Title: Topological Order Series

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Abstract:

- **n-cluster condition**: No off-particle zeros when $c = n$ (or the wave function for the n-clusters is the Laughlin wave function or a n-electron cluster is "kind of trivial")
$\to D_{a+b,n} = D_{a,n} + D_{b,n} \to$

$$S_{a+kn} = S_a + kS_n + \frac{k(k-1)nm}{2} + kma$$

Since $S_1 = 0$, $(m, S_2, \cdots, S_n)$ carries all the information about the pattern of zeros from an n-cluster symmetric polynomial.

- Additional conditions

$$\Delta_2(a,a) = \text{even}, \qquad m > 0, \qquad mn = \text{even}, \qquad 2S_n = 0 \text{ mod } n.$$

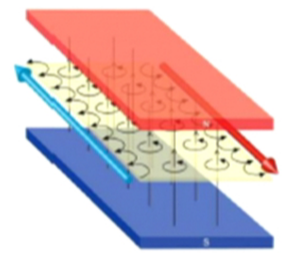
- **A mysterious condition** (the one we want but cannot prove):

$$\Delta_3(a,b,c) = \text{even}$$

Local dancing rule → global dancing pattern

- Local dancing rules of a FQH liquid:
  (1) every electron dances around clock-wise
  (Φ_{FQH} only depends on $z = x + iy$)
  (2) takes exactly three steps to go around each other
  (Relative angular momentum $S_2 = 3$) Wen-Wang arXiv:0803.1016
  → Global dancing pattern $\Phi_{\text{FQH}}(\{z_1, ..., z_N\}) = \prod (z_i - z_j)^3$
- A systematic theory of FQH state – **Pattern of zeros S_a**:
  a-electron cluster has a relative angular momentum S_a

$S_2 = 3$	$S_3 = 9$	$S_4 = 18$	$\nu = 1/3$ Laughlin
$S_2 = 1$	$S_3 = 5$	$S_4 = 10$	$\nu = 1/2$ Pfaffian



- Local dancing rules are enforce by the Hamiltonian to lower energy.
- Only certain sequences S_a correspond to valid FQH states. Which?
- Different POZ S_a give rise to different topological properties

Conditions on pattern of zeros – ground state

- **Concave conditions**

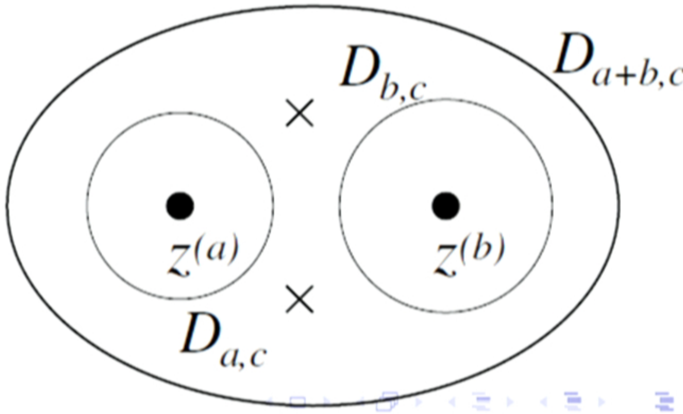
$$\Delta_2(a, b) \equiv S_{a+b} - S_a - S_b = D_{a,b} \geq 0,$$

$$\Delta_3(a, b, c) \equiv S_{a+b+c} - S_{a+b} - S_{b+c} - S_{a+c} + S_a + S_b + S_c \geq 0$$

The second one can be shown by considering $P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \cdots)$ as a function of $z^{(c)}$, which leads to

$$D_{a+b,c} = D_{a,c} + D_{b,c} + \text{ off-particle zeros at “}\times\text{”} \geq D_{a,c} + D_{b,c}$$

- $D_{a,c}, D_{b,c}$= the number of on-particle zeros at $z^{(a)}, z^{(b)}$.
- As $z^{(a)} \to z^{(b)}$, some off-particle zeros at “$\times$” may also approach $z^{(b)}$.
- $D_{a+b,c} \geq D_{a,c} + D_{b,c}$ leads to the second condition.



Primitive solutions for pattern of zeros

The conditions are semi-linear $\to$
if $(m; S_2, \cdots, S_n)$ and $(m'; S'_2, \cdots, S'_n)$ are solutions, then $(m''; S''_2, \cdots, S''_n) = (m; S_2, \cdots, S_n) + (m'; S'_2, \cdots, S'_n)$ is also a solution $\sim P''(\{z_i\}) = P(\{z_i\})P'(\{z_i\})$

1-cluster state: $\nu = 1/m$ Laughlin state

$$P_{1/m}: \qquad \mathbf{S} = (m;),$$
$$(n_0, \cdots, n_{m-1}) = (1, 0, \cdots, 0).$$

2-cluster state: Pfaffian state (Z_2 parafermion state)

$$P_{\frac{2}{2};Z_2}: \qquad (m; S_2) = (2; 0),$$
$$(n_0, \cdots, n_{m-1}) = (2, 0)$$

3-cluster state: Z_3 parafermion state

$$P_{\frac{3}{2};Z_3}: \qquad (m; S_2, S_3) = (2; 0, 0),$$
$$(n_0, \cdots, n_{m-1}) = (3, 0)$$

4-cluster state: Z_4 parafermion state

$$P_{\frac{4}{2};Z_4} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0),$$
$$(n_0, \cdots, n_{m-1}) = (4, 0),$$

5-cluster states: Z_5 (generalized) parafermion state

$$P_{\frac{5}{2};Z_5} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0, 0),$$
$$(n_0, \cdots, n_{m-1}) = (5, 0)$$

$$P_{\frac{5}{8};Z_5^{(2)}} : (m; S_2, \cdots, S_n) = (8; 0, 2, 6, 10),$$
$$(n_0, \cdots, n_{m-1}) = (2, 0, 1, 0, 2, 0, 0, 0)$$

6-cluster state:

$$P_{\frac{6}{2};Z_6} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0, 0, 0),$$
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7-cluster states:

$$P_{\frac{7}{2};Z_7} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0, 0, 0, 0),$$
$$(n_0, \cdots, n_{m-1}) = (7, 0)$$

$$P_{\frac{7}{8};Z_7^{(2)}} : (m; S_2, \cdots, S_n) = (8; 0, 0, 2, 6, 10, 14),$$
$$(n_0, \cdots, n_{m-1}) = (3, 0, 1, 0, 3, 0, 0, 0)$$

$$P_{\frac{7}{18};Z_7^{(3)}} : (m; S_2, \cdots, S_n) = (18; 0, 4, 10, 18, 30, 42),$$
$$(n_0, \cdots, n_{m-1}) = (2, 0, 0, 0, 0, 1, 0, 0, 0, 2, 0, 0, 0, 0, 0)$$

$$P_{\frac{7}{14};C_7} : (m; S_2, \cdots, S_n) = (14; 0, 2, 6, 12, 20, 28),$$
$$(n_0, \cdots, n_{m-1}) = (2, 0, 1, 0, 1, 0, 1, 0, 2, 0, 0, 0, 0, 0)$$

- Also get composite parafermion state $P = P_{Z_{n_1}} P_{Z_{n_2}}$

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How good is the pattern-of-zero classification

Not so bad :-) Not so good :-(

- Every symm. poly. P corresponds to a unique pattern of zero $\{S_a\}$.
  Only some patterns of zero correspond to a unique symm. poly.
- It appears that all the primitive patterns of zero correspond to a unique symm. poly.
- It is known that
  some composite patterns of zero $\to$ a unique symm. polynomial
  some composite patterns of zero $\to$ several symm. polynomials
  So in general, we need more information than $\{n; m; S_a\}$ to fully characterize symmetry polynomial of infinite variables

Topological properties from pattern of zeros

For those patterns of zeros that uniquely characterize a FQH wave function, we should be able to calculate the topological properties of FQH states from the data $(n; m; S_2, \cdots, S_n)$.

Physical properties that we want to get

- The filling fraction (actually given by $\nu = n/m$).
- Topological degeneracy on torus (and other Riemann surface)
- Number of quasiparticle types
- Quasiparticle charges
- Quasiparticle scaling dimensions
- Quasiparticle fusion algebra
- Quasiparticle statistics (Abelian and non-Abelian)
- The counting of edge excitations (central charge c and spectrum)

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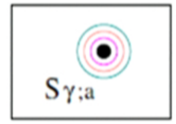
The pattern of zeros for quasiparticles

- A quasiparticle γ in a FQH state can also be quantitatively characterized by pattern of zeros $\{S_{\gamma;a}\}$:



- $P_\gamma(\eta; \{z_i\})$ has a quasiparticle at $z = \eta$
- Let $z_i = \lambda\xi_i + \eta$, $i = 1, 2, \cdots, a$ (bring a electrons to the quasiparticle)

$$P_\gamma(\eta; \{z_i\}) = \lambda^{S_{\gamma;a}} P_\gamma(z^{(a)} = \eta, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1})$$

$S_{\gamma;a}$ is the order of zero of $\Psi_\gamma(\xi, z_i)$ when we bring a electrons to ξ.

Quasi-holes in the $\nu = 1/m$ Laughlin state and fractional charge

- A hole-like excitation = missing an electron, charge = 1

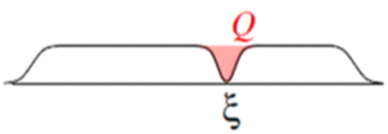
$$\prod_i (\xi - z_i)^m \prod_{i<j} (z_i - z_j)^m \mathrm{e}^{-\sum |z_i|^2/4}$$

which can be splitted into m quasi-hole excitations:

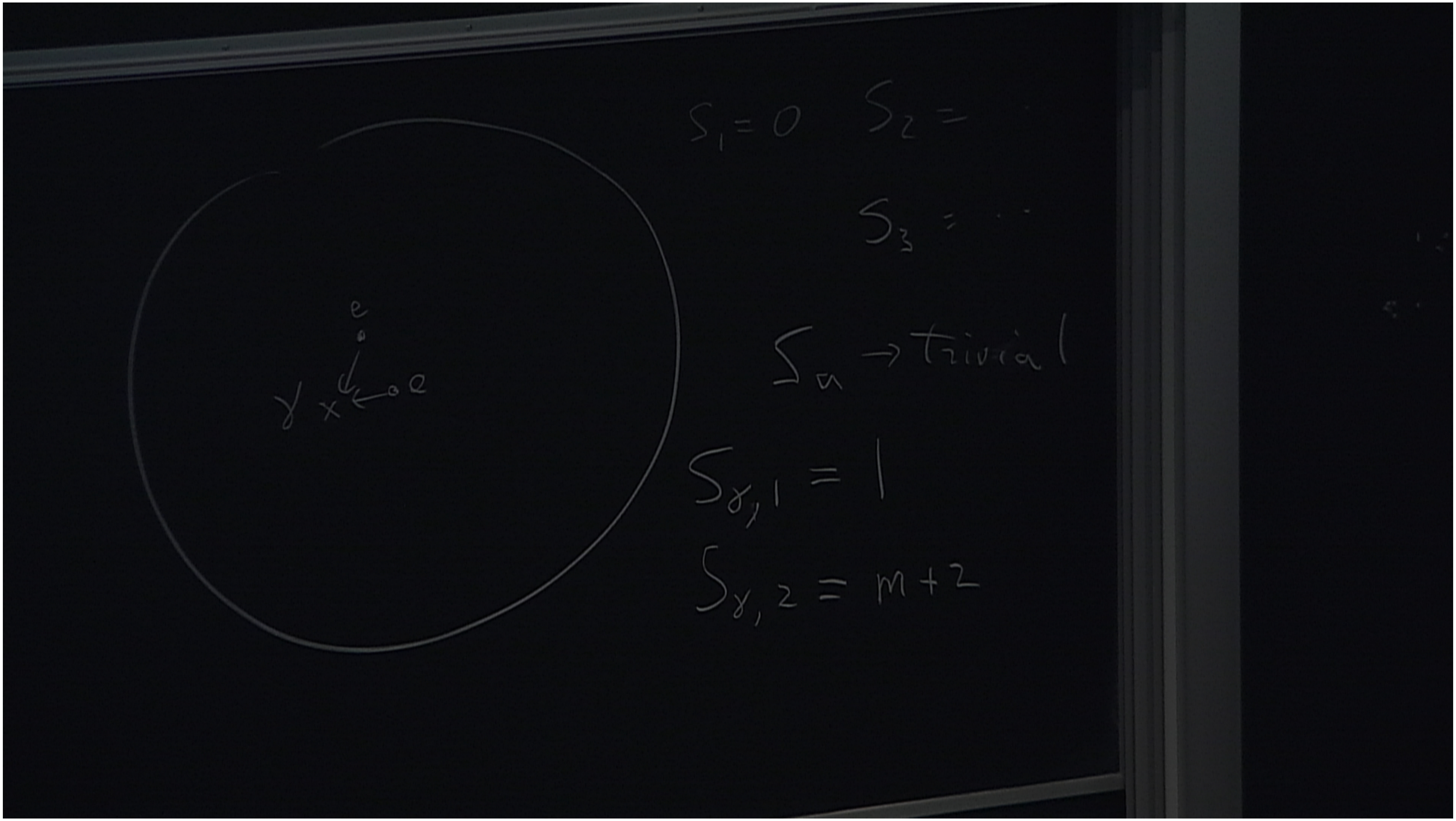
$$\prod_i (\xi_1 - z_i) \cdots \prod_i (\xi_m - z_i) \prod_{i<j} (z_i - z_j)^m \mathrm{e}^{-\sum |z_i|^2/4}$$

- A quasi-hole excitation = minimal excitation, charge = 1/m

$$\prod_i (\xi - z_i) \prod_{i<j} (z_i - z_j)^m \mathrm{e}^{-\sum |z_i|^2/4}$$



- Why the density dip have a small finte size?


S1 = 0   S2 =
S3 = . .
Sa → trivial
Sγ,1 = 1
Sγ,2 = m+2
e
γ x
e

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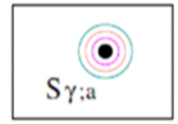

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$$P_\gamma(\eta; \{z_i\}) = \lambda^{S_{\gamma;a}} P_\gamma(z^{(a)} = \eta, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1})$$

$S_{\gamma;a}$ is the order of zero of $\Psi_\gamma(\xi, z_i)$ when we bring a electrons to ξ.
The sequence of integers $\{S_{\gamma;a}\}$ characterizes the quasiparticle γ.

- $\{S_a\}$ correspond to the trivial quasiparticle $\gamma = 0$: $\{S_{0;a}\} = \{S_a\}$

Conditions on $S_{\gamma;a}$

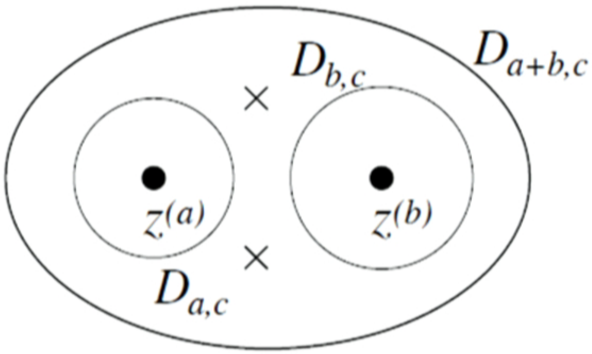
- Concave condition

$$S_{\gamma;a+b} - S_{\gamma;a} - S_b \geq 0,$$
$$S_{\gamma;a+b+c} - S_{\gamma;a+b} - S_{\gamma;a+c} - S_{b+c} + S_{\gamma;a} + S_b + S_c \geq 0$$

- Taking $b = c = 1$, we obtain

$$S_{\gamma;a+2} - 2S_{\gamma;a+1} + S_{\gamma;a} = l_{\gamma;a+2} - l_{\gamma;a+1} \geq S_2, \quad a \geq 0.$$

$l_{\gamma;a}$ **increases with** a: $l_{\gamma;a+1} - l_{\gamma;a} \geq S_2 \geq 0$.



$z^{(a)}$ is a quasiparticle γ fused with a electrons.

For the $\nu = 1$ Pfaffian state ($n = 2$ and $m = 2$)

$$S_1, S_2, \cdots : 0, 0, 2, 4, 8, 12, 18, 24, \cdots$$
$$n_0 n_1 n_2 \cdots : 2020202020202020202 \cdots$$

- Quasiparticle solutions ($S_{\gamma;a} \to l_{\gamma;a} \to n_{\gamma;l}$):

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 2020202020202020202 \cdots \qquad Q_\gamma = 0$$
$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 0202020202020202020 \cdots \qquad Q_\gamma = 1$$
$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 1111111111111111111 \cdots \qquad Q_\gamma = 1/2$$

- All other quasiparticle solutions can be obtained from the above three by removing extra electrons $\to$ only 3 quasiparticle types.
- Ground state degeneracy on torus = number of quasiparticle types
- Charge of quasiparticles

$$Q_\gamma = \frac{1}{m} \sum_{a=1}^{n} (l_{\gamma;a} - l_a)$$

the average increase of $l_{\gamma;a}$ from the ground state values l_a.

Quasi-holes in the $\nu = 1$ Pfaffian state: What is non-Abelian statistics?

- Ground state: $z_1 \approx z_2$, no zero; $z_1 \approx z_2 \approx z_3$, second-order zero;

$$\Psi_{\mathrm{Pf}} = \mathcal{A}\Big(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\frac{1}{z_{N-1} - z_N}\Big) = \mathrm{Pf}\Big(\frac{1}{z_i - z_j}\Big)$$

- A charge-1 quasi-hole state

$$\Psi_{\text{charge-1}} = \prod(\xi - z_i)\mathcal{A}\Big(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\frac{1}{z_{N-1} - z_N}\Big)$$

$$= \mathcal{A}\Big(\frac{(\xi - z_1)(\xi - z_2)}{z_1 - z_2}\frac{(\xi - z_3)(\xi - z_4)}{z_3 - z_4}\cdots\Big) = \mathrm{Pf}\Big(\frac{(\xi - z_i)(\xi - z_j)}{z_i - z_j}\Big)$$

- A state with two charge-$1/2$ quasi-holes

$$\Psi_{(\xi)(\xi')} = \mathcal{A}\Big(\frac{(\xi - z_1)(\xi' - z_2) + (1 \leftrightarrow 2)}{z_1 - z_2}\frac{(\xi - z_3)(\xi' - z_4) + (3 \leftrightarrow 4)}{z_3 - z_4}\cdots\Big)$$

$$= \mathrm{Pf}\Big(\frac{(\xi - z_i)(\xi' - z_j) + (\xi - z_j)(\xi' - z_i)}{z_i - z_j}\Big)$$

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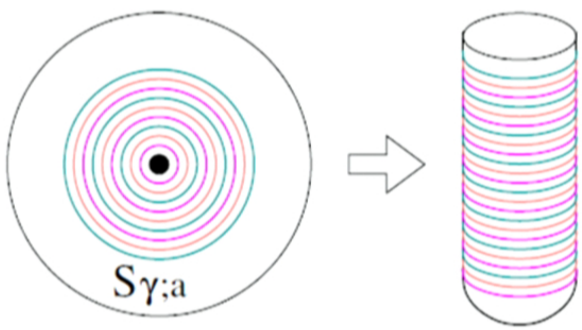
- All other quasiparticle solutions can be obtained from the above three by removing extra electrons $\to$ only 3 quasiparticle types.
- Ground state degeneracy on torus = number of quasiparticle types

Number of quasiparticles types = degeneracy on torus

Consider a quasiparticle γ on the disk at $z = 0$, whose type is determined by $n_{\gamma,l}$ in large l limit.

- On the disk, the wave function has the POZ S_a everywhere with $z \neq 0 \to$ zero energy.
- Deform the dist into a torus
- On the torus, the wave function has the POZ S_a everywhere $\to$ zero energy.
- The number of the zero-energy patterns (wave functions) is the same as the numbers of quaiparticle types.



For the $\nu = 1$ Pfaffian state ($n = 2$ and $m = 2$)

$$S_1, S_2, \cdots : 0, 0, 2, 4, 8, 12, 18, 24, \cdots$$

$$l_1 l_2 l_3 \cdots : 0, 0, 2, 2, 4, 4, 6, 6, 8, 8, \cdots$$

$$n_0 n_1 n_2 \cdots : 20202020202020202 \cdots ,$$

- GPE rule: *(1) Three particles occupy $\geq l_3 + 1 = 3$ obitals or more ($c = 1$). (2) The spread of two particles plus the spread of next two particles $\geq (l_2 - l_1) + (l_3 - l_2) = 2$ ($c = 2$). (3) ...*
- the quasiparticle occupation patterns

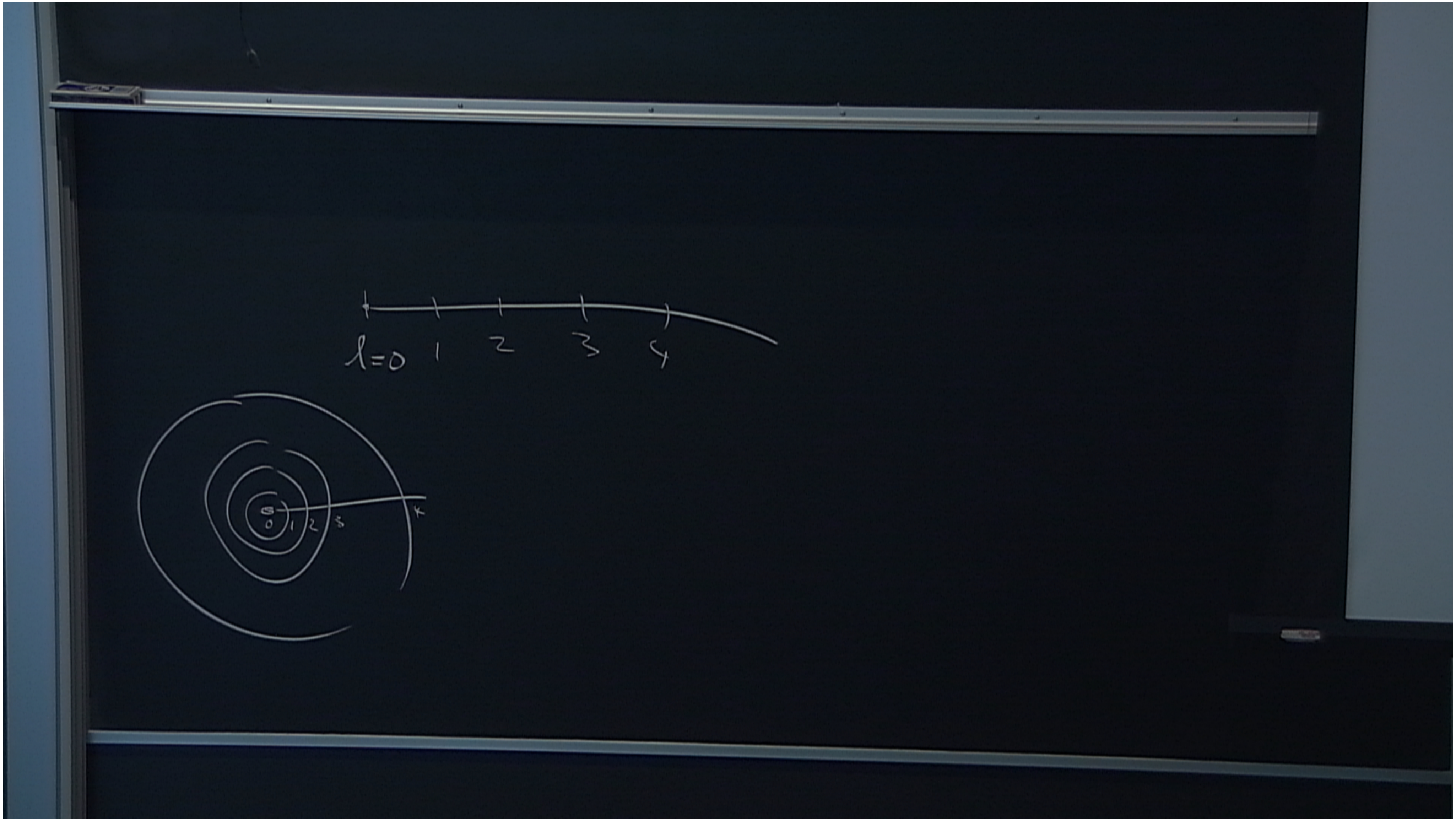
$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 20202020202020202 \cdots \quad Q_\gamma = 0$$

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 02020202020202020 \cdots \quad Q_\gamma = 1$$

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 11111111111111111 \cdots \quad Q_\gamma = 1/2$$

are the only **close-pacted** occupation patterns satisfying the above generalized Pauli exclusion rules.

- **close-pacted**:
(1) add any electrons → violate the GPE rules.
(2) shift any electrons → violate the GPE rules.


l=0 1 2 3 4
0
1
2
3
x

Conditions on $S_{\gamma;a}$

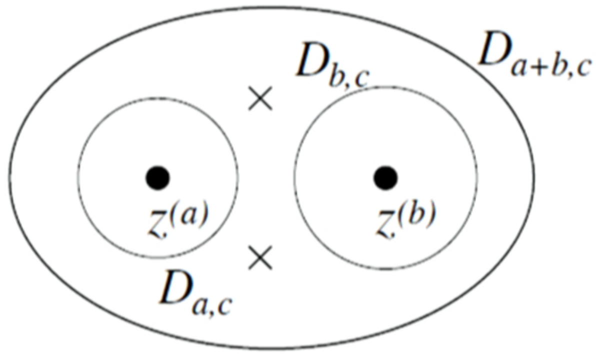
- Concave condition

$$S_{\gamma;a+b} - S_{\gamma;a} - S_b \geq 0,$$
$$S_{\gamma;a+b+c} - S_{\gamma;a+b} - S_{\gamma;a+c} - S_{b+c} + S_{\gamma;a} + S_b + S_c \geq 0$$

- Taking $b = c = 1$, we obtain

$$S_{\gamma;a+2} - 2S_{\gamma;a+1} + S_{\gamma;a} = l_{\gamma;a+2} - l_{\gamma;a+1} \geq S_2, \quad a \geq 0.$$

$l_{\gamma;a}$ **increases with** a**:** $l_{\gamma;a+1} - l_{\gamma;a} \geq S_2 \geq 0$.



$z^{(a)}$ is a quasiparticle γ fused with a electrons.

Pattern of zeros and generalized Pauli exclusion rule

- $l_{\gamma;a+1} - l_{\gamma;a} \geq S_2 \to$ the spacing between any two occupied orbitals by different electrons is no less than S_2
  $\to$ a generalized Pauli exclusion rule

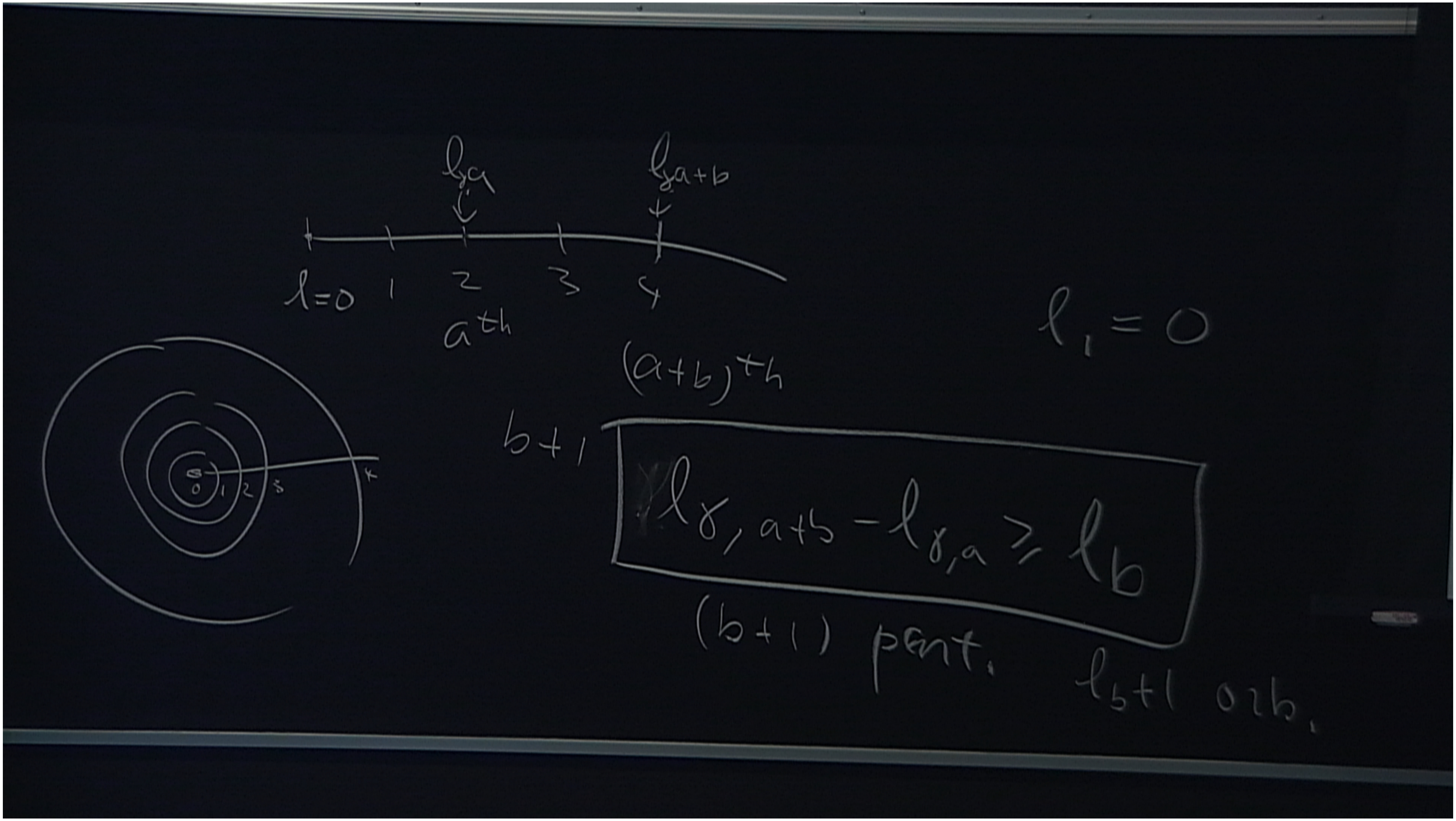
  More general Pauli exclusion (GPE) rule

  In terms of $l_{\gamma;a} = S_{\gamma;a} - S_{\gamma;a-1}$, the concave condition for quasiparticles becomes

$$\sum_{k=1}^{b} l_{\gamma;a+k} \geq S_b = \sum_{k=1}^{b} l_{a+k}, \quad \to \quad l_{\gamma;a} \geq l_a,$$

$$\sum_{k=1}^{c} (l_{\gamma;a+b+k} - l_{\gamma;a+k}) \geq S_{b+c} - S_b - S_c = \sum_{k=1}^{c} (l_{b+k} - l_k)$$

for any $a, b, c \in Z_+$. Note that $l_{\gamma;a+b} - l_{\gamma;a}$ is the spread of $b+1$ electrons. Setting $c = 1 \to$ **The spread of b electrons $\geq l_b$.**
$\to$ **The number of obitals occupied by b electrons $\geq l_b + 1$.**
($l_b =$ the spread of the first b electrons in the ground state.)


lγa
lγa+b
l=0 1 2 3 4
a th
(a+b) th
l1 = 0
0 1 2 3 k
b+1
lγ,a+b − lγ,a ≥ lb
(b+1) pent.
lb+1 orb.

For the $\nu = 1$ Pfaffian state ($n = 2$ and $m = 2$)

$$S_1, S_2, \cdots : 0, 0, 2, 4, 8, 12, 18, 24, \cdots$$

$$l_1 l_2 l_3 \cdots : 0, 0, 2, 2, 4, 4, 6, 6, 8, 8, \cdots$$

$$n_0 n_1 n_2 \cdots : 20202020202020202 \cdots ,$$

- GPE rule: *(1) Three particles occupy* $\geq l_3 + 1 = 3$ *obitals or more* ($c = 1$)*. (2) The spread of two particles plus the spread of next two particles* $\geq (l_2 - l_1) + (l_3 - l_2) = 2$ ($c = 2$)*. (3) ...*
- the quasiparticle occupation patterns

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 20202020202020202 \cdots \qquad Q_\gamma = 0$$

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 02020202020202020 \cdots \qquad Q_\gamma = 1$$

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 11111111111111111 \cdots \qquad Q_\gamma = 1/2$$

are the only **close-pacted** occupation patterns satisfying the above generalized Pauli exclusion rules.

- **close-pacted**:
  (1) add any electrons $\rightarrow$ violate the GPE rules.
  (2) shift any electrons $\rightarrow$ violate the GPE rules.

7-cluster states:

$$P_{\frac{7}{2};Z_7} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0, 0, 0, 0),$$
$$(n_0, \cdots, n_{m-1}) = (7, 0)$$

$$P_{\frac{7}{8};Z_7^{(2)}} : (m; S_2, \cdots, S_n) = (8; 0, 0, 2, 6, 10, 14),$$
$$(n_0, \cdots, n_{m-1}) = (3, 0, 1, 0, 3, 0, 0, 0)$$

$$P_{\frac{7}{18};Z_7^{(3)}} : (m; S_2, \cdots, S_n) = (18; 0, 4, 10, 18, 30, 42),$$
$$(n_0, \cdots, n_{m-1}) = (2, 0, 0, 0, 0, 1, 0, 0, 0, 2, 0, 0, 0, 0, 0)$$

$$P_{\frac{7}{14};C_7} : (m; S_2, \cdots, S_n) = (14; 0, 2, 6, 12, 20, 28),$$
$$(n_0, \cdots, n_{m-1}) = (2, 0, 1, 0, 1, 0, 1, 0, 2, 0, 0, 0, 0, 0)$$

- Also get composite parafermion state $P = P_{Z_{n_1}} P_{Z_{n_2}}$

Pattern of zeros and generalized Pauli exclusion rule

- $l_{\gamma;a+1} - l_{\gamma;a} \geq S_2 \to$ the spacing between any two occupied orbitals by different electrons is no less than S_2
  $\to$ a generalized Pauli exclusion rule

  More general Pauli exclusion (GPE) rule

  In terms of $l_{\gamma;a} = S_{\gamma;a} - S_{\gamma;a-1}$, the concave condition for quasiparticles becomes

$$\sum_{k=1}^{b} l_{\gamma;a+k} \geq S_b = \sum_{k=1}^{b} l_{a+k}, \quad \to \quad l_{\gamma;a} \geq l_a,$$

$$\sum_{k=1}^{c} (l_{\gamma;a+b+k} - l_{\gamma;a+k}) \geq S_{b+c} - S_b - S_c = \sum_{k=1}^{c} (l_{b+k} - l_k)$$

for any $a, b, c \in Z_+$. Note that $l_{\gamma;a+b} - l_{\gamma;a}$ is the spread of $b+1$ electrons. Setting $c = 1 \to$ **The spread of b electrons $\geq l_b$.**
$\to$ **The number of obitals occupied by b electrons $\geq l_b + 1$.**
(l_b = the spread of the first b electrons in the ground state.)

Number of quasiparticle types from pattern of zeros

For the parafermion states $P_{\nu=\frac{n}{2};Z_n}$ $(m=2)$,

$P_{\frac{2}{2};Z_2}$	$P_{\frac{3}{2};Z_3}$	$P_{\frac{4}{2};Z_4}$	$P_{\frac{5}{2};Z_5}$	$P_{\frac{6}{2};Z_6}$	$P_{\frac{7}{2};Z_7}$	$P_{\frac{8}{2};Z_8}$	$P_{\frac{9}{2};Z_9}$	$P_{\frac{10}{2};Z_{10}}$
3	4	5	6	7	8	9	10	11

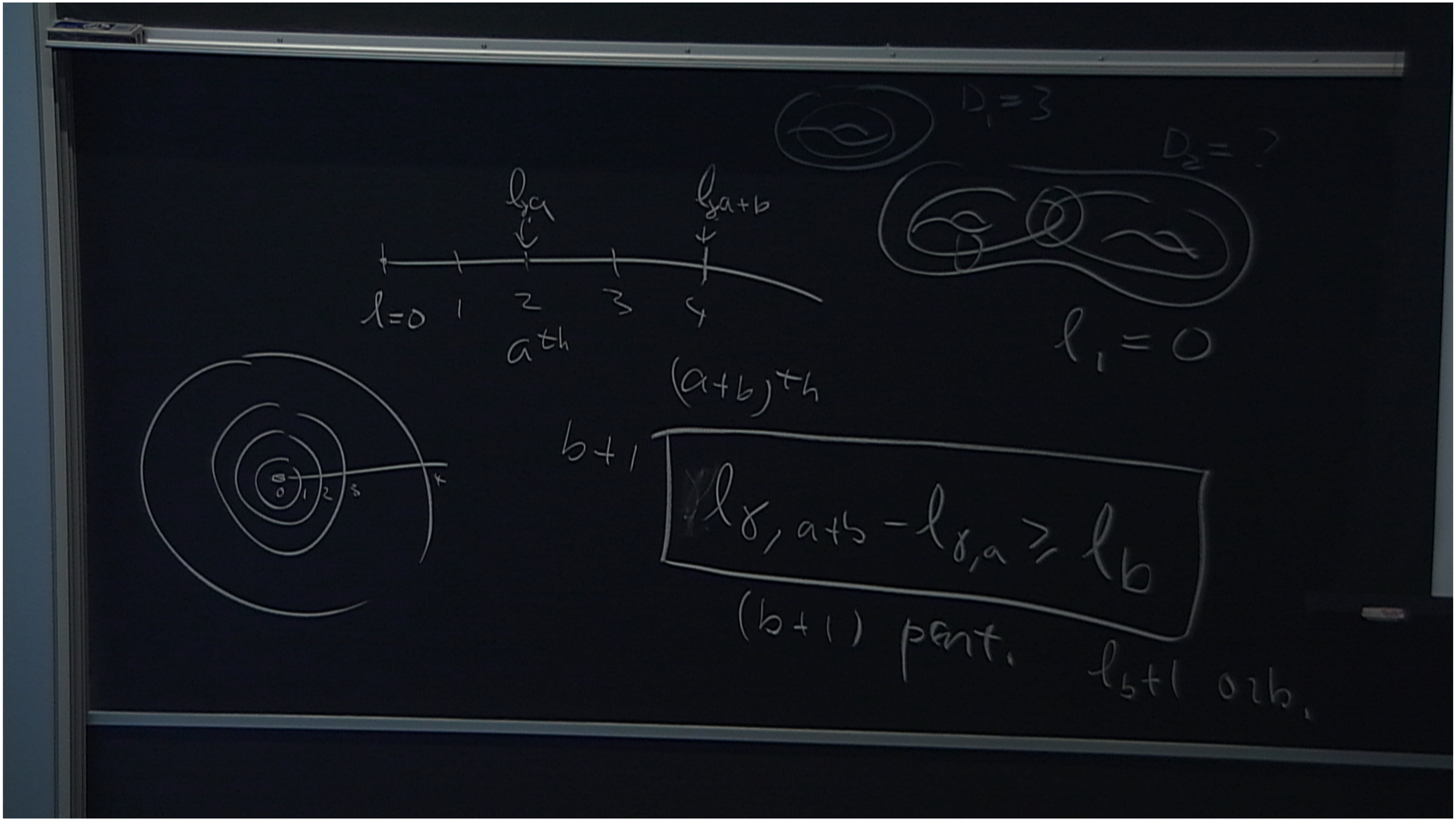
For the parafermion states $P_{\nu=\frac{n}{2+2n};Z_n}$ $(m=2+2n)$

$P_{\frac{2}{6};Z_2}$	$P_{\frac{3}{8};Z_3}$	$P_{\frac{4}{10};Z_4}$	$P_{\frac{5}{12};Z_5}$	$P_{\frac{6}{14};Z_6}$	$P_{\frac{7}{16};Z_7}$	$P_{\frac{8}{18};Z_8}$	$P_{\frac{9}{20};Z_9}$	$P_{\frac{10}{22};Z_{10}}$
9	16	25	36	49	64	81	100	121

For the generalized parafermion states $P_{\nu=\frac{n}{m};Z_n^{(k)}}$

$P_{\frac{5}{8};Z_5^{(2)}}$	$P_{\frac{5}{18};Z_5^{(2)}}$	$P_{\frac{7}{8};Z_7^{(2)}}$	$P_{\frac{7}{22};Z_7^{(2)}}$	$P_{\frac{7}{18};Z_7^{(3)}}$	$P_{\frac{7}{32};Z_7^{(3)}}$	$P_{\frac{8}{18};Z_8^{(3)}}$	$P_{\frac{9}{8};Z_9^{(2)}}$
24	54	32	88	72	128	81	40

where k and n are coprime.


$D_1 = 3$
$D_2 = ?$
$\ell_{\gamma a}$
$\ell_{\gamma a+b}$
$l=0$ 1 2 3 4
a^{th}
$(a+b)^{th}$
$\ell_1 = 0$
b+1
$\ell_{\gamma, a+b} - \ell_{\gamma, a} \geq \ell_b$
(b+1) pent.
ℓ_{b+1} orb.

For the $\nu = 1$ Pfaffian state ($n = 2$ and $m = 2$)

$$S_1, S_2, \cdots : 0, 0, 2, 4, 8, 12, 18, 24, \cdots$$

$$l_1 l_2 l_3 \cdots : 0, 0, 2, 2, 4, 4, 6, 6, 8, 8, \cdots$$

$$n_0 n_1 n_2 \cdots : 20202020202020202 \cdots ,$$

- GPE rule: *(1) Three particles occupy* $\geq l_3 + 1 = 3$ *obitals or more* $(c = 1)$*. (2) The spread of two particles plus the spread of next two particles* $\geq (l_2 - l_1) + (l_3 - l_2) = 2$ $(c = 2)$*. (3) ...*
- the quasiparticle occupation patterns

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 20202020202020202 \cdots \qquad Q_\gamma = 0$$

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 02020202020202020 \cdots \qquad Q_\gamma = 1$$

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 11111111111111111 \cdots \qquad Q_\gamma = 1/2$$

are the only **close-pacted** occupation patterns satisfying the above generalized Pauli exclusion rules.

- **close-pacted**:
(1) add any electrons $\to$ violate the GPE rules.
(2) shift any electrons $\to$ violate the GPE rules.

GPE rule and edge spectrum for the Pfaffian state

First kind of edge:

$$M_0 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : \; 2020202020|00000000 \cdots$$
$$M_0 + 1 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : \; 2020202011|10000000 \cdots$$
$$M_0 + 1 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : \; 2020202012|00000000 \cdots \text{ not allowed}$$
$$M_0 + 2 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : \; 2020202011|01000000 \cdots$$
$$M_0 + 2 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : \; 2020202010|20000000 \cdots$$
$$M_0 + 2 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : \; 2020202011|10000000 \cdots$$
$$M_0 + 3 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : \; 2020202011|00100000 \cdots$$
$$M_0 + 3 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : \; 2020202010|11000000 \cdots$$
$$M_0 + 3 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : \; 2020202011|01000000 \cdots$$
$$M_0 + 3 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : \; 2020202011|20000000 \cdots$$
$$M_0 + 3 : n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : \; 2020201111|10000000 \cdots$$

M	M_0	M_0+1	M_0+2	M_0+3	M_0+4
# of states D_n	1	1	3	5	10

Edge states = zero-energy states of ideal Hamiltonian

- For ideal Hamiltonian $V_{1/3}(z_1, z_2) = -\partial_{z_1}\delta(z_1 - z_2)\partial_{z_1}$, the N electron state $P_{1/3} = \prod_{i<j}(z_i - z_j)^3$, is the zero-energy state with minimal angular momentum (the order of z_i's) $M_0 = N(N-1)$.
- Other zero-energy state has higher anglar momenta. Those zero energy states are the so called **edge states**:

The energy spectrum of 100 lowest levels of the ideal Hamiltonian with 6 electrons