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Abstract:

Infinitesimal Generators = Vector Fields

In differential geometry, one identifies vector fields with
first order differential operators (derivations).

In local coordinates, the vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\,\frac{\partial}{\partial x^i} \;+\; \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\,\frac{\partial}{\partial u^\alpha}$$

on the space of independent and dependent variables, generates the one-parameter group (flow)

$$\frac{dx^i}{d\varepsilon} = \xi^i(x,u) \qquad \frac{du^\alpha}{d\varepsilon} = \varphi^\alpha(x,u)$$

Invariance

A function $F : M \to \mathbb{R}$ is invariant if it is not affected by the group transformations:

$$F(g \cdot z) = F(z)$$

for all $g \in G$ and $z \in M$.

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \varphi \frac{\partial}{\partial u} = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u}$$

Characteristic:

$$Q(x, u, u_x) = \varphi - u_x \xi = x + u\, u_x$$

By the prolongation formula, the infinitesimal generator is

$$\mathrm{pr}\, \mathbf{v} = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} + (1 + u_x^2) \frac{\partial}{\partial u_x} + 3 u_x u_{xx} \frac{\partial}{\partial u_{xx}} + \cdots$$

★ The solutions to the characteristic equation

$$Q(x, u, u_x) = x + u\, u_x = 0$$

are circular arcs — rotationally invariant curves.

Functional Gradient

Functional

$$F[u] = \int L(x, u^{(n)})\, dx$$

Variation $v = \delta u$:

$$F[u+v] = F[u] + \langle \delta F\,; v \rangle + \text{ h.o.t.}$$

$$= \int L(u, u_t, u_{tt}, \ldots)\, dt + \int \left(\frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u_t} v_t + \frac{\partial L}{\partial u_{tt}} v_{tt} \right) dt + \cdots$$

Integration by parts:

$$\int \left(\frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u_t} v_t + \frac{\partial L}{\partial u_{tt}} v_{tt} \right) dt = \int \left(\frac{\partial L}{\partial u} - D_t \frac{\partial L}{\partial u_t} + D_t^2 \frac{\partial L}{\partial u_{tt}} - \cdots \right) v\, dt$$

Euler–Lagrange equations:

$$\delta F = E(L) = \frac{\partial L}{\partial u} - D_t \frac{\partial L}{\partial u_t} + D_t^2 \frac{\partial L}{\partial u_{tt}} - \cdots = 0$$

Conservation Laws — Dynamics

In continua, a conservation law states that the temporal rate of change of a quantity P in a region of space D is governed by the associated flux through its boundary:

$$\frac{\partial}{\partial t}\int_D T\,dx = \oint_{\partial D} X$$

or, in differential form,

$$\mathrm{D}_t T = \mathrm{Div}\, X$$

- In particular, if the flux X vanishes on the boundary ∂D, then the total density $\int_D T\,dx$ is conserved — constant.

Conservation Laws — Statics

In statics, a conservation law corresponds to a path- or surface-independent integral $\oint_C X = 0$ — in differential form,

$$\mathrm{Div}\, X = 0$$

Thus, in fracture mechanics, one can measure the conserved quantity near the tip of a crack by evaluating the integral at a safe distance.

Conservation Laws in Analysis

- ⋆ In modern mathematical analysis, most existence theorems, stability results, scattering theory, etc., for partial differential equations rely on the existence of suitable conservation laws.
- ⋆ Completely integrable systems can be characterized by the existence of infinitely many higher order conservation laws.
- ⋆ In the absence of symmetry, Noether's Identity is used to construct divergence identities that take the place of conservation laws in analysis.

Noether's First Theorem

Theorem. If $\mathbf{v}$ generates a one-parameter group of variational symmetries of a variational problem, then the characteristic Q of $\mathbf{v}$ is the characteristic of a conservation law of the Euler-Lagrange equations:

$$\mathrm{Div}\, P = Q\, E(L)$$

Proof: Noether's Identity = Integration by Parts

$$\boxed{\mathrm{pr}\, \mathbf{v}(L) + L\, \mathrm{Div}\, \xi = Q\, E(L) - \mathrm{Div}\, P}$$

$\mathrm{pr}\, \mathbf{v}$ — prolonged vector field (infinitesimal generator)

Q — characteristic of $\mathbf{v}$

P — boundary terms resulting from the integration by parts computation

Conservation of Energy

Infinitesimal generator and characteristic:

$$\mathbf{v} = \frac{\partial}{\partial u} \qquad\qquad Q = -u_t$$

Euler–Lagrange equations:

$$E(L) = \frac{\partial L}{\partial u} - D_t \frac{\partial L}{\partial u_t} + D_t^2 \frac{\partial L}{\partial u_{tt}} - \cdots = 0$$

Conservation law:

$$0 = Q\,E(L) = u_t \left(\frac{\partial L}{\partial u} - D_t \frac{\partial L}{\partial u_t} + D_t^2 \frac{\partial L}{\partial u_{tt}} - \cdots \right)$$

$$= D_t \left(L - u_t \frac{\partial L}{\partial u_t} + \cdots \right)$$

Conservation Law $\Longrightarrow$ Symmetry

$$\boxed{\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = Q\, E(L) - \operatorname{Div} P}$$

Conversely, if

$$\operatorname{Div} A = Q\, E(L)$$

is any conservation law, assumed, without loss of generality, to be in characteristic form, and Q is the characteristic of the vector field $\mathbf{v}$, then

$$\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = \operatorname{Div}(A - P) = \operatorname{Div} B$$

and hence $\mathbf{v}$ generates a divergence symmetry group.

Integrable Systems

In the 1960's, the discovery of the soliton in Kruskal and Zabusky's numerical studies of the Korteweg–deVries equation, a model for nonlinear water waves, which was motivated by the Fermi–Pasta–Ulam problem, provoked a revolution in the study of nonlinear dynamics.

The theoretical justification of their observations came through the study of the associated symmetries and conservation laws.

Indeed, integrable systems like the Korteweg–deVries equation, nonlinear Schrödinger equation, sine-Gordon equation, KP equation, etc. are characterized by their admitting an infinite number of higher order symmetries – as first defined by Noether — and, through Noether's theorem, higher order conservation laws!

The Strong Version

Noether's First Theorem. Let $\Delta = 0$ be a normal system of Euler-Lagrange equations. Then there is a one-to-one correspondence between nontrivial conservation laws and nontrivial variational symmetries.

⋆ A system of partial differential equations is normal if, under a change of variables, it can be written in Cauchy–Kovalevskaya form.

Noether's Second Theorem

Theorem. A system of Euler-Lagrange equations is under-determined, and hence admits a nontrivial differential relation if and only if it admits an infinite dimensional variational symmetry group depending on an arbitrary function.
The associated conservation laws are trivial.

Open Question: Are there over-determined systems of Euler–Lagrange equations for which trivial symmetries give non-trivial conservation laws?

A Very Simple Example:

Variational problem:

$$I[u,v] = \iint (u_x + v_y)^2 \, dx \, dy$$

Variational symmetry group:

$$(u,v) \longmapsto (u + \varphi_y, v - \varphi_x)$$

Euler-Lagrange equations:

$$\Delta_1 = E_u(L) = u_{xx} + v_{xy} = 0$$

$$\Delta_2 = E_v(L) = u_{xy} + v_{yy} = 0$$

Differential relation:

$$D_x \Delta_2 - D_y \Delta_2 \equiv 0$$