Title: Topological Order Series

Date: Jun 11, 2015 03:30 PM

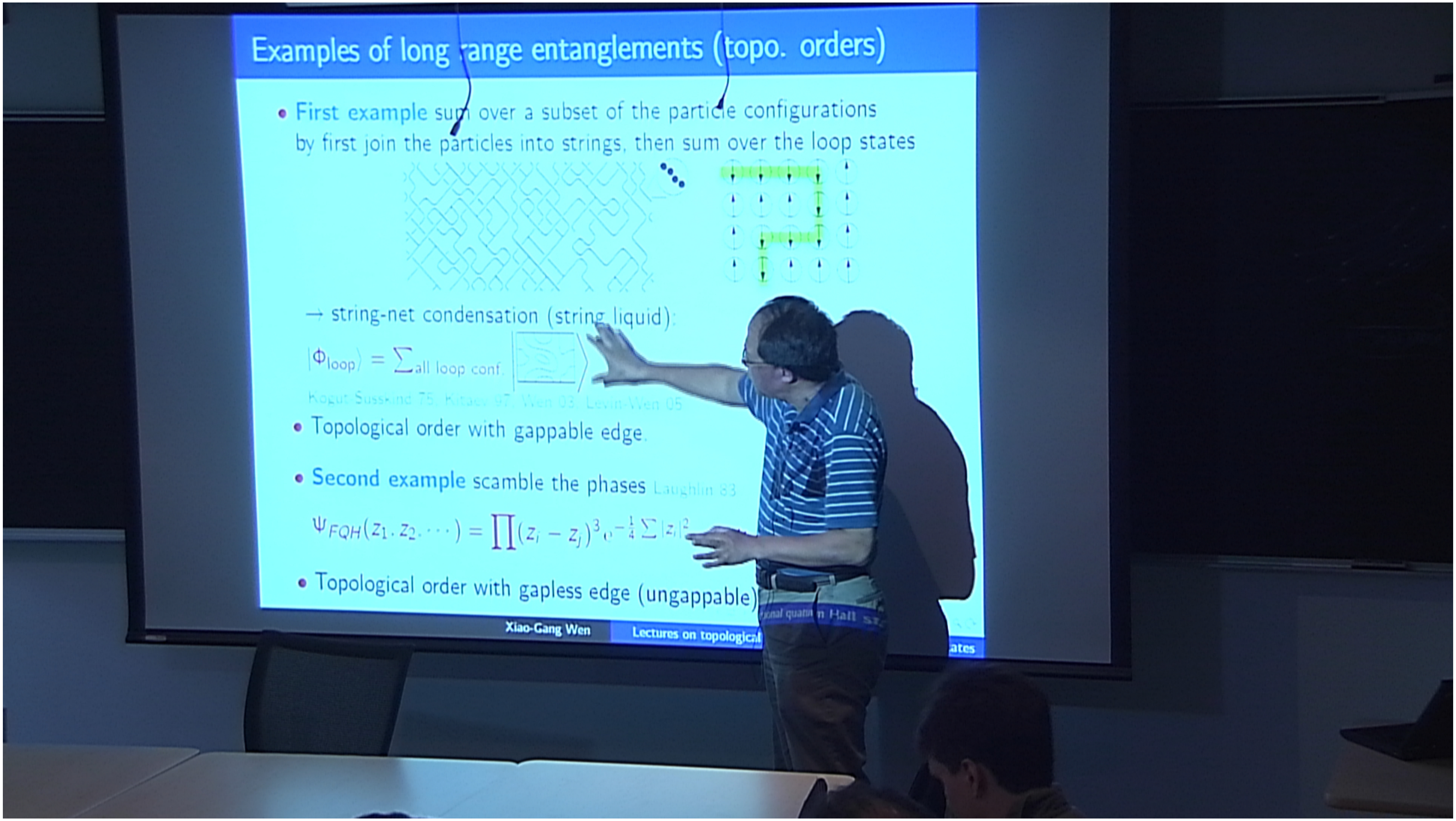
URL: http://pirsa.org/15060016

Abstract:

Lectures on topological order:
Fractional quatum Hall states

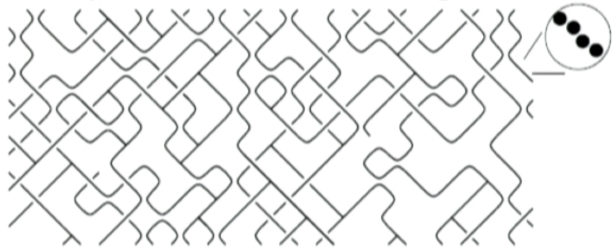
Xiao-Gang Wen

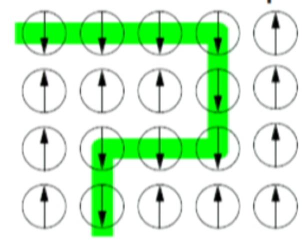
2015/6/11


Examples of long range entanglements (topo. orders)
First example sum over a subset of the particle configurations
by first join the particles into strings, then sum over the loop states
→ string-net condensation (string liquid):
|Φloop⟩ = ∑all loop conf.
Kogut Susskind 75; Kitaev 97; Wen 03; Levin-Wen 05
Topological order with gappable edge.
Second example scamble the phases Laughlin 83
ΨFQH(z1, z2, · · ·) = ∏(zi − zj)^3 e^−¼ ∑|zi|²
Topological order with gapless edge (ungappable)
Xiao-Gang Wen
Lectures on topological

Examples of long range entanglements (topo. orders)

- **First example** sum over a subset of the particle configurations by first join the particles into strings, then sum over the loop states





$\rightarrow$ string-net condensation (string liquid):

$|\Phi_{\text{loop}}\rangle = \sum_{\text{all loop conf.}} \Big|$  $\Big\rangle$

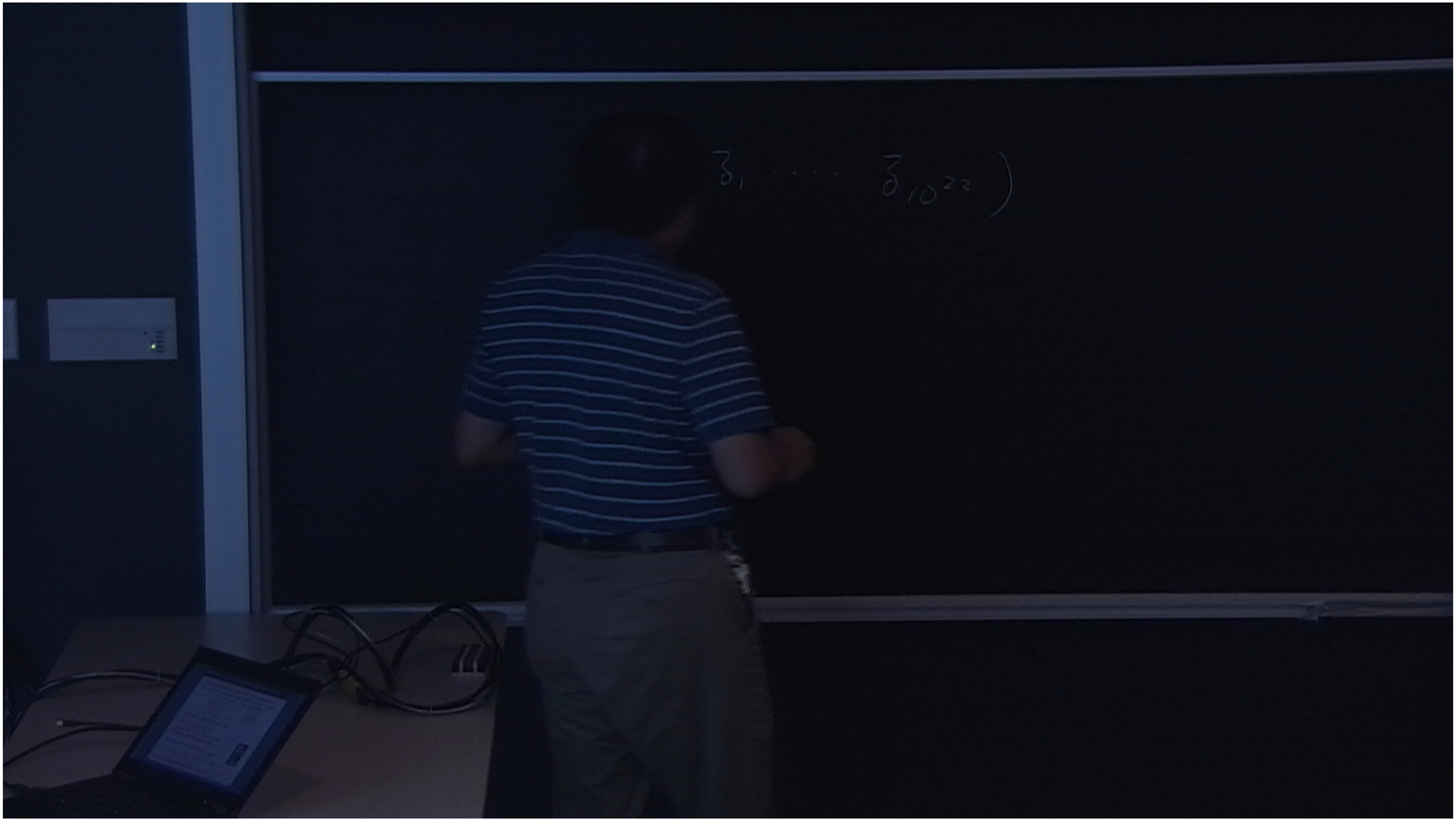
Kogut-Susskind 75, Kitaev 97, Wen 03, Levin-Wen 05

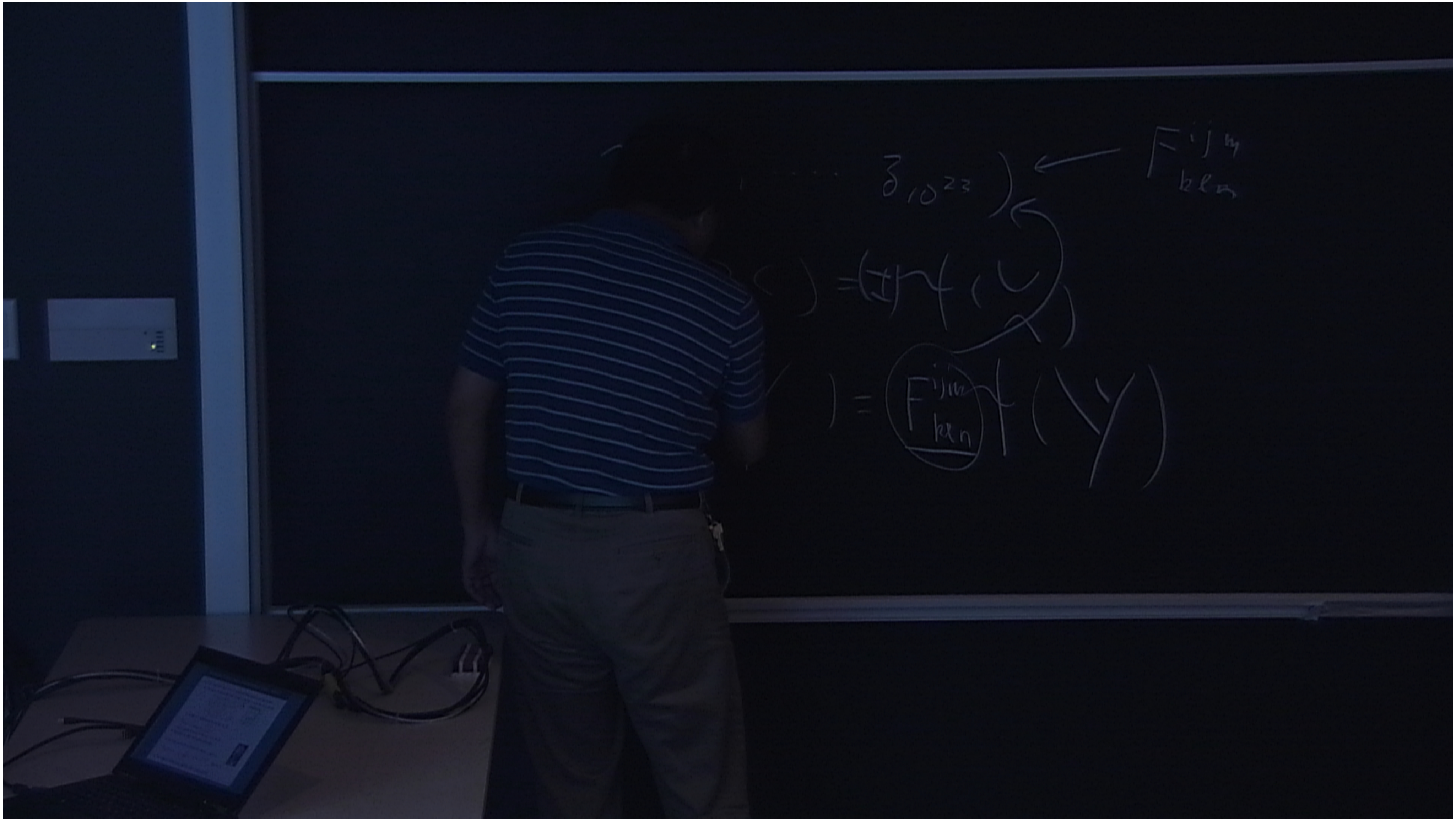
- Topological order with gappable edge.

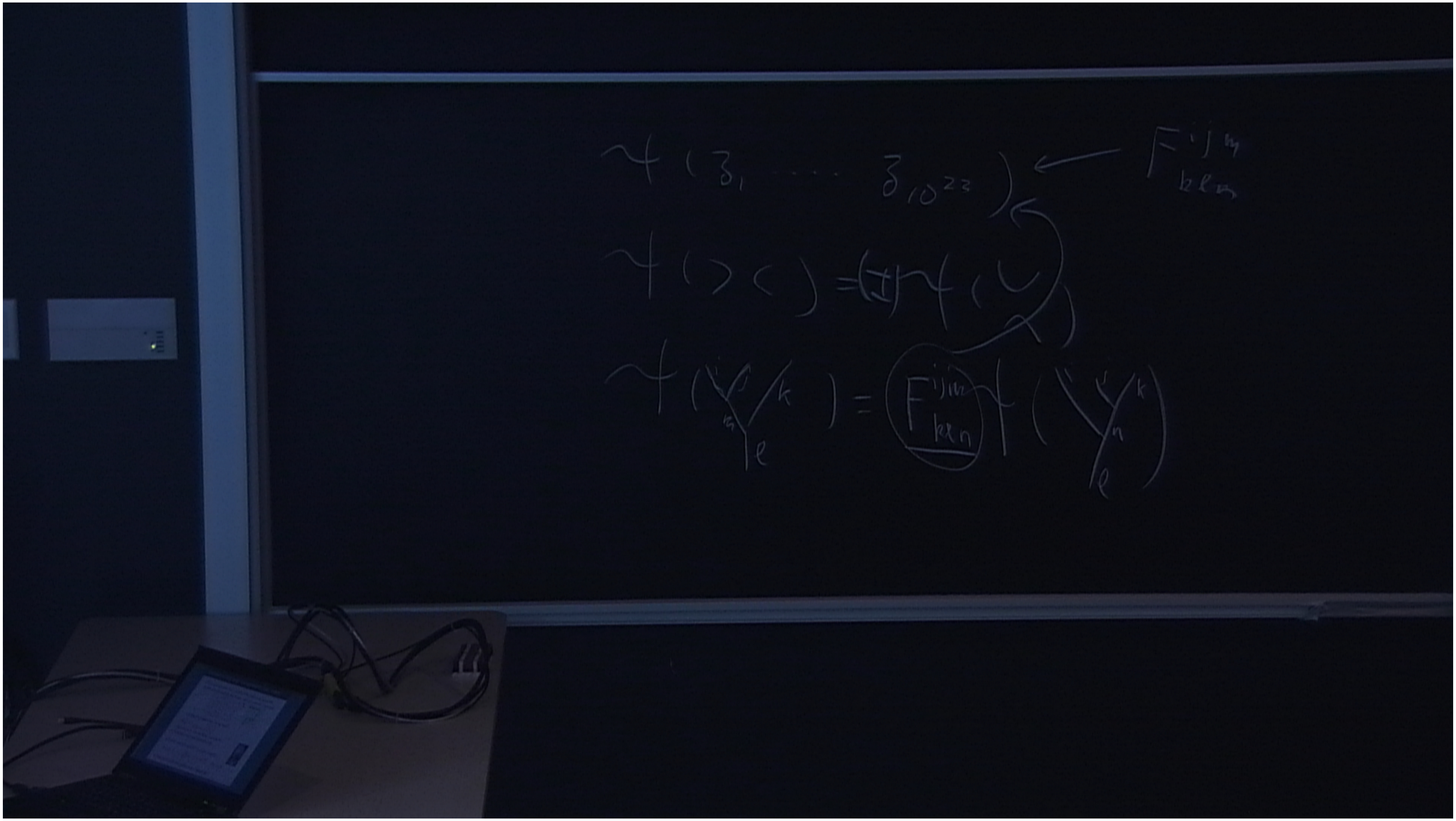
- **Second example** scamble the phases Laughlin 83

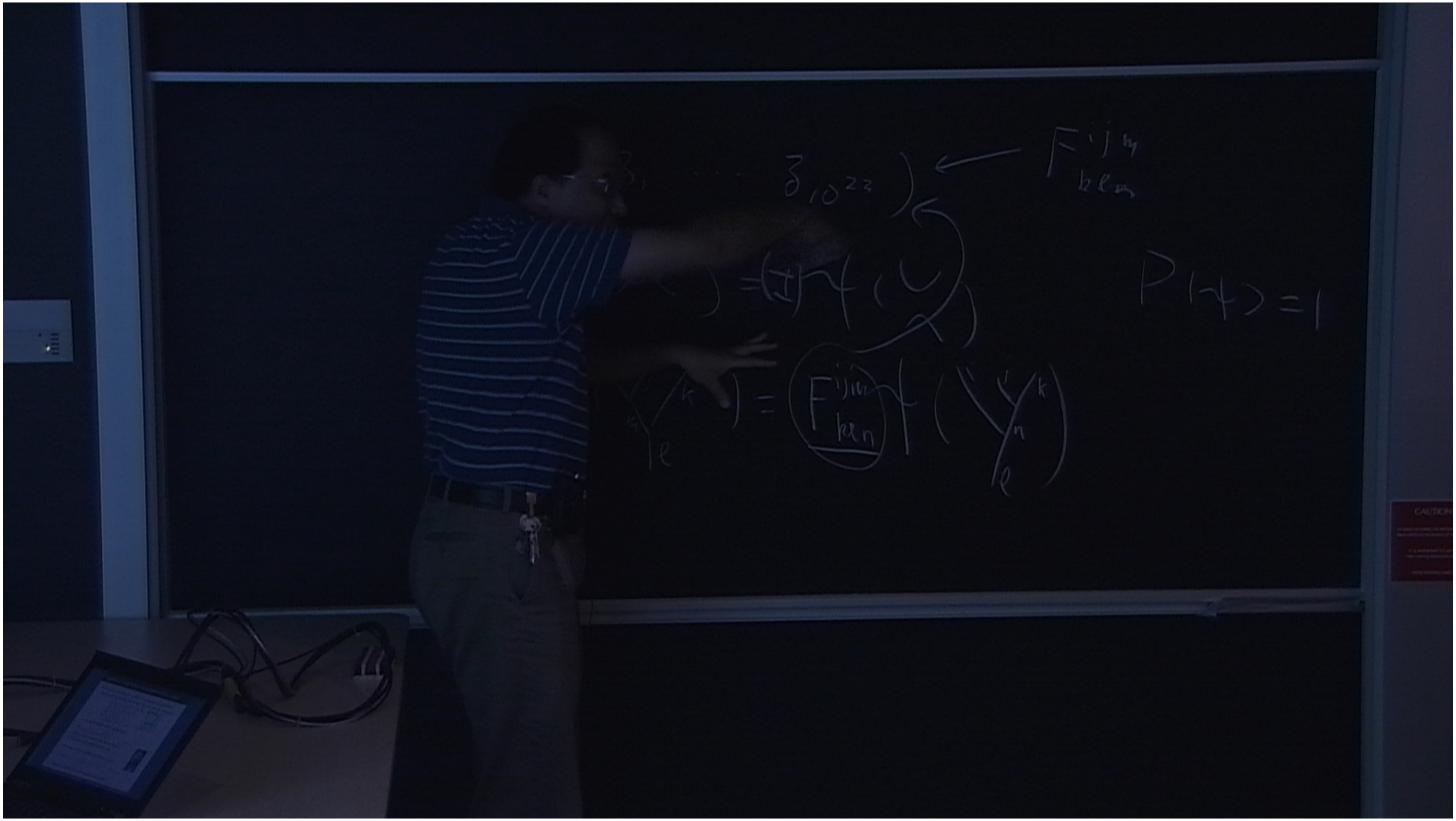
$$\Psi_{FQH}(z_1, z_2, \cdots) = \prod (z_i - z_j)^3 e^{-\frac{1}{4}\sum |z_i|^2}, \quad \Psi_{SF}(z_1, z_2, \cdots) = 1.$$

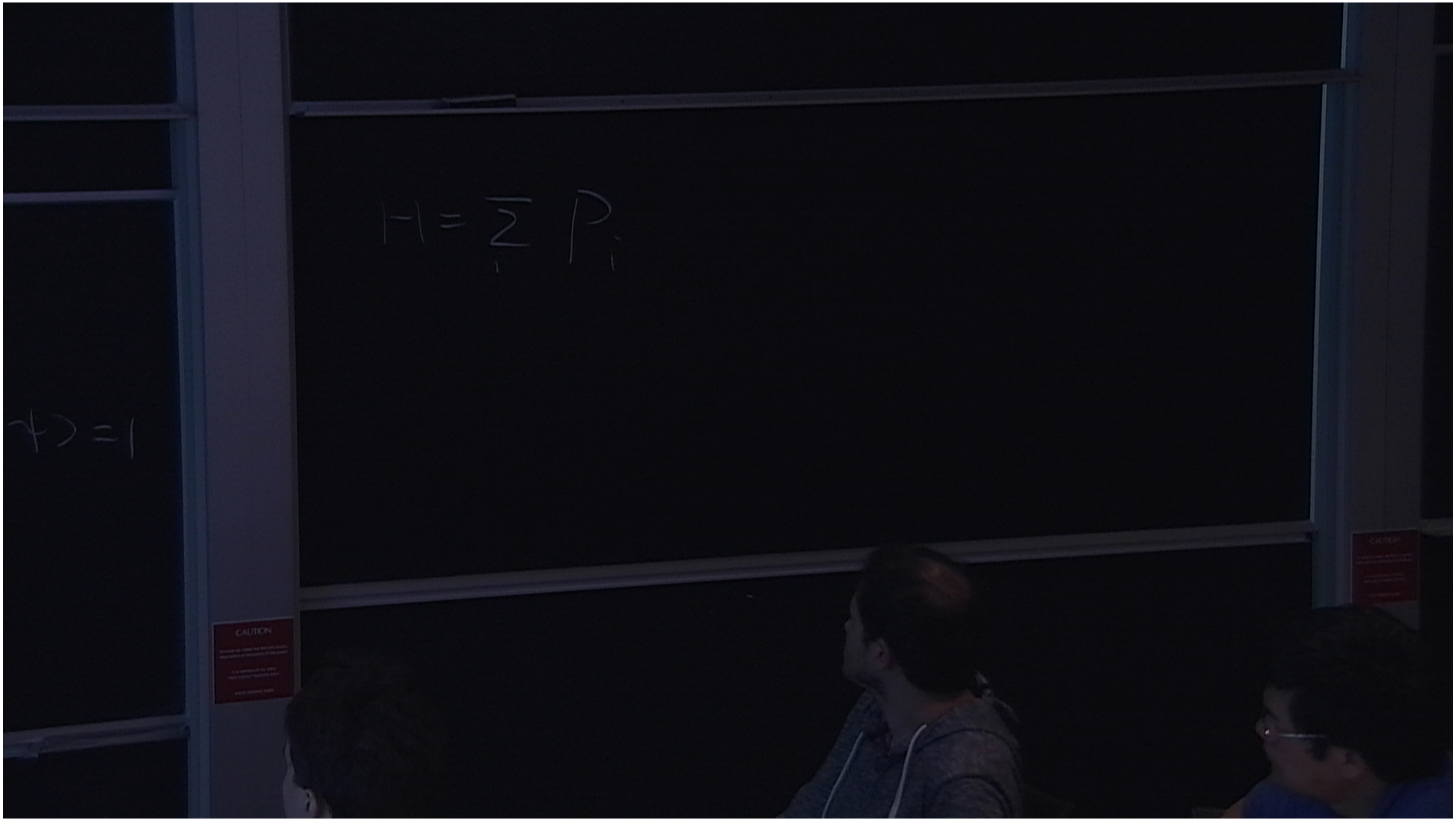
- Topological order with gapless edge (ungappable).

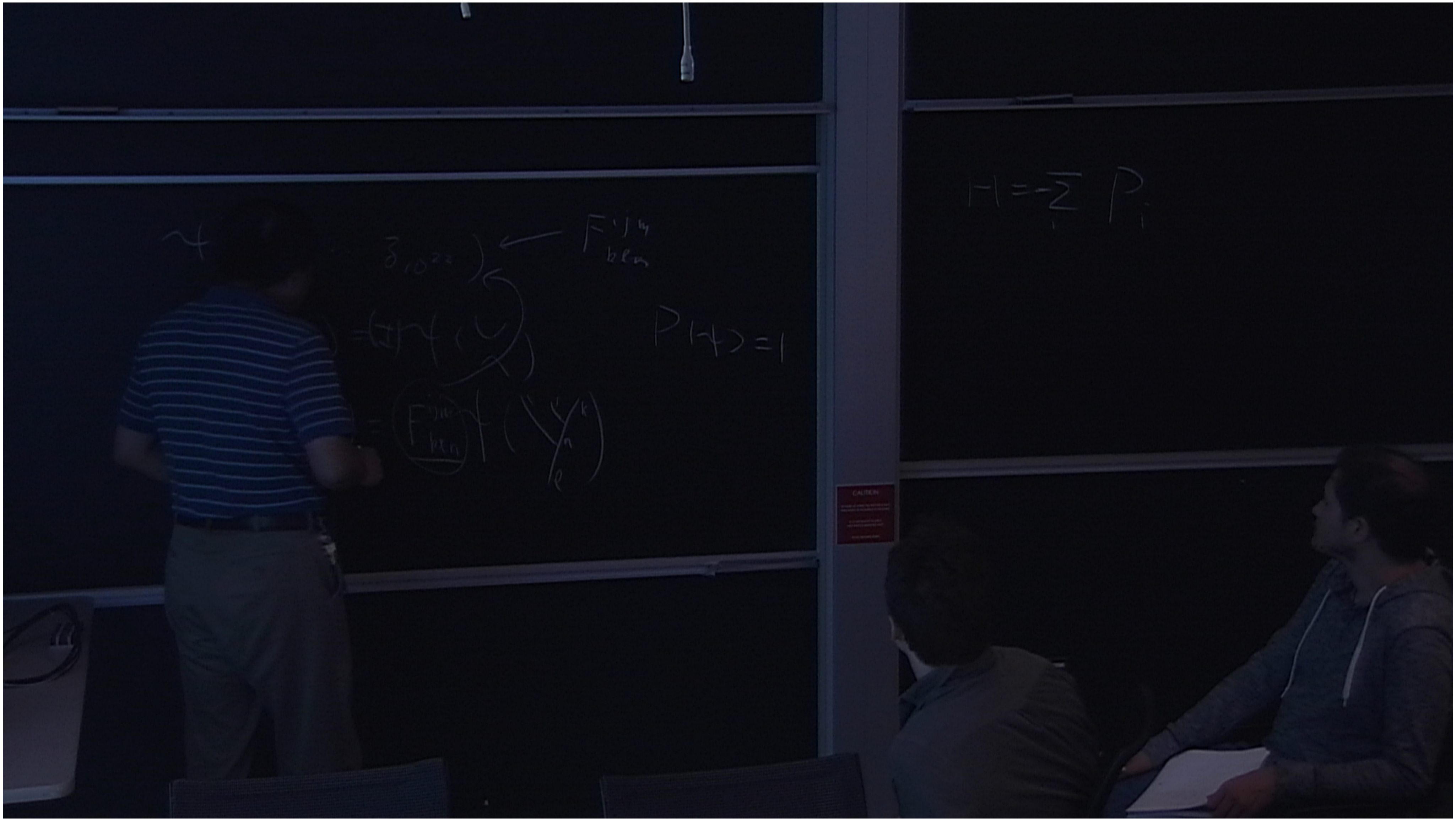

$z_1 \cdots\cdots z_{10^{23}}$)

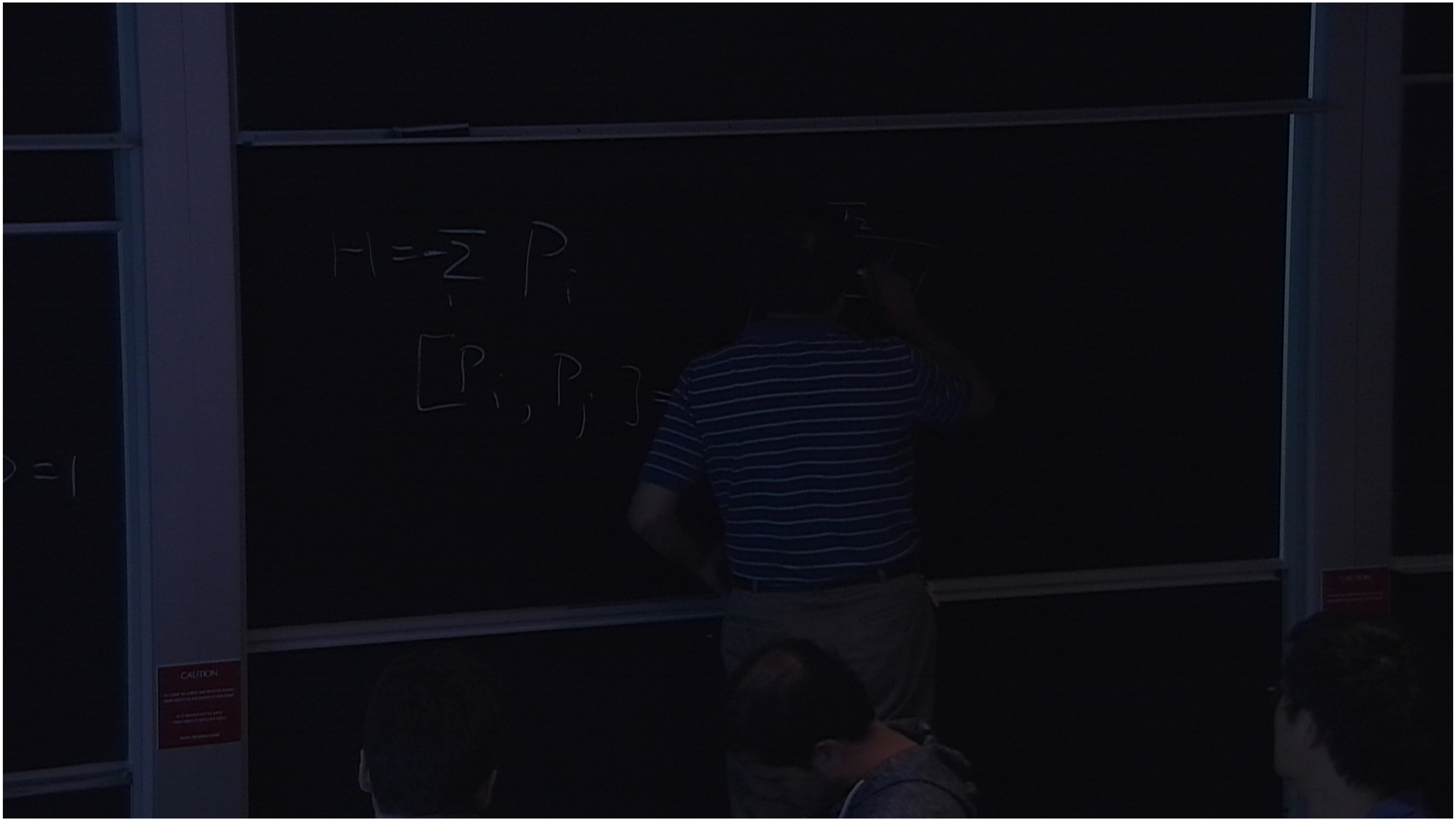







H = Σ_i P_i
ψ⟩ = 1
CAUTION


H = -Σ P_i
P|ψ> = 1


H = -Σ_i P_i
[P_i, P_j] =
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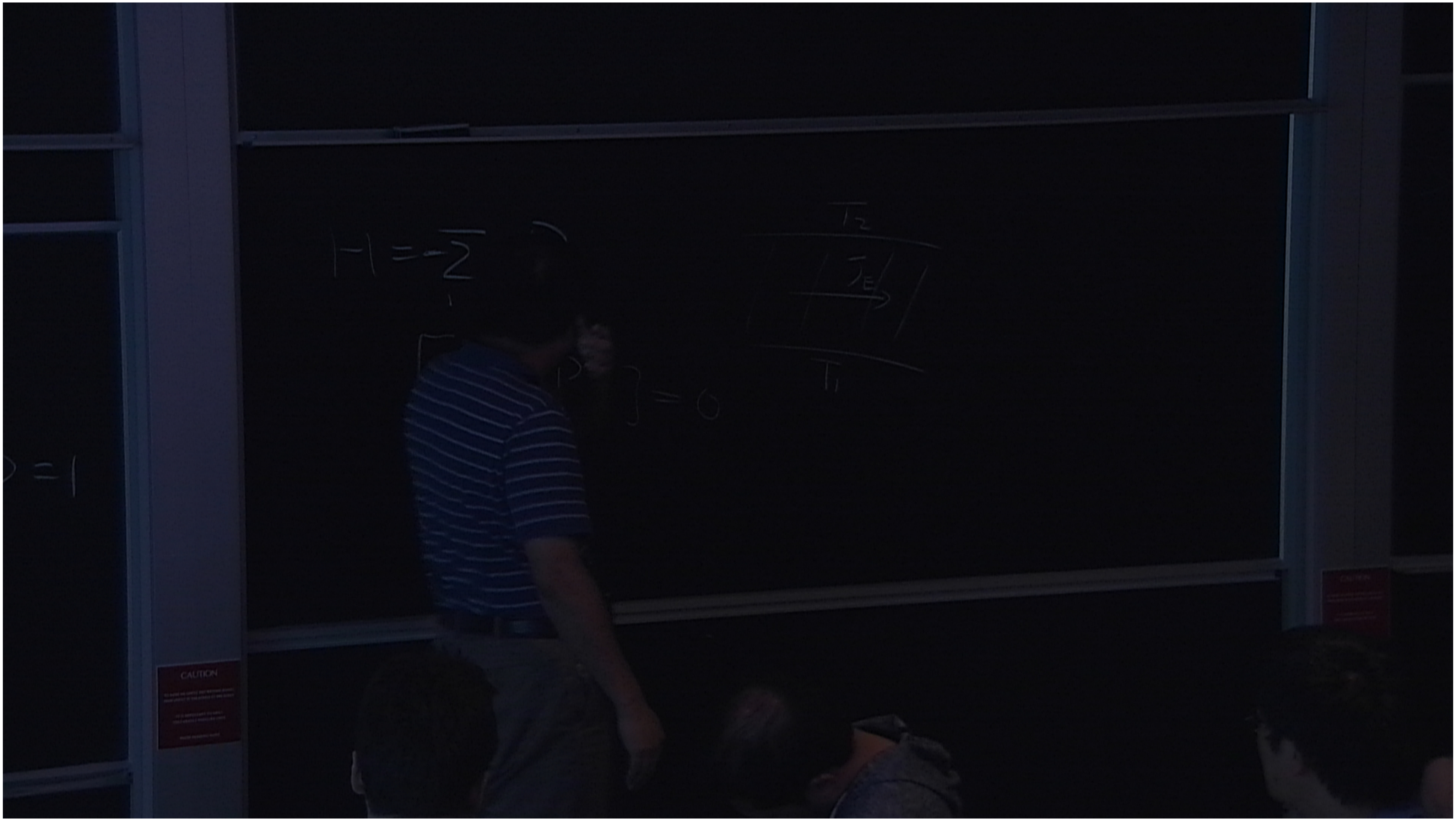
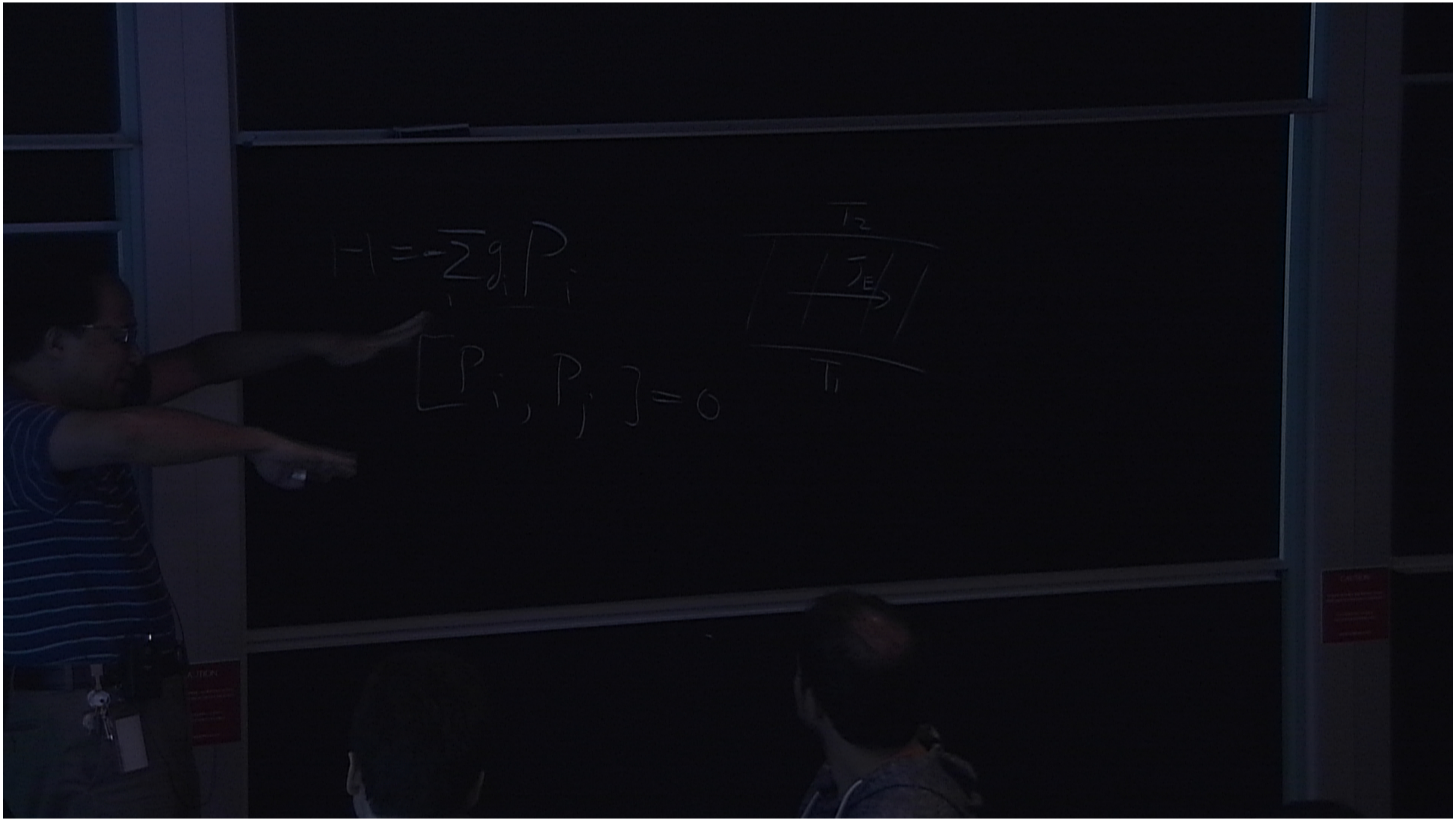
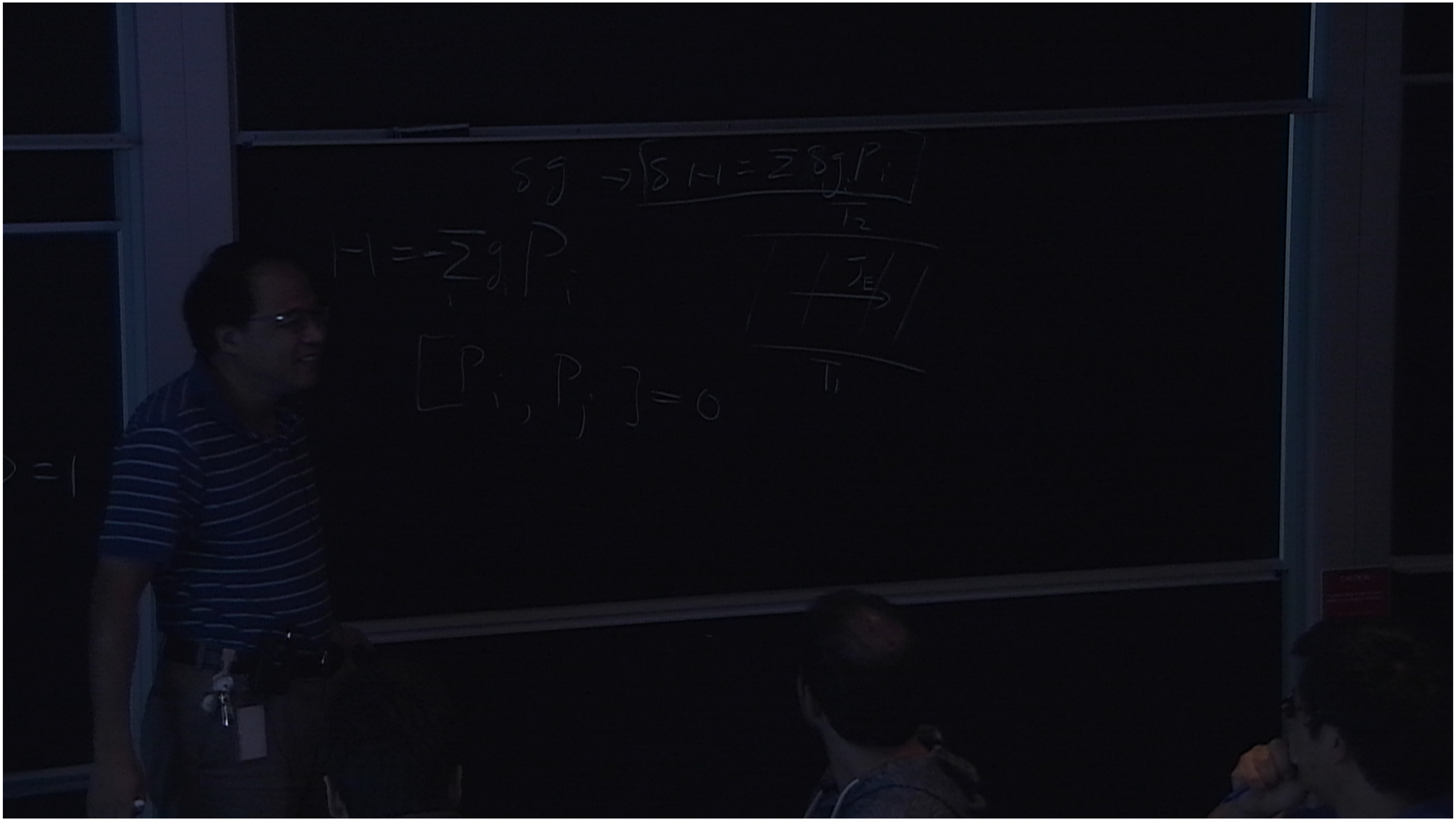
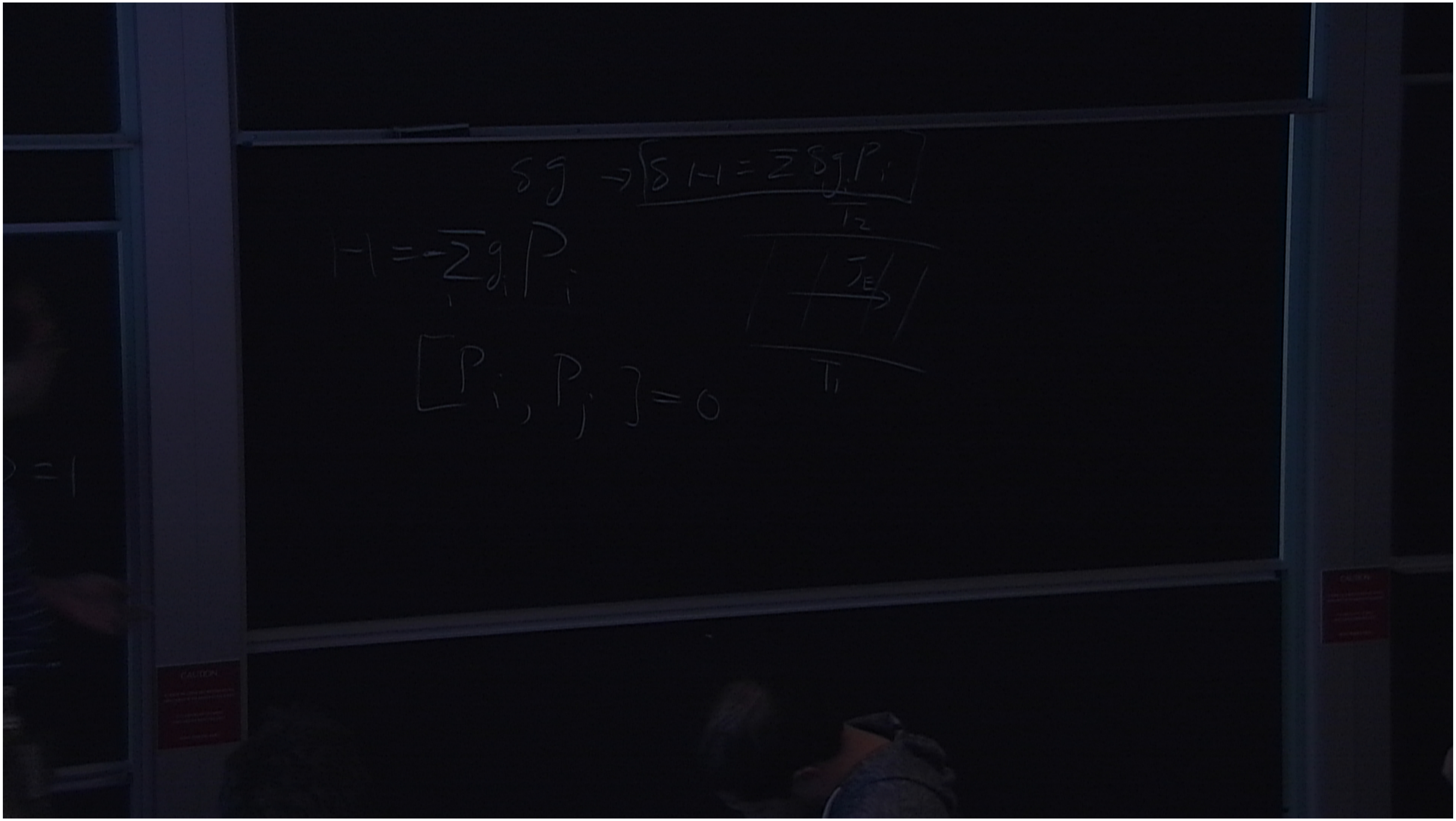

CAUTION


$H = -\sum_i g_i P_i$
$[P_i, P_j] = 0$
T_2
T_E
T_1



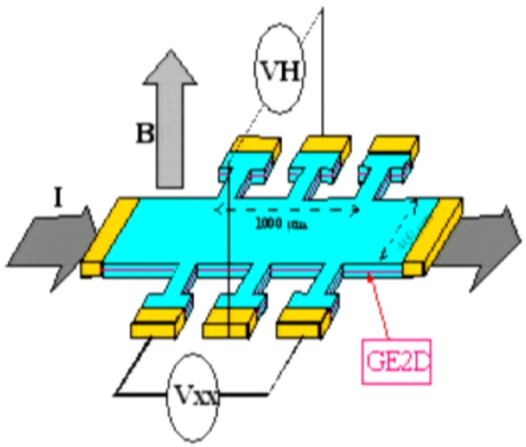

δg → δH = Σ δg_i P_i
H = -Σ g_i P_i
[P_i , P_j] = 0
T_2
J_E
T_1

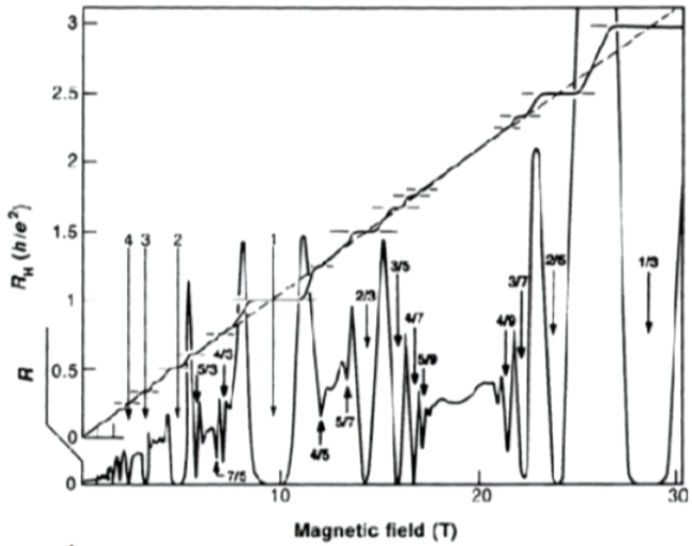
The experimental discovery of FQH effect

- Quantum Hall states (1980's)
  Quantized Hall conductance:

$$\sigma_{xy} = \frac{I}{V_H} = \frac{m}{n}\frac{e^2}{h} = \frac{1}{R_H}$$

$$\frac{m}{n} = \nu = \frac{\#\text{ of electrons}}{\#\text{ of flux quanta}}$$





- FQH states states have different phases
  with no symmetry breaking, no crystal order, no spin order, ...
  so they must have a new order – **topological order** Wen 89

Introduction of fractional quantum Hall (FQH) states

- One-particle in magnatic field (choose $B = 1$ and $z = x + \mathrm{i}y$):
  $H_0 = -\sum(\partial_z - \frac{B}{4}z^*)(\partial_{z^*} + \frac{B}{4}z)$
  Lowest energy eigenstates: $P(z)\mathrm{e}^{-\frac{1}{4}|z|^2}$, $P(z) = \sum a_l z^l$
  since $\mathrm{e}^{\frac{1}{4}zz^*}(\mathrm{i}\partial_z - \mathrm{i}\frac{1}{4}z^*)(\mathrm{i}\partial_{z^*} + \mathrm{i}\frac{1}{4}z)\mathrm{e}^{-\frac{1}{4}zz^*} = (\mathrm{i}\partial_z - \mathrm{i}\frac{1}{2}z^*)\mathrm{i}\partial_{z^*}$

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- N-electrons (fermionic or bosonic) in a magnetic field:

$$H(g_1, g_2) = \sum_{i=1}^{N}(\mathrm{i}\partial_{z_i} - \mathrm{i}\frac{B}{4}z_i^*)(\mathrm{i}\partial_{z_i^*} + \mathrm{i}\frac{B}{4}z_i) + \sum_{i<j} V_{g_1,g_2}(x_i - x_j, y_i - y_j)$$

- When $V_{g_1,g_2} = 0$, there are many minimal energy wave functions
  $\Psi = P(z_1, \cdots, z_N)\mathrm{e}^{-\frac{1}{4}\sum_{i=1}^N z_i z_i^*}$, $P =$ a (anti-)symmetric polynomial

  all have zero energy (for any P):

$$\Big[\sum_{i=1}^{N}(\mathrm{i}\partial_{z_i} - \mathrm{i}\frac{B}{4}z_i^*)(i\partial_{z_i^*} + \mathrm{i}\frac{B}{4}z_i)\Big]P(z_1, \cdots, z_N)\mathrm{e}^{-\frac{1}{4}\sum_{i=1}^N z_i z_i^*} = 0$$

3 ideal FQH states: the exact zero-energy ground states

- $\nu = 1/2$ bosonic Laughlin state: $z_1 \approx z_2$, second order zero

$$P_{1/2} = \prod_{i<j} (z_i - z_j)^2, \quad V_{1/2}(z_1, z_2) = \delta(z_1 - z_2).$$

$$[\sum_{i<j} V_{1/2}(z_i - z_j)] P_{1/2} = 0.$$

All other states have finite energies in $N \to \infty$ limit (gapped).

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- When $V_{g_1,g_2}=0$, there are many minimal energy wave functions
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  all have zero energy (for any P):
  $$\Big[\sum_{i=1}^{N}(\mathrm{i}\partial_{z_i} - \mathrm{i}\frac{B}{4}z_i^*)(i\partial_{z_i^*} + \mathrm{i}\frac{B}{4}z_i)\Big] P(z_1,\cdots,z_N)\mathrm{e}^{-\frac{1}{4}\sum_{i=1}^{N} z_i z_i^*} = 0$$
- For small non-zero V_{g_1,g_2}, there is only one minimal energy wave function P whose form is determined by V_{g_1,g_2}.

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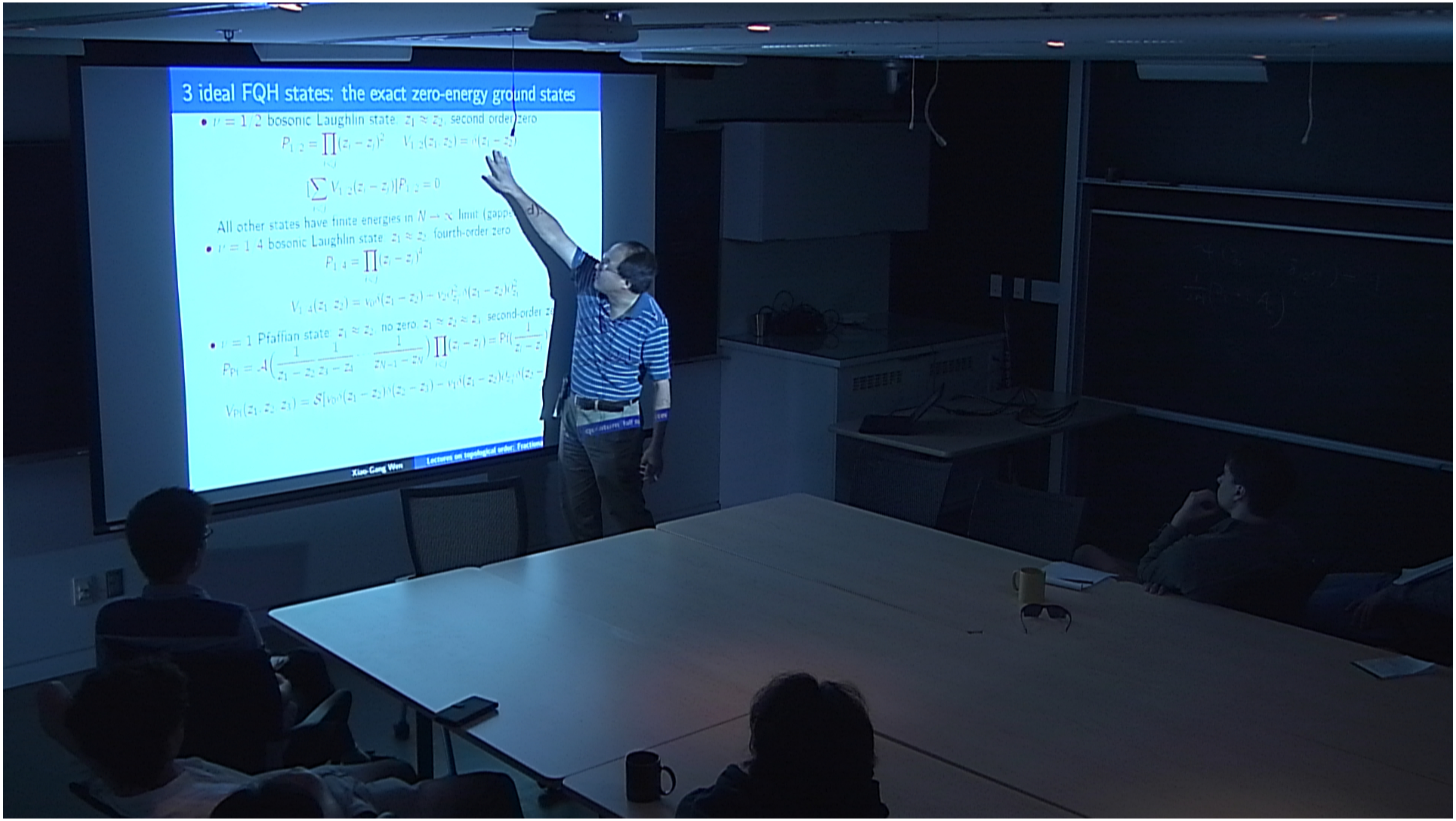
All other states have finite energies in $N \to \infty$ limit (gapped).

- $\nu = 1/4$ bosonic Laughlin state: $z_1 \approx z_2$, fourth-order zero

$$P_{1/4} = \prod_{i<j}(z_i - z_j)^4$$

$$V_{1/4}(z_1, z_2) = v_0\delta(z_1 - z_2) + v_2\partial^2_{z_1^*}\delta(z_1 - z_2)\partial^2_{z_1}$$


3 ideal FQH states: the exact zero-energy ground states
ν = 1/2 bosonic Laughlin state: z1 ≈ z2, second order zero
All other states have finite energies in N → ∞ limit (gapped)
ν = 1/4 bosonic Laughlin state: z1 ≈ z2, fourth-order zero
ν = 1 Pfaffian state: z1 ≈ z2, no zero, z1 ≈ z2 ≈ z3, second-order zero
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Lectures on topological order: Fractional

3 ideal FQH states: the exact zero-energy ground states

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$$P_{1/4} = \prod_{i<j}(z_i - z_j)^4$$

$$V_{1/4}(z_1, z_2) = v_0\delta(z_1 - z_2) + v_2\partial^2_{z_1^*}\delta(z_1 - z_2)\partial^2_{z_1}$$

- $\nu = 1$ Pfaffian state: $z_1 \approx z_2$, no zero; $z_1 \approx z_2 \approx z_3$, second-order zero:

$$P_{\text{Pf}} = \mathcal{A}\Big(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\frac{1}{z_{N-1} - z_N}\Big)\prod_{i<j}(z_i - z_j) = \text{Pf}\big(\frac{1}{z_i - z_j}\big)\prod_{i<j}(z_i - z_j)$$

$$V_{\text{Pf}}(z_1, z_2, z_3) = \mathcal{S}[v_0\delta(z_1 - z_2)\delta(z_2 - z_3) - v_1\delta(z_1 - z_2)\partial_{z_3^*}\delta(z_2 - z_3)\partial_{z_3}]$$

- $\nu = 1$ fermionic IQH state: $z_1 \approx z_2$, first-order zero:

$$P_1 = \prod_{i<j}(z_i - z_j); \quad V_1(z_1, z_2) = 0$$

From wavefunction to physical properties

- **What are the physical properties of those FQH states described by $P_{1/2}$, $P_{1/4}$, and $P_{\rm Pf}$?**
  Are they they really belong to different phases? What are the fractional charge/statistics of quasiparticles, edge excitations, *etc* ?
- The densities of gapped FQH states are quantized as rational-number (filling fraction) $\nu \times \frac{1}{2\pi}$:

$$\rho_e(z) = \frac{\int d^2z_2...d^2z_N \, |P(z, z_2, ..., z_N)|^2 e^{-\frac{1}{2}\sum |z_i|^2}}{\int d^2z_1 d^2z_2...d^2z_N \, |P(z_1, z_2, ..., z_N)|^2 e^{-\frac{1}{2}\sum |z_i|^2}} = \Big|_{|z|<r_N} \nu \frac{1}{2\pi}$$

Hall conductance $\sigma_{xy} = \nu \frac{e^2}{h}$

$P_1 = \prod (z_i - z_j) \to \nu = 1.$

$P_{1/2} = \prod (z_i - z_j)^2 \to \nu = 1/2,$

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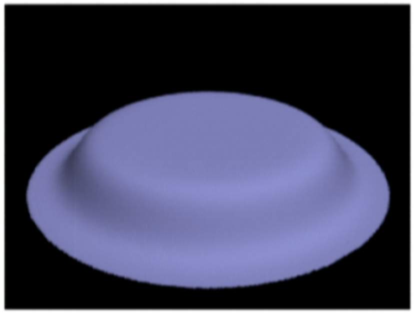
$P_{\rm Pf} = {\rm Pf}(\frac{1}{z_i - z_j}) \prod (z_i - z_j) \to \nu = 1.$



$\nu = \frac{\sigma_{xy} h}{e^2}$ **is quantized as exact an rational number as** $N \to \infty$

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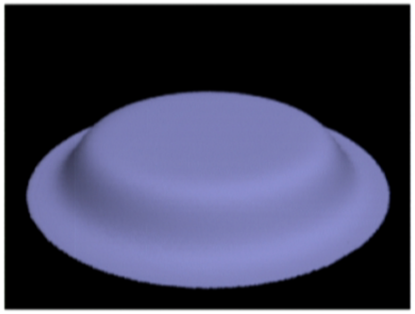
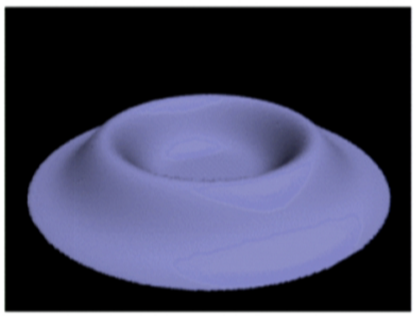


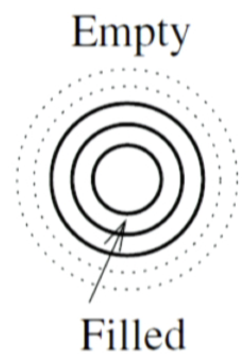
$\nu = \frac{\sigma_{xy} h}{e^2}$ **is quantized as exact an rational number as** $N \to \infty$

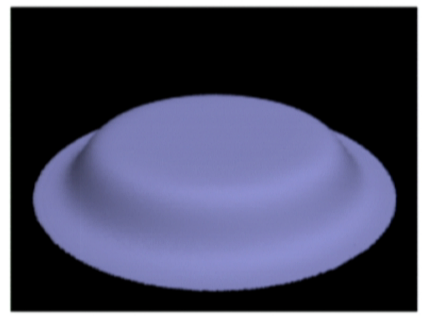
Why $\nu = 1$ for state $\Psi_1 = \prod_{i<j}(z_i - z_j)\mathrm{e}^{-\sum |z_i|^2/4}$

One-particle eigenstate (orbital) for $H_0 = -\sum(\partial_z - \frac{B}{4}z^*)(\partial_{z^*} + \frac{B}{4}z)$:
$z^l \mathrm{e}^{-\frac{1}{4}|z|^2} \to$ a ring-like wave function with
a radius $r_l = \sqrt{2l}$ and angular momentum l.
The $\nu = 1$ many-fermion state is obtained by filling the orbitals:

$$\Psi = \prod_{i<j}(z_i - z_j)\mathrm{e}^{-\frac{1}{4}\sum |z_i|^2} = \mathcal{A}[(z_1)^0(z_2)^1 ...]\mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}$$







l electrons within radius $r_l \to$ one electron per $\pi r_l^2/l = 2\pi$ area.
$\to \nu = 1$.

Quasi-holes in the $\nu = 1/m$ Laughlin state and fractional charge

- A hole-like excitation = missing an electron, charge = 1

$$\prod_i (\xi - z_i)^m \prod_{i<j} (z_i - z_j)^m \mathrm{e}^{-\sum |z_i|^2/4}$$

which can be splitted into m quasi-hole excitations:

$$\prod_i (\xi_1 - z_i) \cdots \prod_i (\xi_m - z_i) \prod_{i<j} (z_i - z_j)^m \mathrm{e}^{-\sum |z_i|^2/4}$$

- A quasi-hole excitation = minimal excitation, charge = 1/m

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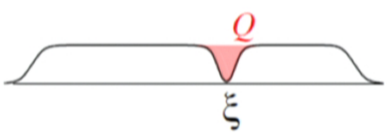

- Why the density dip have a small finte size?

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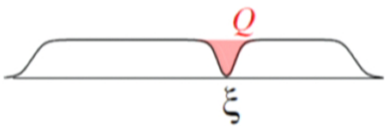

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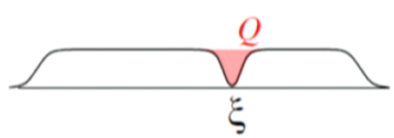

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Quasi-holes in the $\nu = 1/m$ Laughlin state and fractional charge

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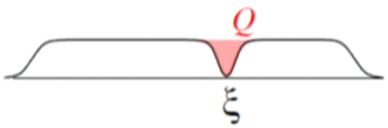

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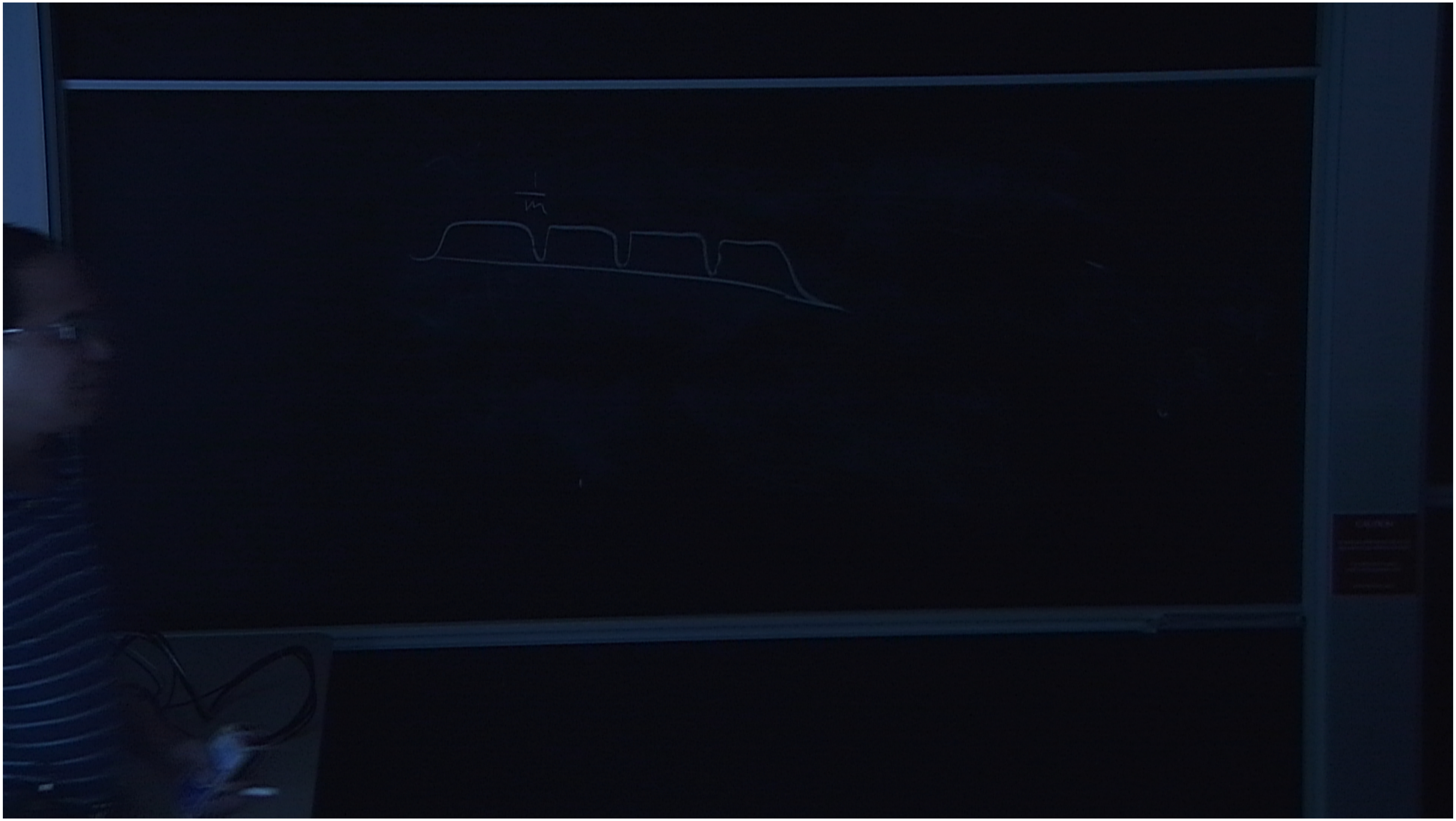

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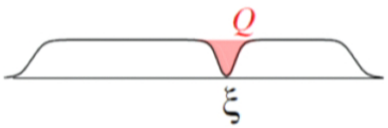

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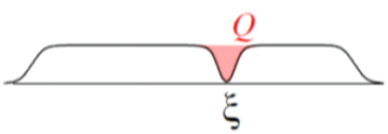

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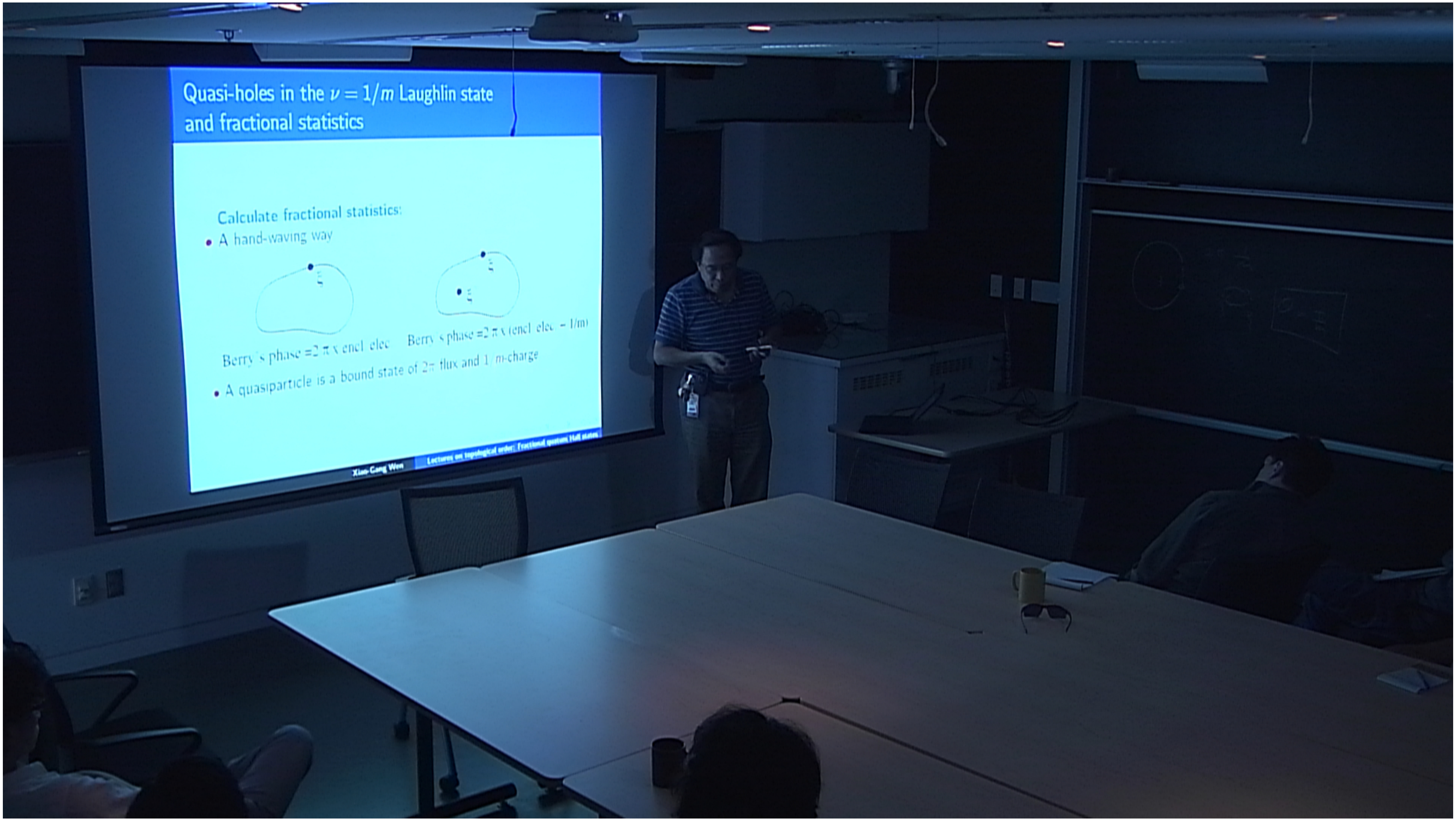
Lectures on topological order: Fractional quantum Hall states



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Quasi-holes in the $\nu = 1$ Pfaffian state: What is non-Abelian statistics?

- Ground state: $z_1 \approx z_2$, no zero; $z_1 \approx z_2 \approx z_3$, second-order zero;

$$\Psi_{\text{Pf}} = \mathcal{A}\Big(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\frac{1}{z_{N-1} - z_N}\Big) = \text{Pf}\Big(\frac{1}{z_i - z_j}\Big)$$

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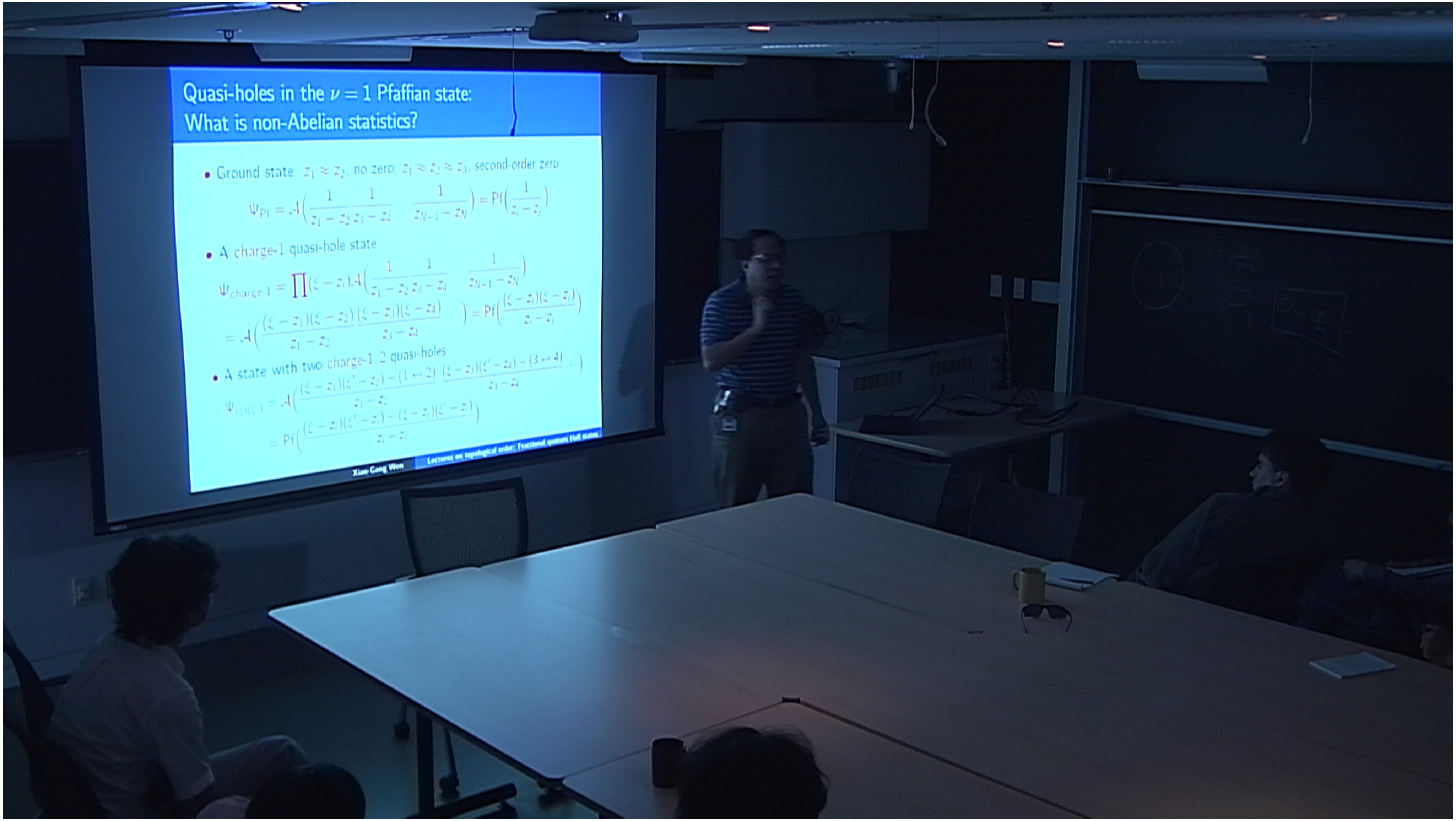
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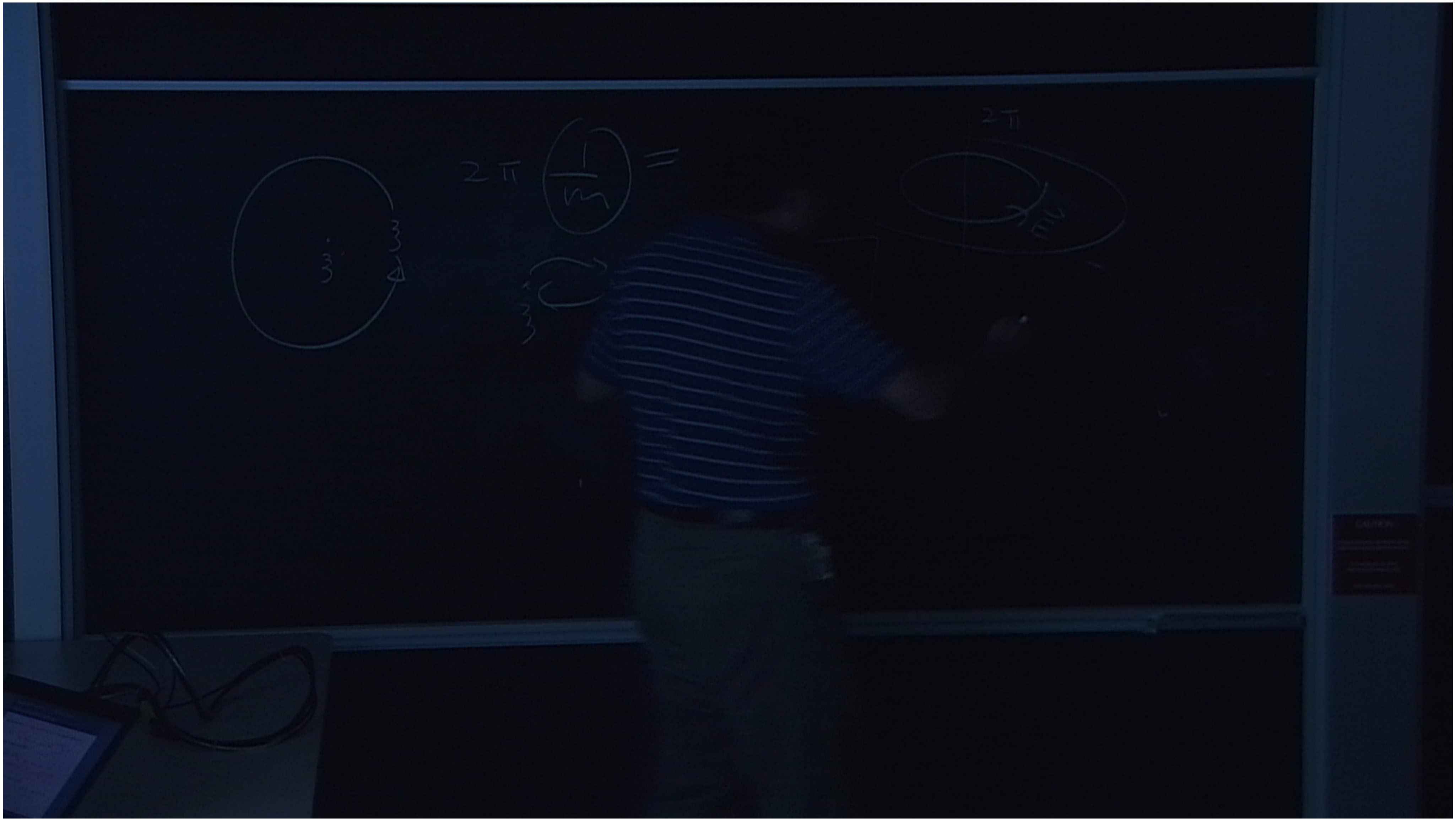
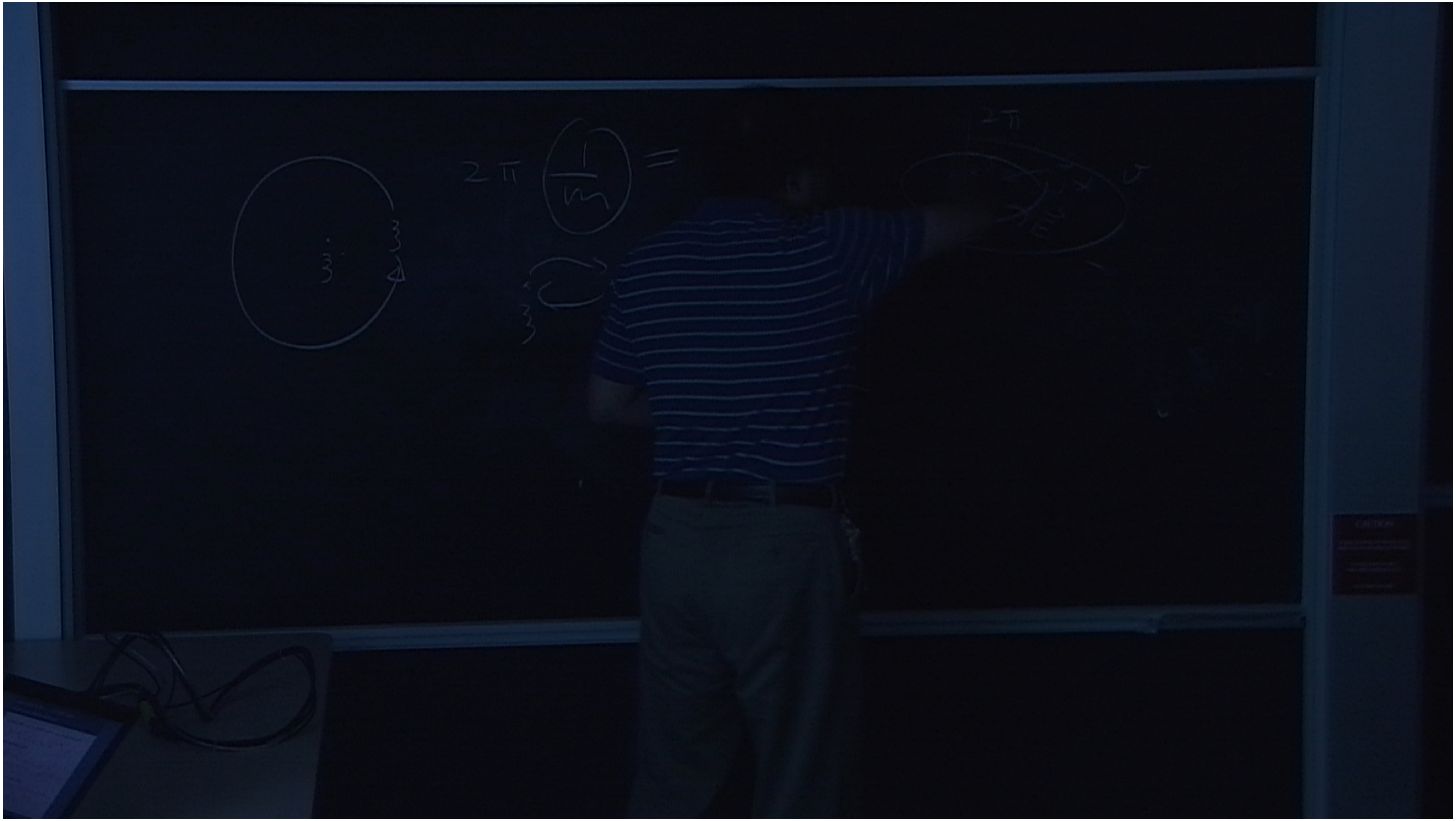
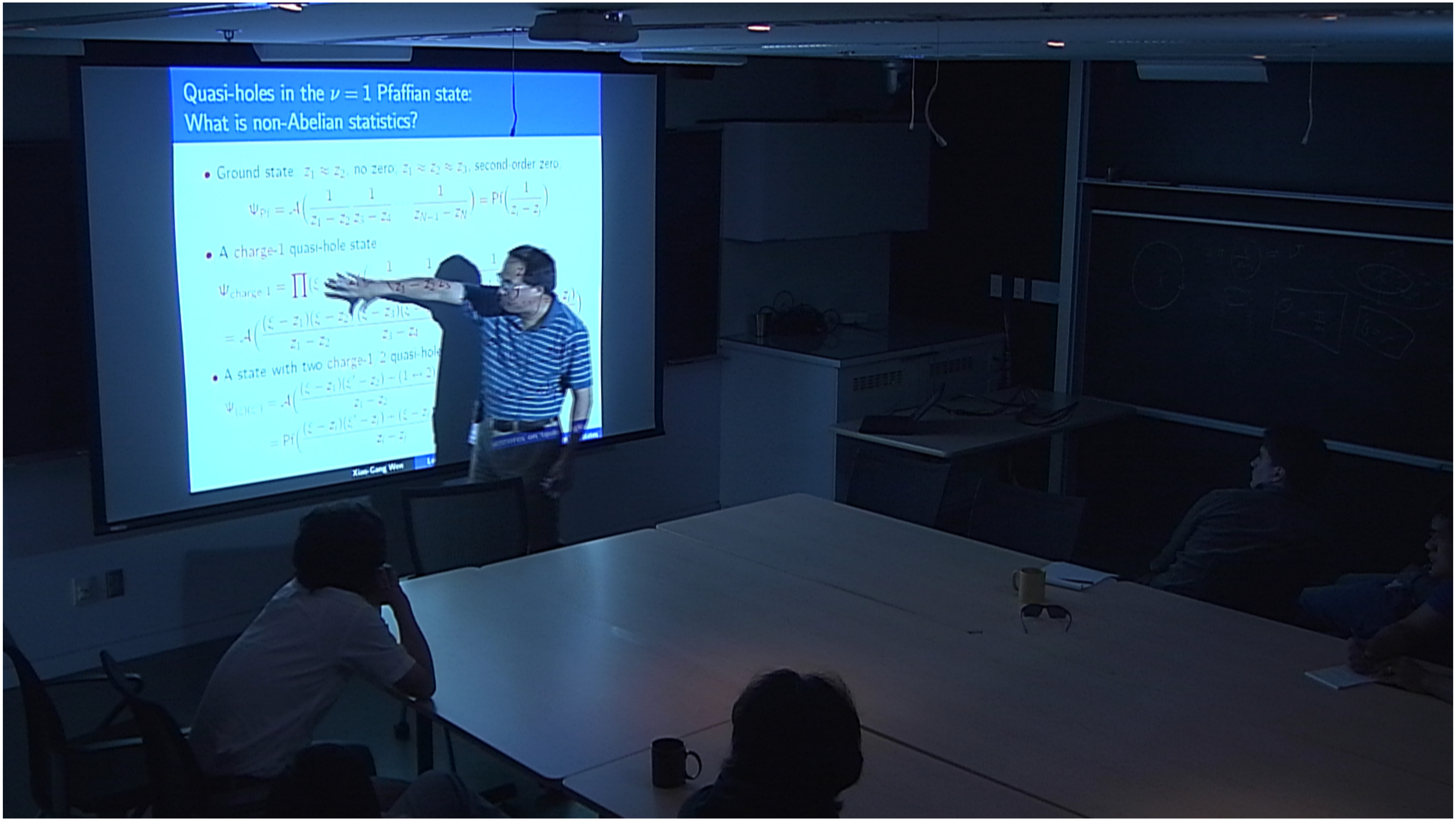
Lectures on topological order: Fractional quantum Hall states



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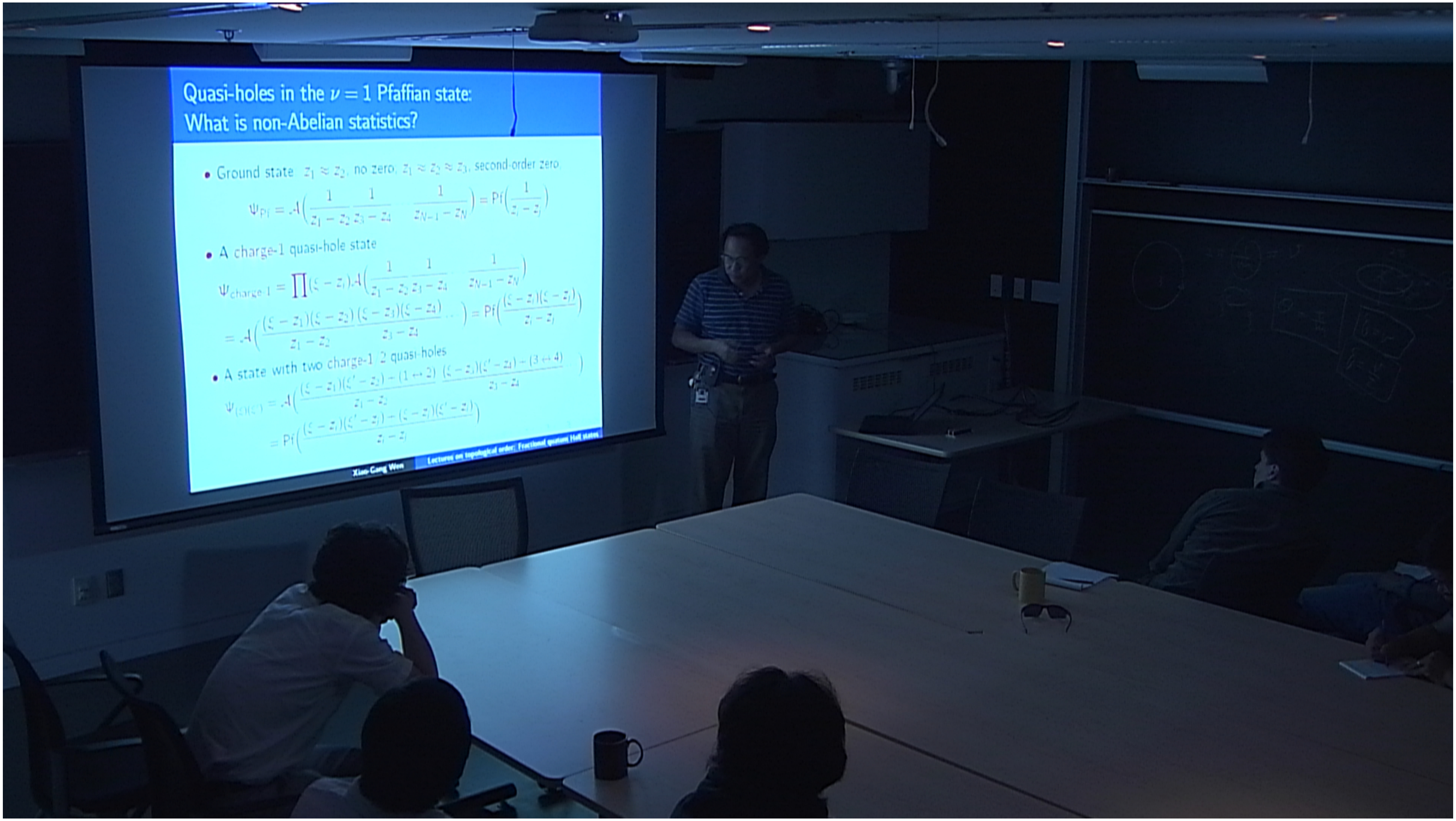
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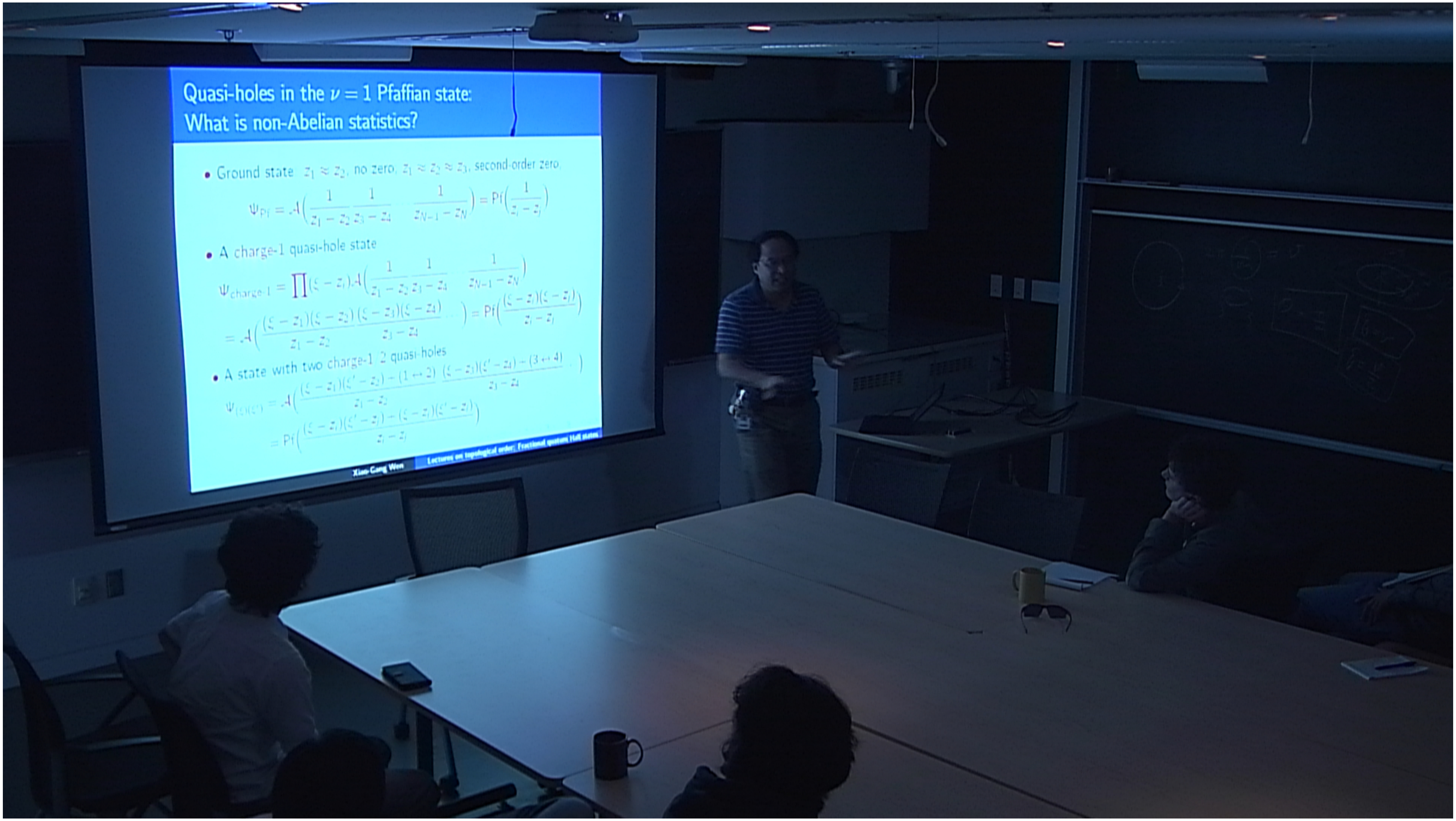
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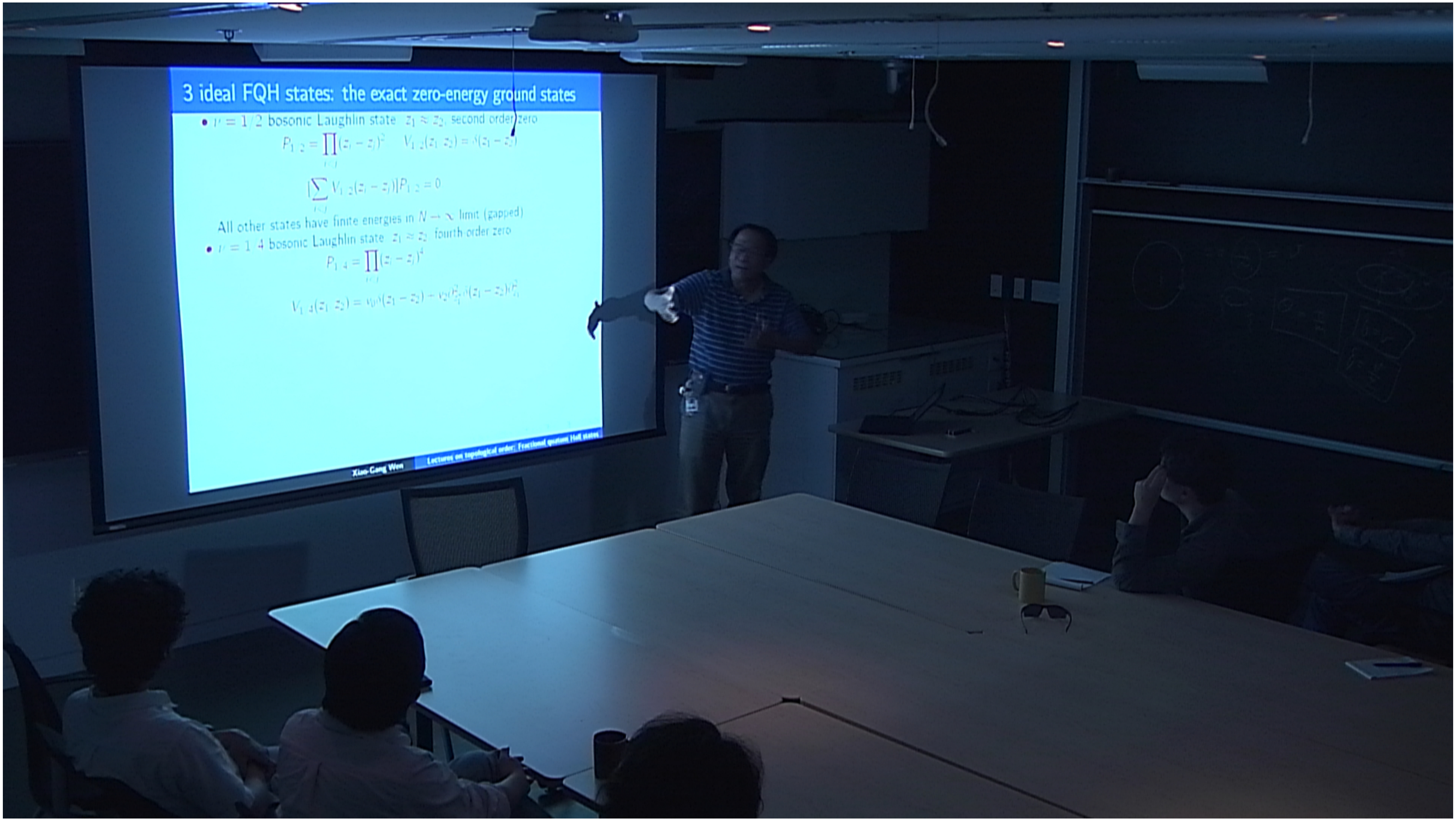

3 ideal FQH states: the exact zero-energy ground states
• $\nu = 1/2$ bosonic Laughlin state $z_1 \approx z_2$, second order zero
$P_{1/2} = \prod_{i<j}(z_i - z_j)^2 \quad V_{1/2}(z_1, z_2) = \delta(z_1 - z_2)$
$[\sum_{i<j} V_{1/2}(z_i - z_j)]P_{1/2} = 0$
All other states have finite energies in $N \to \infty$ limit (gapped)
• $\nu = 1/4$ bosonic Laughlin state $z_1 \approx z_2$, fourth-order zero
$P_{1/4} = \prod_{i<j}(z_i - z_j)^4$
$V_{1/4}(z_1, z_2) = v_0\delta(z_1 - z_2) + v_2\partial_{z_1}^2\delta(z_1 - z_2)\partial_{z_1}^2$