x \rho = 0, \qquad \rho(x) = nh(x)$$

- The Hamiltonian (i.e. the energy) of the edge waves:

$$H = \int \mathrm{d}x\ \frac{1}{2}e\rho E h = \int \mathrm{d}x\ \pi\frac{v}{\nu}\rho^2$$

Low energy effective theory of edge excitations

- In momentum space $\rho(x) = \sum_k L^{-1/2} \mathrm{e}^{\mathrm{i}kx} \rho_k$ (L = length of ring):

$$\dot{\rho}_k = -\mathrm{i} v k \rho_k, \quad H = 2\pi \frac{v}{\nu} \sum_{k>0} \rho_{-k}\rho_k, \quad k = \frac{2\pi}{L} \times \text{integer}.$$

- If we identify $\rho_k|_{k>0}$ as the 'coordinates' and $p_k = \mathrm{i}2\pi\rho_{-k}/\nu k$ as the corresponding canonical 'momenta', then the standard Hamiltonian equation will reproduce the equation of motion

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \quad \to \quad \dot{\rho}_k = \mathrm{i} v k \rho_k.$$

- The phase-space Lagrangian

$$S = \sum_{k>0} (p_k \dot{\rho}_k - 2\pi \frac{v}{\nu} \rho_{-k}\rho_k) = \int dx \frac{\pi}{\nu} \frac{1}{\partial_x} \rho \dot{\rho} - \int dx \frac{\pi v}{\nu} \rho^2$$

$$= -\frac{m}{4\pi} \int dx\, \partial_x \phi (\partial_t + v\partial_x)\phi, \quad \rho = \frac{1}{2\pi} \partial_x \phi$$

reproduces the Hamlitonian and equation of motion

$$H = \int \mathrm{d}x\, \frac{v}{4\pi\nu} (\partial_x \phi)^2 \qquad \partial_t \rho + v \partial_x \rho = 0.$$

Low energy effective theory of edge excitations

- In momentum space $\rho(x) = \sum_k L^{-1/2} \mathrm{e}^{\mathrm{i}kx} \rho_k$ (L = length of ring):

$$\dot{\rho}_k = -\mathrm{i} v k \rho_k, \quad H = 2\pi \frac{v}{\nu} \sum_{k>0} \rho_{-k}\rho_k, \quad k = \frac{2\pi}{L} \times \text{integer}.$$

- If we identify $\rho_k|_{k>0}$ as the 'coordinates' and $p_k = \mathrm{i}2\pi\rho_{-k}/\nu k$ as the corresponding canonical 'momenta', then the standard Hamiltonian equation will reproduce the equation of motion

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \quad \to \quad \dot{\rho}_k = \mathrm{i} v k \rho_k.$$

- The phase-space Lagrangian

$$S = \sum_{k>0} (p_k \dot{\rho}_k - 2\pi \frac{v}{\nu} \rho_{-k}\rho_k) = \int dx \frac{\pi}{\nu} \frac{1}{\partial_x} \rho\dot{\rho} - \int dx \frac{\pi v}{\nu} \rho^2$$

$$= -\frac{m}{4\pi} \int dx\, \partial_x \phi (\partial_t + v\partial_x)\phi, \quad \rho = \frac{1}{2\pi} \partial_x \phi$$

reproduces the Hamlitonian and equation of motion

$$H = \int \mathrm{d}x\, \frac{v}{4\pi\nu} (\partial_x \phi)^2 \qquad \partial_t \rho + v\partial_x \rho = 0.$$

Electron/quasiparticle operators and correlations

$[\rho_k, \rho_{k'}] = \frac{\nu}{2\pi} k \delta_{k+k'} \to [\rho(x), \rho(y)] = -\mathrm{i}\frac{\nu}{2\pi}\delta'(x-y)$
$\to [\rho(x), \phi(y)] = \mathrm{i}\nu\delta(x-y) \to [\phi(x), \phi(y)] = \mathrm{i}\pi\nu \mathrm{sgn}(x-y)$

- **Electron operator**: The electron operator on the edge creates a localized charge

$$[\rho(x), c(x')] = -\delta(x-x')c(x'), \qquad c \propto \mathrm{e}^{\mathrm{i}\frac{1}{\nu}\phi}, \quad \rho = \frac{1}{2\pi}\partial_x\phi$$

- **Quasiparticle operator**: The quasiparticle operator on the edge creates a localized charge

$$[\rho(x), \psi_q^\dagger(x')] = -\nu\delta(x-x')\psi_q^\dagger(x'), \quad \psi_q \propto \mathrm{e}^{\mathrm{i}\phi}, \quad \rho = \frac{1}{2\pi}\partial_x\phi$$

- Correlations and conformal dimensions:

$\langle \rho(x,t)\rho(0,0)\rangle = -\frac{1}{(2\pi)^2 m}\frac{1}{(x+\mathrm{i}vt)^2} \to \langle \phi(x,t)\phi(0,0)\rangle = -\frac{\ln(x+\mathrm{i}vt)}{m}$
$\to$
$\langle c(z)c^\dagger(z')\rangle = \langle \mathrm{e}^{\mathrm{i}m\phi(z)}\mathrm{e}^{-\mathrm{i}m\phi(z')}\rangle \sim \frac{1}{(z-z')^m}, \quad g_e = 2\mathrm{dim}(c) = m.$
$\to$
$\langle \psi_q(z)\psi_q^\dagger(z')\rangle = \langle \mathrm{e}^{\mathrm{i}\phi(z)}\mathrm{e}^{-\mathrm{i}\phi(z')}\rangle \sim \frac{1}{(z-z')^{1/m}}, \quad g_q = 2\mathrm{dim}(\psi_q) = \frac{1}{m}.$

Edge CFT → bulk CFT

- The above theory $\frac{m}{4\pi}\partial_x\phi(\partial_t + v\partial_x)\phi$ for edge states is a chiral conformal field theory – a Gaussian model or a $U(1)$ current algebra, with electron operator $c(z) \propto e^{\mathrm{i}m\phi(z)}$
- The correlation of the N electron operators in the CFT produce the N-electron bulk wave function:

$$P(\{z_i\}) = \prod(z_i - z_j)^m \propto \langle [c^\dagger(z_\infty)]^N_{\text{point-split}} \prod c(z_i)\rangle$$

The above can be shown by using

$$\langle e^{-\mathrm{i}mN\phi(z_\infty)} \prod e^{\mathrm{i}m\phi(z_i)}\rangle = \langle e^{-\mathrm{i}mN\phi(z_\infty)+\sum \mathrm{i}m\phi(z_i)}\rangle$$

$$= e^{\frac{1}{2}\langle[-\mathrm{i}mN\phi(z_\infty)+\sum \mathrm{i}m\phi(z_i)]^2\rangle} \propto e^{-m^2\sum_{i<j}\langle\phi(z_i)\phi(z_j)\rangle} = \prod_{i<j}(z_i - z_j)^m$$

If the bulk wave function can be written as a correlation of "electron operators in a "bulk CFT", then the same CFT will also "describe" the gapless edge states.

Edge CFT $\to$ bulk CFT

- The above theory $\frac{m}{4\pi}\partial_x\phi(\partial_t + v\partial_x)\phi$ for edge states is a chiral conformal field theory – a Gaussian model or a $U(1)$ current algebra, with electron operator $c(z) \propto e^{i m\phi(z)}$
- The correlation of the N electron operators in the CFT produce the N-electron bulk wave function:

$$P(\{z_i\}) = \prod (z_i - z_j)^m \propto \langle [c^\dagger(z_\infty)]^N_{\text{point-split}} \prod c(z_i)\rangle$$

The above can be shown by using

$$\langle e^{-i mN\phi(z_\infty)} \prod e^{i m\phi(z_i)}\rangle = \langle e^{-i mN\phi(z_\infty)+\sum i m\phi(z_i)}\rangle$$

$$= e^{\frac{1}{2}\langle[-i mN\phi(z_\infty)+\sum i m\phi(z_i)]^2\rangle} \propto e^{-m^2\sum_{i<j}\langle\phi(z_i)\phi(z_j)\rangle} = \prod_{i<j}(z_i - z_j)^m$$

If the bulk wave function can be written as a correlation of "electron operators in a "bulk CFT", then the same CFT will also "describe" the gapless edge states.

Application to Laughline FQH states

- CFT $= \frac{1}{4\pi}\partial_x\phi(\partial_t + v\partial_x)\phi \to \langle e^{i\alpha\phi(z_1)} e^{-i\alpha\phi(z_1)}\rangle \propto \frac{1}{(z_1 - z_2)^{\alpha^2}}$
  $\to$ Scaling dimension of $e^{i\alpha\phi(z_1)}$: $h = [e^{i\alpha\phi(z_1)}] = \frac{\alpha^2}{2}$
- OPE:
$$e^{i\alpha_1\phi(z_1)} e^{i\alpha_2\phi(z_2)} \propto \frac{e^{i(\alpha_1+\alpha_2)\phi(z_2)}}{(z_1 - z_2)^{\frac{\alpha_1^2}{2}+\frac{\alpha_1^2}{2}-\frac{(\alpha_1+\alpha_2)^2}{2}}} + \cdots$$
- If we choose the electron operator as $c(z) = e^{i\sqrt{m}\phi(z)}$, the electron operators will be mutually local respect to each other, since $\frac{m}{2} + \frac{m}{2} - \frac{4m}{2} = -m \in \mathbb{Z}$.
- For the above choice of electron operator, the quasiparticle operator must be of a form $\psi_q^n(z) = e^{i\frac{n}{\sqrt{m}}\phi(z)}$, since $\frac{n^2}{2m} + \frac{m}{2} - \frac{(\sqrt{m}+\frac{n}{\sqrt{m}})^2}{2} = -n \in \mathbb{Z}$
- We see that $\psi_q \sim c^{1/m}$ as expected.
- If we replace ϕ by $\sqrt{m}\phi$, we will recover the previous description.

"Electron" operator and "quasiparticle" operators in CFT

- The electron operator $c(z)$, by definition, is an operator in CFT that give rise to single-valued correlation function

$$P(\{z_i\}) \propto \langle O(z_\infty) \prod c(z_i) \rangle$$

Single-valued respect to z_i's $\to$ Electron operators c are mutually local respect to each other.

- A quasiparticle operator $\psi_q(\xi)$, by definition, is an operator in CFT that give rise to single-valued correlation function respect to the electron operator $c(z)$

$$P_q(\xi_1, \xi_2\{z_i\}) \propto \langle O(z_\infty)\psi_{q_1}(\xi_1)\psi_{q_2}(\xi_2) \prod c(z_i) \rangle.$$

Single-valued respect to z_i's, and may not be single-valued respect to ξ, ξ'. $\to$ Quasiparticle operators ψ_q are local respect to the electron operator c; Quasiparticle operator ψ_{q_1} may not be local respect to another quasiparticle operator ψ_{q_2}.

$\mathrm{CFT}_{\mathrm{minimal}}$ + electron operator $c \leftrightarrow$ FQH state.

Application to Laughline FQH states

- CFT $= \frac{1}{4\pi}\partial_x\phi(\partial_t + v\partial_x)\phi \to \langle e^{i\alpha\phi(z_1)} e^{-i\alpha\phi(z_1)}\rangle \propto \frac{1}{(z_1-z_2)^{\alpha^2}}$
  $\to$ Scaling dimension of $e^{i\alpha\phi(z_1)}$: $h = [e^{i\alpha\phi(z_1)}] = \frac{\alpha^2}{2}$
- OPE:
$$e^{i\alpha_1\phi(z_1)} e^{i\alpha_2\phi(z_2)} \propto \frac{e^{i(\alpha_1+\alpha_2)\phi(z_2)}}{(z_1 - z_2)^{\frac{\alpha_1^2}{2}+\frac{\alpha_1^2}{2}-\frac{(\alpha_1+\alpha_2)^2}{2}}} + \cdots$$
- If we choose the electron operator as $c(z) = e^{i\sqrt{m}\phi(z)}$, the electron operators will be mutually local respect to each other, since $\frac{m}{2} + \frac{m}{2} - \frac{4m}{2} = -m \in \mathbb{Z}$.
- For the above choice of electron operator, the quasiparticle operator must be of a form $\psi_q^n(z) = e^{i\frac{n}{\sqrt{m}}\phi(z)}$, since
  $\frac{n^2}{2m} + \frac{m}{2} - \frac{(\sqrt{m}+\frac{n}{\sqrt{m}})^2}{2} = -n \in \mathbb{Z}$
- We see that $\psi_q \sim c^{1/m}$ as expected.
- If we replace ϕ by $\sqrt{m}\phi$, we will recover the previous description.

Electron/quasiparticle operators for Pfaffian state

- Edge effective theory of filling fraction $\nu = 1/m$ Pfaffian state:

$$\mathcal{L} = \frac{m}{4\pi}\partial_x\phi(\partial_t + v\partial_x)\phi + \lambda(\partial_t + v\partial_x)\lambda$$

- Electron operator: Ferminic electron $\to \nu = 1/\text{even}$

$$c(z) = \lambda(z)\mathrm{e}^{\mathrm{i}\, m\phi(z)}, \quad \langle c^\dagger(z)c(0)\rangle \sim 1/z^{m+1}, \quad g_e = m+1$$

- Quasiparticle operator

$$\psi_q(z) = \sigma(z)\mathrm{e}^{\mathrm{i}\,\frac{1}{2}\phi(z)}, \quad \lambda(z)\sigma(0) \sim \frac{1}{z^{1/2}}\sigma(0)$$

$$\mathrm{e}^{\mathrm{i}\,\frac{m}{\phi}(z)}\mathrm{e}^{\mathrm{i}\,\frac{1}{2}\phi(0)} \sim \frac{1}{z^{\frac{m^2}{2m}+\frac{(\frac{1}{2})^2}{2m}-\frac{(m+\frac{1}{2})^2}{2m}}}\mathrm{e}^{\mathrm{i}\,(m+\frac{1}{2})\phi} = z^{1/2}\mathrm{e}^{\mathrm{i}\,(m+\frac{1}{2})\phi}$$

- σ is the boundary-condition-changing operator with scaling dimension $h_\sigma = \frac{1}{16}$. On a ring $E_{tot} = \epsilon L + \frac{2\pi v}{L}(-\frac{c}{24} + h_{exe})$
The chiral Majorona model= $\frac{1}{2}$ 1D transverse Ising model $-\sum(\sigma_i^x\sigma_{i+1}^x + h\sigma^z)$ at critical point and $\sigma_L\sigma_R = \sigma^x$.
- Quasiparticle correlation $\langle\psi_q^\dagger(z)\psi_q(0)\rangle \sim 1/z^{\frac{1}{4m}+\frac{1}{8}}$, $g_q = \frac{1}{4m} + \frac{1}{8}$