Title: Emmy Noether: Her life, work, and influence

Date: Jun 22, 2015 08:00 PM

URL: http://pirsa.org/15060040

Abstract: Emmy Noether was a giant of mathematics whose work tied together two fundamental concepts: conservation laws and symmetries in nature. But who was she, and why does her work still have such impact? Mathematician Peter Olver explores Noetherâ€™s life and career, and delves into the curious history of her famous theorems. Physicist Ruth Gregory looks at the lasting impact of Noetherâ€™s theorem, and how it connects with the Standard Model and Einsteinâ€™s general relativity.

# *Emmy Noether: Symmetry and Conservation; History and Impact*

Peter J. Olver

University of Minnesota

http://www.math.umn.edu/~olver

# References

[1] E. Noether, Invariante Variationsprobleme, *Nachr. König. Gesell. Wissen. Göttingen, Math.–Phys. Kl.* (1918) 235–257.

[2] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, Graduate Texts in Math., vol. 107, Springer–Verlag, New York, 1993.

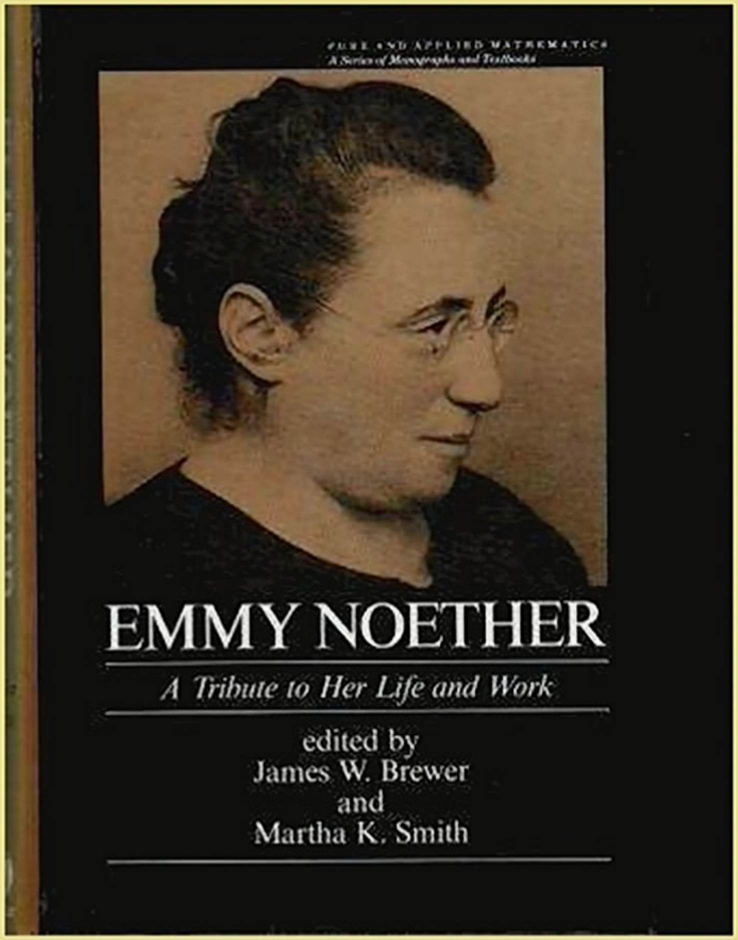
[3] Y. Kosmann-Schwarzbach, *The Noether Theorems. Invariance and Conservation Laws in the Twentieth Century*, Springer, New York, 2011.

Includes English translation of [1]

*Emmy Noether was one of the most influential mathematicians of the century. The development of abstract algebra, which is one of the most distinctive innovations of twentieth century mathematics, is largely due to her — in published papers, in lectures, and in personal influence on her contemporaries.*

— Nathan Jacobson

# Noether the Mathematician




III NOETHER'S MATHEMATICS

6. Galois Theory — 115

*Richard G. Swan*

1. Introduction / 2. Constructing Galois Extensions / 3. Generic Galois Extensions / 4. Fischer's Theorem / 5. Galois Descent / 6. A Birational Invariant / 7. Stable Equivalence / 8. Lenstra's Theorem / 9. Tori / 10. Steenrod's Problem / References

7. The Calculus of Variations — 125

*E. J. McShane*

8. Commutative Ring Theory — 131

*Robert Gilmer*

*Idealtheorie in Ringbereichen* / *Abstrakter Aufbau der Idealtheorie in algebraischen Zahl- und Funktionenkörpern* / Emmy Noether's Influence in Commutative Ring Theory / Acknowledgments

9. Representation Theory — 145

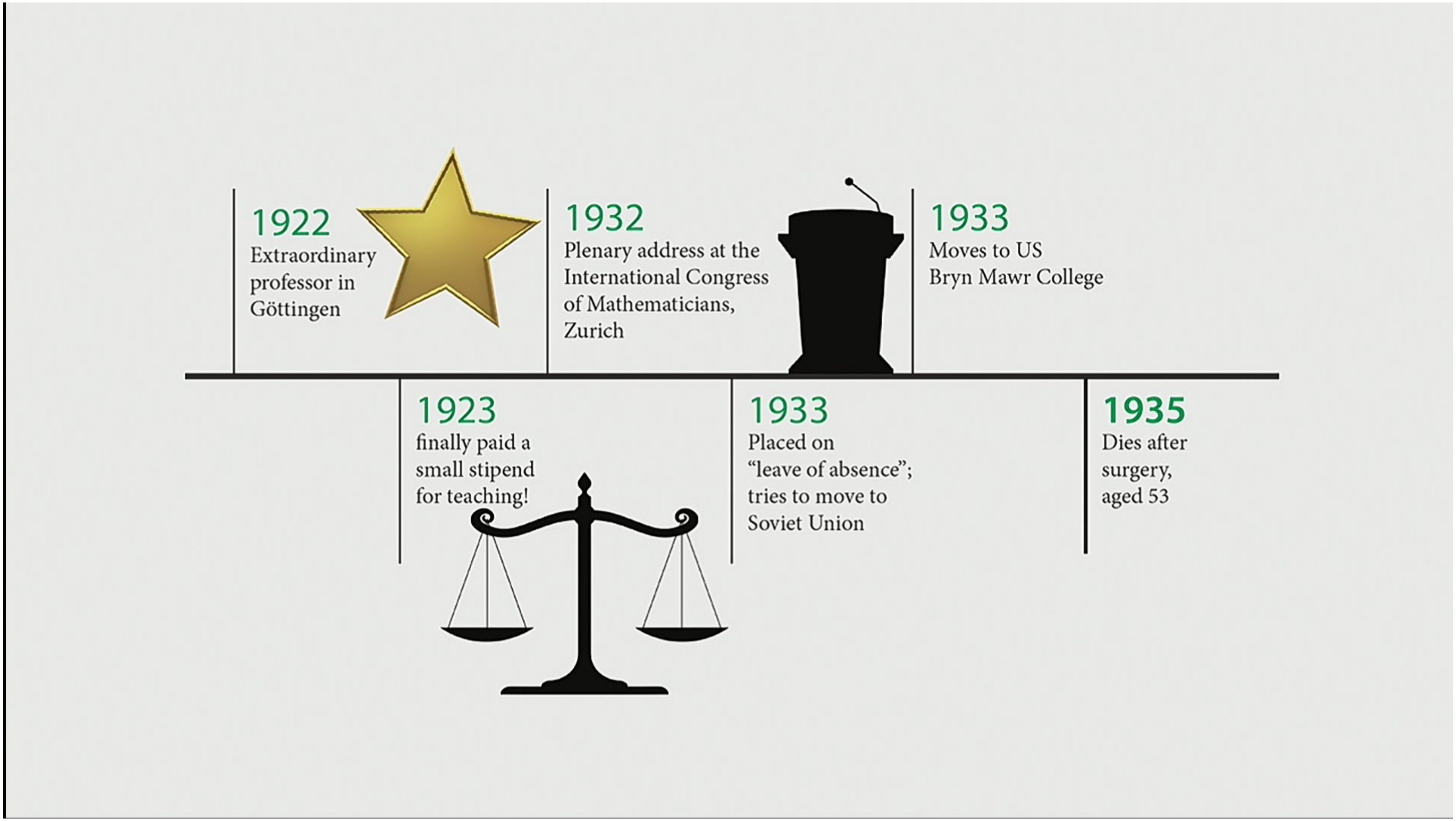
*T. Y. Lam*

Notes / References

10. Algebraic Number Theory — 157

*A. Fröhlich*

Galois Module Structure / Cohomology, or Central Simple Algebras / References


1922
Extraordinary professor in Göttingen
1932
Plenary address at the International Congress of Mathematicians, Zurich
1933
Moves to US
Bryn Mawr College
1923
finally paid a small stipend for teaching!
1933
Placed on "leave of absence"; tries to move to Soviet Union
1935
Dies after surgery, aged 53

# Noether's Three Fundamental Contributions to Analysis and Physics

**First Theorem.** There is a one-to-one correspondence between symmetry groups of a variational problem and conservation laws of its Euler–Lagrange equations.

---

**Second Theorem.** An infinite-dimensional variational symmetry group depending upon an arbitrary function corresponds to a nontrivial differential relation among its Euler–Lagrange equations.

---

★ The conservation laws associated with the variational symmetries in the Second Theorem are trivial — this resolved Hilbert's original paradox in relativity that was the reason he and Klein invited Noether to Göttingen.

# Noether's Three Fundamental Contributions to Analysis and Physics

**First Theorem.** There is a one-to-one correspondence between symmetry groups of a variational problem and conservation laws of its Euler–Lagrange equations.

---

**Second Theorem.** An infinite-dimensional variational symmetry group depending upon an arbitrary function corresponds to a nontrivial differential relation among its Euler–Lagrange equations.

---

**Introduction** of higher order generalized symmetries.

$\Longrightarrow$ later (1960's) to play a fundamental role in the discovery and classification of integrable systems and solitons.

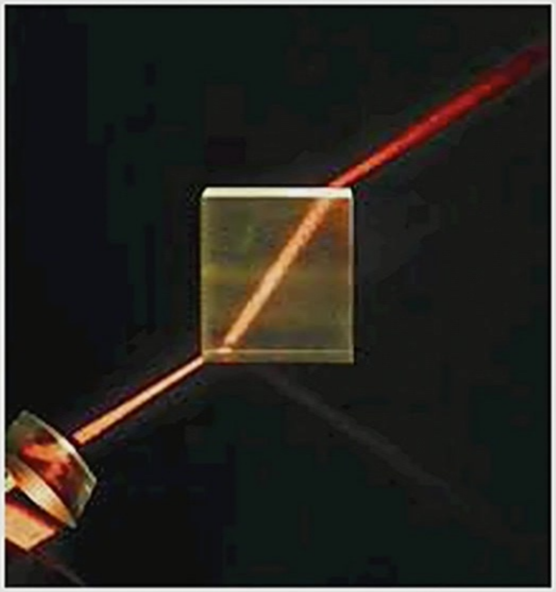
## The Calculus of Variations

*[Leibniz] conceives God in the creation of the world like a mathematician who is solving a minimum problem, or rather, in our modern phraseology, a problem in the calculus of variations — the question being to determine among an infinite number of possible worlds, that for which the sum of necessary evil is a minimum.*

— Paul du Bois-Reymond

# Fermat's Principle in Optics

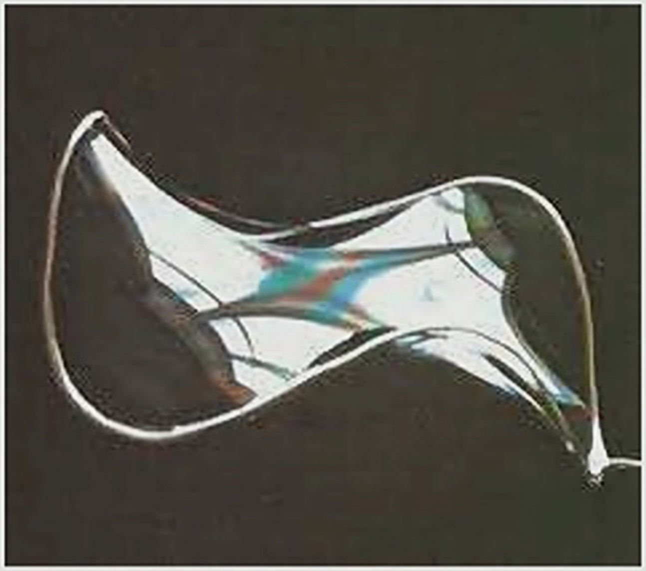
Light travels along the path that takes the least time:



$\Longrightarrow$ Snell's Law = Loi de Descartes

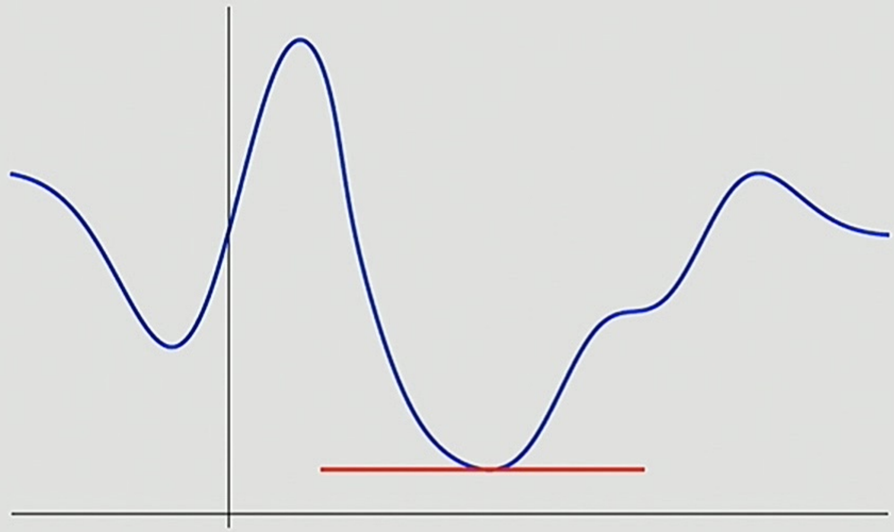
# Plateau's Problem

The surface of least area spanning a space curve
is a minimal surface. $\Longrightarrow$ soap films



# Minimization

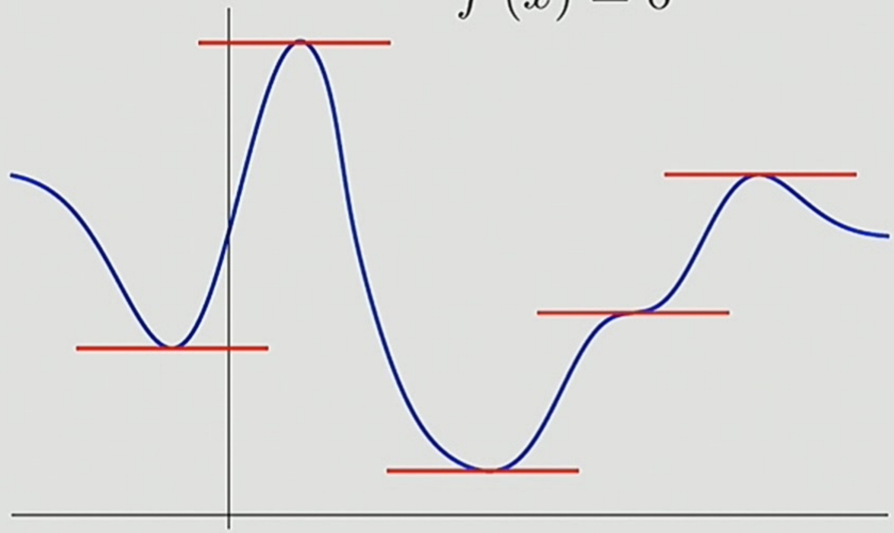
- At any minimum of a function the tangent line is horizontal:



## The First Derivative Test

A minimum of a (nice) function of one variable $f(x)$ satisfies

$$f'(x) = 0$$



♣ But this also holds at maxima and inflection points!

⋆ Distinguishing minima from maxima from inflection points requires the second derivative test — not used here!

# How do you find the peaks in a mountain range?

A better solution:

The gradient of the height function is the vector that points in the direction of steepest increase, i.e., uphill

# The Variational Principle

In general, a variational problem requires minimizing a function $\mathcal{F}$ over an infinite-dimensional space, in the form of an action functional, which depends on the space/time coordinates and the physical fields.

★ The functional gradient vanishes at the minima: $\delta\mathcal{F} = 0$.

$\Longrightarrow$ This gives a system of differential equations, whose solutions are the minimizers.

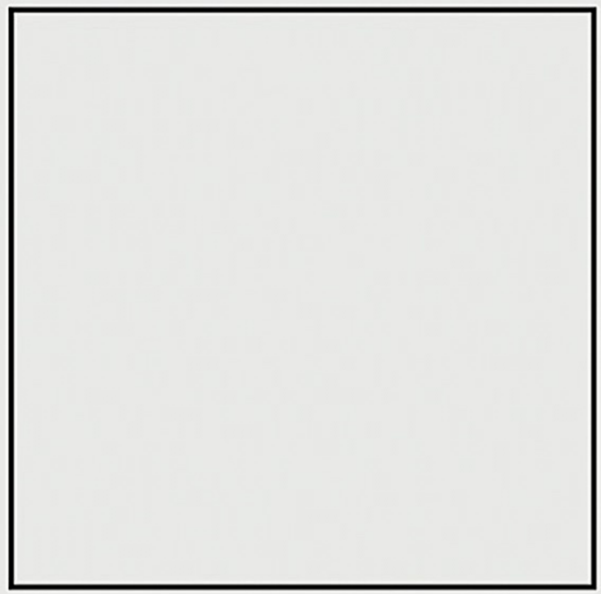
◇ **Modern Physics:** The action functional should incorporate all of the symmetries of Nature.

# A Brief History of Symmetry

**Symmetry $\Longrightarrow$ Group Theory!**

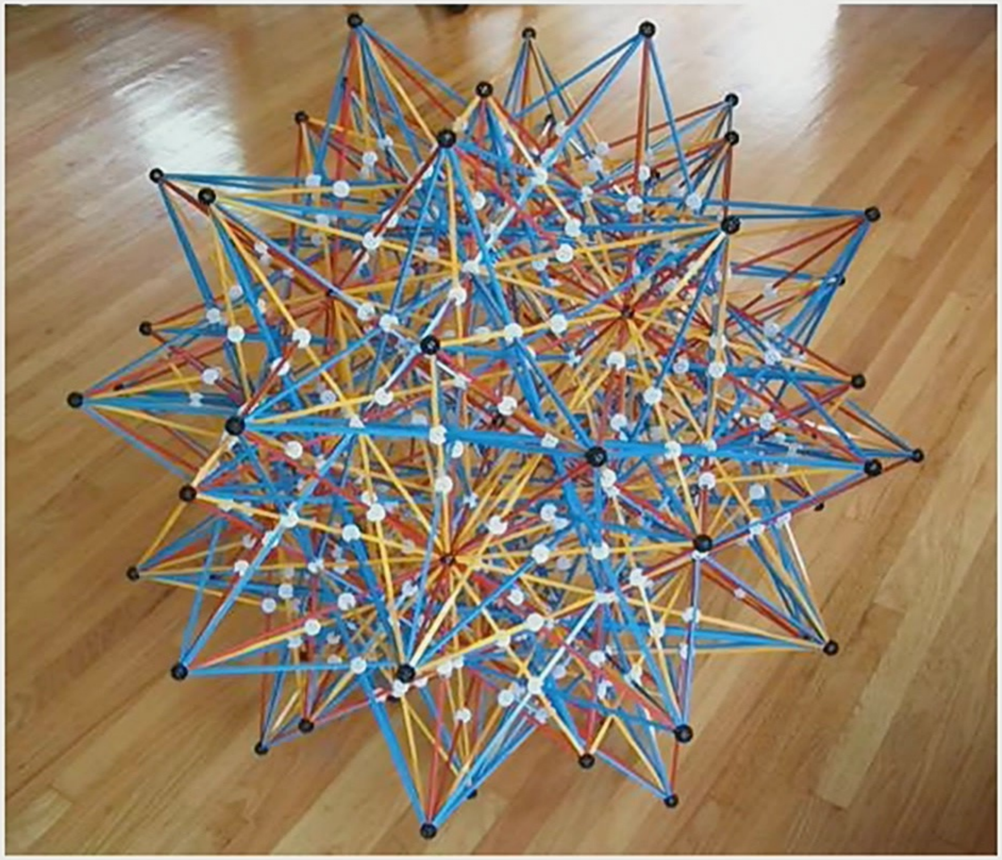
- Abel, Galois — polynomials
- Lie — differential equations and variational principles
- Noether — conservation laws and higher order symmetries
- Weyl, Wigner, etc. — quantum mechanics
  "der Gruppenpest" (J. Slater)

# Discrete Symmetry Group

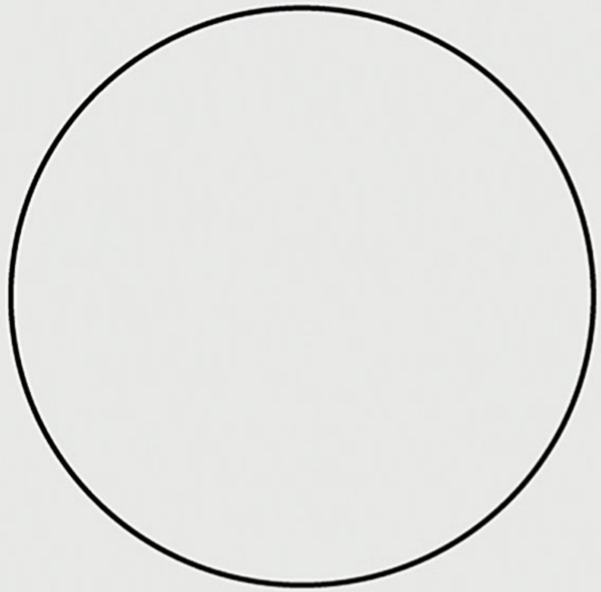


Symmetry group = rotations by $0^\circ, 90^\circ, 180^\circ, 270^\circ$

# Discrete Symmetry Group



# Continuous Symmetry Group



Symmetry group = *all* rotations

⋆ A continuous group is known as a Lie group
— in honor of Sophus Lie (1842–1899)

# A Brief History of Conservation Laws

In physics, a conservation law asserts that a particular measurable property $P$ of an isolated physical system does not change as the system evolves.

---

Conservation of momentum: Wallis (1670), Newton (1687)

Conservation of mass: Lomonosov (1748), Lavoisier (1774)

Conservation of energy: Lagrange (1788), Helmholtz (1847), Rankine (1850), also: Mohr, Mayer, Joule, Faraday, ...

## In Summary . . .

Noether's Theorem states that to each continuous symmetry group of the action functional there is a corresponding conservation law of the physical equations and vice versa.

# The Modern Manual for Physics

♠ To construct a physical theory:

**Step 1:** Determine the allowed group of symmetries:

- translations
- rotations
- conformal (angle-preserving) transformations
- Galilean boosts
- Poincaré transformations (relativity)
- gauge transformations
- CPT (charge, parity, time reversal) symmetry
- supersymmetry
- $\mathrm{SU}(3),\ \mathrm{G}_2,\ \mathrm{E}_8 \times \mathrm{E}_8,\ \mathrm{SO}(32),\ \ldots$
- etc., etc.

**Step 2:** Construct a variational principle ("energy") that admits the given symmetry group.

---

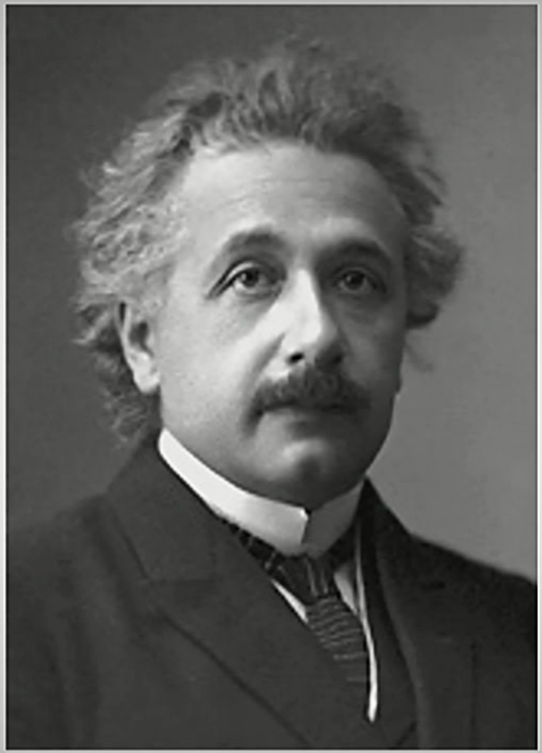
**Step 3:** Invoke Nature's obsession with minimization to determine the corresponding field equations associated with the variational principle.

---

**Step 4:** Use Noether's First and Second Theorems to write down $(a)$ conservation laws, and $(b)$ differential identities satisfied by the field equations.

---

**Step 5:** Try to solve the field equations.
Even special solutions are of immense interest
$\Longrightarrow$ black holes.



"Fraulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began." —*Albert Einstein.*

It is hard to overestimate the importance of Noether's work in modern physics.

Her basic insights on symmetry underlie our methods, our theories and our intuition.

The link between symmetry and conservation is how we describe our world.

# Conservation Laws

Conservation laws mean something rather specific in physics – charge is conserved, momentum, or energy. But we are used to the idea of conservation laws in life:


No such thing as a free lunch

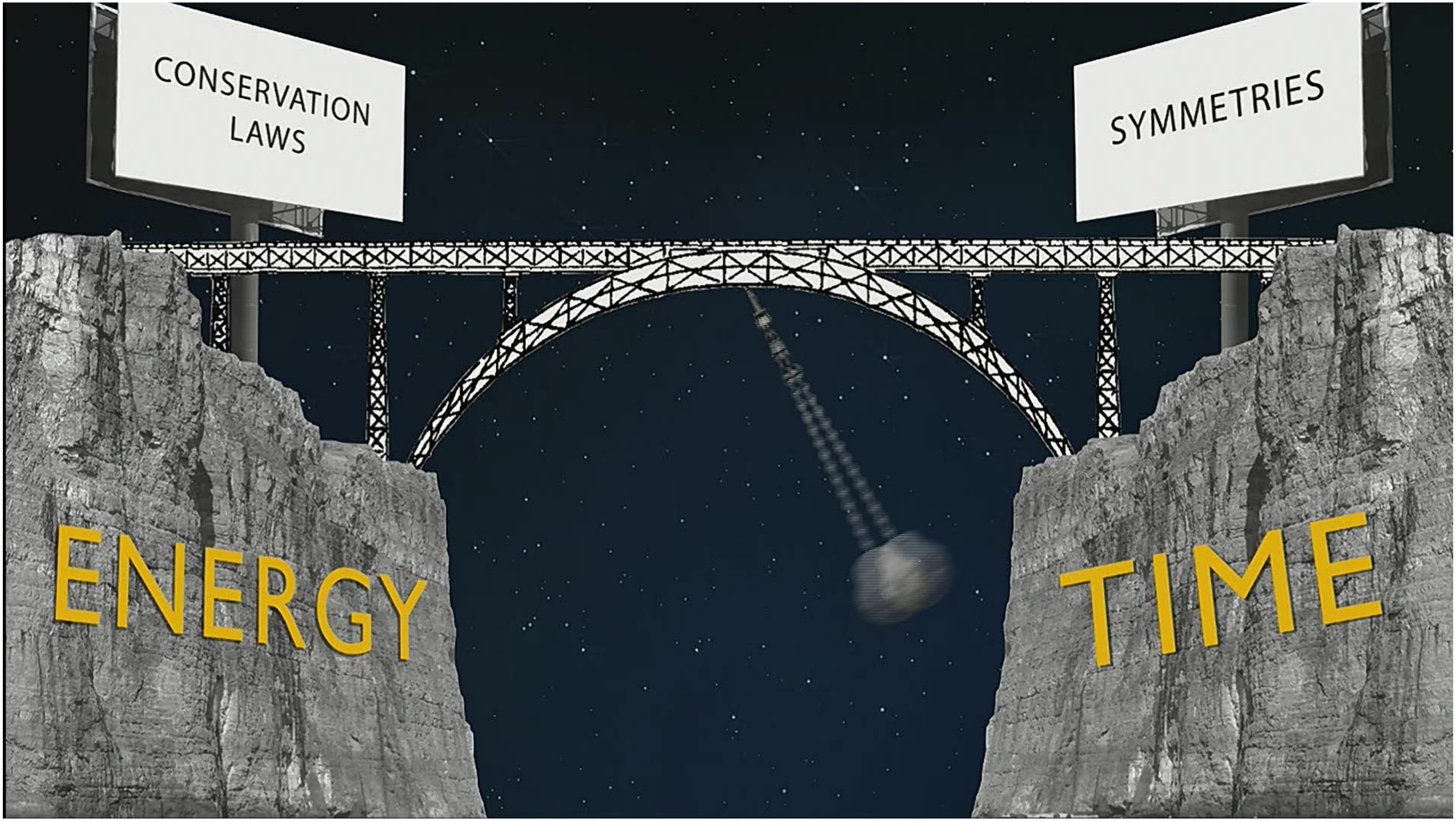
What goes around comes around

You get what you pay for

# Symmetry

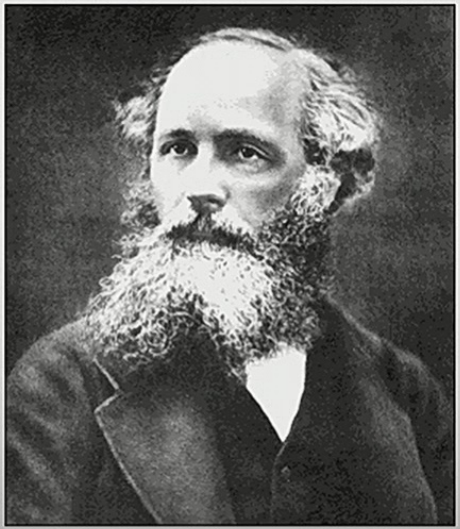
- An object, that turns out to be symmetric.

*H. Weyl*


CONSERVATION
LAWS
SYMMETRIES
ENERGY
TIME

# Electric Symmetry?

Faraday and Maxwell noticed that electricity and magnetism were related – one could look like the other if in motion. Maxwell's equations expressed this mathematically.

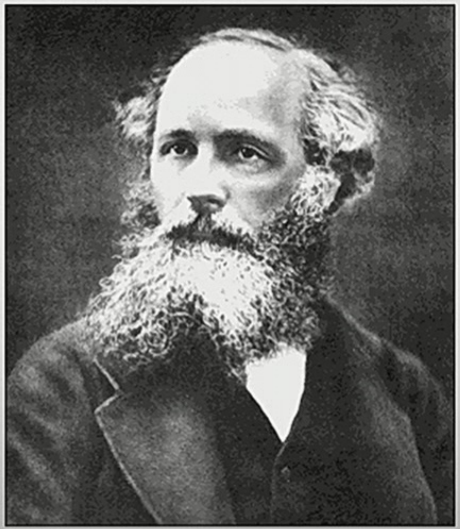


$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

# ELECTRIC SYMMETRY?

Faraday and Maxwell noticed that electricity and magnetism were related – one could look like the other if in motion. Maxwell's equations expressed this mathematically.



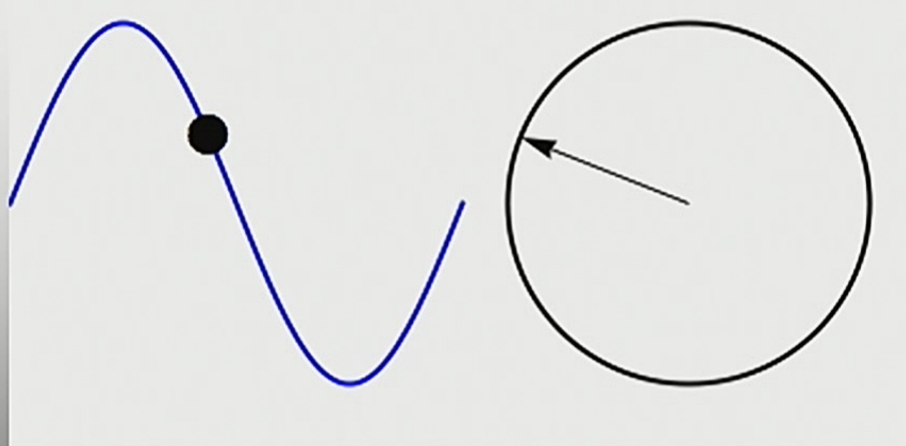
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

# PHASE

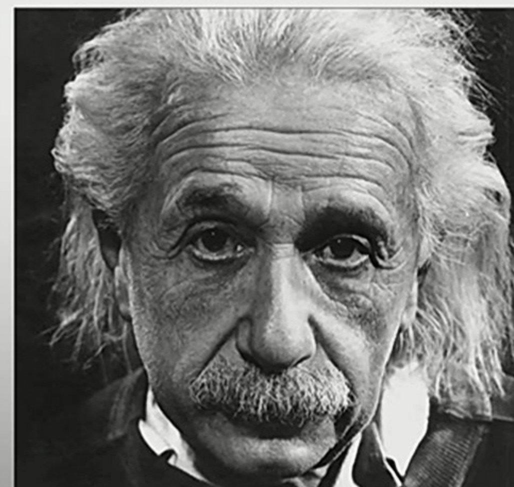
This *internal* symmetry we call *phase* – changing this phase shuffles the details of how the electromagnetic field is found, but not the field itself.

Phase is like an internal angle – we can visualise by thinking about a wave – the phase tells us where we are on the wave:



# Gauge Symmetry

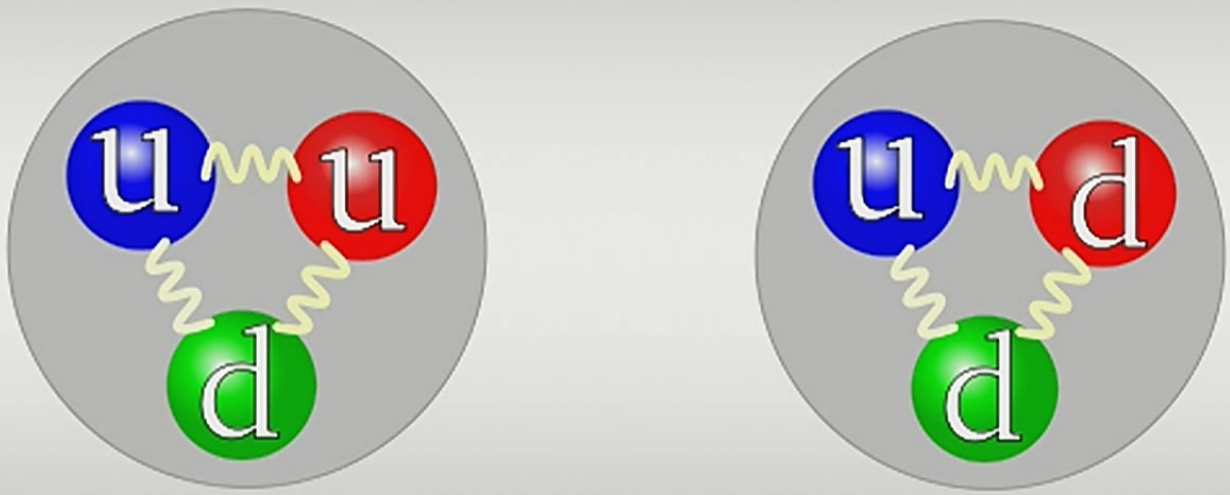
This is the basic underlying theme behind the **STANDARD MODEL** of particle interactions, the Higgs mechanism and Einstein's **GENERAL RELATIVITY**.



# PARTICLE PHYSICS

Like conservation of charge, there are other symmetries in nature even more hidden – we only know about them from “laws”, rules that tell us certain interactions can or can’t happen.

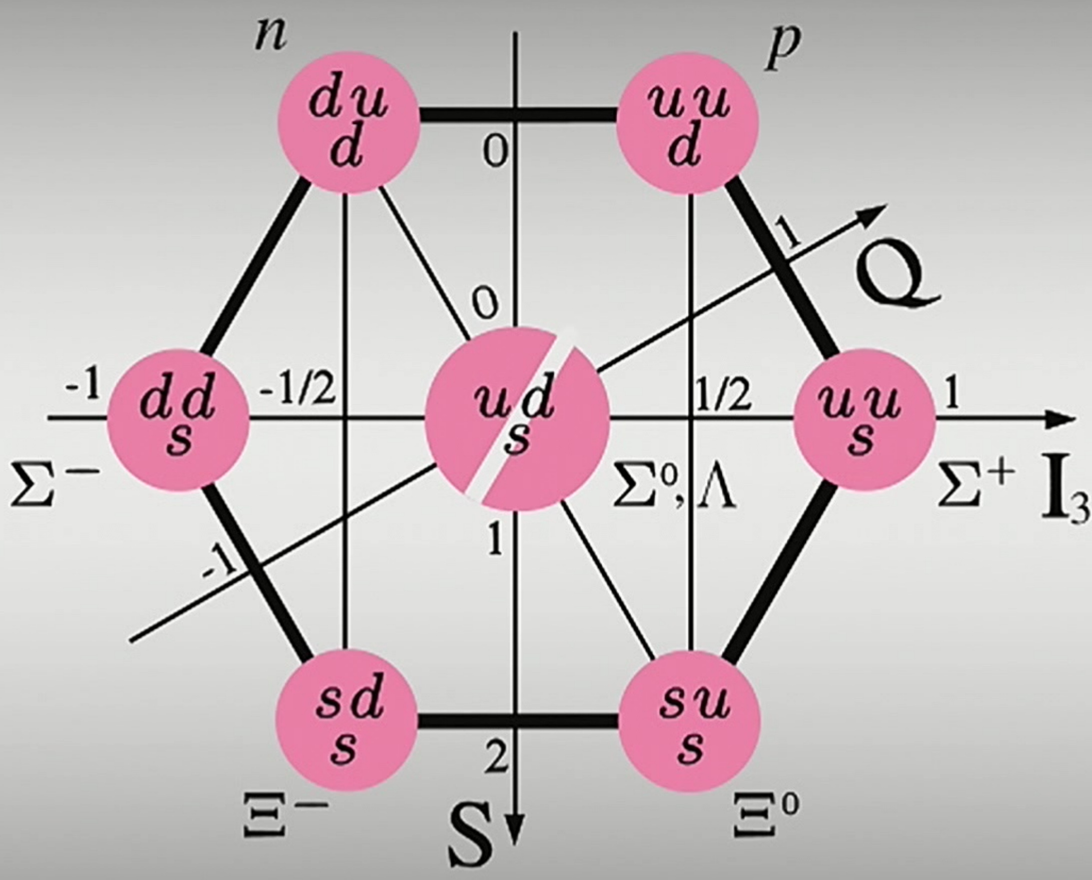
Although the proton and neutron seem different, they are very nearly the same … thinking of them in this way led Heisenberg to isospin…




Proton

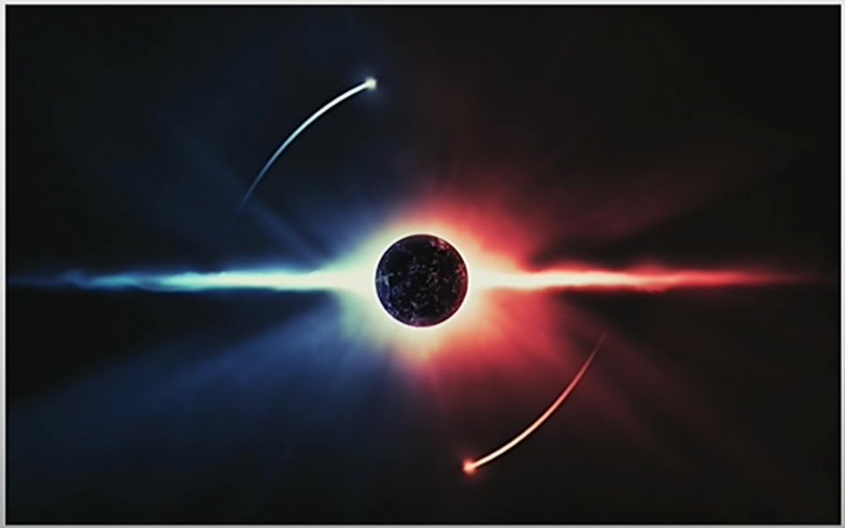
Neutron

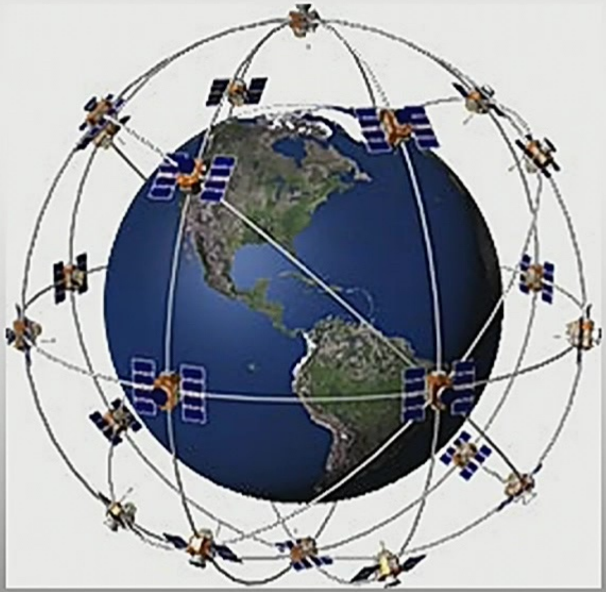
By building up patterns of new particles and their allowed interactions, physicists learned what charges were conserved, and what symmetries underlay those charges.



# General Relativity

Einstein's theory of gravity – explains the universe from less than nanoseconds old to now. Tested in our solar system and used in GPS.





General Relativity: “Space tells matter how to move; matter tells space how to curve”.

Problem is – gravity is highly nonlinear.

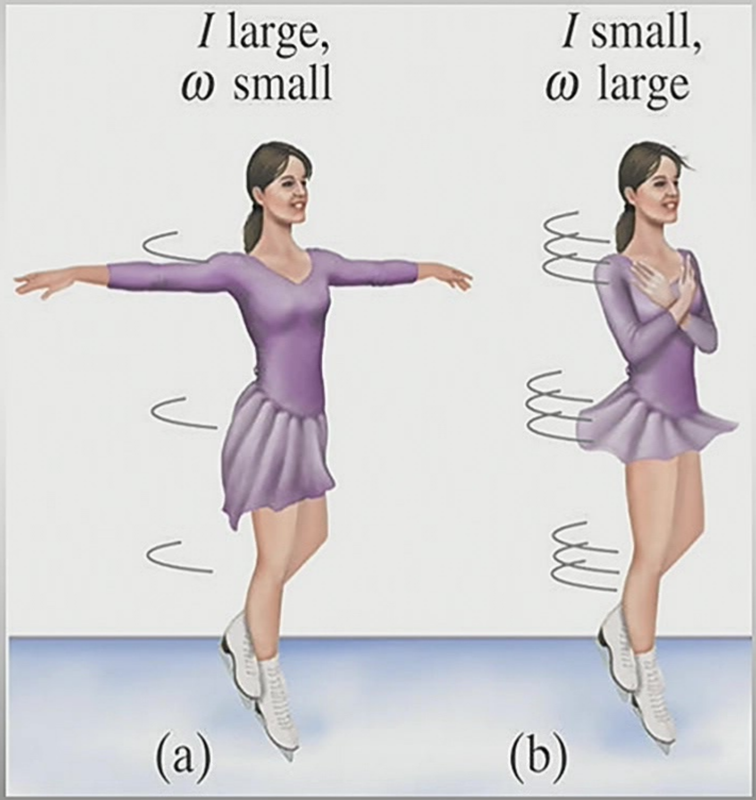
Noether helps us solve physical problems, like how planets move around the sun, or finding gravitational solutions, such as a black hole.

# Noether in Action

Local rotational symmetry of the laws of physics give conservation of angular momentum.

We see this with ice skaters spinning around – pulling their arms in reduces their moment of inertia, so speeds up their spin.

It's the same with the planets!



# Who was Emmy?

Completely unegotistical and free of vanity, she never claimed anything for herself, but promoted the works of her students above all.

*Bartel van de Waerden*

# EMMY NOETHER

Let's celebrate a wonderful lady and remember to recognize outstanding achievement in all its forms!