Title: Topological Order Series

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Abstract:

Lectures on topological order: fusion category and modular tensor category theories of topological excitations

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Category theory – a theory of relations (morphism)

A category $\mathcal{C}$ is a set $\{1, \alpha, \beta, \cdots\}$ of "objects" (particles), with morphism (relation) $\alpha \to \beta$:

- Morphism $\alpha \to \alpha$ exists.
- If morphisms $\alpha \to \beta$ and $\beta \to \gamma$ exist, $\to$ morphism $\alpha \to \gamma$ exists.

α is a simple object if only $\alpha \to \alpha$ exists.

- Example:

  Objects = particles carrying a $SO(3)$ representation.

  Morphism = local symmetric preserving deformation/evolutions.

  There is no morphism between spin-1 and spin-2 particles.

  There is a morphism between (spin-1$\oplus$spin-2) and (spin-2$\oplus$spin-3) particles.

  In this example, morphism is a complicated but physical way to characterize the group representation structure carried by the particle. Simple object = particle carrying irreducible representations.

Philosophy of category theory

Usually, when we try to understand an object, we like to divide the object into smaller pieces (or more basic components). If we can do that, we say that we gain a better understanding of the object. This is the reductionist approach.

But there is another way of understanding. We do not think about the internal structure of the object, and pretend the internal structure is not there. (Maybe the internal structure really does not exist.) We try to understand an object through all its relations with all other objects. In fact, we use all those relations (morphisms) to define the object. In other words, there are no objects, just relations. A collection of relations defines the notion (and all the properties) of an object.

- The notion of phase (object) is defined via phase transition (morphism)
- The notion of quantum wave function (object) is defined via measurement (morphism)

Tensor (fusion) category theory

Tensor category: a category with fusion $\alpha \otimes \beta$.

- Example:
  two particles (spin-1) and (spin-2), from far aways, can be viewed as one particle:

$$(\text{spin-1}) \otimes (\text{spin-2}) = (\text{spin-1} \oplus \text{spin-2} \oplus \text{spin-3})$$

We see that simple-object $\otimes$ simple-object = composite object.

Braided fusion category theory

Braided fusion category (BFC): a tensor (fusion) cetegory with baiding $\alpha \otimes \beta \to \beta \otimes \alpha$.

- Example:
  All spin-s particle are bosons. Such an example Rep($SO(3)$) is a BFC that describes a gapped bosonic state with $SO(3)$ symmetry and trivial topological order.
- Another example:

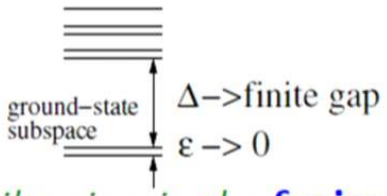
- BFC sRep($SU(2)$) has particles labeled by the $SU(2)$ representation.
- The half-integer spins are fermions and integer spins are bosons.
- Such a BFC describes a gapped spin-1/2 fermionic state with $SU(2)$ symmetry and trivial topological order.

- sRep($SU(2)$) $\neq$ Rep($SU(2)$). The BFC Rep($SU(2)$) describes a gapped bosonic state with $SU(2)$ symmetry and trivial topological order.

Theory of non-Abelian statistics: fusion space of topo. exc.

What are the most general properties of the topological excitations? can be boson, can be fermion, can be semion, ...

A state with quasiparticles $|i_1, i_2, i_3, \cdots\rangle$
at $\vec{x}_1, \vec{x}_2, \vec{x}_3, \cdots$ = a gapped ground state of
$H + \delta H_{i_1}^{\text{trap}}(\vec{x}_1) + \delta H_{i_2}^{\text{trap}}(\vec{x}_2) + \cdots$



- *The ground state subspace of the above Hamiltonian is the* **fusion space** $\mathcal{V}^F(i_1, i_2, i_3, \cdots)$ *of the quasiparticles* $i_1, i_2, i_3, \cdots$.
- If the ground state degeneracy is stable arbitary purterbations around $\vec{x}_1, \vec{x}_2, \vec{x}_3, \cdots$, then the traped quasiparticles are **simple**.
- If the ground state degeneracy is not stable against some perturbations $\delta H(\vec{x}_1)$ near $\vec{x}_1$, then the traped quasiparticle i_1 at $\vec{x}_1$ is **composite**.
- If i_1 is composite, we can add $\delta H(\vec{x}_1)$ to split the ground state subspace:
$$\mathcal{V}^F(i_1, i_2, i_3, \cdots) \to \mathcal{V}^F(j_1, i_2, i_3, \cdots) \oplus \mathcal{V}^F(k_1, i_2, i_3, \cdots) \oplus \cdots$$
We denote $i_1 = j_1 \oplus k_1 \oplus \cdots$.

Fusion algebra of (non-Abelian) topological excitations

- For simple i, j, if we view (i, j) as one particle, it may correspond to a composite particle:

$$\mathcal{V}^F(i,j,l_1,l_2,\cdots) = \oplus_{\tilde{k}} \mathcal{V}^F(\tilde{k},l_1,l_2,\cdots)$$
$$= \oplus_k \oplus_{\alpha_k^{ij}=1}^{N_k^{ij}} \mathcal{V}^F_{\alpha_k^{ij}}(k,l_1,l_2,\cdots)$$

$i \otimes j = \oplus_k N_k^{ij} k \rightarrow$ the *fusion algebra*.

- **Associativity**:
$(i \otimes j) \otimes k = i \otimes (j \otimes k) = \oplus_l N_l^{ijk} l, \quad N_l^{ijk} = \sum_m N_m^{ij} N_l^{mk} = \sum_n N_n^{jk} N_l^{in}$
- **Topologically protected non-local degrees of freedom**:
For simple quasiparticles, $i, j, \cdots$, we cannot view their fusion space $\mathcal{V}^F(i,j,k,\cdots)$ as $\mathcal{V}(i) \otimes \mathcal{V}(j) \otimes \mathcal{V}(k) \otimes \cdots$, where the space $\mathcal{V}(i)$ describes the local degrees of freedom of the quasiparticle-i.
If so, we can add local perturbations near i to split the degeneracy.

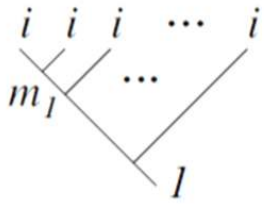
For simple quasiparticles, the degrees of freedom described by their fusion space $\mathcal{V}^F(i,j,k,\cdots)$ are non-local and topologically protected.

Quantum dimension and "fractional" degree of freedom

Vector space fractionalization:

- In general, $\dim[\mathcal{V}^F(i,i,i,\cdots)] \neq d_i^n,\ d_i \in \mathbb{Z}$.
  Quasiparticle i may carry fractional degree freedom.
  $\dim[\mathcal{V}^F(i,i,\cdots,i)] = \sum_{m_i} N_{m_1}^{ii} N_{m_2}^{m_1 i} \cdots N_1^{m_{n-2} i} = (\mathbf{N}^i)_{i1}^{n-1} \sim d_i^n$
  where the matrix $(\mathbf{N}^i)_{jk} = N_k^{ji}$, and d_i the largest eigenvalue of $\mathbf{N}^i$:

$$\dim[\mathcal{V}^F(i,i)] = N_1^{ii}, \quad \dim[\mathcal{V}^F(i,i,i)] = N_{m_1}^{ii} N_1^{m_1 i},$$
$$\dim[\mathcal{V}^F(i,i,i,i)] = N_{m_1}^{ii} N_{m_2}^{m_1 i} N_1^{m_2 i}.$$



- d_i is called the *quantum dimension* of the quasiparticle i.
  Abelian particle $\to d_i = 1$. Non-Abelian particle $\to d_i \neq 1$.

Relation between fusion spaces and the F-matrix

- Two different ways to fuse $i, j, k \to l$:
  $(i \otimes j) \otimes k = i \otimes (j \otimes k) = \oplus_l N_l^{ijk} l \to$
  $N_l^{ijk} = \dim \mathcal{V}^F(i,j,k,l^*) = \sum_m N_m^{ij} N_l^{mk} = \sum_n N_n^{jk} N_l^{in}$
  Two ways to get the fusion space $\mathcal{V}^F(i,j,k,l^*) \to$ Two basis sets:

$$\mathcal{V}^F(i,j,k,l^*) = \oplus_m \oplus_{\alpha_m^{ij}=1}^{N_m^{ij}} \mathcal{V}^F_{\alpha_m^{ij}}(m,k,l^*)$$

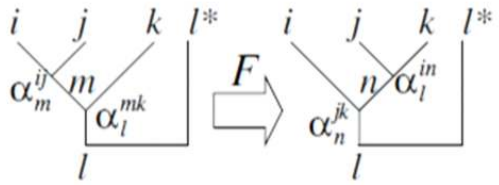
$$= \oplus_m \oplus_{\alpha_m^{ij}=1}^{N_m^{ij}} \oplus_{\alpha_l^{mk}=1}^{N_l^{mk}} \mathcal{V}^F_{\alpha_m^{ij};\alpha_l^{mk},m}(l,l^*)$$

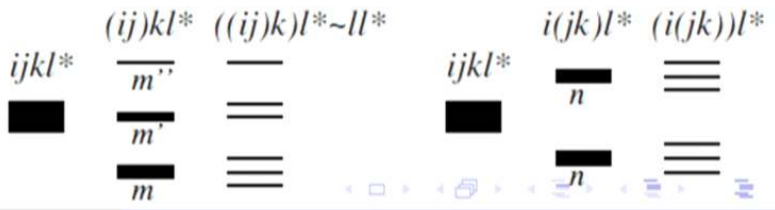
$$= \{|\alpha_m^{ij}, \alpha_l^{mk}, m\rangle_{(ij)k\to l}\}$$

$$\mathcal{V}^F(i,j,k,l^*) = \oplus_n \oplus_{\alpha_n^{jk}=1}^{N_n^{jk}} \mathcal{V}^F_{\alpha_n^{jk}}(i,n,l^*)$$

$$= \oplus_n \oplus_{\alpha_n^{jk}=1}^{N_n^{jk}} \oplus_{\alpha_l^{in}=1}^{N_l^{in}} \mathcal{V}^F_{\alpha_n^{jk};\alpha_l^{in},n}(l,l^*)$$

$$= \{|\alpha_n^{jk}, \alpha_l^{in}, n\rangle_{i(jk)\to l}\}$$





Relation between fusion spaces and the F-matrix

- $|\alpha_n^{jk}, \alpha_l^{in}, m\rangle_{(ij)k\to l} = \sum_{n,\alpha_n^{jk},\alpha_l^{in}} F^{ijk;m,\alpha_m^{ij},\alpha_l^{mk}}_{l;n,\alpha_n^{jk},\alpha_l^{in}} |\alpha_n^{jk}, \alpha_l^{in}, n\rangle_{i(jk)\to l}$

  where $\mathbf{F}_l^{ijk}$ is an unitary matrix.

Relation between fusion spaces and the F-matrix

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  where $\mathbf{F}_l^{ijk}$ is an unitary matrix.

- Two different ways to fuse: $(i,j),k,\cdots$ and : $i,(j,k),\cdots$:

$$V^F(i,j,k,\cdots) = \oplus_m \oplus_{\alpha_m^{ij}=1}^{N_m^{ij}} V^F_{\alpha_m^{ij}}(m,k,\cdots)$$
$$= \oplus_m \oplus_{\alpha_m^{ij}=1}^{N_m^{ij}} \oplus_l \oplus_{\alpha_l^{mk}=1}^{N_l^{mk}} V^F_{\alpha_m^{ij};\alpha_l^{mk},m}(l,\cdots)$$
$$= \oplus_l \{|\alpha_m^{ij}, \alpha_l^{mk}, m\rangle_{(ij)k\to l}\} \otimes V^F(l,\cdots)$$

$$V^F(i,j,k,\cdots) = \oplus_n \oplus_{\alpha_n^{jk}=1}^{N_n^{jk}} V^F_{\alpha_n^{jk}}(i,n,\cdots)$$
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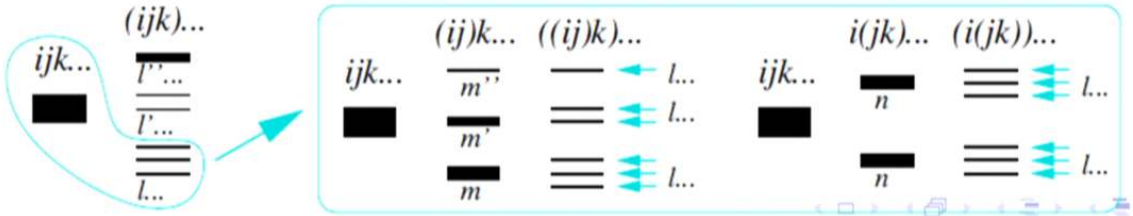
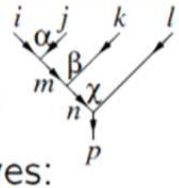
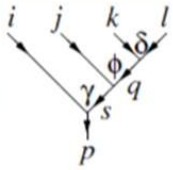
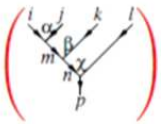
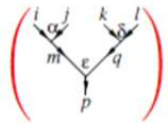
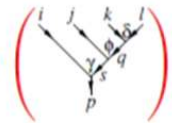
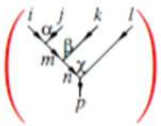
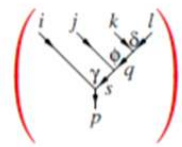


Consistent conditions for $F^{ijk;m\alpha\beta}_{l;n\chi\delta}$ and UFC

Two different ways of fusion  and  are related via two different paths of F-moves:

$\Phi\Big($  $\Big) = \sum_{q,\delta,\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon}\,\Phi\Big($  $\Big) = \sum_{q,\delta,\epsilon;s,\phi,\gamma} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma}\,\Phi\Big($  $\Big)$,

$\Phi\Big($  $\Big) = \sum_{t,\eta,\varphi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi}\,\Phi\Big($ $\Big) = \sum_{t,\eta,\varphi;s,\kappa,\gamma} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma}\,\Phi\Big($ $\Big)$

$= \sum_{t,\eta,\kappa;\varphi;s,\kappa,\gamma;q,\delta,\phi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi}\,\Phi\Big($  $\Big)$.

The two paths should lead to the same unitary trans.:

$$\sum_{t,\eta,\varphi,\kappa} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi} = \sum_{\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma}$$

Such a set of non-linear algebraic equations is the famous pentagon identity. Moore-Seiberg 89

N^{ij}_k, $F^{ijk;m\alpha\beta}_{l;n\chi\delta}$ $\rightarrow$ **Unitary fusion category (UFC)**