Title: Topological Order Series

Date: Jun 16, 2015 03:00 PM

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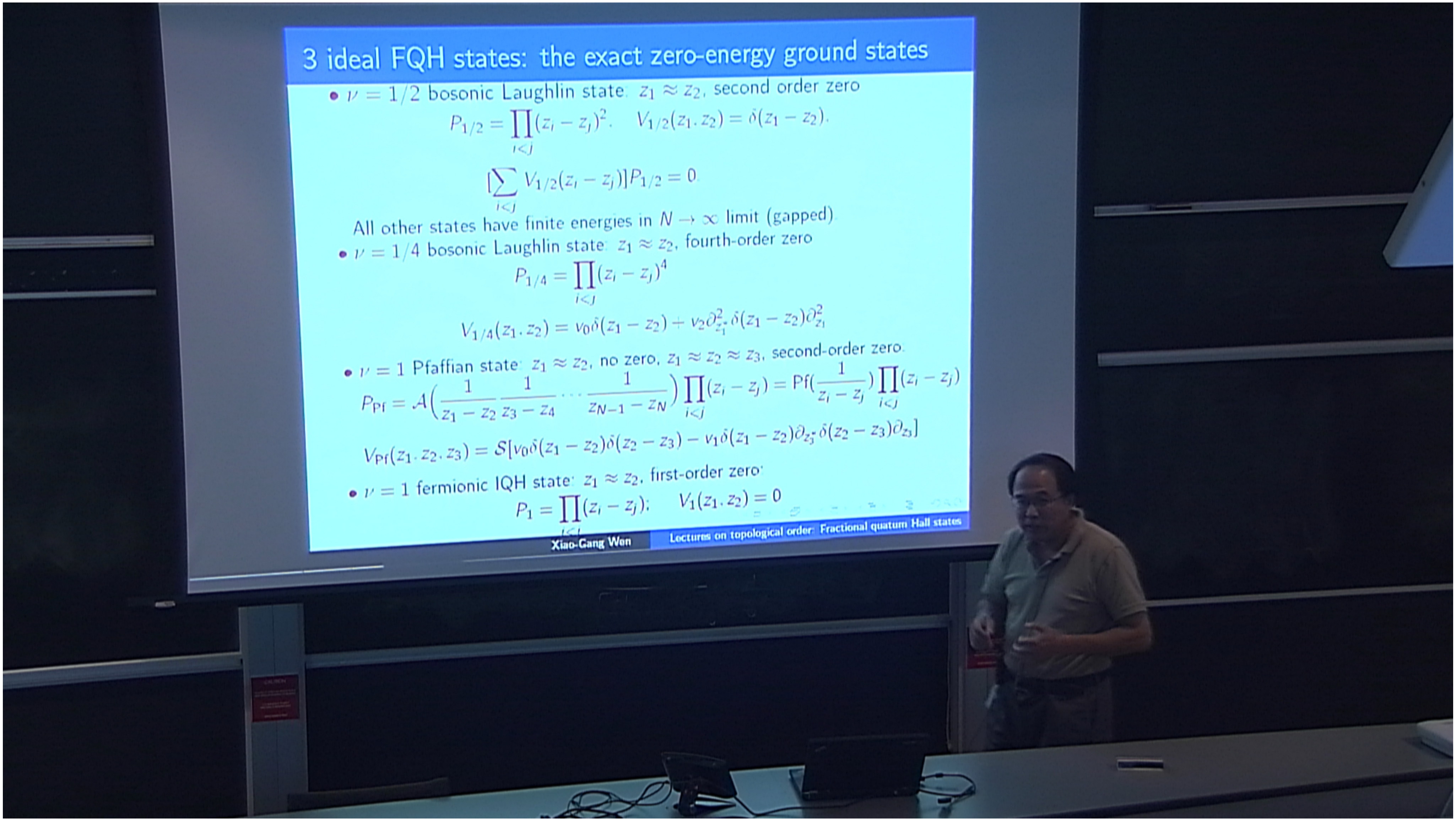
Abstract:


3 ideal FQH states: the exact zero-energy ground states
• $\nu = 1/2$ bosonic Laughlin state: $z_1 \approx z_2$, second order zero
$P_{1/2} = \prod_{i<j}(z_i - z_j)^2$, $V_{1/2}(z_1, z_2) = \delta(z_1 - z_2)$.
$[\sum_{i<j} V_{1/2}(z_i - z_j)] P_{1/2} = 0$.
All other states have finite energies in $N \to \infty$ limit (gapped).
• $\nu = 1/4$ bosonic Laughlin state: $z_1 \approx z_2$, fourth-order zero
$P_{1/4} = \prod_{i<j}(z_i - z_j)^4$
$V_{1/4}(z_1, z_2) = v_0\delta(z_1 - z_2) + v_2\partial^2_{z_1^*}\delta(z_1 - z_2)\partial^2_{z_1}$
• $\nu = 1$ Pfaffian state: $z_1 \approx z_2$, no zero, $z_1 \approx z_2 \approx z_3$, second-order zero.
$P_{\text{Pf}} = \mathcal{A}\Big(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\frac{1}{z_{N-1} - z_N}\Big)\prod_{i<j}(z_i - z_j) = \text{Pf}(\frac{1}{z_i - z_j})\prod_{i<j}(z_i - z_j)$
$V_{\text{Pf}}(z_1, z_2, z_3) = \mathcal{S}[v_0\delta(z_1 - z_2)\delta(z_2 - z_3) - v_1\delta(z_1 - z_2)\partial_{z_3^*}\delta(z_2 - z_3)\partial_{z_3}]$
• $\nu = 1$ fermionic IQH state: $z_1 \approx z_2$, first-order zero:
$P_1 = \prod_{i<j}(z_i - z_j)$; $V_1(z_1, z_2) = 0$
Xiao-Gang Wen
Lectures on topological order: Fractional quatum Hall states

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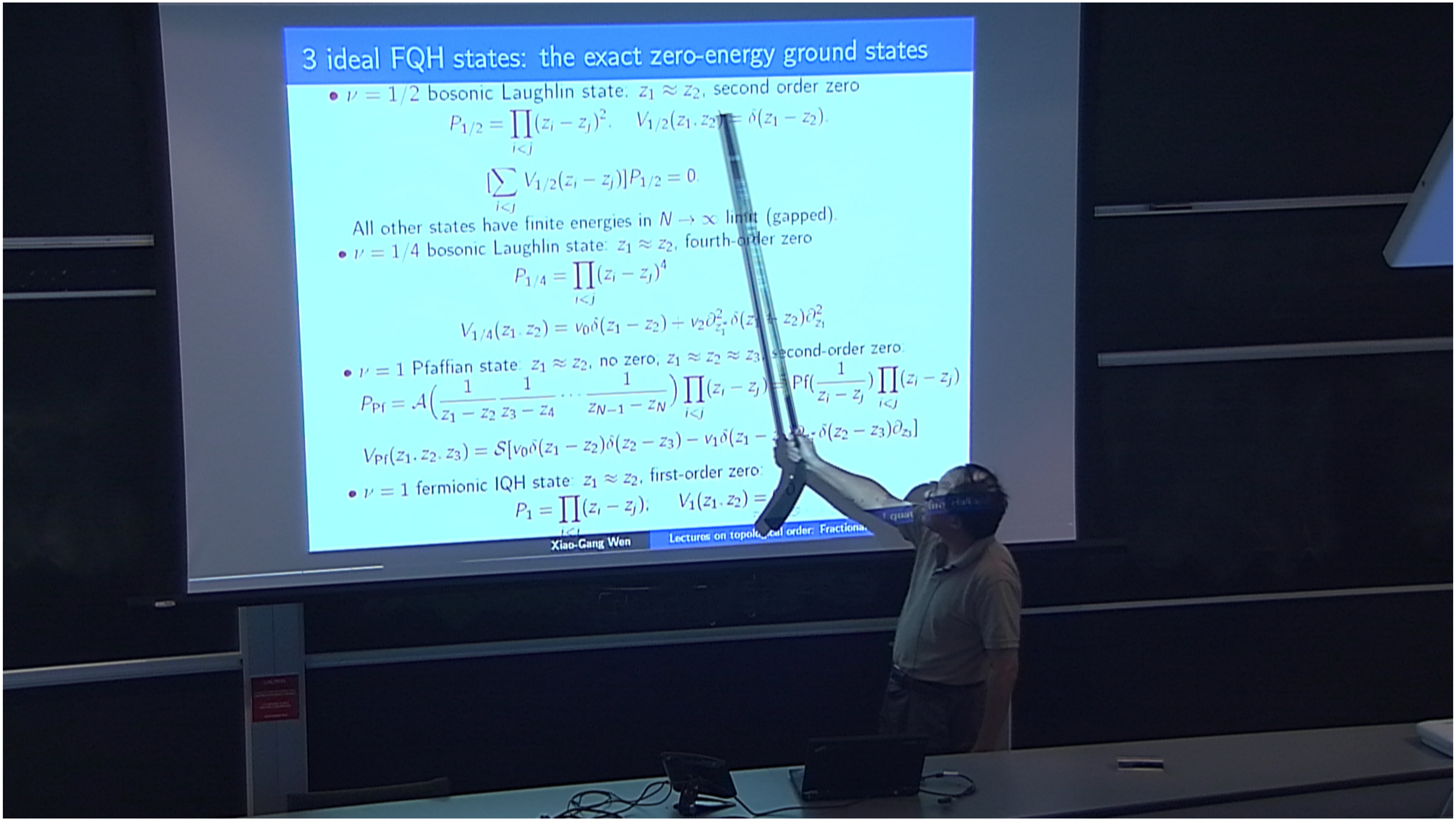
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Quasi-holes in the $\nu = 1/m$ Laughlin state and fractional charge

- A hole-like excitation = missing an electron, charge = 1

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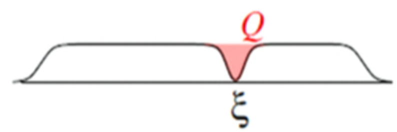

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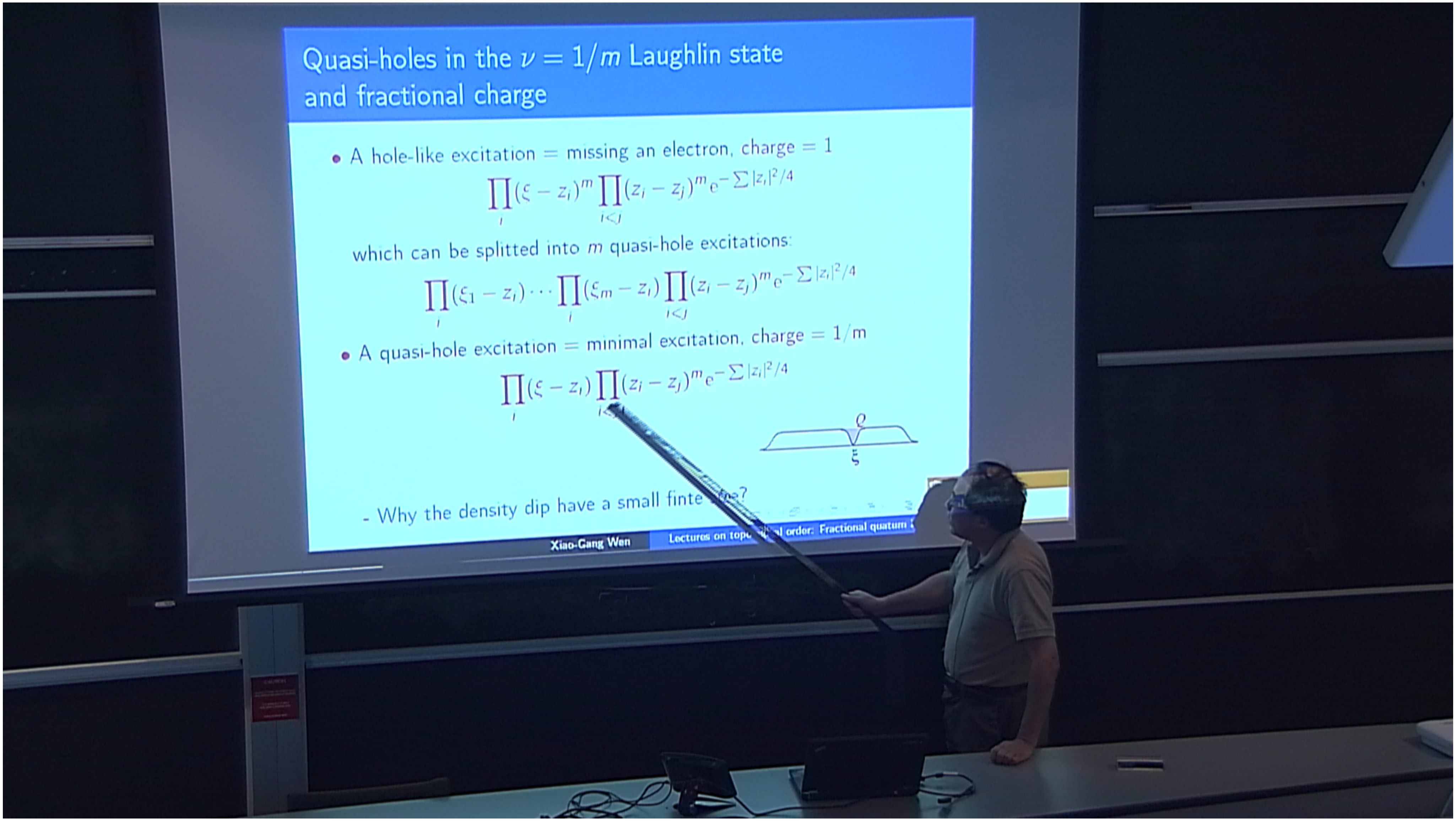
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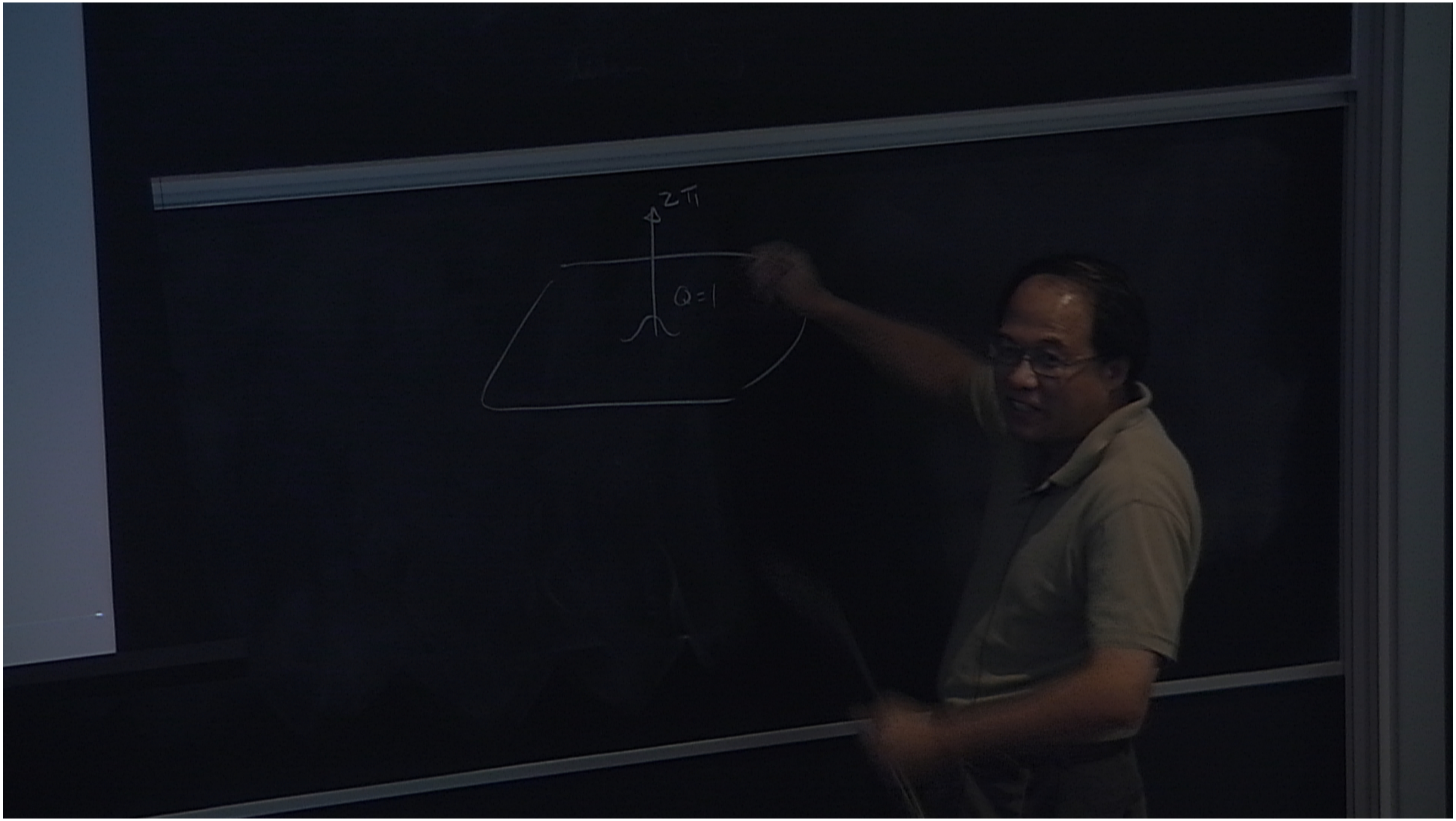

$2T_1$
$Q=1$

Quasi-holes in the $\nu = 1$ Pfaffian state: What is non-Abelian statistics?

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- A charge-1 quasi-hole state

$$\Psi_{\text{charge-1}} = \prod(\xi - z_i)\mathcal{A}\Big(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\frac{1}{z_{N-1} - z_N}\Big)$$

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- A state with two charge-$1/2$ quasi-holes

$$\Psi_{(\xi)(\xi')} = \mathcal{A}\Big(\frac{(\xi - z_1)(\xi' - z_2) + (1 \leftrightarrow 2)}{z_1 - z_2}\frac{(\xi - z_3)(\xi' - z_4) + (3 \leftrightarrow 4)}{z_3 - z_4}\cdots\Big)$$

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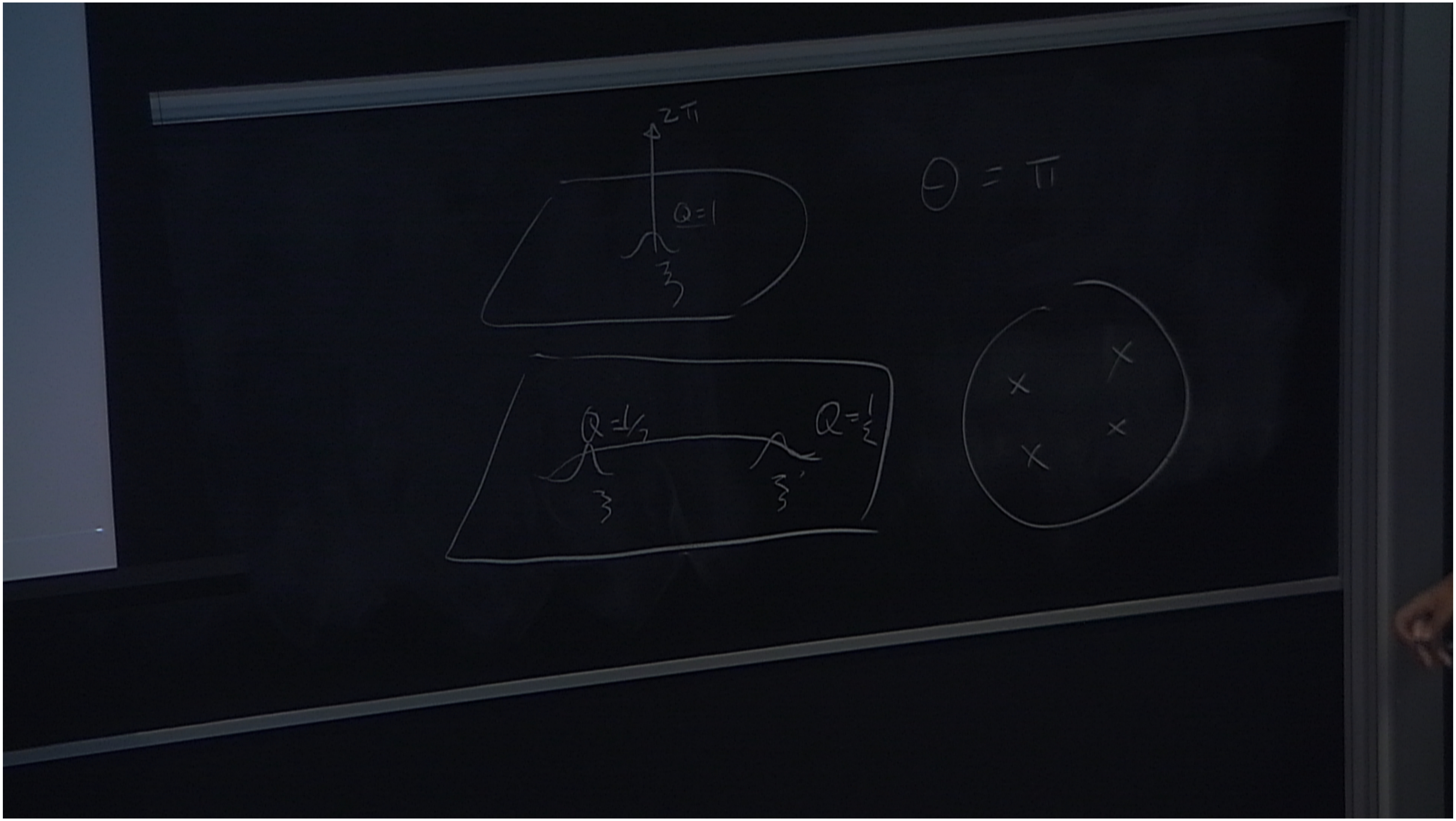

2π
Q=1
ξ
θ = π
Q=-1/2
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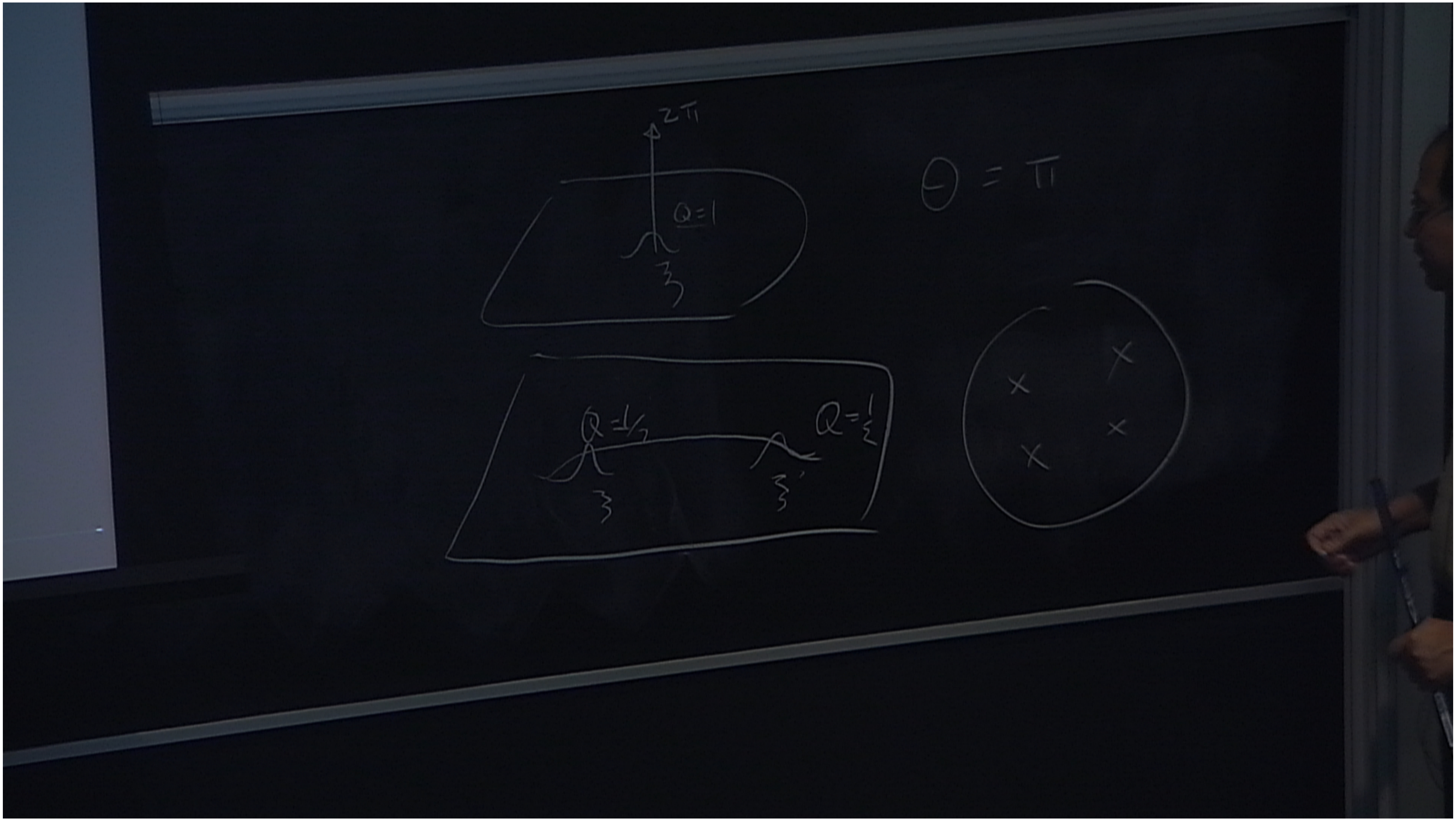
ξ
ξ'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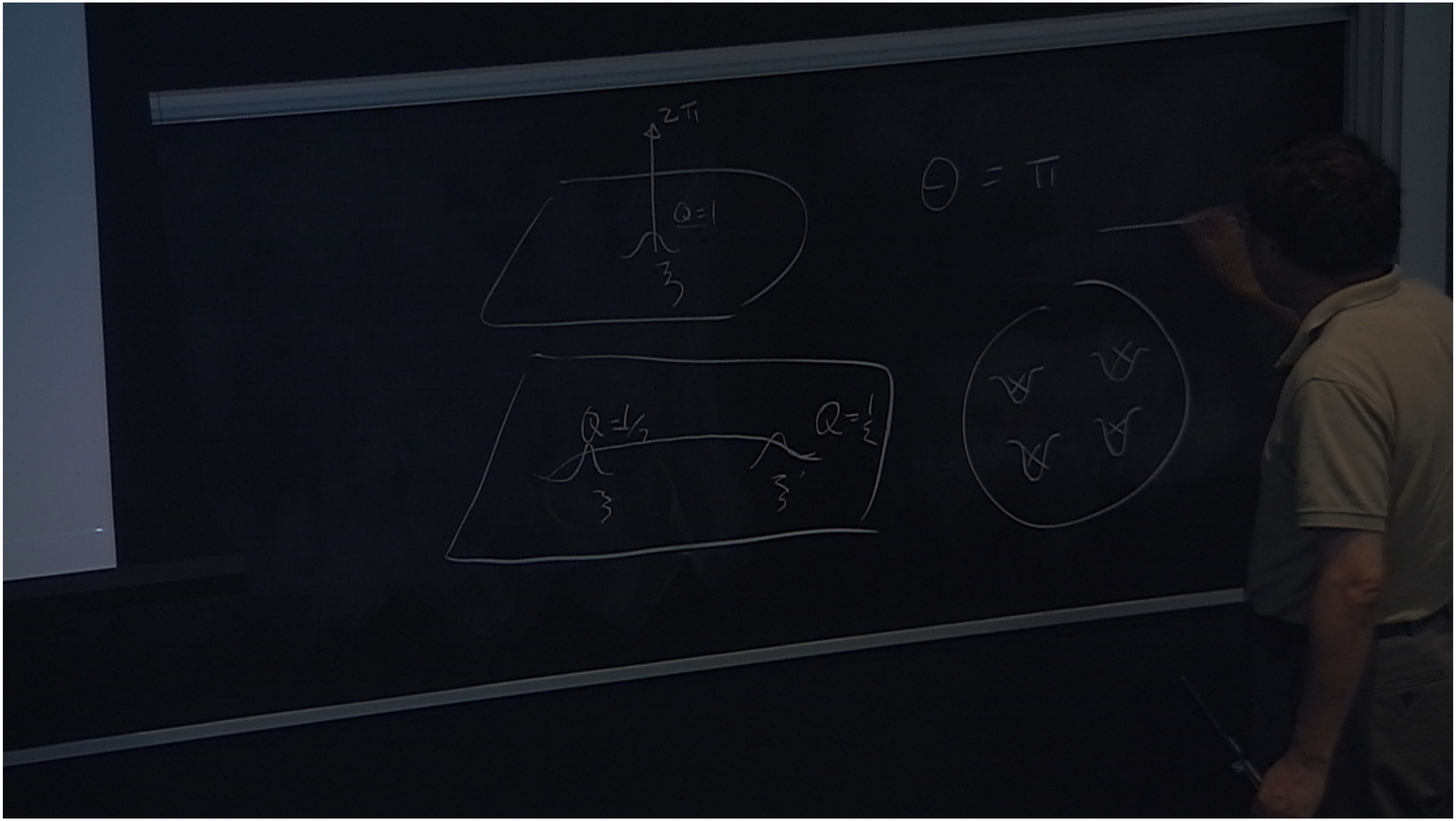

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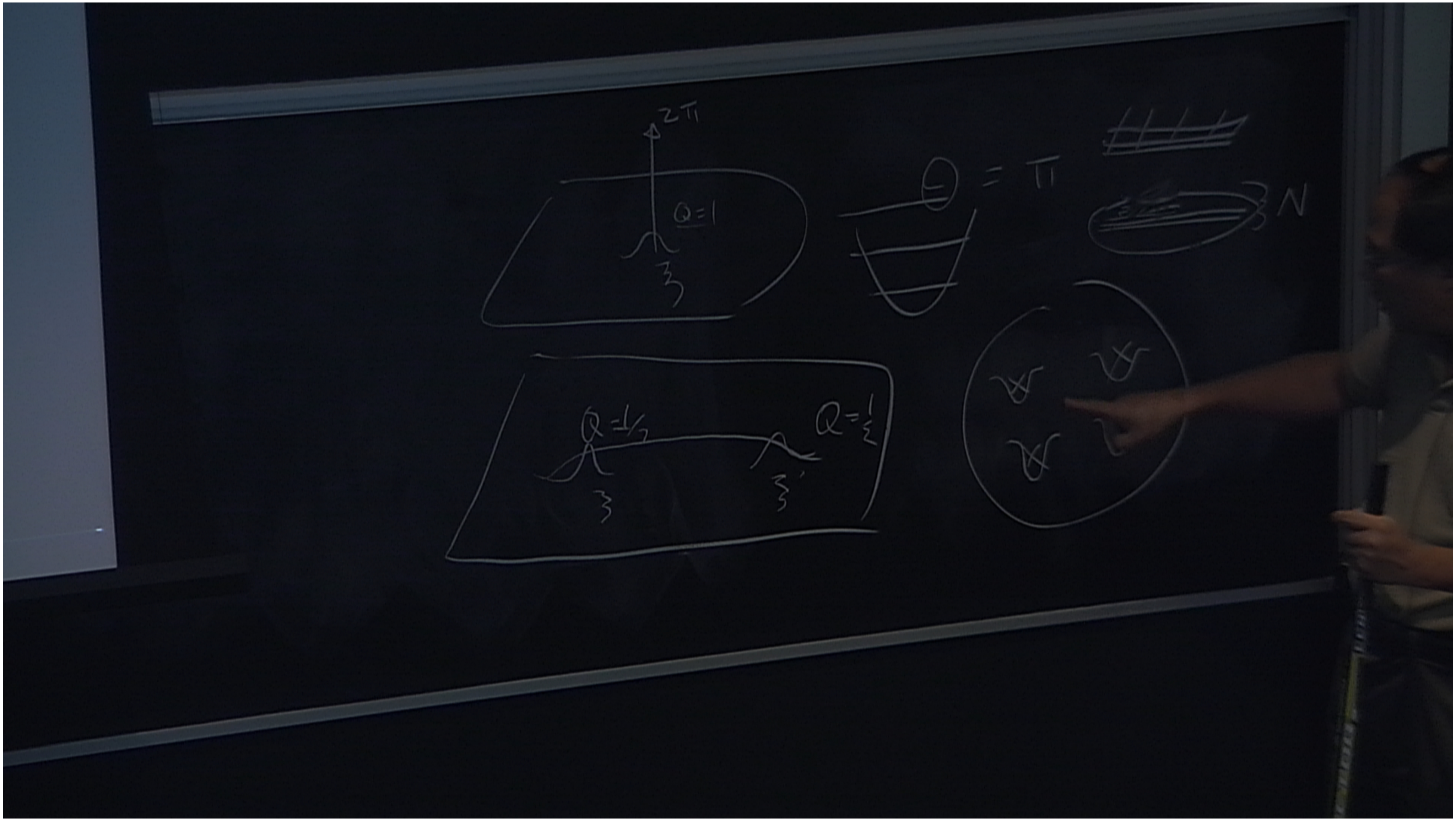

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How many states with four charge-1/2 quasi-holes?

- One of the state with four charge-1/2 quasi-holes

$$P_{(12)(34)} = \mathrm{Pf}\Big(\frac{(\xi_1 - z_i)(\xi_2 - z_i)(\xi_3 - z_j)(\xi_4 - z_j) + (i \leftrightarrow j)}{z_i - z_j}\Big)\cdots$$

$$= \mathrm{Pf}\Big(\frac{[12,34]_{z_i z_j}}{z_i - z_j}\Big)\cdots$$

The other two are $P_{(13)(14)}$, $P_{(14)(23)}$.

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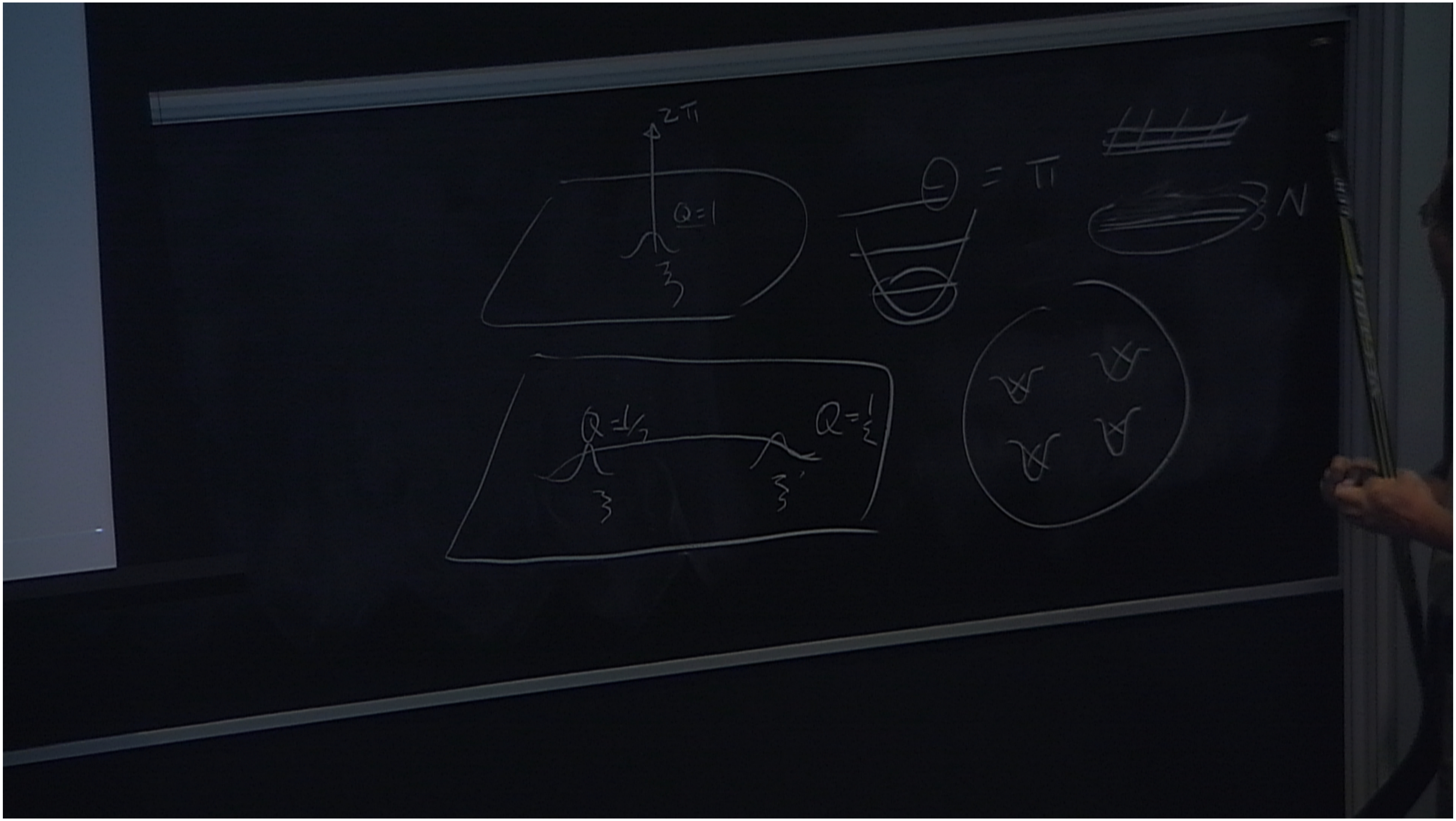
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- **But only two linearly independent states**. Using Nayak & Wilzcek

$$[12,34]_{z_i z_j} - [13,24]_{z_i z_j} = (z_i - z_j)^2(\xi_1 - \xi_4)(\xi_2 - \xi_3) = z_{ij}^2 \xi_{14}\xi_{23}$$

we find (with $z_{12} = z_1 - z_2$, $\xi_{12} = \xi_1 - \xi_2$, etc)

$$P_{(13)(24)} = \mathcal{A}\Big(\frac{[12,34]_{z_1 z_2} - z_{12}^2 \xi_{14}\xi_{23}}{z_{12}} \frac{[12,34]_{z_3 z_4} - z_{34}^2 \xi_{14}\xi_{23}}{z_{34}}\cdots\Big)$$
$$= P_{(12)(34)} - N_{pair}\mathcal{A}\Big(z_{12}\xi_{14}\xi_{23}\frac{[12,34]_{z_3 z_4}}{z_{34}}\cdots\Big)$$

$$P_{(12)(34)} - P_{(13)(24)} = N_{pair}\xi_{14}\xi_{23}\mathcal{A}\Big(z_{12}\frac{[12,34]_{z_3z_4}}{z_{34}}\cdots\Big)$$

Similarly

$$P_{(12)(34)} - P_{(14)(23)} = N_{pair}\xi_{13}\xi_{24}\mathcal{A}\Big(z_{12}\frac{[12,34]_{z_3z_4}}{z_{34}}\cdots\Big)$$

So

$$\frac{P_{(12)(34)} - P_{(13)(24)}}{\xi_{14}\xi_{23}} = \frac{P_{(12)(34)} - P_{(14)(23)}}{\xi_{13}\xi_{24}}$$

- Two states for four charge-1/2 quasiholes, even if we fixed their positions. The two states are topologically degenerate.
  $D_n = \frac{1}{2}(\sqrt{2})^n$ **states for** n **charge-1/2 quasiholes.**
  $\sqrt{2}$ **states per charge-1/2 quasihole !!!**
  Quantum dimension for the charge-1/2 quasihole $d = \sqrt{2}$
  The charge-1/2 quasihole has a non-Abelian statistics.
  For Abelian anyons $d = 1$.
- *In general, for* n *particles with non-Abelian statistics, there are* D_n *degenerate states even after we fix the positions of the particle:*
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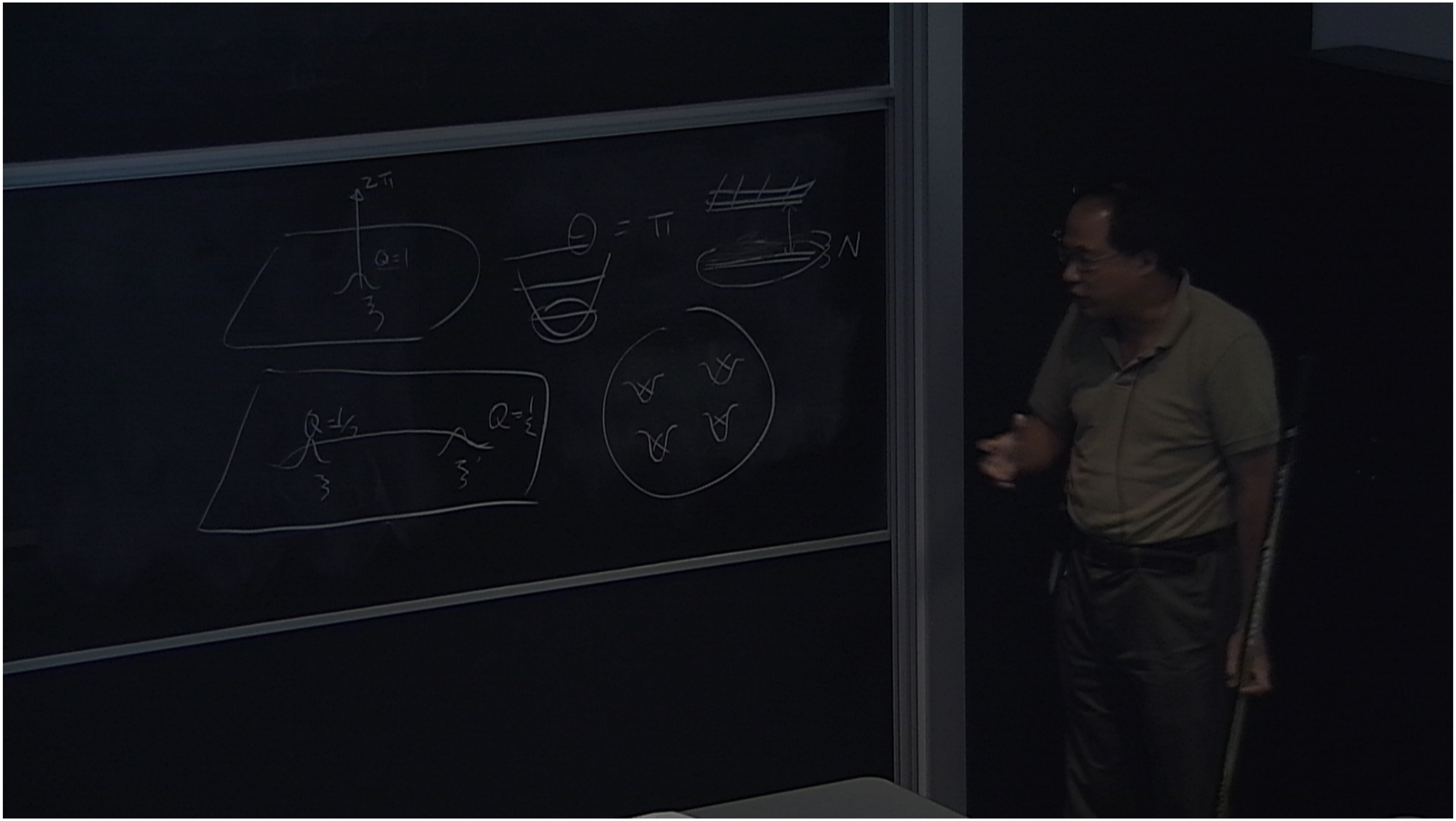

2π
$Q=1$
$\Theta=\pi$
N
$Q=-1/2$
$Q=\frac{1}{2}$

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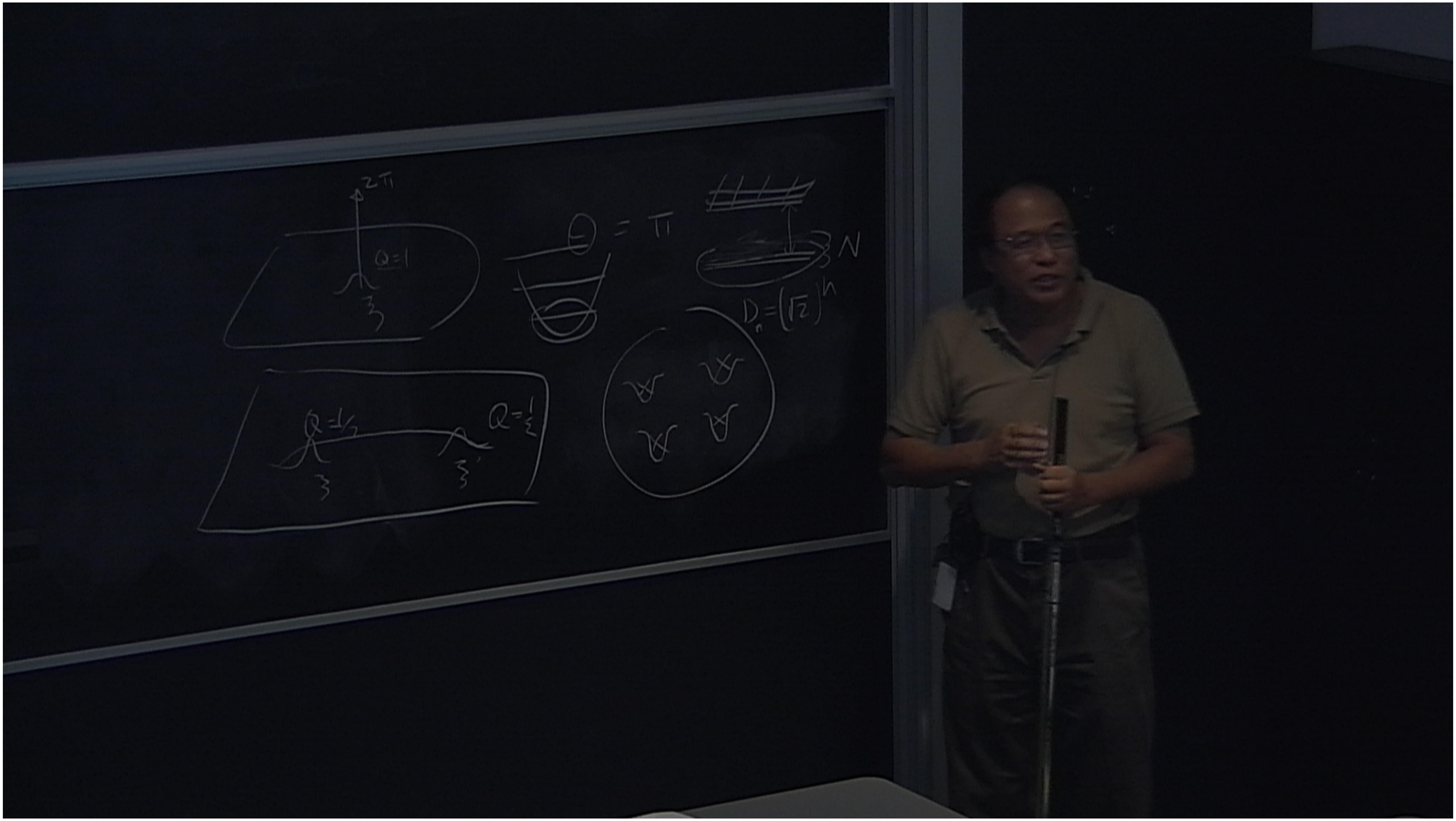
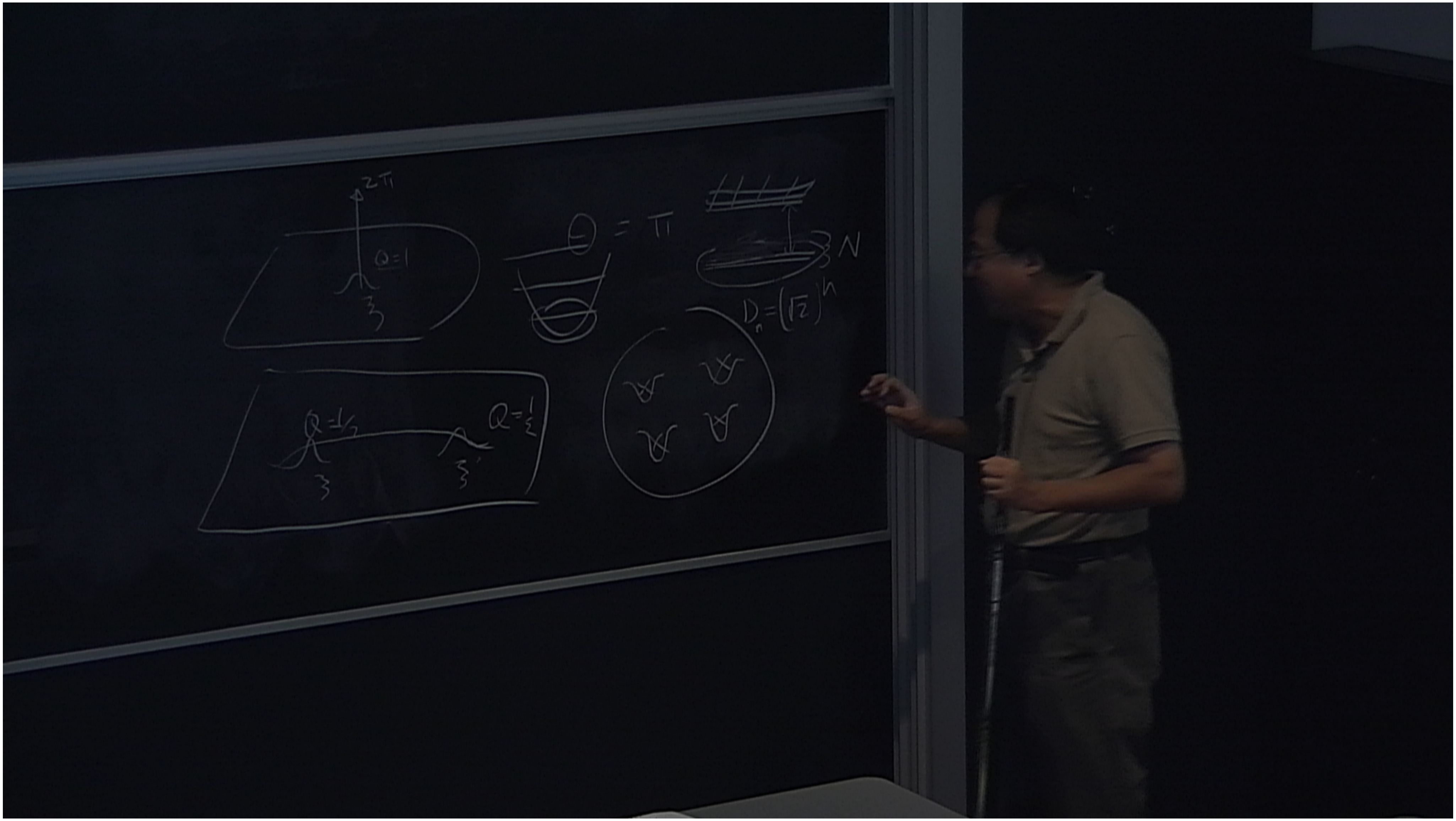
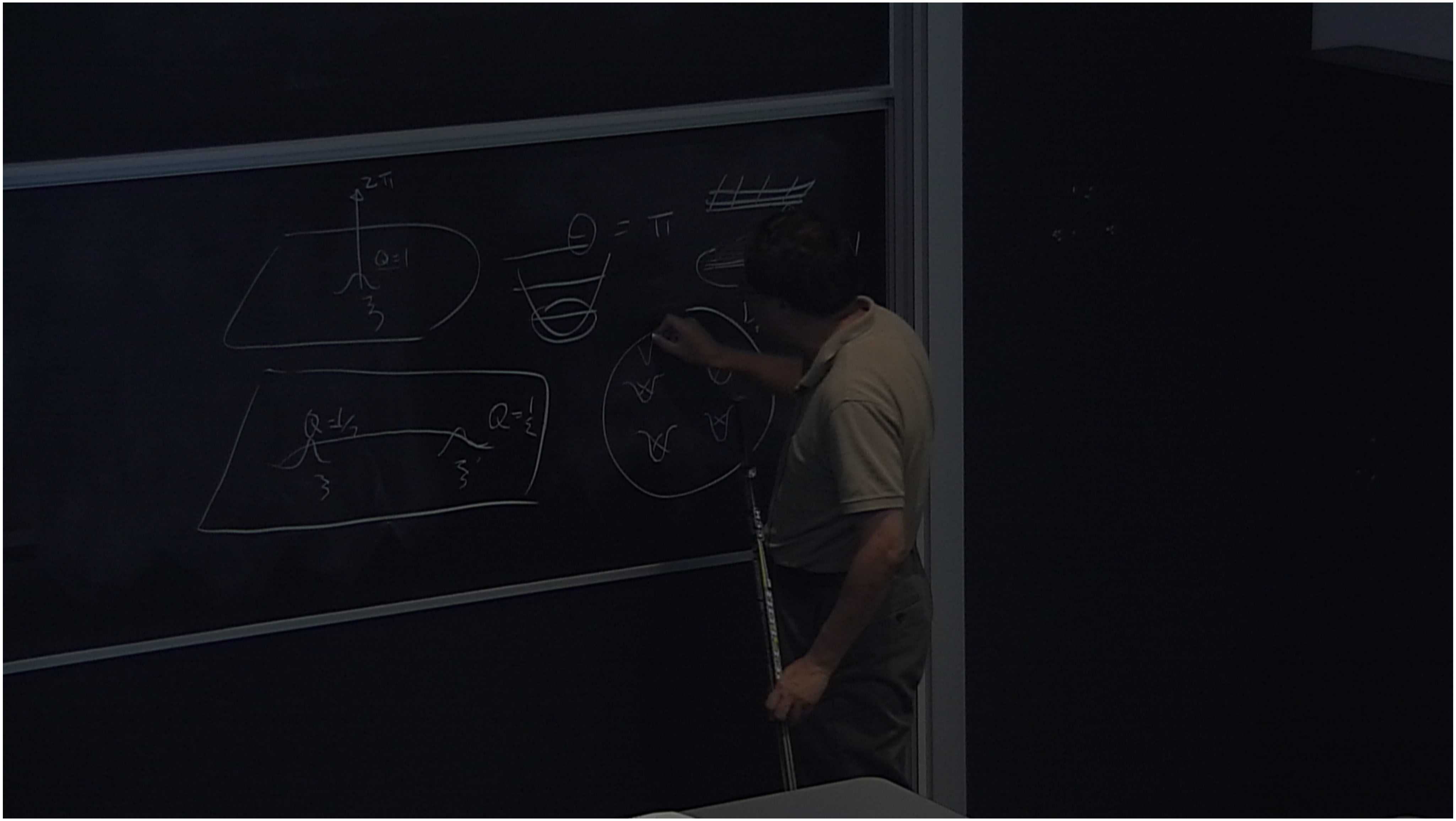
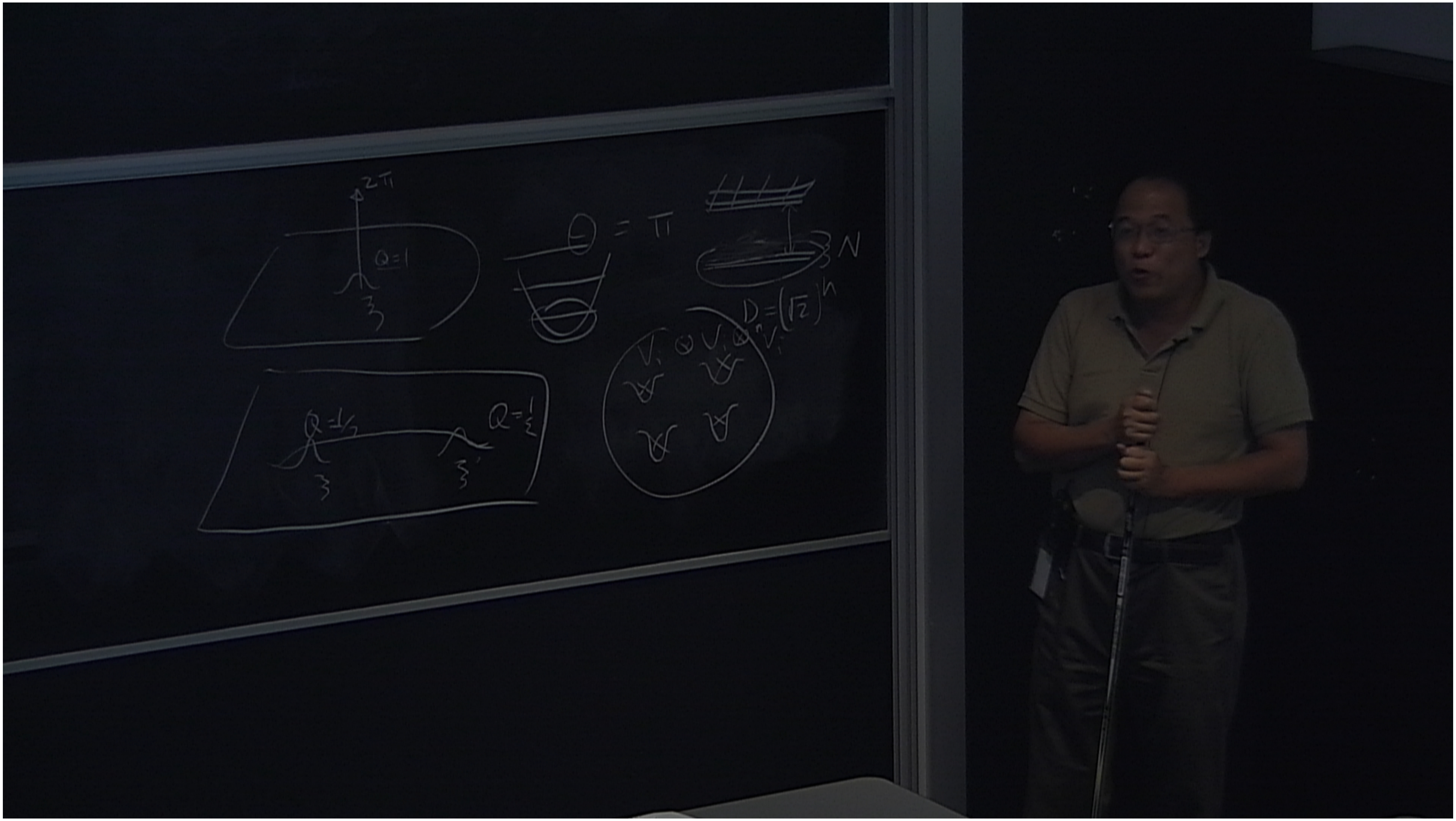

2π
Q=1
ξ
Θ = π
N
$D_n = (\sqrt{2})^n$
Q=-½
Q=½
ξ
ξ'



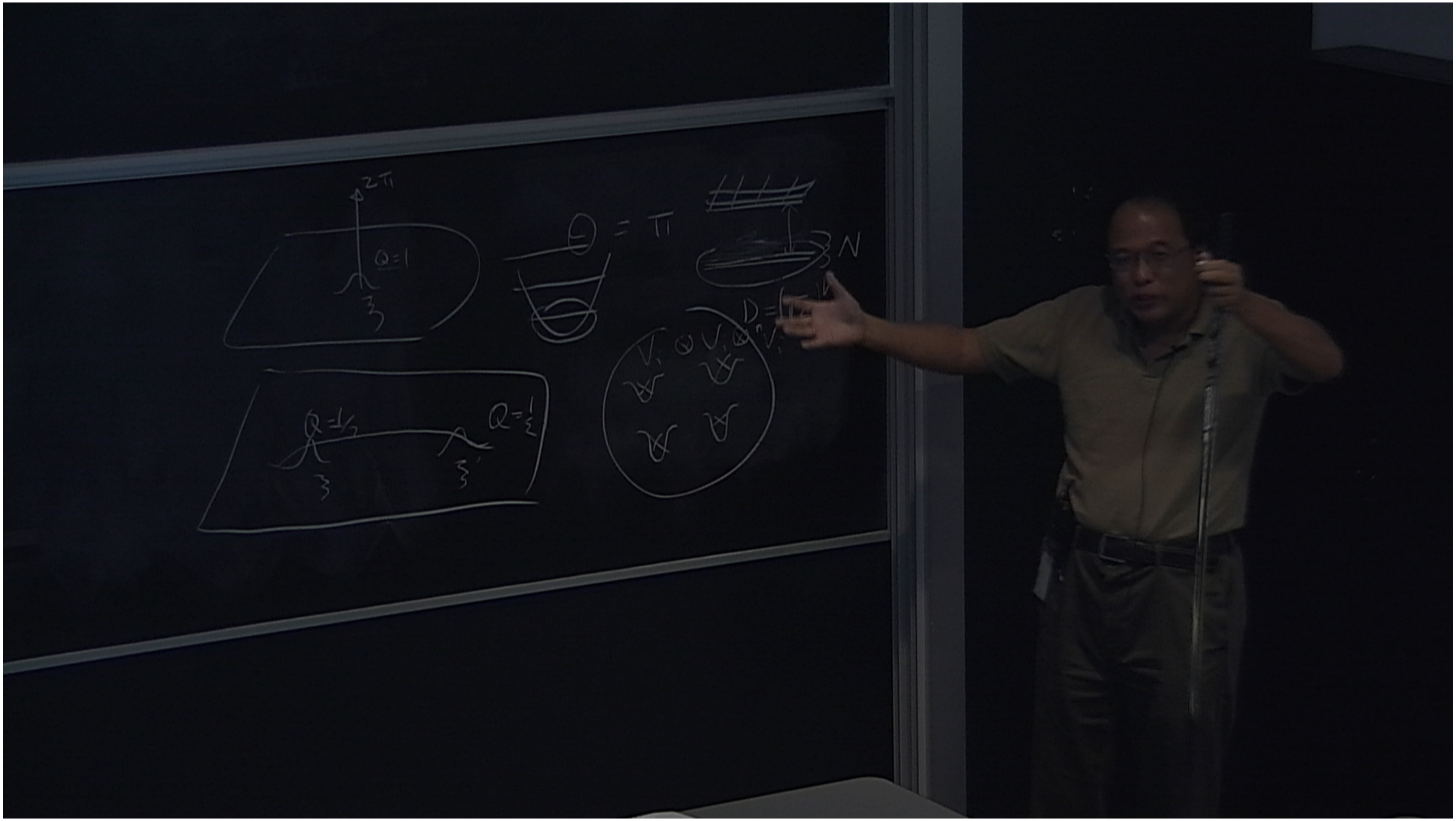
$2\pi_1$
Q=1
$\Theta = \pi_1$
N
$D_n = (\sqrt{2})^n$
Q=-1/2
Q=1/2



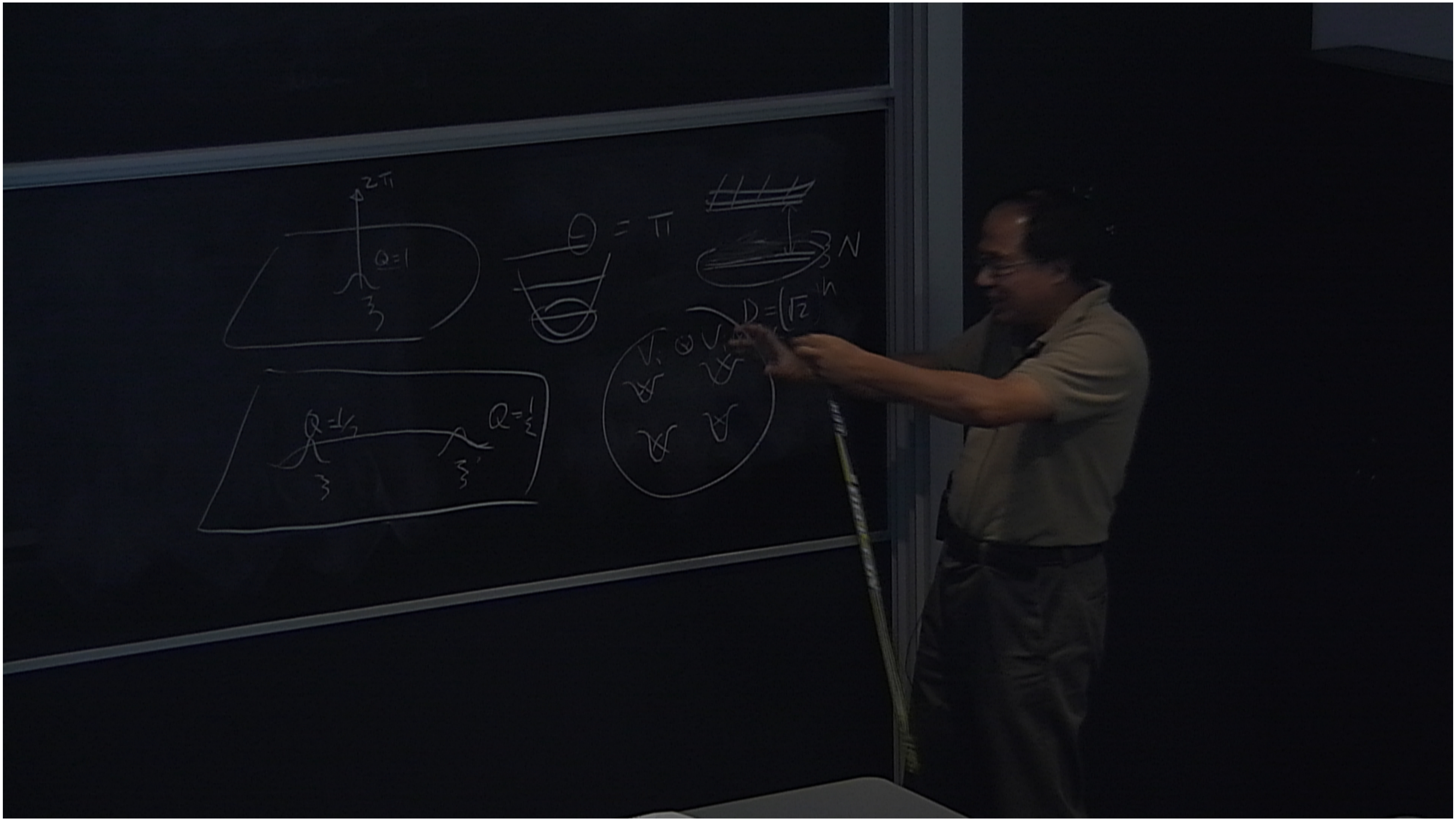
2π
Q=1
θ = π
Q=-1/2
Q=1/2



2π
Q=1
Θ = π
N
$D_n=(\sqrt{2})^n$
Q=1/2
Q=1/2

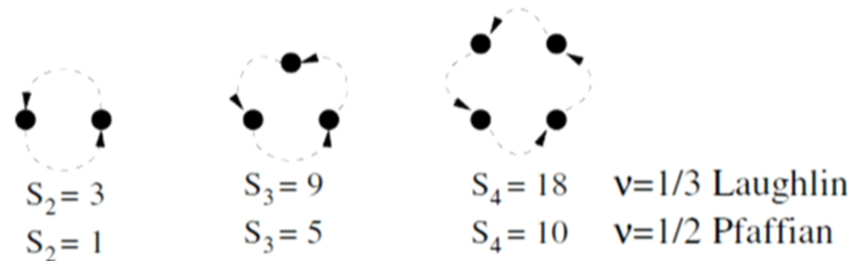


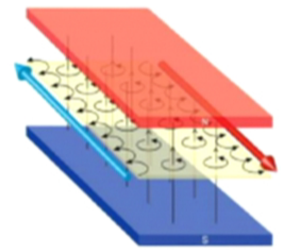
2π
$Q=1$
ξ
$\Theta = \pi$
N
$Q=1/2$
$Q=\frac{1}{2}$
ξ
ξ'


2π
Q=1
Θ = π
N
Q=½
Q=½
V₁ ⊗ V₂

Local dancing rule → global dancing pattern

- Local dancing rules of a FQH liquid:
  (1) every electron dances around clock-wise
  (Φ_{FQH} only depends on $z = x + \mathrm{i}y$)
  (2) takes exactly three steps to go around each other
  (Relative angular momentum $S_2 = 3$) Wen-Wang arXiv:0803.1016
  → Global dancing pattern $\Phi_{\mathrm{FQH}}(\{z_1, ..., z_N\}) = \prod(z_i - z_j)^3$
- A systematic theory of FQH state – **Pattern of zeros S_a**:
  a-electron cluster has a relative angular momentum S_a





- Local dancing rules are enforce by the Hamiltonian to lower energy.
- Only certain sequences S_a correspond to valid FQH states. Which?
- Different POZ S_a give rise to different topological properties

How many states with four charge-1/2 quasi-holes?

- One of the state with four charge-1/2 quasi-holes

$$P_{(12)(34)} = \text{Pf}\Big(\frac{(\xi_1 - z_i)(\xi_2 - z_i)(\xi_3 - z_j)(\xi_4 - z_j) + (i \leftrightarrow j)}{z_i - z_j}\Big)\cdots$$

$$= \text{Pf}\Big(\frac{[12,34]_{z_i z_j}}{z_i - z_j}\Big)\cdots$$

The other two are $P_{(13)(14)}$, $P_{(14)(23)}$.

- **But only two linearly independent states**. Using Nayak & Wilzcek

$$[12,34]_{z_i z_j} - [13,24]_{z_i z_j} = (z_i - z_j)^2(\xi_1 - \xi_4)(\xi_2 - \xi_3) = z_{ij}^2 \xi_{14}\xi_{23}$$

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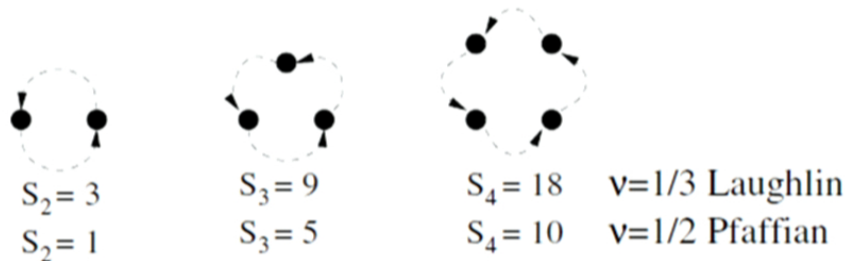

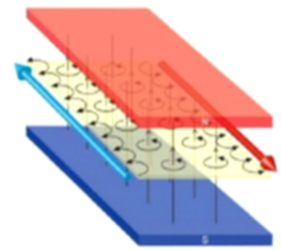

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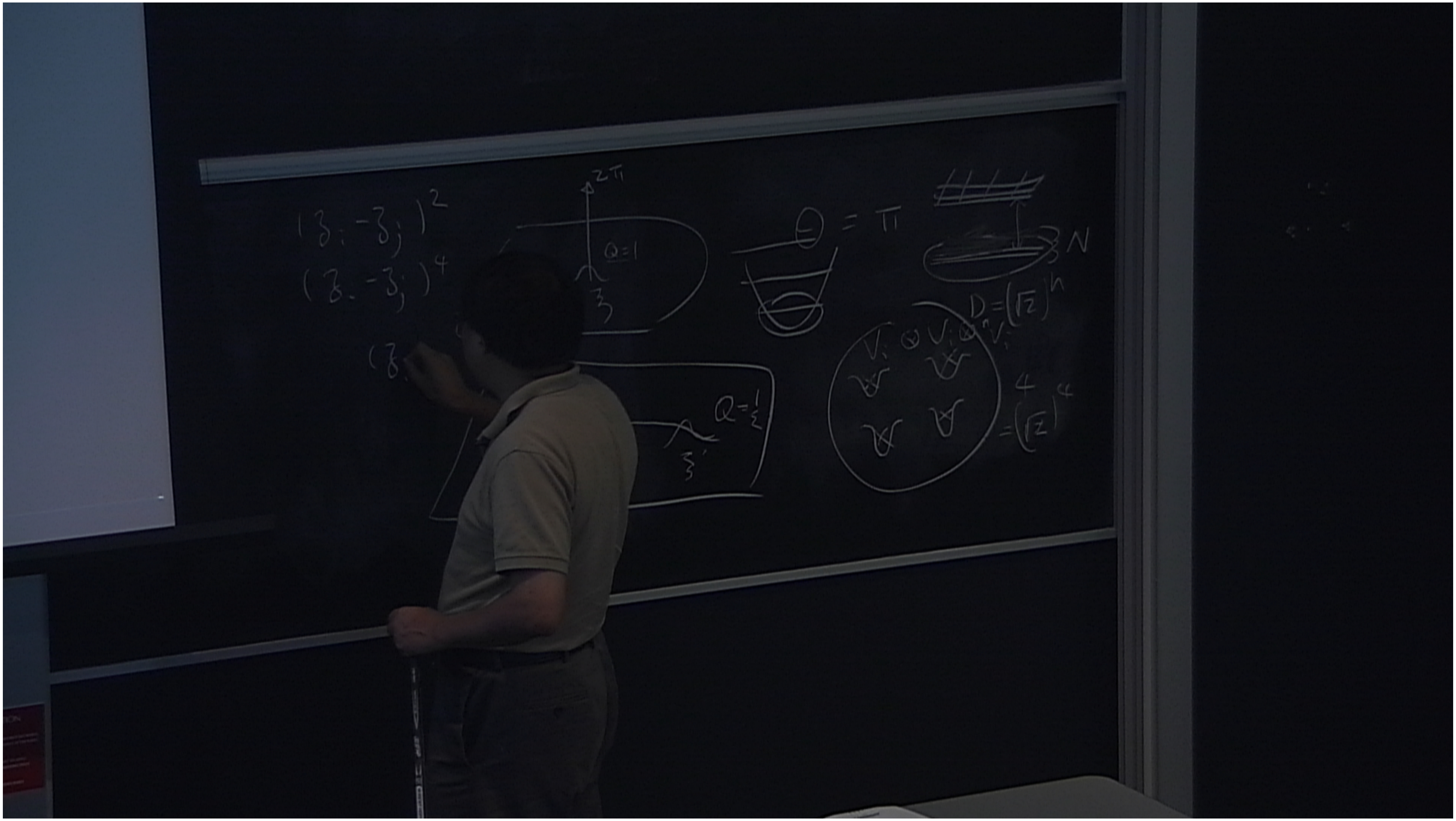

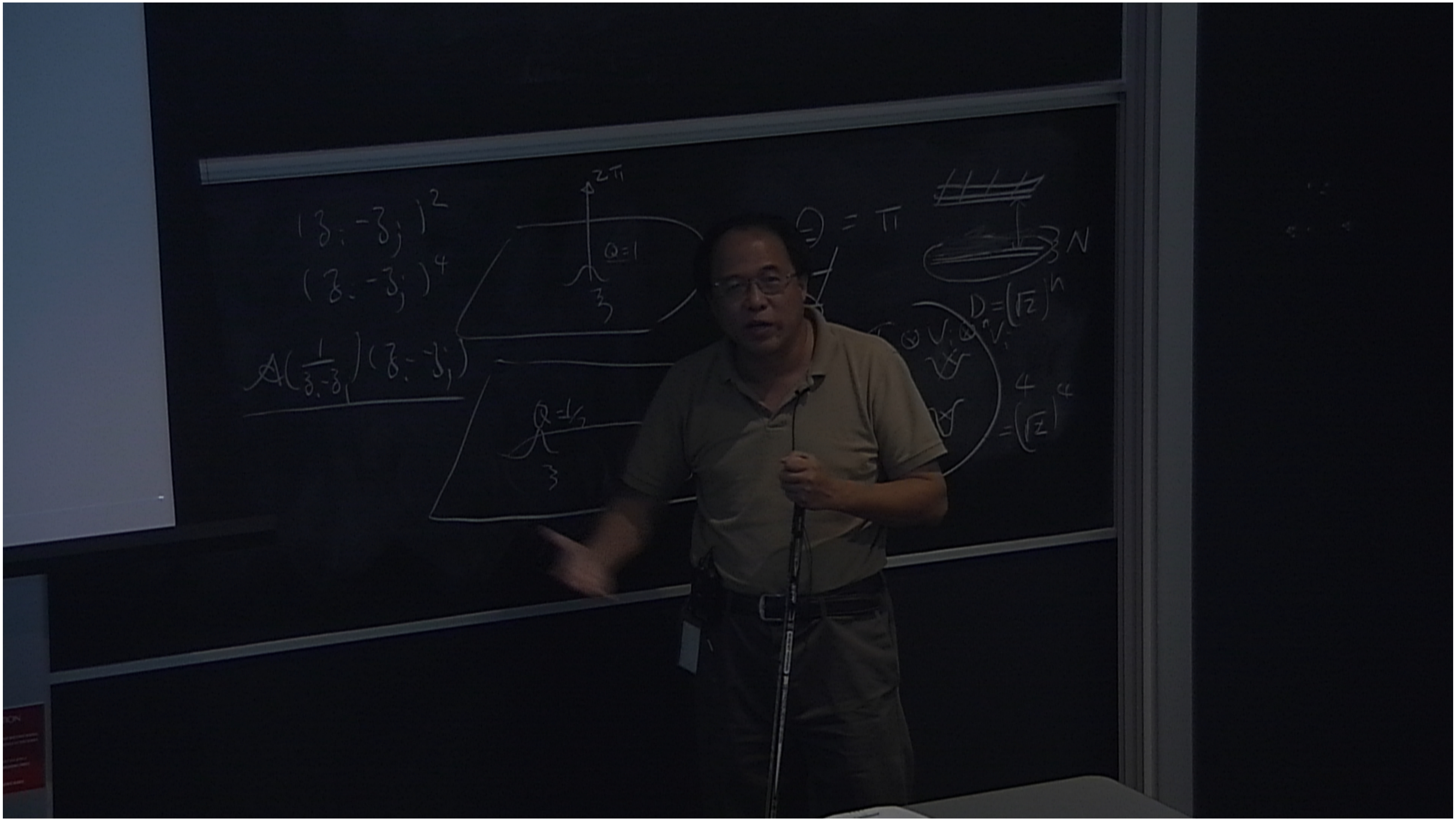

A general theory for non-Abelian states – Pattern of zeros

Let $z_i = \lambda\xi_i + z^{(a)}$, $i = 1, 2, \cdots, a$

$$P(\{z_i\}) = \lambda^{S_a}\tilde{P}(\xi^1, ..., \xi^a; z^{(a)}, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1})$$

- The sequence of integers $\{S_a\}$ characterizes the polynomial $P(\{z_i\})$ and is called the pattern of zeros.
- S_a is the relative angular momentum of an a-electron cluster.

$$\Psi_1 = \prod_{i<j}(z_i - z_j) \to S_1, S_2, = 0, 1, 3, 6, 10, ...$$

$$\Psi_{1/2} = \prod_{i<j}(z_i - z_j)^2 \to S_1, S_2, = 0, 2, 6, 12, 20, ...$$

$$\Psi_{1/4} = \prod_{i<j}(z_i - z_j)^4 \to S_1, S_2, = 0, 4, 12, 24, 40, ...$$

$$P_{\mathrm{Pf}} = \mathcal{A}\Big(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\Big)\prod_{i<j}(z_i - z_j) \to S_1, S_2, = 0, 0, 2, 4, 8, 12, 18, 24,$$

The physical meaning of pattern of zeros: pattern of zeros and orbital distribution

- S_a is the power of z's in $P(z_1, \cdots, z_a)$ if the polynomial has a variables.
- Relation to monomial symmetric polynomial ("highest weight"?): Let $l_a = S_a - S_{a-1}$ or $S_a = \sum_{i=1}^{a} l_i$, then

$$P(\{z_i\}) \sim \mathcal{S}[z_1^{l_1} z_2^{l_2} \cdots] + \cdots, \quad = m_{\{l_1, l_2, \ldots\}}(z_i, .., z_N) + \ldots$$

  The monomial symmetric polynomial can produces any sequences S_a if we treat S_a as the POZ at $z = 0$.
- **Pattern of zeros and occupation distribution**
  Convert l_a to n_l, the occupation distribution: $n_l = \sum_{a=1}^{\infty} \delta_{l_a, l}$

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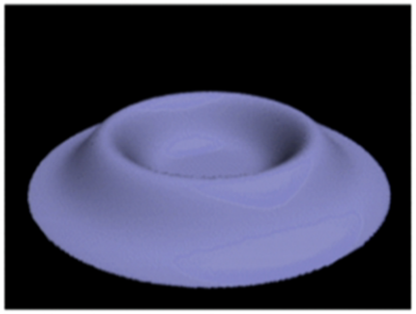
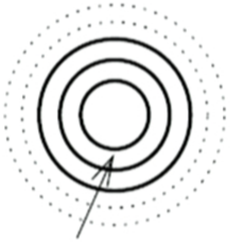
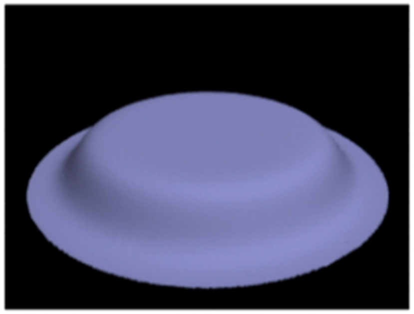
$$z^l \mathrm{e}^{-\frac{1}{4}|z|^2}, \qquad \Psi = \prod_{i<j}(z_i - z_j)\,\mathrm{e}^{-\frac{1}{4}\sum|z_i|^2} = \mathcal{A}[(z_1)^0(z_2)^1 ...]\,\mathrm{e}^{-\frac{1}{4}\sum|z_i|^2}$$



$$\prod_{i<j}(z_i - z_j) = \mathcal{A}[(z_1)^0(z_2)^1 ...] \to S_1, S_2, ... = 0, 1, 3, 6, 10, ...$$

$$l_1, l_2, \cdots : 0, 1, 2, 3, 4, \cdots, \qquad n_0 n_1 n_2 \cdots : \ 111111111111 \cdots$$

For $P_{1/2}$:

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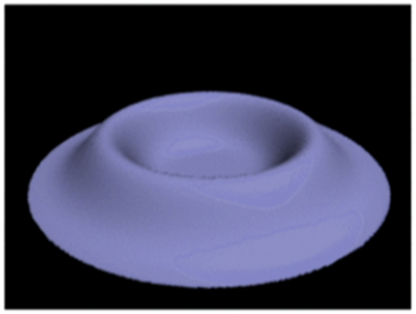

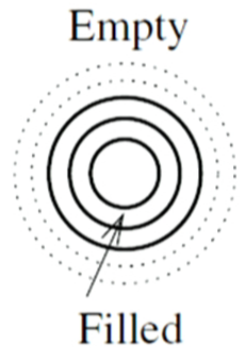

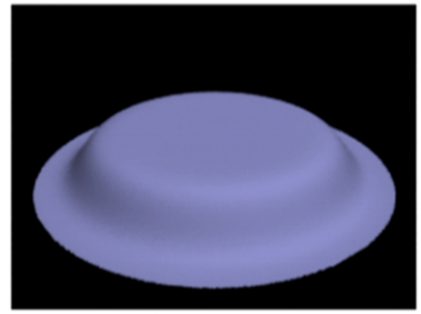

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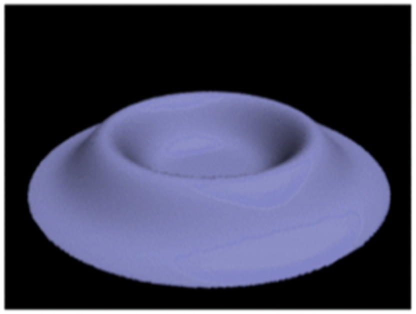

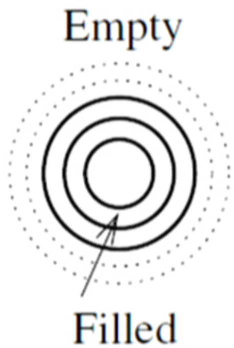

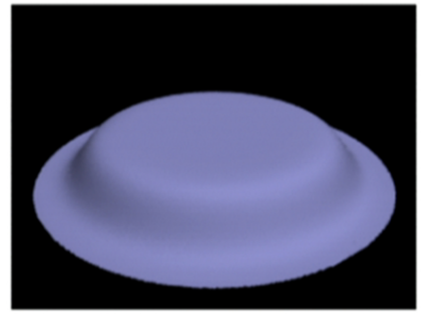

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- $\nu = 1/4$ Laughlin state $P_{1/4} = \prod_{i<j}(z_i - z_j)^2$

$$S_1, S_2, \cdots : 0, 4, 12, 24, 40, 60, 84, \cdots$$
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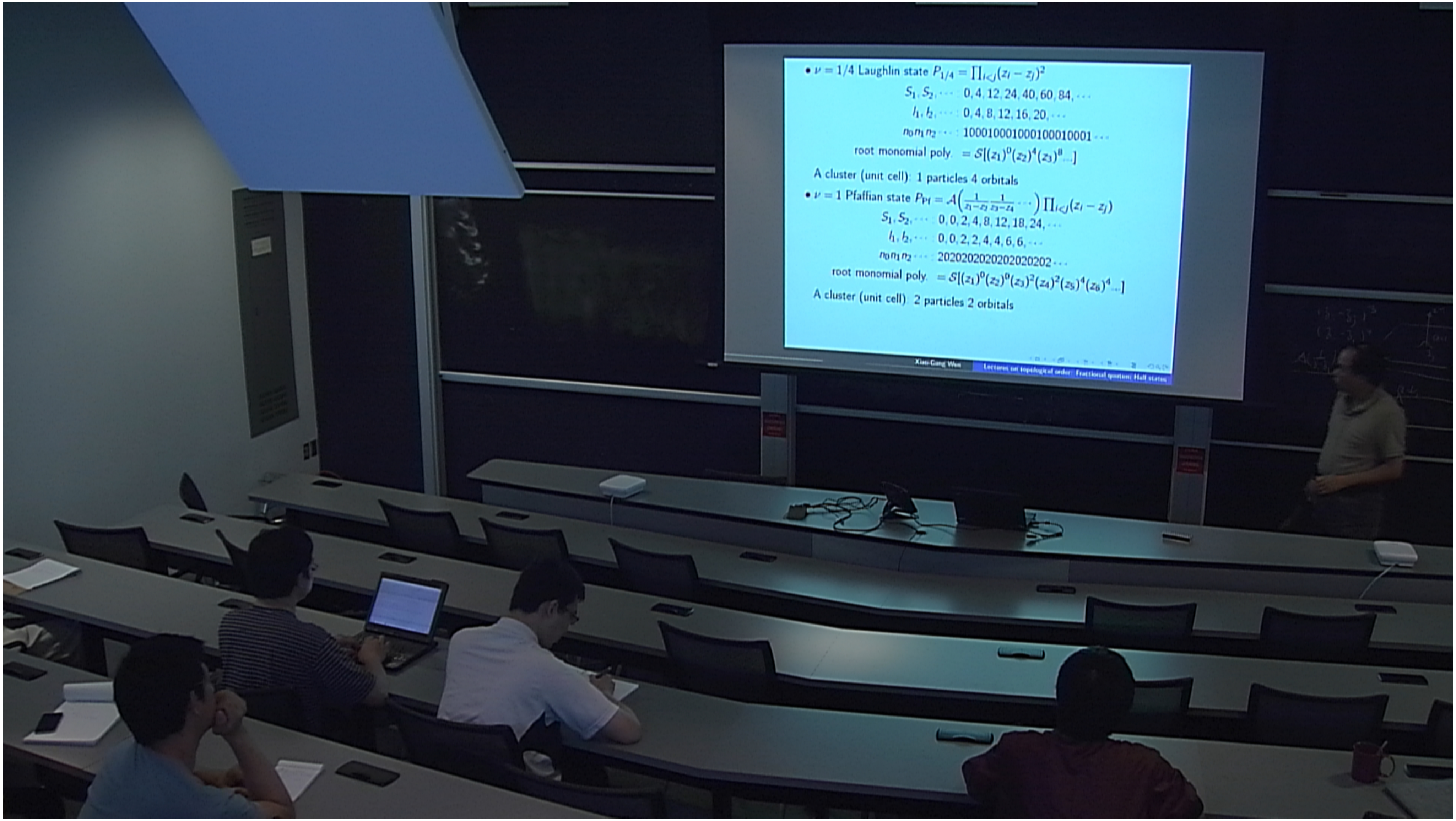

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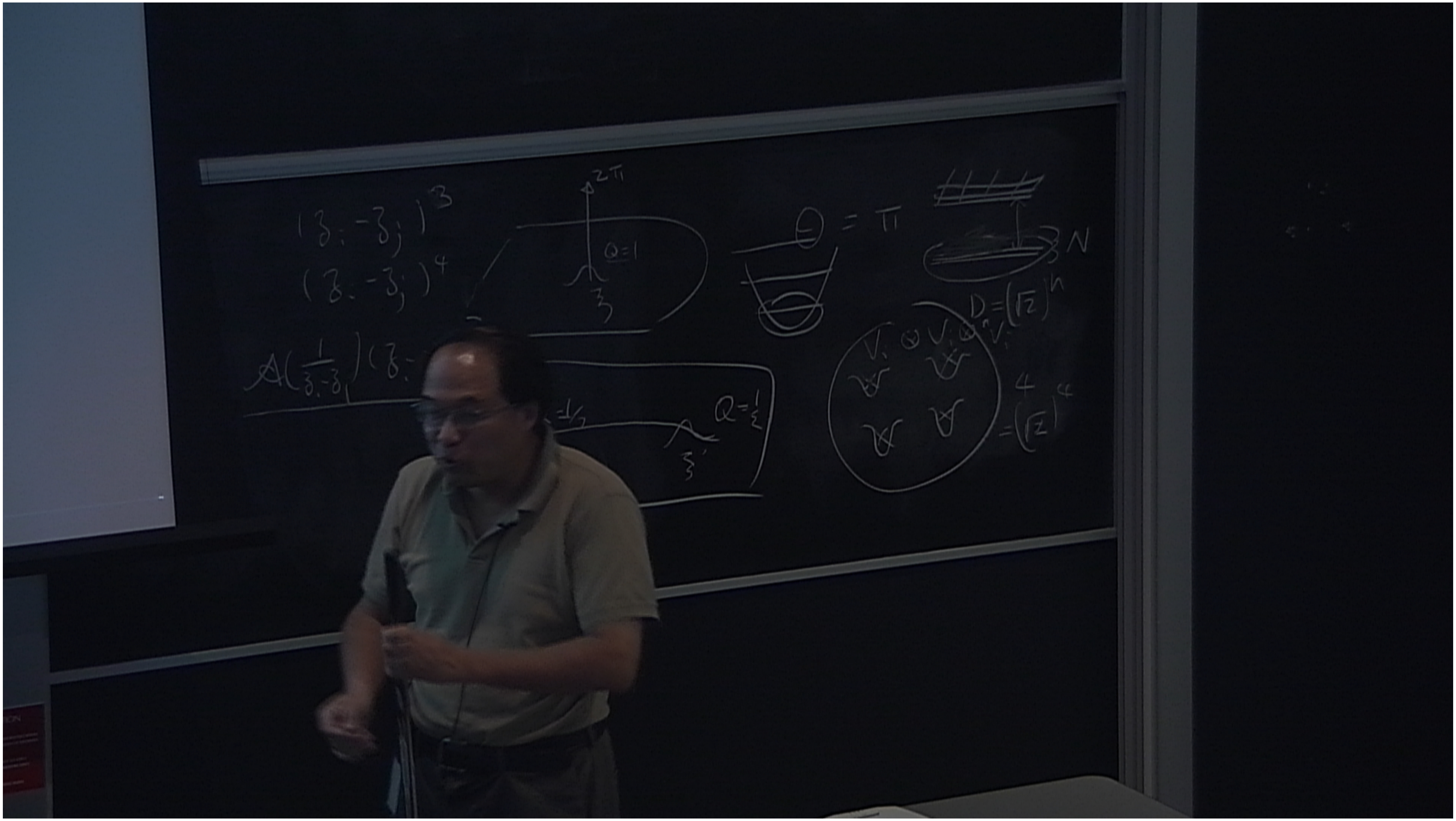
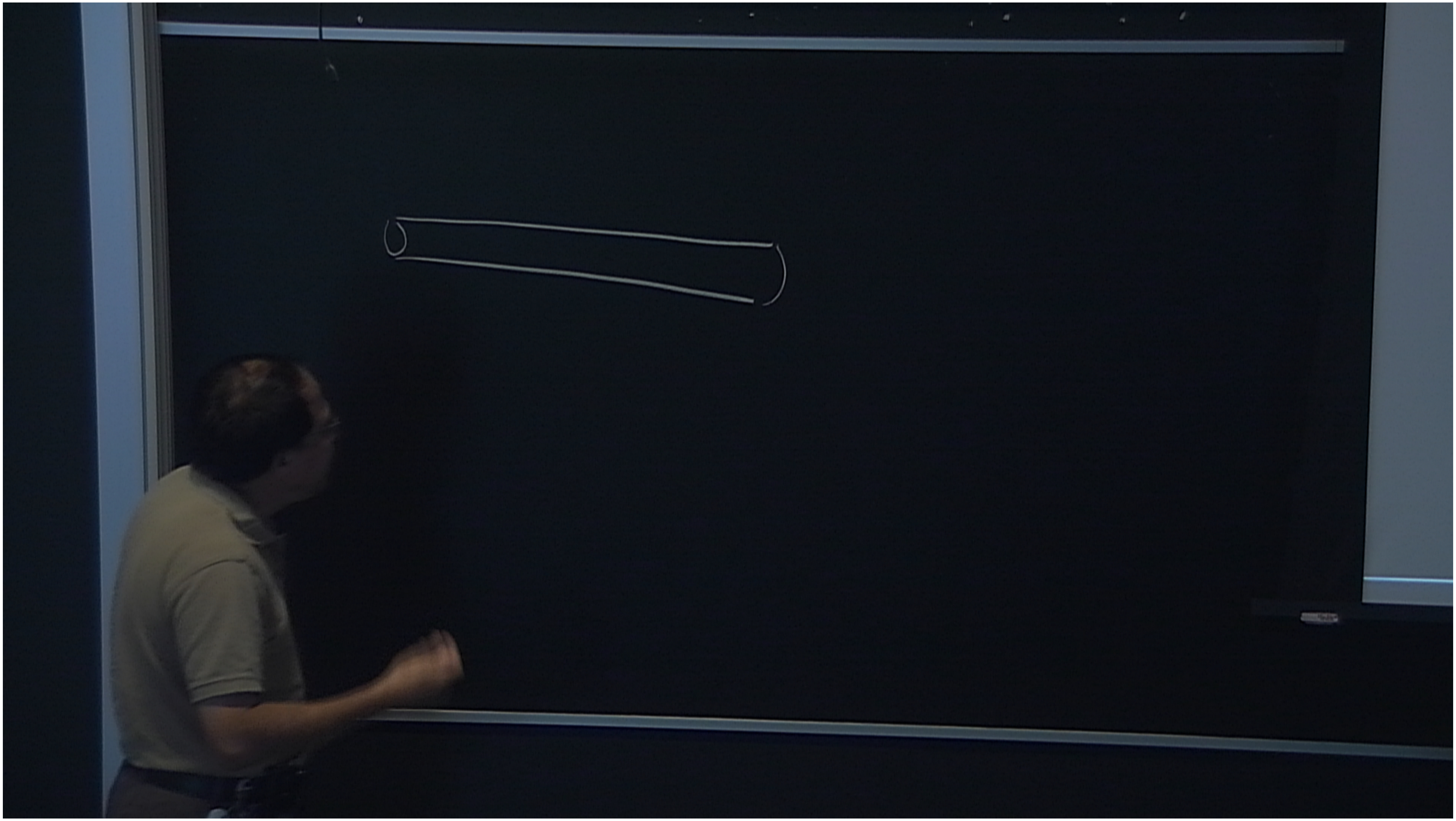
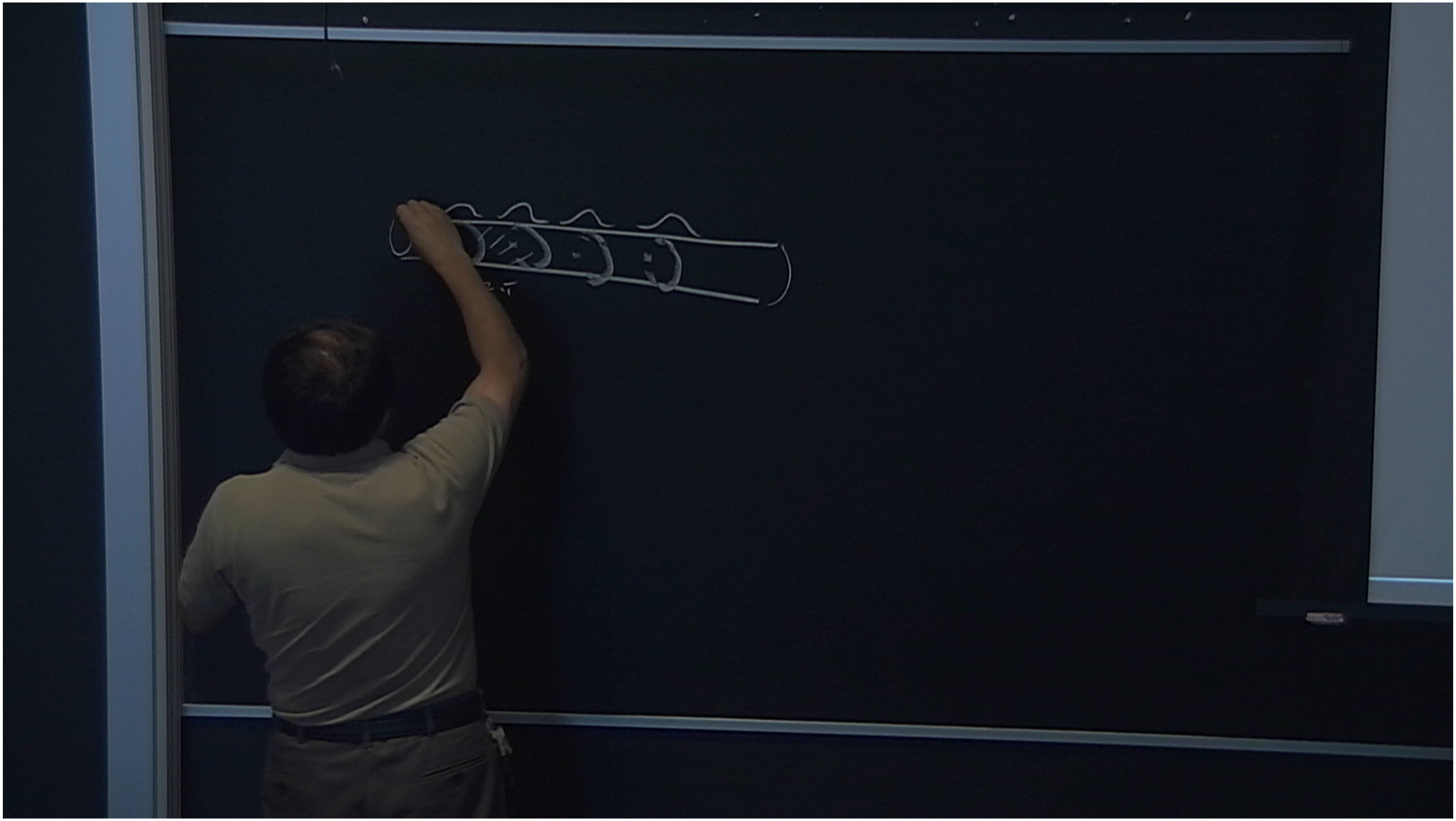
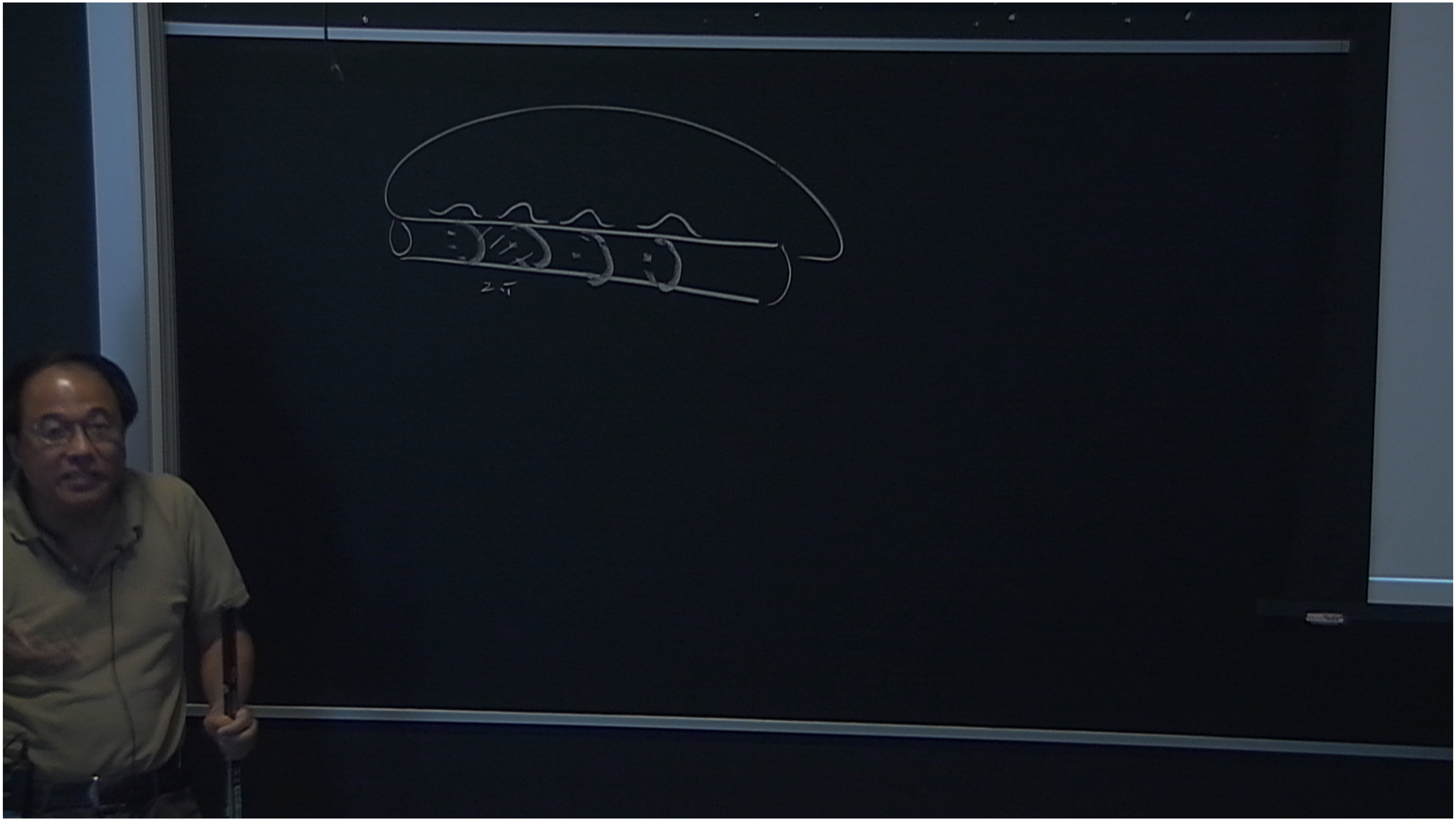
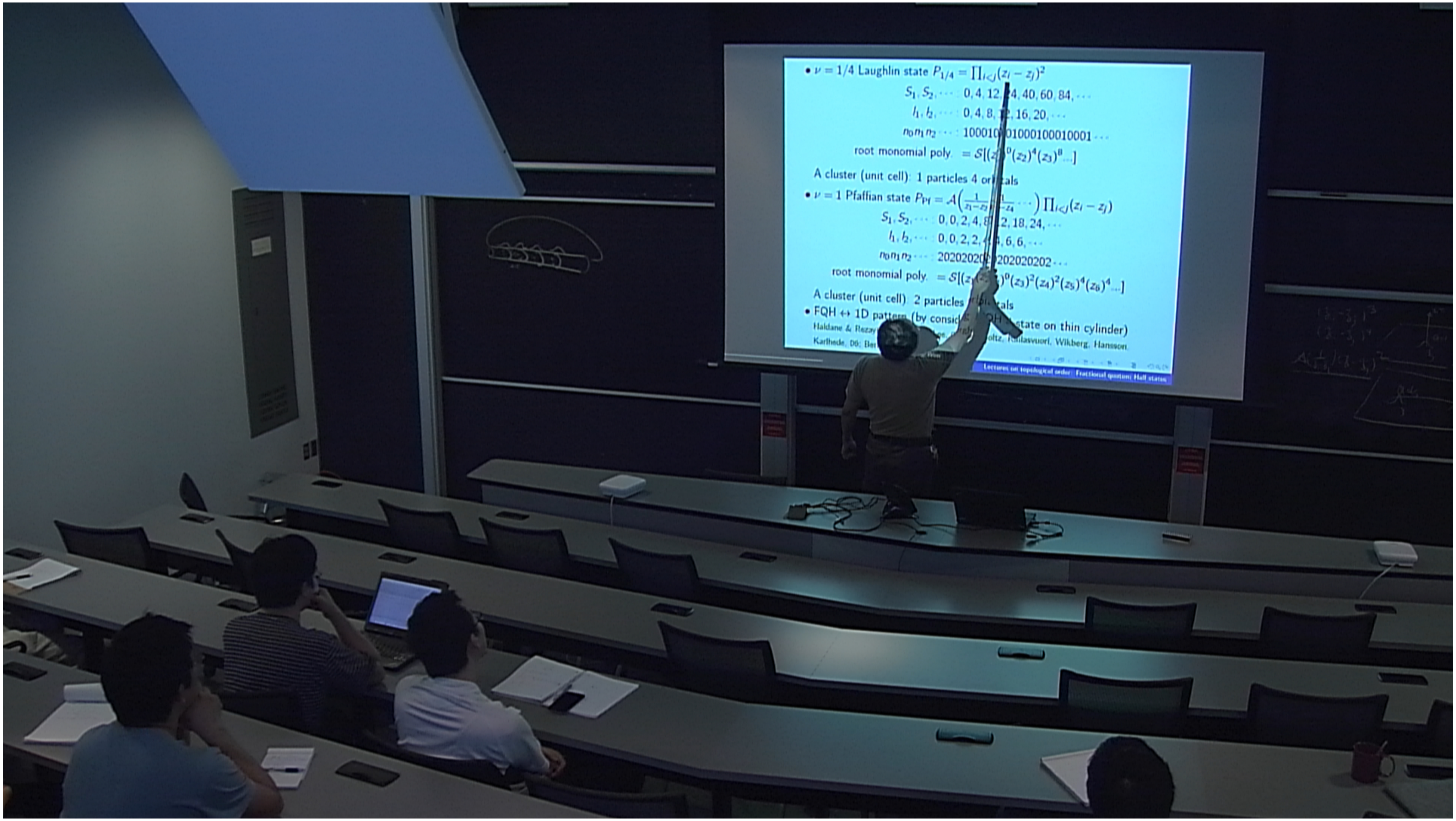
root monomial poly. $= \mathcal{S}[(z_1)^0 (z_2)^0 (z_3)^2 (z_4)^2 (z_5)^4 (z_6)^4 ...]$
A cluster (unit cell): 2 particles 2 orbitals
Xiao-Gang Wen
Lectures on topological order: Fractional quantum Hall states











• $\nu = 1/4$ Laughlin state $P_{1/4} = \prod_{i<j}(z_i - z_j)^2$
$S_1, S_2, \cdots$ 0, 4, 12,
4, 40, 60, 84, ⋯
$l_1, l_2, \cdots$ 0, 4, 8,
, 16, 20, ⋯
$n_0 n_1 n_2 \cdots$ 100010
01000100010001 ⋯
root monomial poly.
A cluster (unit cell): 1 particles 4 or
• $\nu = 1$ Pfaffian state
$S_1, S_2, \cdots$ 0, 0, 2, 4,
2, 18, 24, ⋯
$l_1, l_2, \cdots$ 0, 0, 2, 2,
, 6, 6, ⋯
$n_0 n_1 n_2 \cdots$ 2020202
202020202 ⋯
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A cluster (unit cell): 2 particles
• FQH ↔ 1D patt
state on thin cylinder)
Wikberg, Hansson
Lectures on topological order: Fractional quantum Hall states

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A cluster (unit cell): 1 particles 4 orbitals

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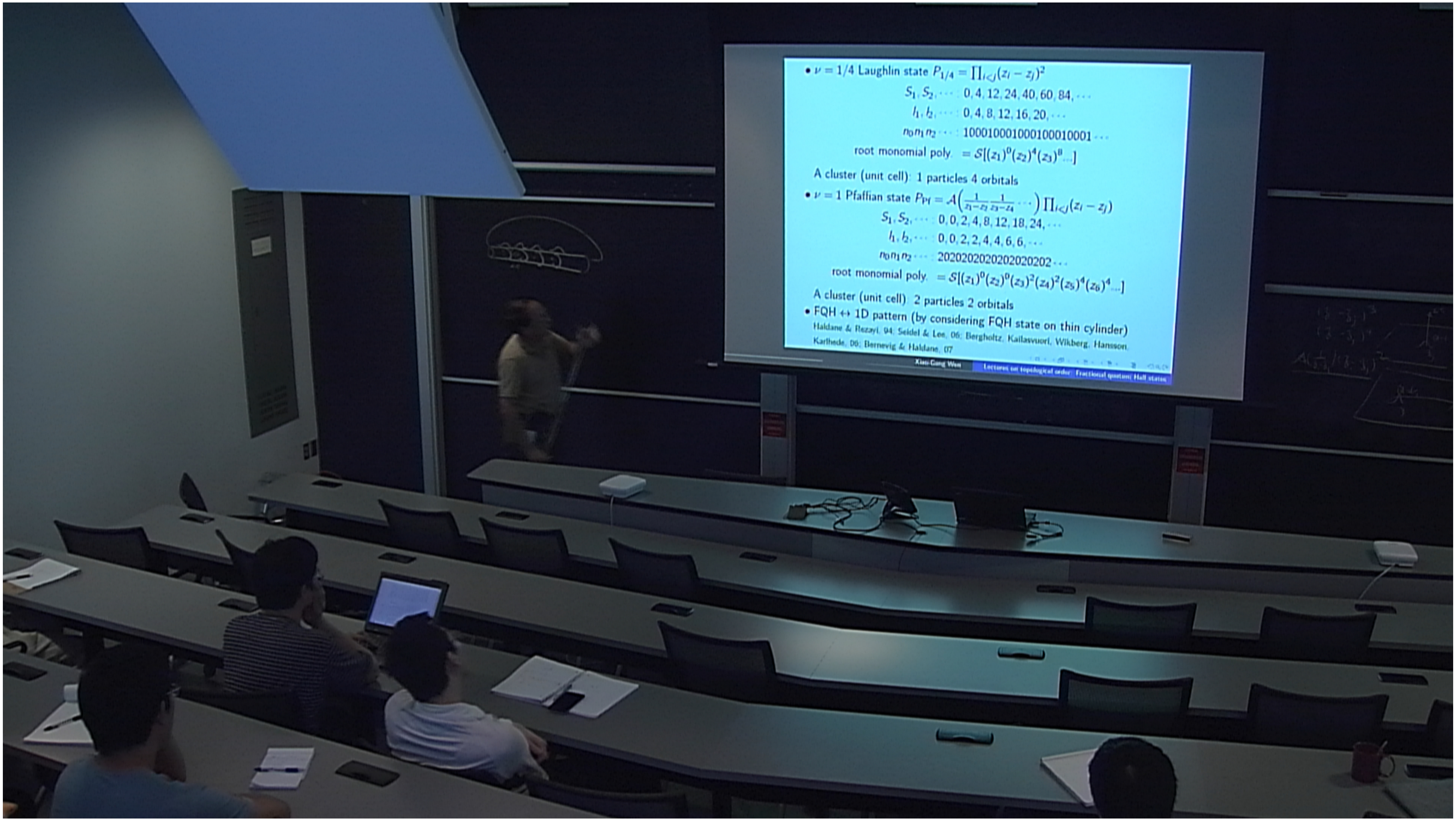
- FQH $\leftrightarrow$ 1D pattern (by considering FQH state on thin cylinder)

Haldane & Rezayi, 94; Seidel & Lee, 06; Bergholtz, Kailasvuori, Wikberg, Hansson, Karlhede, 06; Bernevig & Haldane, 07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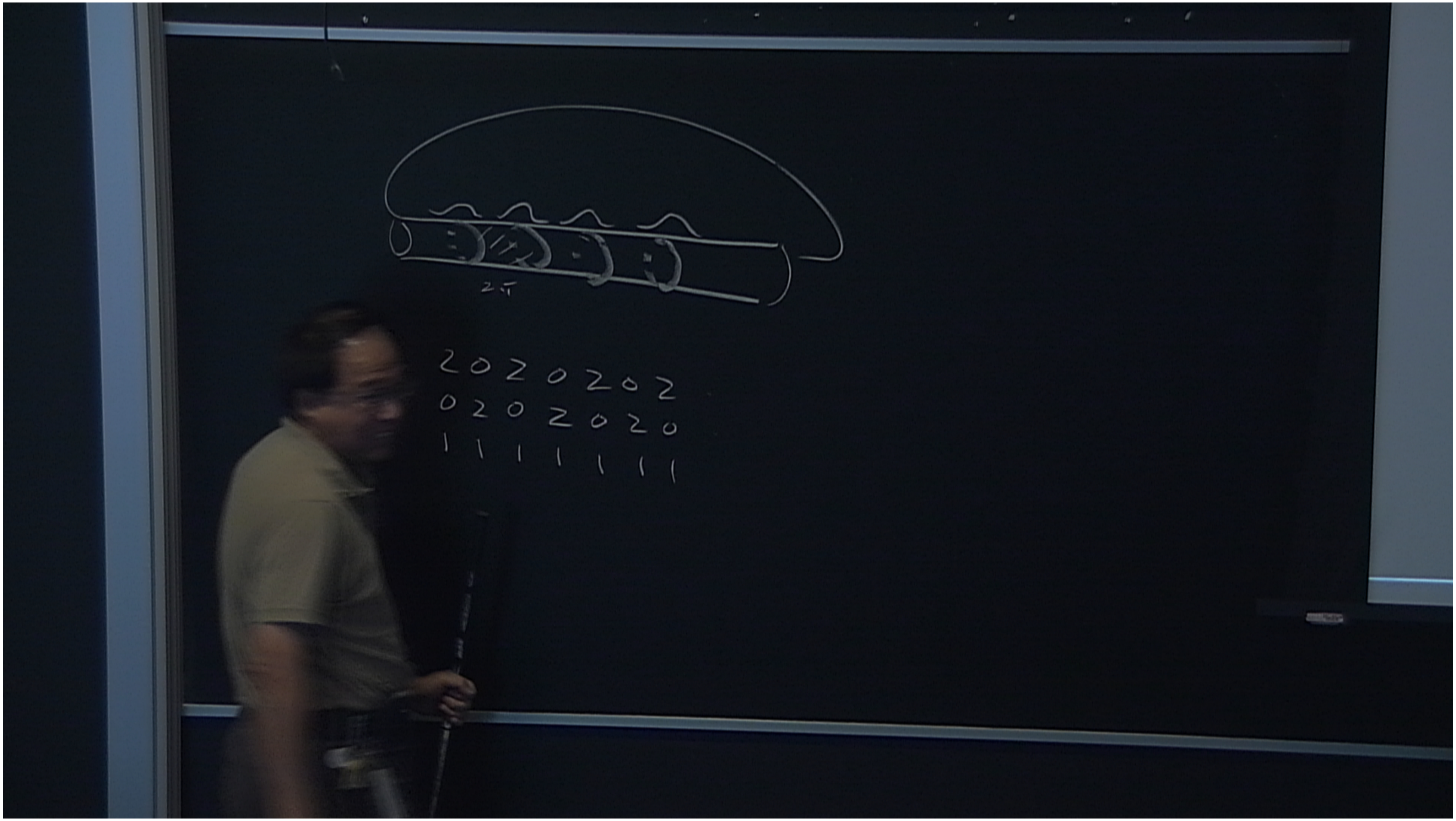
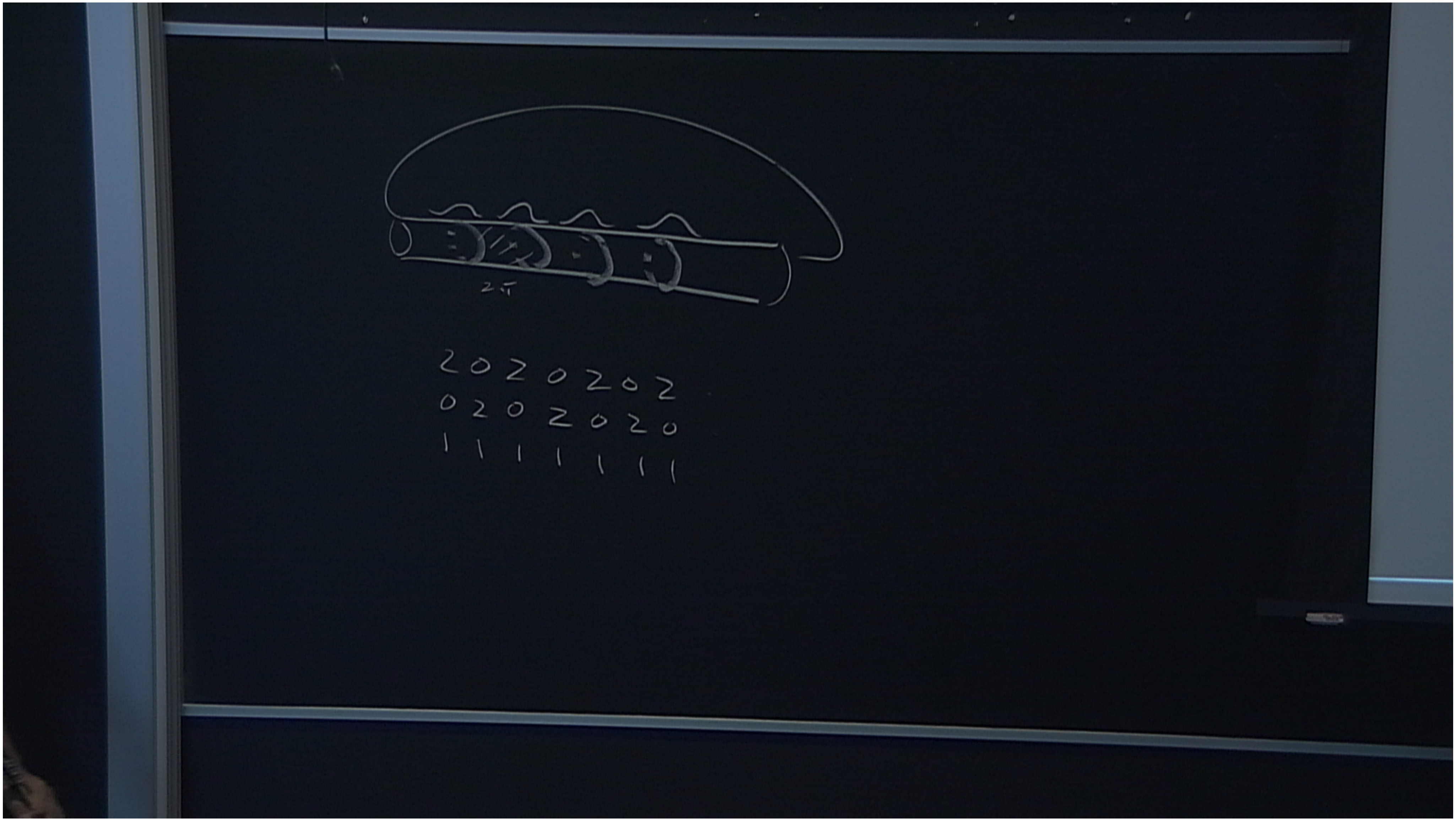
Lectures on topological order: Fractional quantum Hall states


2π
2 0 2 0 2 0 2
0 2 0 2 0 2 0
1 1 1 1 1 1 1


2π
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1111111

A classification problem

- We have seen that
  each symmetric polynomial $P(\{z_i\}) \to \{S_a\}$ a pattern of zeros.
  But each sequence of integers $\{S_a\} \not\to P(\{z_i\})$
- Find all the **conditions** a sequence $\{S_a\}$ must satisfy, such that $\{S_a\}$ describe a symmetric polynomial that satisfies the unique fusion condition. $\to$
  A classification of symmetric polynomials (FQH states) through pattern of zeros.

Derived polynomials

- **Unique fusion cond.**: Let $z_i = \lambda\xi_i + z^{(a)}$, $i = 1, 2, \cdots, a$
$$P(\{z_i\}) = \lambda^{S_a}\tilde{P}(\xi^1, ..., \xi^a; z^{(a)}, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1}),$$
  $\tilde{P}$ does not depend on the "shape" $\{\xi^i\}$
  $\tilde{P}(\{\xi^i\}; z^{(a)}, z_{a+1}, z_{a+2}, \cdots) = F(\{\xi^i\})P_{\text{derived}}(z^{(a)}, z_{a+1}, z_{a+2}, \cdots)$
- So, we can rewrite the above as
$$P(\{z_i\}) = \lambda^{S_a}P_{\text{derived}}(z^{(a)}, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1})$$
- If we repeat the above, we get a derived polynomial $P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \cdots)$.

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Conditions on pattern of zeros – ground state

Input: DVI - 1280x720p@60Hz
Output: SDI - 1920x1080i@60Hz

- **Concave conditions**

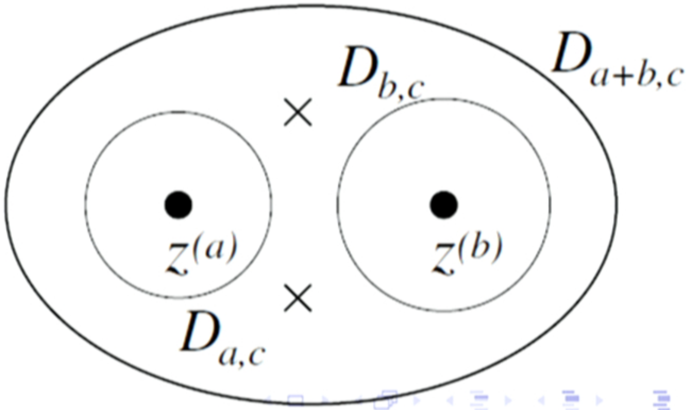
$$\Delta_2(a,b) \equiv S_{a+b} - S_a - S_b = D_{a,b} \geq 0,$$

$$\Delta_3(a,b,c) \equiv S_{a+b+c} - S_{a+b} - S_{b+c} - S_{a+c} + S_a + S_b + S_c \geq 0$$

The second one can be shown by considering $P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \cdots)$ as a function of $z^{(c)}$, which leads to

$$D_{a+b,c} = D_{a,c} + D_{b,c} + \text{ off-particle zeros at "}\times\text{"} \geq D_{a,c} + D_{b,c}$$

- $D_{a,c}, D_{b,c}$ = the number of on-particle zeros at $z^{(a)}, z^{(b)}$.
- As $z^{(a)} \to z^{(b)}$, some off-particle zeros at "$\times$" may also approach $z^{(b)}$.
- $D_{a+b,c} \geq D_{a,c} + D_{b,c}$ leads to the second condition.



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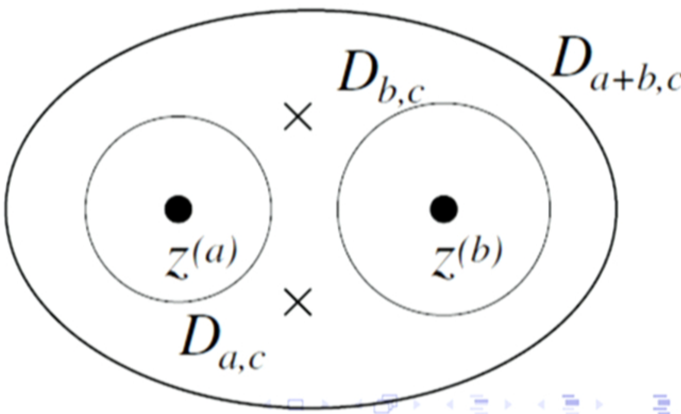

Primitive solutions for pattern of zeros

The conditions are semi-linear $\to$
if $(m; S_2, \cdots, S_n)$ and $(m'; S'_2, \cdots, S'_n)$ are solutions, then
$(m''; S''_2, \cdots, S''_n) = (m; S_2, \cdots, S_n) + (m'; S'_2, \cdots, S'_n)$ is also a solution $\sim P''(\{z_i\}) = P(\{z_i\})P'(\{z_i\})$

1-cluster state: $\nu = 1/m$ Laughlin state

$$P_{1/m}: \quad \boldsymbol{S} = (m;),$$
$$(n_0, \cdots, n_{m-1}) = (1, 0, \cdots, 0).$$

2-cluster state: Pfaffian state (Z_2 parafermion state)

$$P_{\frac{2}{2};Z_2}: \quad (m; S_2) = (2; 0),$$
$$(n_0, \cdots, n_{m-1}) = (2, 0)$$

3-cluster state: Z_3 parafermion state

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$$S_1, S_2, \cdots : 0, 4, 12, 24, 40, 60, 84, \cdots$$
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A cluster (unit cell): 1 particles 4 orbitals

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$$(n_0, \cdots, n_{m-1}) = (2, 0)$$

3-cluster state: Z_3 parafermion state

$$P_{\frac{3}{2};Z_3}: \quad (m; S_2, S_3) = (2; 0, 0),$$
$$(n_0, \cdots, n_{m-1}) = (3, 0)$$

4-cluster state: Z_4 parafermion state

$$P_{\frac{4}{2};Z_4} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0),$$
$$(n_0, \cdots, n_{m-1}) = (4, 0),$$

5-cluster states: Z_5 (generalized) parafermion state

$$P_{\frac{5}{2};Z_5} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0, 0),$$
$$(n_0, \cdots, n_{m-1}) = (5, 0)$$

$$P_{\frac{5}{8};Z_5^{(2)}} : (m; S_2, \cdots, S_n) = (8; 0, 2, 6, 10),$$
$$(n_0, \cdots, n_{m-1}) = (2, 0, 1, 0, 2, 0, 0, 0)$$

6-cluster state:

$$P_{\frac{6}{2};Z_6} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0, 0, 0),$$
$$(n_0, \cdots, n_{m-1}) = (6, 0)$$