Title: Topological Order Series

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Abstract:

Lectures on topological order:
Long range entanglement and topological excitations

Xiao-Gang Wen

2015/5/28

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and topological excitations

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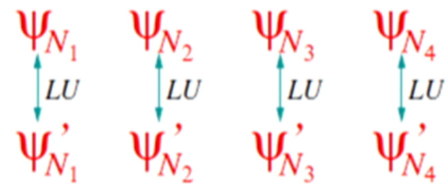
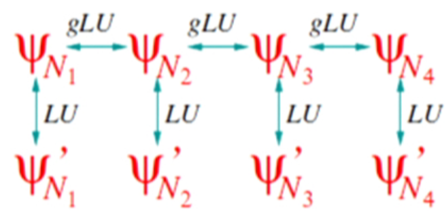
Local quantum systems and gapped quantum systems
A local quantum system is described by $(\mathcal{V}_N, H_N)$
$\mathcal{V}_N$: a Hilbert space with a tensor structure $\mathcal{V}_N = \otimes_{i=1}^N \mathcal{V}_i$
H_N: a local Hamiltonian acting on $\mathcal{V}_N$:
$H_N = \sum O_{ij}$
$\mathcal{V}_{grnd\ space} \subset \mathcal{V}_N$

A gapped quantum liquid phase: [Zeng-Wen, arXiv:1406.5090]

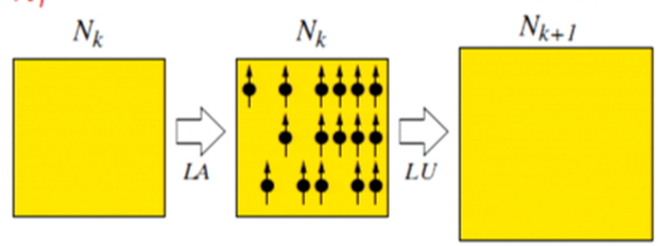
- **A gapped quantum phase:**
  $H_{N_1}, H_{N_2}, H_{N_3}, H_{N_4}, \cdots$
  $H'_{N_1}, H'_{N_2}, H'_{N_3}, H'_{N_4}, \cdots$
  OK definition only for translation invariant systems.

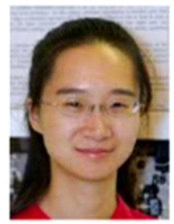


- **A gapped quantum liquid phase:**
  $H_{N_1}, H_{N_2}, H_{N_3}, H_{N_4}, \cdots$
  $H'_{N_1}, H'_{N_2}, H'_{N_3}, H'_{N_4}, \cdots$
  $N_{k+1} = sN_k, \ s \sim 2$



- $\Psi_{N_{i+1}} \overset{LA}{\sim} \Psi_{N_i} \otimes \Psi^{dp}_{N_{i+1}-N_i}$. Generalized local unitary (gLU) trans.
  where
  $\Psi^{dp}_N = \otimes_{i=1}^N |\uparrow\rangle$

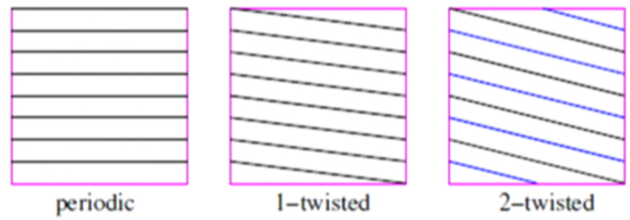




Examples

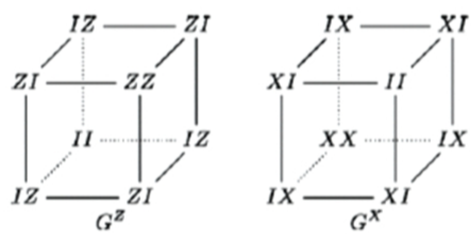
- Transverse Ising model in symmetry breaking phase
  $\to$ a gaped quantum liquid. Ground state degeneracy $GSD = 2$

- Stacking 2+1D FQH states $\to$ gapped quantum state, but not liquids.
  - Layered $\nu = 1/m$ FQH state:
    Ground state degeneracy can be
    $GSD = m^{L_z}, m, m^2$

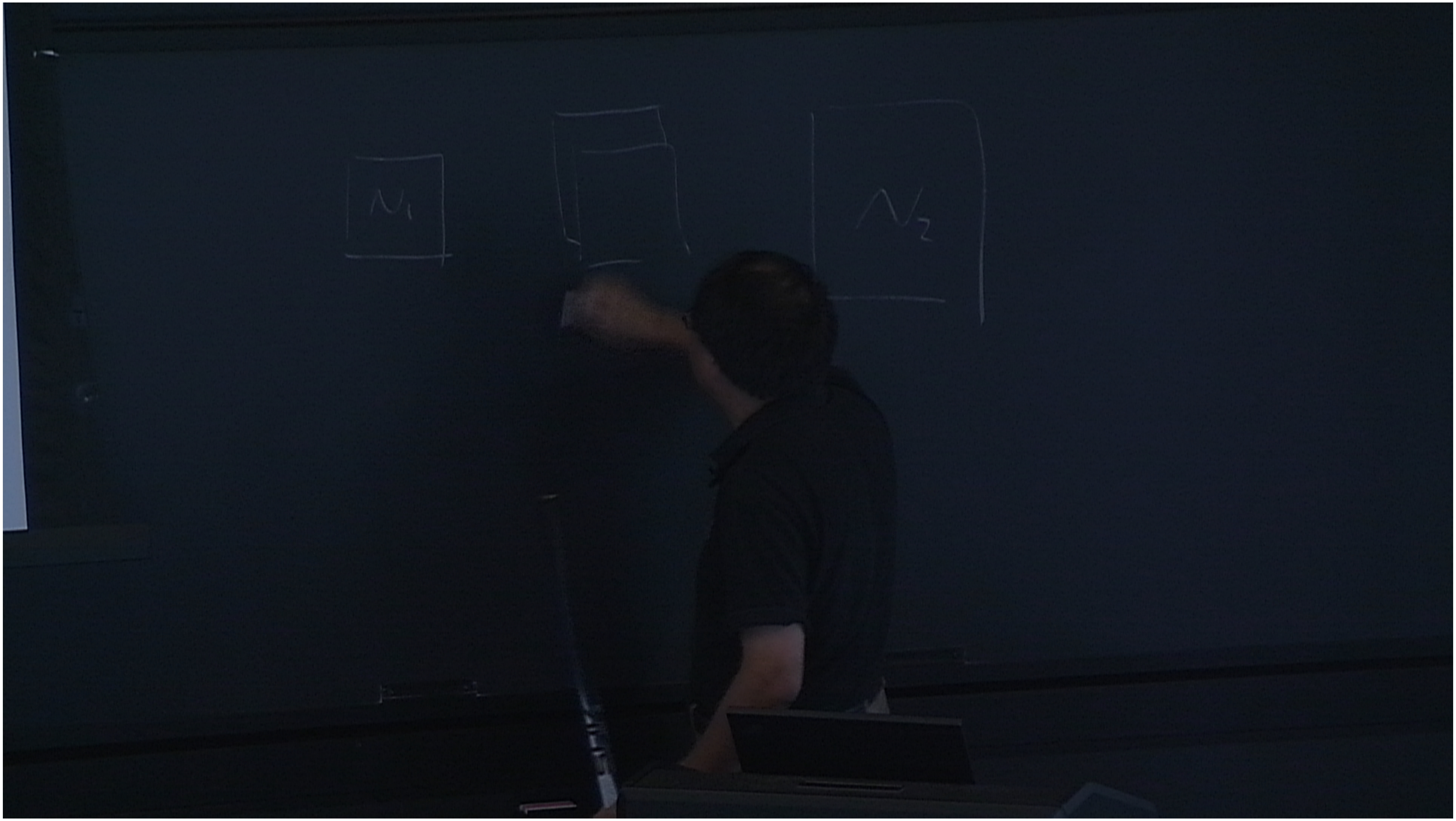


- Haah's cubic code on 3D cubic lattice:

$$H = -\sum_{cubes} (G^Z + G^X),$$



Jeongwan Haah, Phys. Rev. A 83, 042330 (2011) arXiv:1101.1962



N_1
N_2

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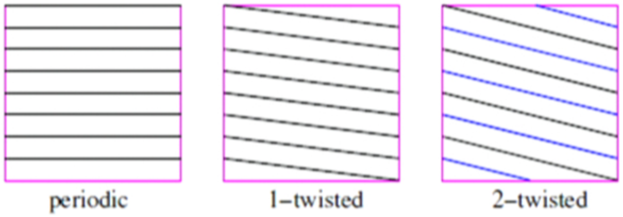

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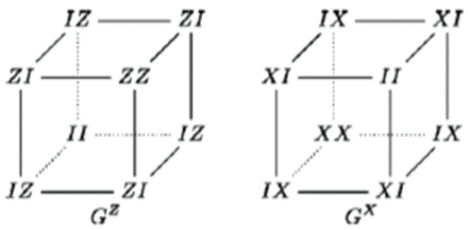

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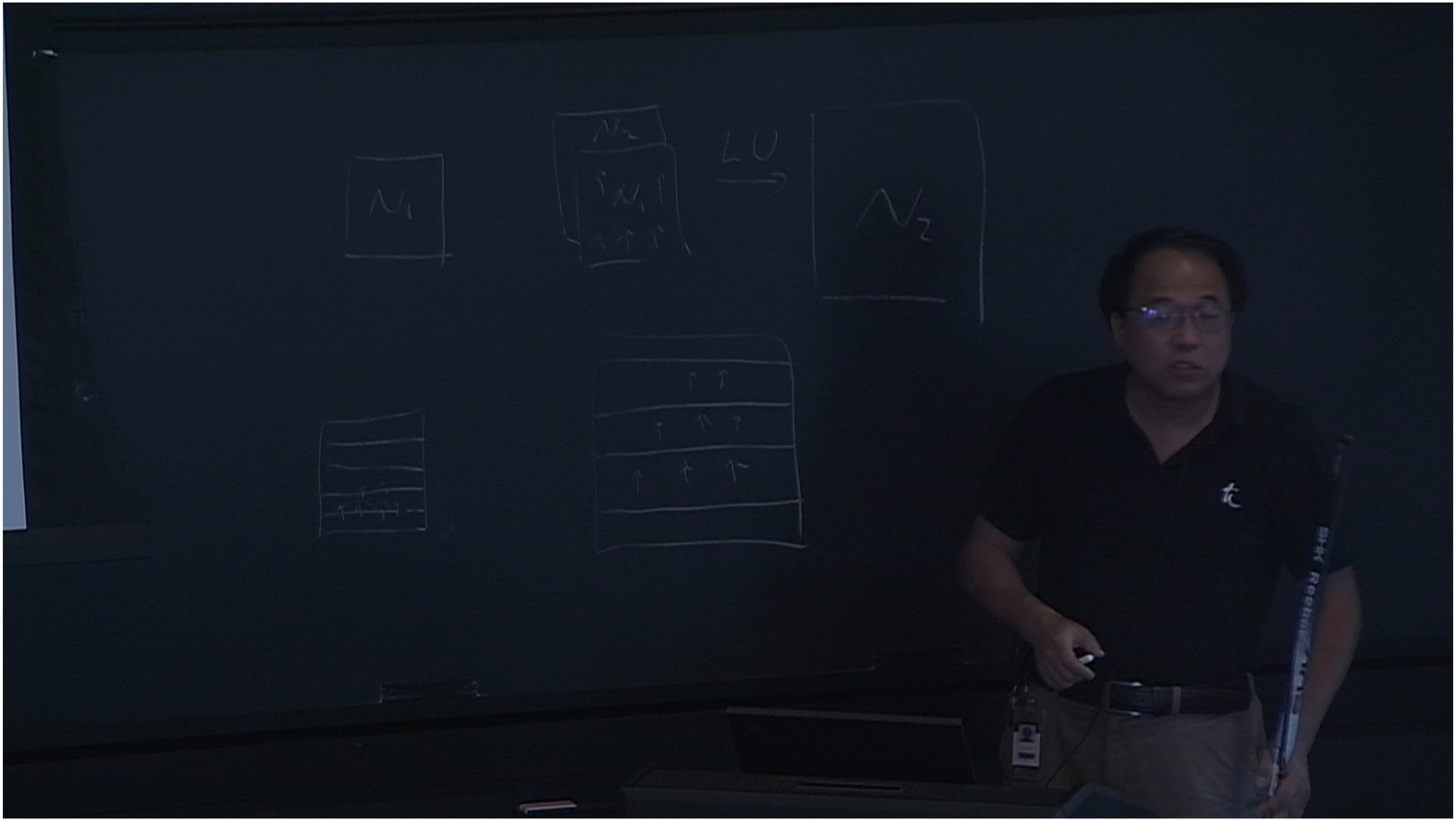
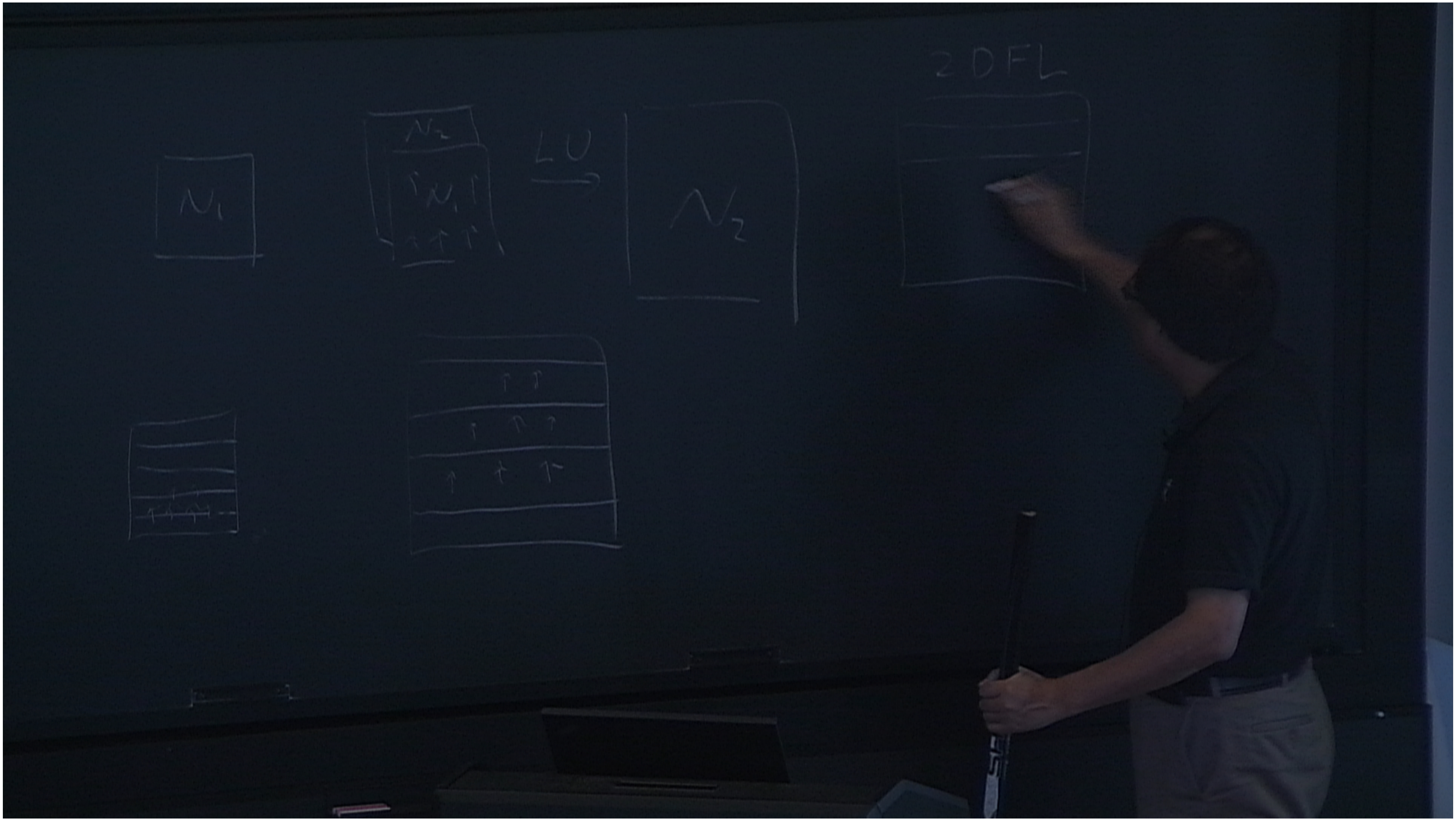


N1
N2
N1
LU
N2


2DFL
LU

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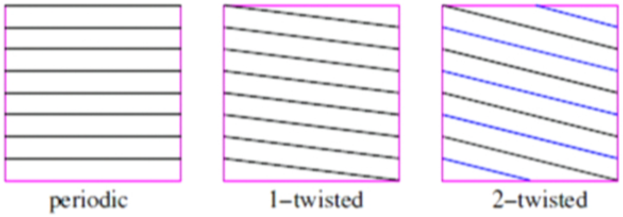

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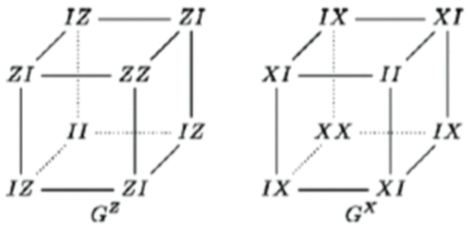

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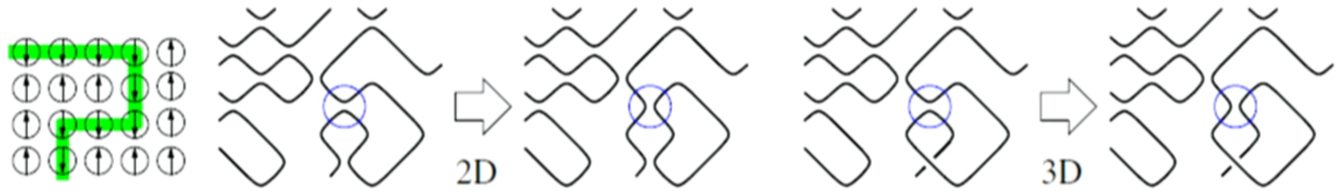
Sum over a subset: local rule → global wave function



- Local rules of a string liquid:
  (1) Dance while holding hands (no open ends)
  (2) $\Phi_{\text{str}}\Big($$\Big) = \Phi_{\text{str}}\Big($$\Big)$, $\Phi_{\text{str}}\Big($$\Big) = \Phi_{\text{str}}\Big($$\Big)$
  → Global wave function $\Phi_{\text{str}}\Big($$\Big) = 1$
- Local rules of another string liquid:
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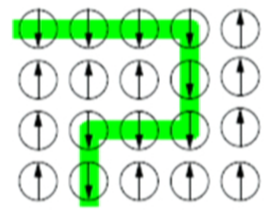
  → Global wave function $\Phi_{\text{str}}\Big($$\Big) = (-)^{\#\text{ of loops}}$
- Two topo. orders: Z_2 topo. order Read-Sachdev PRL 66, 1773 (91), Wen PRB 44, 2664 (91), Moessner-Sondhi PRL 86 1881 (01) and double-semion topo. order. Freedman etal cond-mat/0307511, Levin-Wen cond-mat/0404617

Example of Topologically ordered states

To make topological order, we need to sum over many different product states, but we should not sum over everything.

$\sum_{\text{all spin config.}} |\uparrow\downarrow ..\rangle = |\rightarrow\rightarrow ..\rangle$

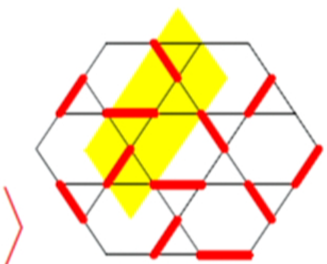


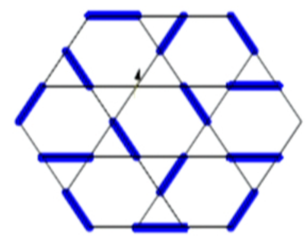
- *sum* over a subset of spin config.:

$|\Phi^{Z_2}_{\text{loops}}\rangle = \sum \left| \text{loops} \right\rangle$

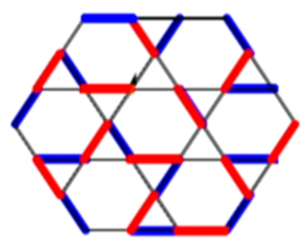
$|\Phi^{DS}_{\text{loops}}\rangle = \sum (-)^{\#\text{ of loops}} \left| \text{loops} \right\rangle$

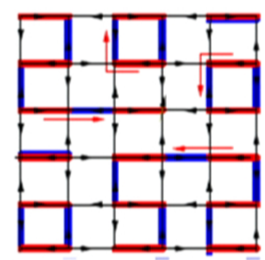
$|\Phi^{\theta}_{\text{loops}}\rangle = \sum (e^{i\theta})^{\#\text{ of loops}} \left| \text{loops} \right\rangle$





- Can the above wavefunction be the ground states of local Hamiltonians?





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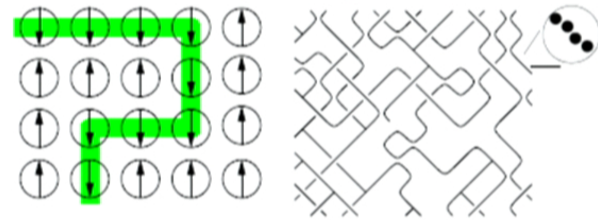

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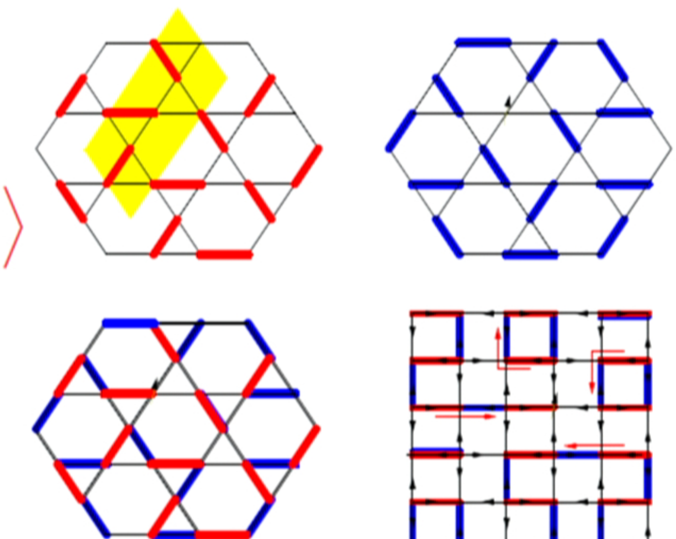

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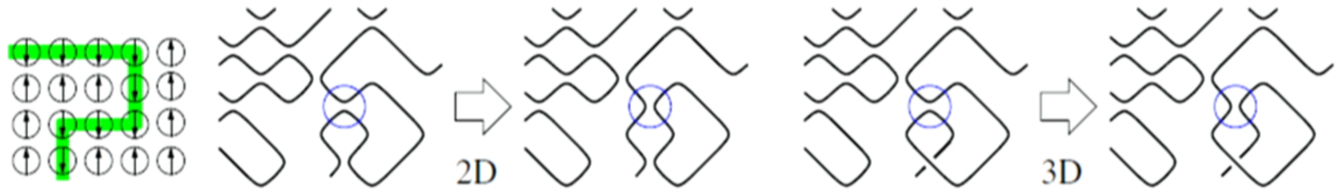
Sum over a subset: local rule → global wave function



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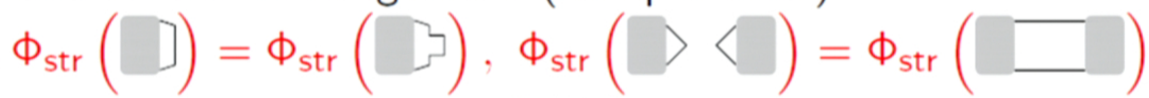
  (2) 

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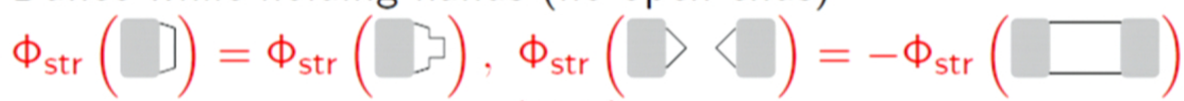
  (2) 

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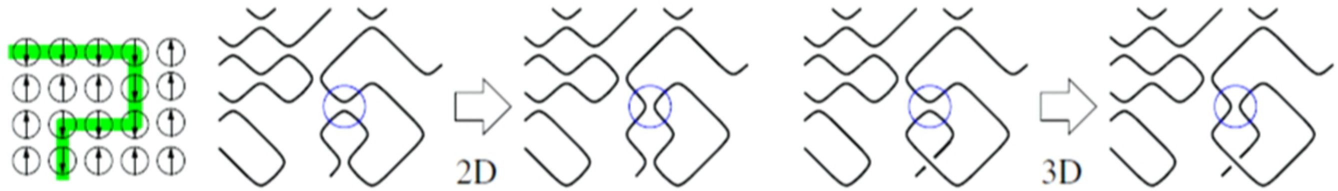

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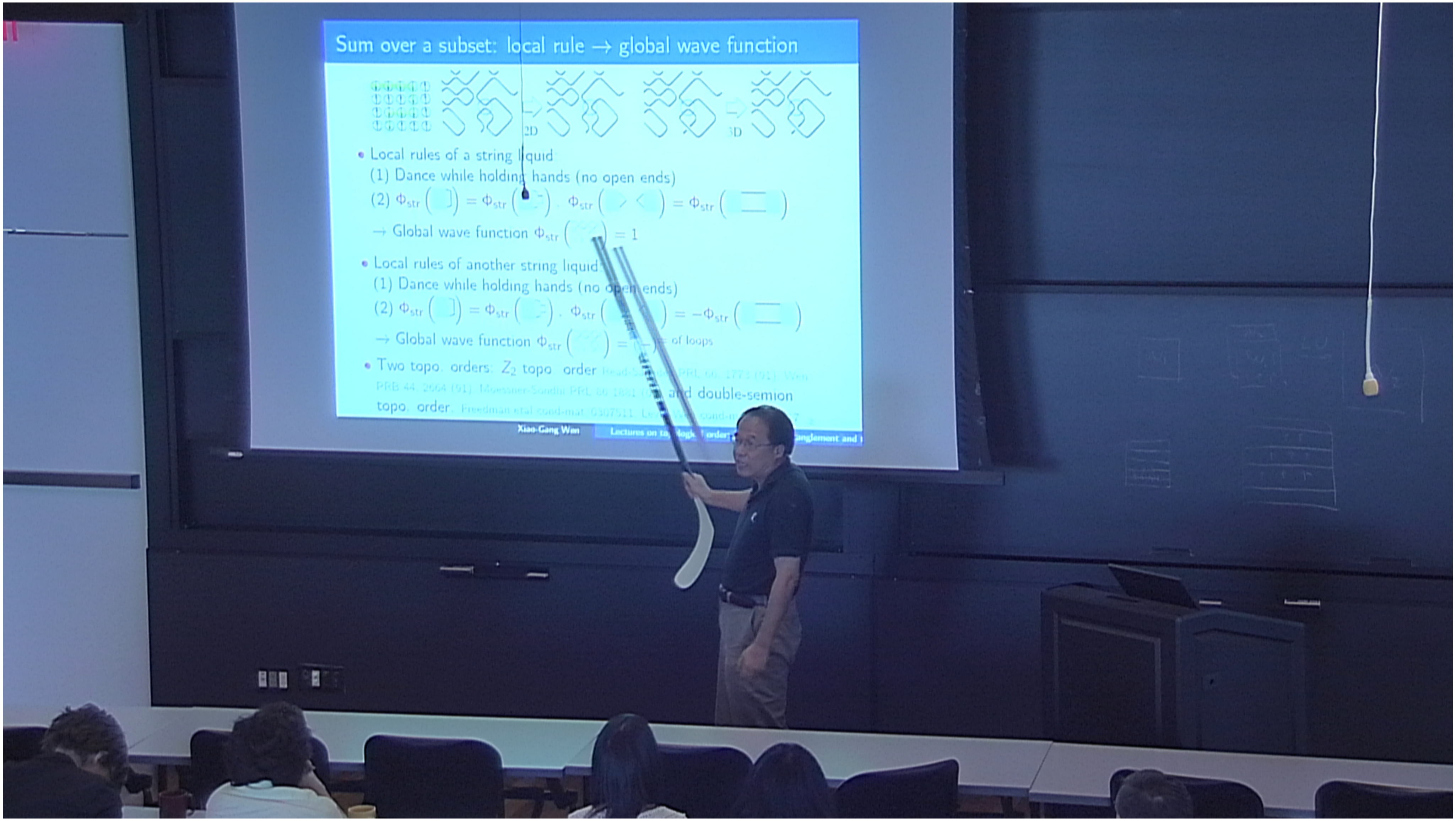

Sum over a subset: local rule → global wave function
2D
3D
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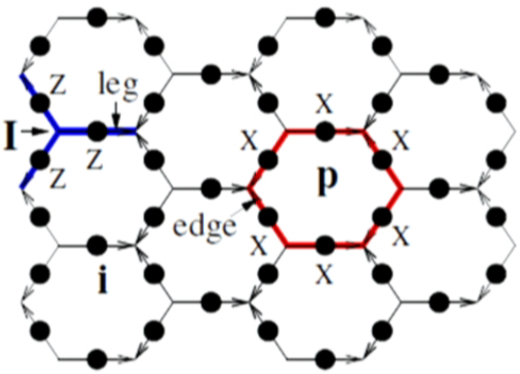
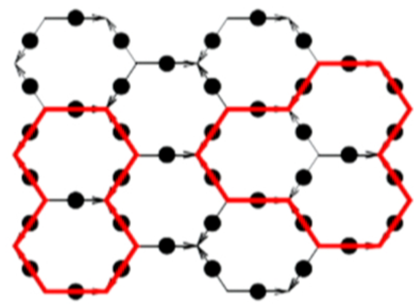
Two topo. orders: Z_2 topo. order and double-semion topo. order
Xiao-Gang Wen

Toric-code model – Z_2 topological order, Z_2 gauge theory

Local rule $\Phi_{\text{str}}(\,) - \Phi_{\text{str}}(\,) = \Phi_{\text{str}}(\,) - \Phi_{\text{str}}(\,) = 0$
$\to$ local Hamiltonian $\hat{P}\Phi_{\text{str}} = 0$.

- The Hamiltonian to enforce the local rules: Kitaev quant-ph/9707021





$$H = -U\sum_{I} Q_I - g\sum_{p} F_p, \quad Q_I = \prod_{\text{legs of } I} \sigma_i^z, \quad F_p = \prod_{\text{edges of } p} \sigma_i^x$$

- The hamiltonian is a sum of commuting operators
  $[F_p, F_{p'}] = 0$, $[Q_I, Q_{I'}] = 0$, $[F_p, Q_I] = 0$. $F_p^2 = Q_I^2 = 1$
- Ground state $|\Psi_{\text{grnd}}\rangle$: $F_p|\Psi_{\text{grnd}}\rangle = Q_I|\Psi_{\text{grnd}}\rangle = |\Psi_{\text{grnd}}\rangle$
  $\to (1 - Q_I)\Phi_{\text{grnd}} = (1 - F_p)\Phi_{\text{grnd}} = 0$.

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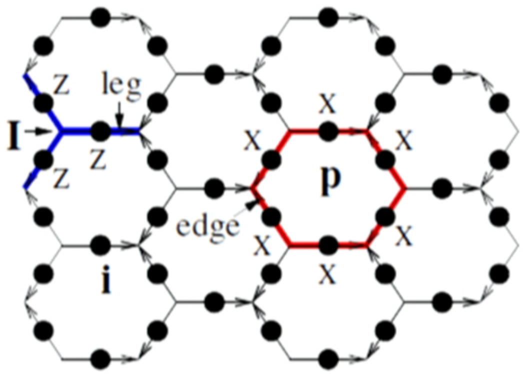

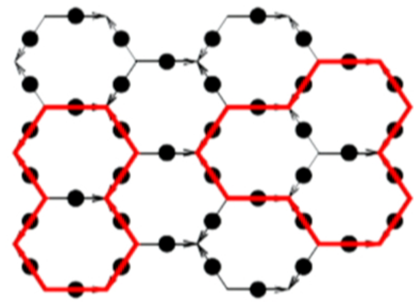

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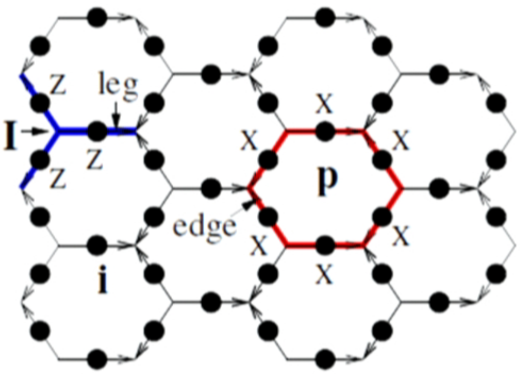

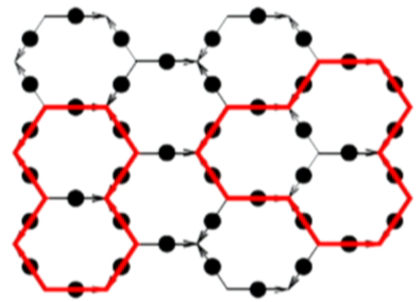

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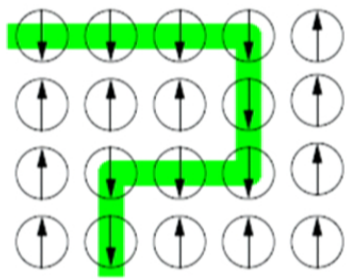
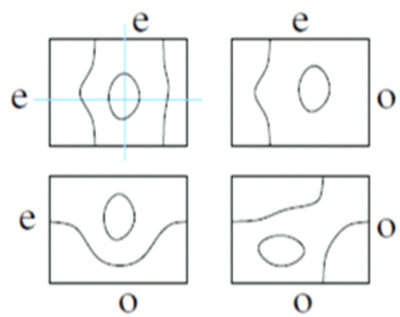
$\to (1 - Q_I)\Phi_{\text{grnd}} = (1 - F_p)\Phi_{\text{grnd}} = 0$.

Many-body energy spectrum of toric code model

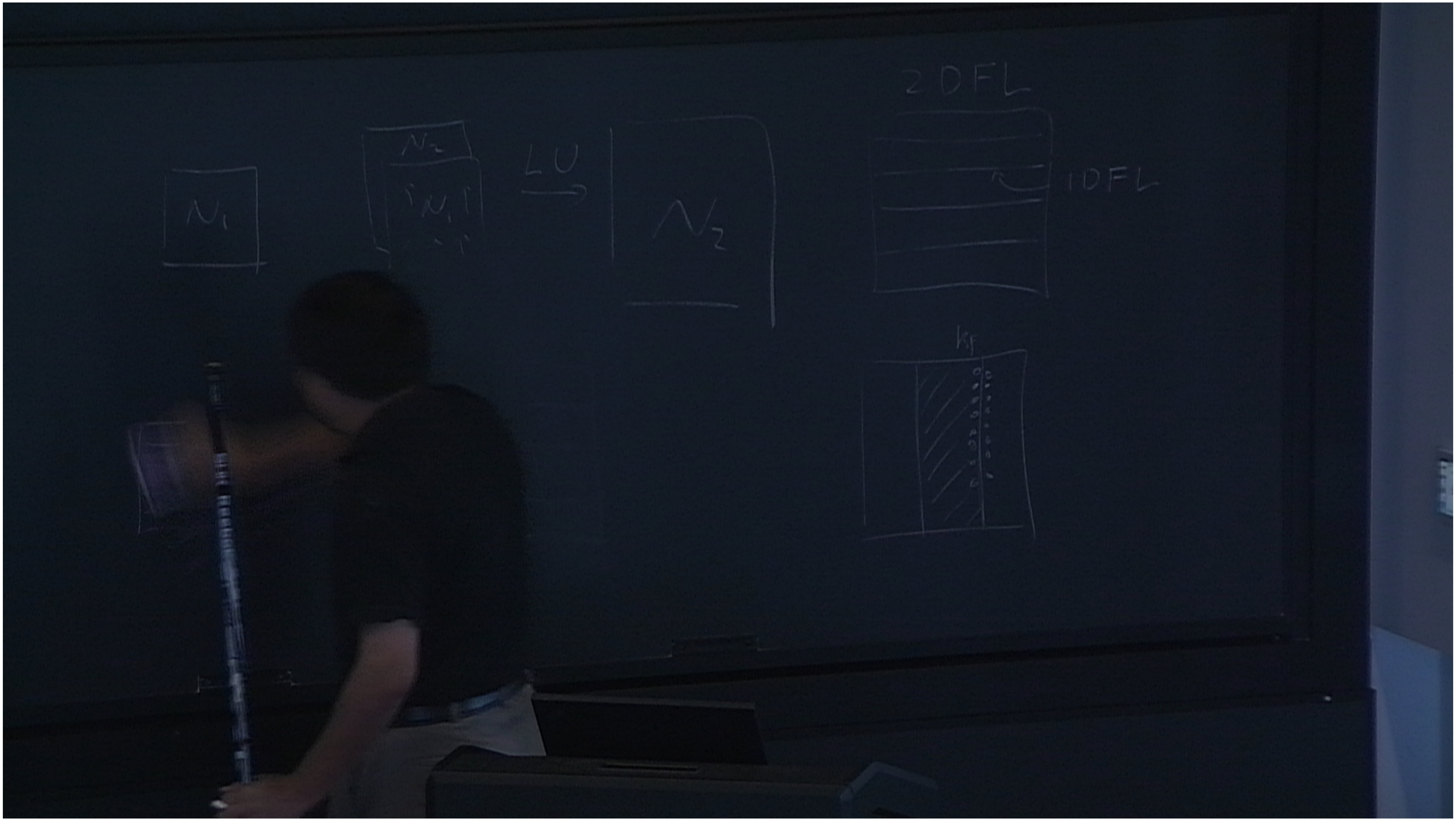
- The $-U\sum_{I} Q_I$ enforce closed-string ground state.
- F_p adds a small loop and generates a permutation among the loop states |  ⟩ $\to$ Ground states on torus $|\Psi^{\alpha}_{\text{grnd}}\rangle = \sum_{\text{loops}}$ |  ⟩
- There are four degenerate ground states $\alpha = ee, eo, oe, oo$





$D^{\text{tor}}=4$

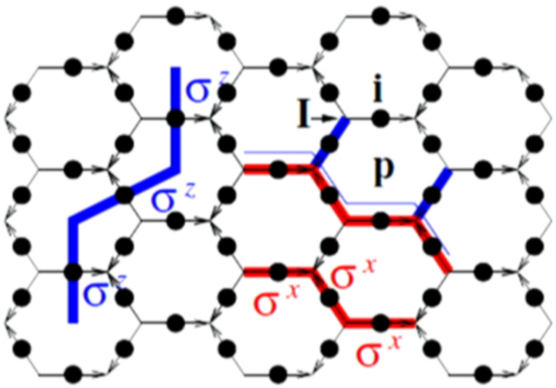
- On genus g surface, ground state degeneracy $D_g = 4^g$
- Quasiparticle excitation energy gap $\Delta^Q_p = 2U$, $\Delta^F_p = 2g$
  Spectrum energy gap $\Delta^Q = 4U$, $\Delta^F = 4g$



N_1
N_2
LU
N_2
2DFL
1DFL
k_f

The string operators and ground state degeneracy

- Toric code model:
  $H = -U \sum_{I} Q_{I} - g \sum_{p} F_{p}$
  $Q_{I} = \prod_{\text{legs of } I} \sigma_{i}^{z}$,
  $F_{p} = \prod_{\text{edges of } p} \sigma_{i}^{x}$
- Topological excitations:
  e-type: $Q_{I} = 1 \to Q_{I} = -1$
  m-type: $F_{p} = 1 \to F_{p} = -1$



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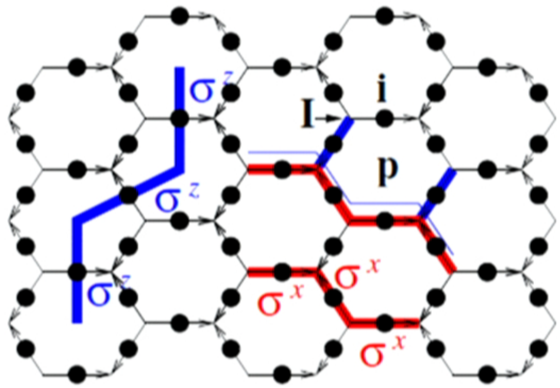
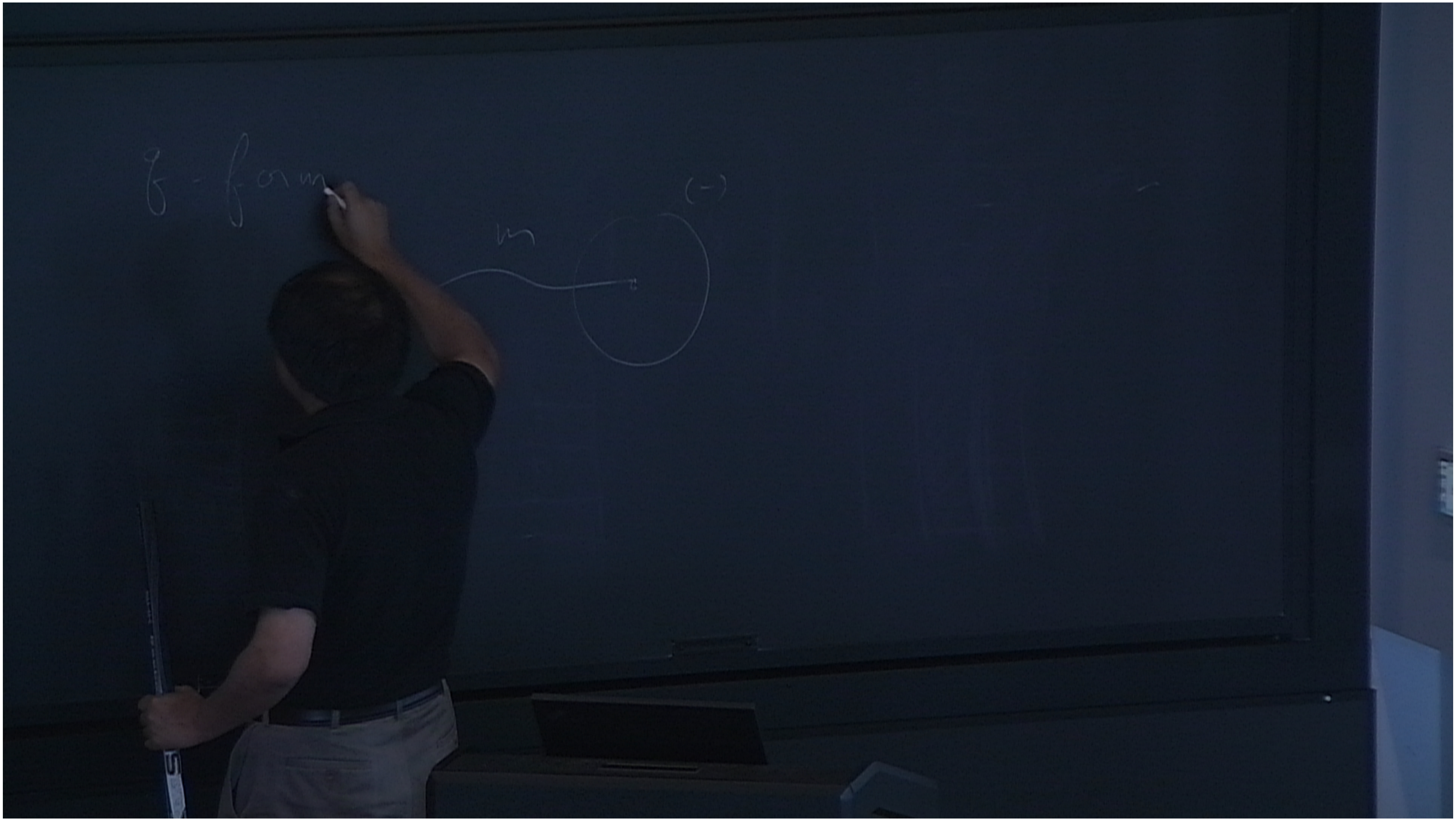


- Type-e string operator $W_e = \prod_{\text{string}} \sigma^x_i$
- Type-m string operator $W_m = \prod_{\text{string}*} \sigma^z_i$
- Type-ϵ string op. $W_\epsilon = \prod_{\text{string}} \sigma^x_i \prod_{\text{legs}} \sigma^z_i$

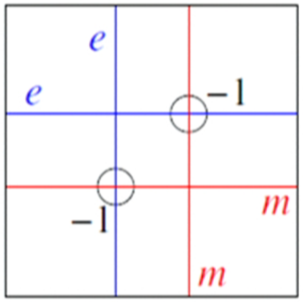
- $[H, W_e^{\text{closed}}] = [H, W_m^{\text{closed}}] = 0$. $\to$ Closed strings cost no energy
- $[Q_{\boldsymbol{I}}, W_e^{\text{open}}] \neq 0$ flip $Q_{\boldsymbol{I}} \to -Q_{\boldsymbol{I}}$, $[F_{\boldsymbol{p}}, W_m^{\text{open}}] \neq 0$ flip $F_{\boldsymbol{p}} \to -F_{\boldsymbol{p}}$
  $\to$ open-string create a pair of topo. excitations at their ends.



q-form
m
(-)

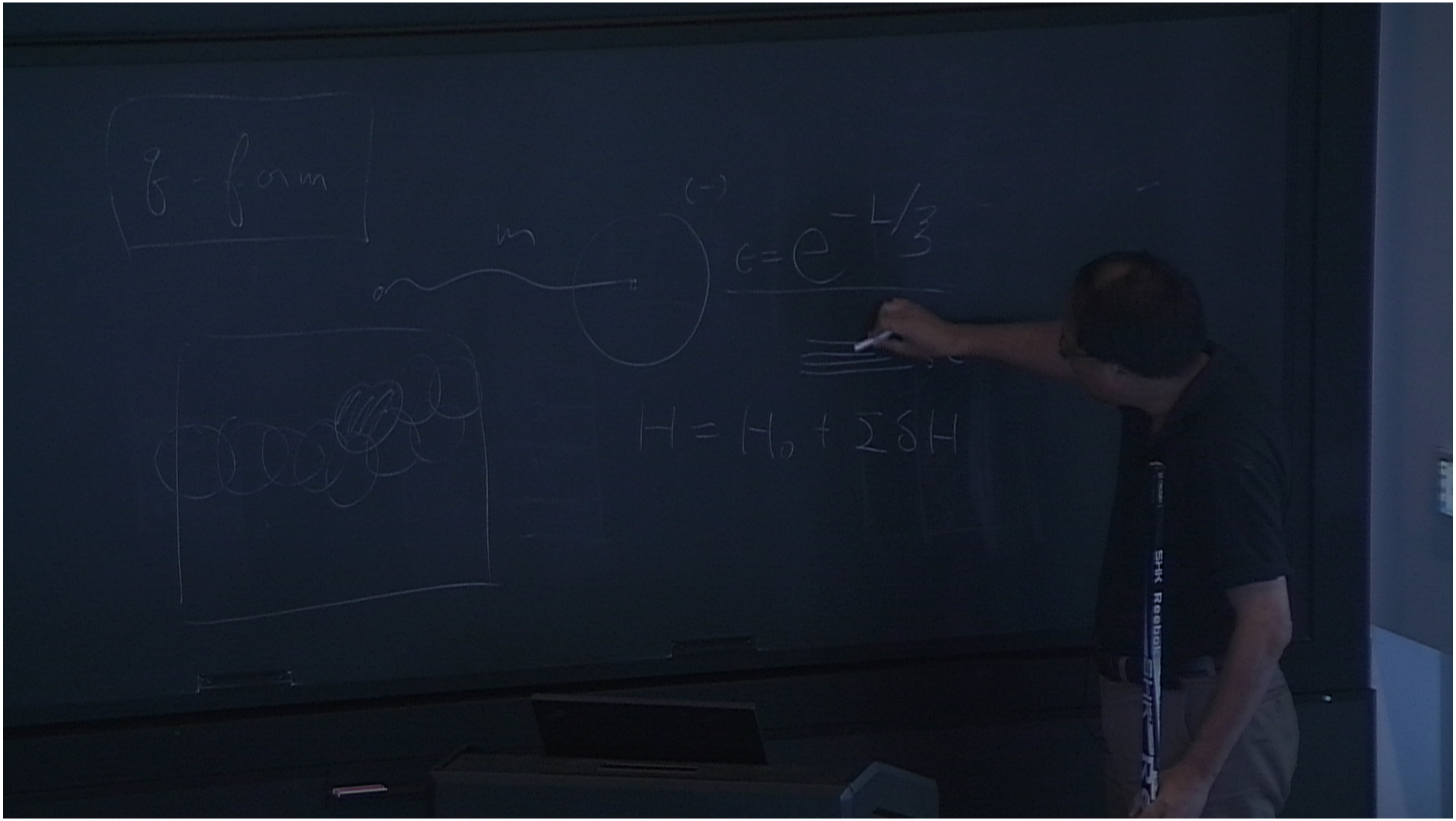
Topological ground state degneracy and code distance

- When strings cross,
  $W_e W_m = (-)^{\#\text{ of cross}} W_m W_e \to$
  4^g degeneracy on genus g surface
  $\to$ **Topological degneracy**
  Degeneracy remain exact for any perturbations localized in a finite region.



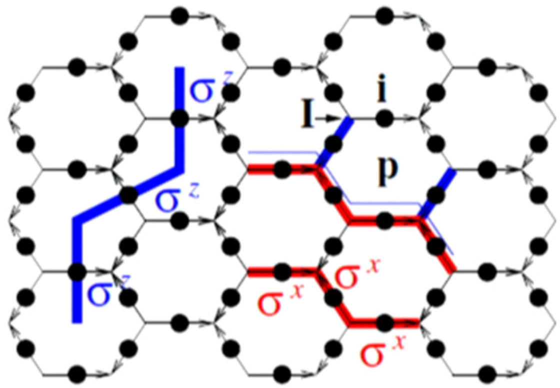
- The above degenerate ground states form a "code", which has a large **code distance** of order L (the linear size of the system).
- Two states $|\psi\rangle$ and $|\psi'\rangle$ that can be connected by first-order local perturbation δH: $\langle\psi'|\delta H|\psi\rangle > O(|\delta H|), \quad L \to \infty$
  $\to$ code distance $= 1$.

  Two states $|\psi\rangle$ and $|\psi'\rangle$ that can be connected by n^{th}-order local perturbation $\to$ code distance $= n$.
- Symmetry breaking ground states in d-dim have
  code distance $\sim L^d$ respected to symmetry preserving perturbation.
  code distance ~ 1 respected to symmetry breaking perturbation.


q-form
m
(−)
ε=e^{−L/3}
H = H_0 + Σ δH

The string operators and ground state degeneracy

- Toric code model:
  $H = -U\sum_{I} Q_{I} - g\sum_{p} F_{p}$
  $Q_{I} = \prod_{\text{legs of } I} \sigma^z_{i}$,
  $F_{p} = \prod_{\text{edges of } p} \sigma^x_{i}$
- Topological excitations:
  e-type: $Q_{I} = 1 \to Q_{I} = -1$
  m-type: $F_{p} = 1 \to F_{p} = -1$



- Type-e string operator $W_e = \prod_{\text{string}} \sigma^x_i$ $\to$ e-type. $e \times e = 1$
- Type-m string operator $W_m = \prod_{\text{string}^*} \sigma^z_i$ $\to$ m-type. $m \times m = 1$
- Type-ϵ string op. $W_\epsilon = \prod_{\text{string}} \sigma^x_i \prod_{\text{legs}} \sigma^z_i$ $\to$ ϵ-type $= e \times m$

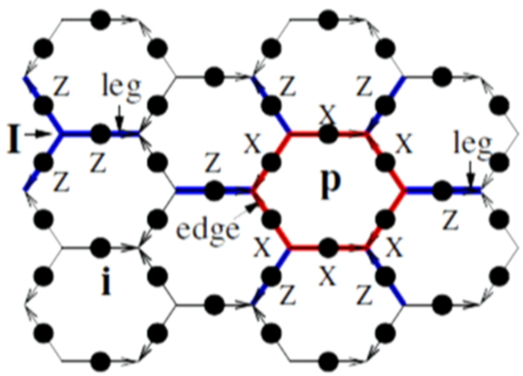
- $[H, W_e^{\text{closed}}] = [H, W_m^{\text{closed}}] = 0$. $\to$ Closed strings cost no energy
- $[Q_{I}, W_e^{\text{open}}] \neq 0$ flip $Q_{I} \to -Q_{I}$, $[F_{p}, W_m^{\text{open}}] \neq 0$ flip $F_{p} \to -F_{p}$
  $\to$ open-string create a pair of topo. excitations at their ends.

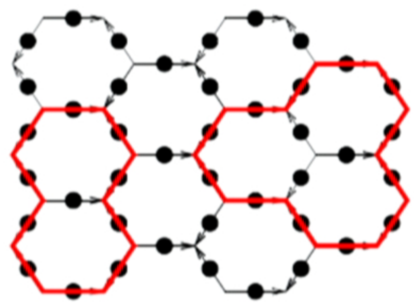
- **Fusion algebra** of string operators $\to$ fusion of topo. excitations:
  $W_e^2 = W_m^2 = W_\epsilon^2 = W_e W_m W_\epsilon = 1$ when strings are parallel

Double-semion model

Local rules: Levin-Wen cond-mat/0404617

- The Hamiltonian to enforce the local rules:





$H = -U\sum_I Q_I - \frac{g}{2}\sum_p (F_p + h.c.)$,

$Q_I = \prod_{\text{legs of } I} \sigma^z_i$, $\quad F_p = (\prod_{\text{edges of } p} \sigma^x_j)(-\prod_{\text{legs of } p} \mathrm{i}^{\frac{1-\sigma^z_j}{2}})$

- Ground state wave function $\Phi(X) = (-)^{X_c}$, where X_c is the number of loops in the string configuration X.

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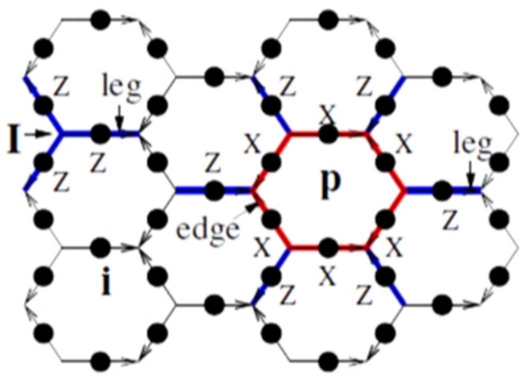

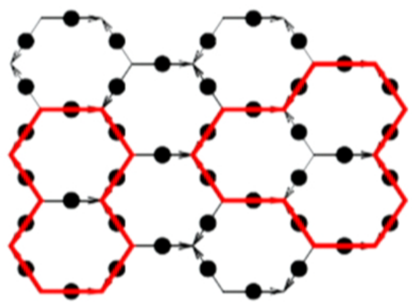

$H = -U \sum_{I} Q_{I} - \frac{g}{2} \sum_{p} (F_{p} + h.c.)$,

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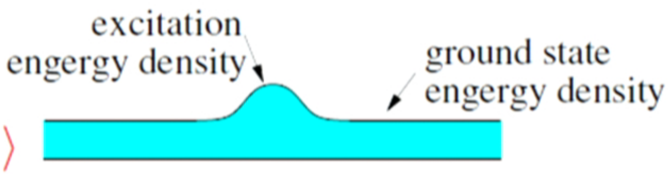
Local and topological quasiparticle excitations

In a system: $H = \sum_{\vec{x}} H_{\vec{x}}$



- a particle-like excitation:
  energy density $= \langle \Psi_{exc} | H_{\vec{x}} | \Psi_{exc} \rangle$
  $|\Psi_{exc}\rangle$ is the gapped ground state of $H + \delta H^{\text{trap}}(\vec{x})$.
- Local quasiparticle excitation: $|\Psi_{exc}\rangle = \hat{O}(\vec{x}) |\Psi_{grnd}\rangle$
- Topological quasiparticle excitations $|\Psi_{exc}\rangle \neq \hat{O}(\vec{x}) |\Psi_{grnd}\rangle$ for any local operators $\hat{O}(\vec{x})$
- Topological quasiparticle types: if $|\Psi'_{exc}\rangle = \hat{O}(\vec{x}) |\Psi_{exc}\rangle$, then $|\Psi'_{exc}\rangle$ and $|\Psi_{exc}\rangle$ belong to the same type.
- *Number of topological quasiparticle types is an important topological invariant that characterizes the topological order.*
  Only topological quasiparticles can carry fractional statistics and fractional quantum numbers.
- **Example**: Open string operators create pairs of topo. excitations. Open string operators are hopping operators of topo. excitations
- $W_e^2 = W_m^2 = W_\epsilon^2 = W_e W_m W_\epsilon = 1 \to$ fusion of topo. excitations

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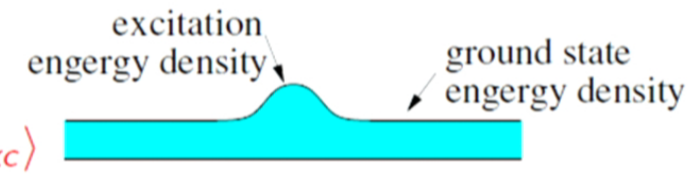

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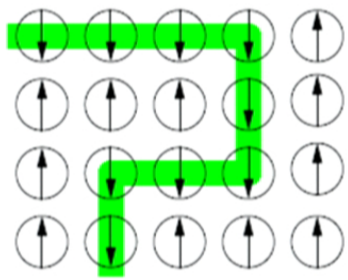
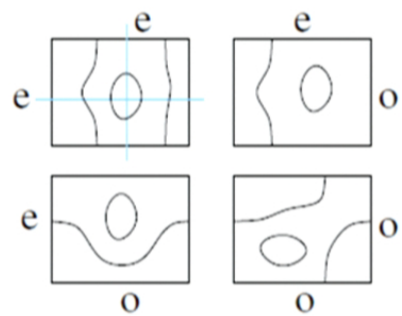
Many-body energy spectrum of toric code model

- The $-U\sum_I Q_I$ enforce closed-string ground state.
- F_p adds a small loop and generates a permutation among the loop states $|\ \rangle \to$ Ground states on torus $|\Psi^{\alpha}_{\text{grnd}}\rangle = \sum_{\text{loops}} |\ \rangle$
- There are four degenerate ground states $\alpha = ee, eo, oe, oo$





$D^{\text{tor}}=4$

- On genus g surface, ground state degeneracy $D_g = 4^g$
- Quasiparticle excitation energy gap $\Delta^Q_p = 2U$, $\Delta^F_p = 2g$
  Spectrum energy gap $\Delta^Q = 4U$, $\Delta^F = 4g$

Emergence of fractional spin/statistics

- Why electron carry spin-1/2 and Fermi statistics?
- Ends of strings (type-I) are point-like excitations, which can carry spin-1/2 and Fermi statistics?

  Fidkowski-Freedman-Nayak-Walker-Wang cond-mat/0610583

- $\Phi_{\text{str}}\left(\ \right) = 1$ string liquid $\Phi_{\text{str}}\left(\ \right) = \Phi_{\text{str}}\left(\ \right)$

  360° rotation: $\rightarrow$ and $=$ $\rightarrow$: $R_{360^\circ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  $+$ $\equiv e$ spin 0 mod 1. $-$ $\equiv \epsilon$ spin $1/2$ mod 1.

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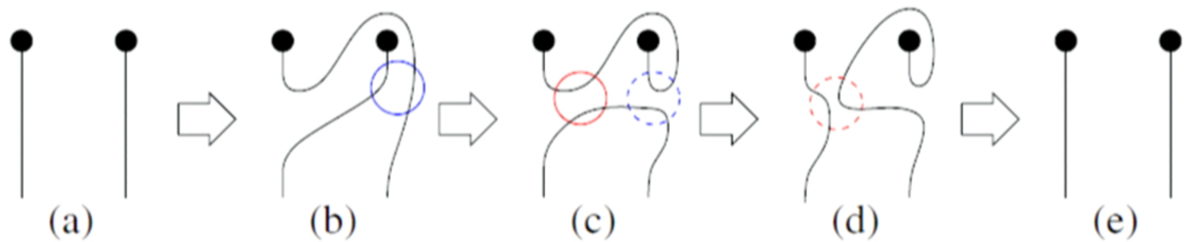
Spin-statistics theorem

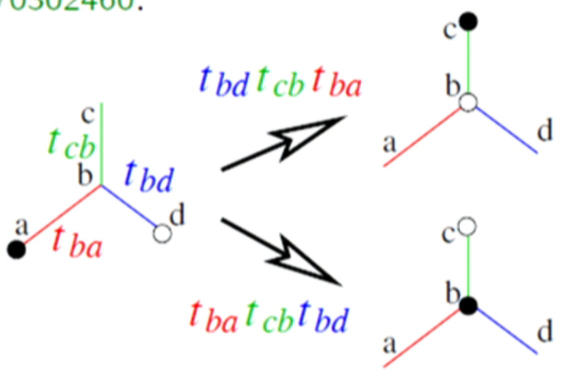


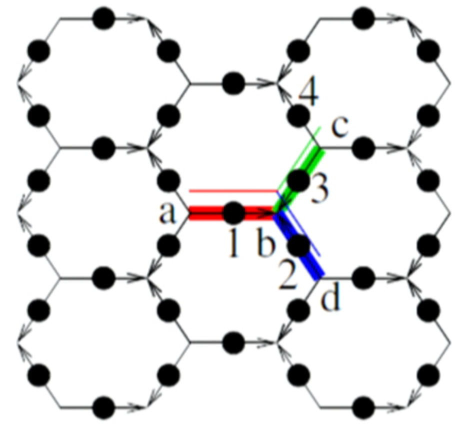
- (a) $\to$ (b) = exchange two string-ends.
- (d) $\to$ (e) = 360° rotation of a string-end.
- Amplitude (a) = Amplitude (e)
- Exchange two string-ends plus a 360° rotation of one of the string-end generate no phase.

$\to$ **Spin-statistics theorem**

Statistics of ends of strings

- The statistics is determined by particle hopping operators Levin-Wen cond-mat/0302460:





- An open string operator is a hopping operator of the 'ends'. The algebra of the open string operator determine the statistics.
- For type-I string: $t_{ba} = \sigma_1^x$, $t_{cb} = \sigma_3^x$, $t_{bd} = \sigma_2^x$
  We find $t_{bd}t_{cb}t_{ba} = t_{ba}t_{cb}t_{bd}$
  The ends of type-I string are bosons
- For type-III strings: $t_{ba} = \sigma_1^x$, $t_{cb} = \underline{\sigma_3^x}\sigma_4^z$, $t_{bd} = \sigma_2^x\underline{\sigma_3^z}$
  We find $t_{bd}t_{cb}t_{ba} = -t_{ba}t_{cb}t_{bd}$
  The ends of type-III strings are fermions

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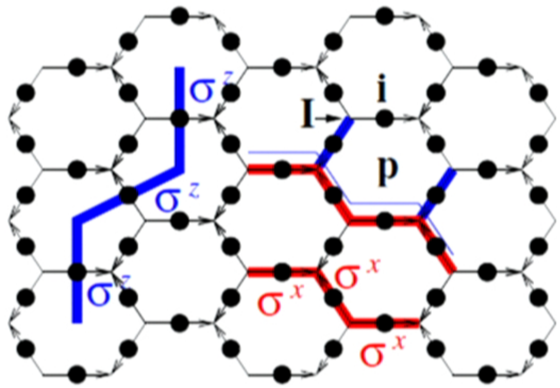

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- $[Q_{I}, W_e^{\text{open}}] \neq 0$ flip $Q_{I} \to -Q_{I}$, $[F_{p}, W_m^{\text{open}}] \neq 0$ flip $F_{p} \to -F_{p}$
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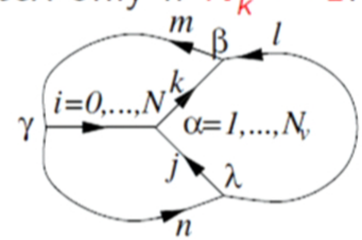
More general patterns of long-range entanglement

Generslize the Z_2/double-semion local rule:

- **More general wave functions are defined on graphs**:
  - There are $N+1$ states on each link labeled by $i, j, k, \cdots$
  - There are states on each vertex labeled by $\alpha, \beta, \cdots$
- Within the ground state, i, j, k can join at a vertex only if $N_k^{ij} = 1$.

  i, j, k cannot join at a vertex if $N_k^{ij} = 0$.

  N_k^{ij} is called fusion coefficient, which describe the string-net structure (generalize the close-string condition)

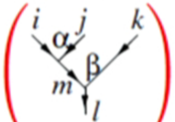


  - $N_k^{ij} = 0$, vertex i, j, k is not allowed in the ground state. The vector space at the vertex i, j, k has zero dimension in the ground state.
  - $N_k^{ij} = 1$, vertex i, j, k is allowed in the ground state. The vector space at the vertex i, j, k has 1-dimension in the ground state.
  - $N_k^{ij} = 2$, vertex i, j, k is allowed in the ground state. The vector space at the vertex i, j, k has 2-dimension labeled by α.

More general patterns of long-range entanglement

We still have Φ_{str} () $= \Phi_{\text{str}}$ ()

More general reconnection rule: F-move

F-move: Φ (

) $= \sum_{n=0}^{N} \sum_{\chi=1}^{N_{kjn}} \sum_{\delta=1}^{N_{nil}} F^{ijk;m\alpha\beta}_{l;n\chi\delta} \Phi$ (

)

- The matrix $F^{ijk}_{l} \to (F^{ijk}_{l})^{m\alpha\beta}_{n\chi\delta}$ = local unitary transformation

- The wave functions $\Psi_{ijkl,\cdots}(m,\alpha,\beta) = \Psi(i,j,k,l,m,\alpha,\beta,\cdots)$ with fixed $ijkl$ but different $\cdots$ span a local space – *local spport space of the ground state wave function.*

- The wave functions $\Psi'_{ijkl,\cdots}(n,\chi,\delta) = \Psi'(i,j,k,l,n,\chi,\delta,\cdots)$ with fixed $ijkl$ but different $\cdots$ span another local space.

- The two local suport spaces have the same dim. $\sum_m N^{ij}_m N^{mk}_l = \sum_n N^{in}_l N^{jk}_n$ and are related by an unitary F^{ijk}_l.



- **Data $N, N^{ij}_k, F^{ijk}_l \to$ wave function**

Levin-Wen cond-mat/0404617; Chen-Gu-Wen arXiv:1004.3835

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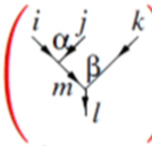
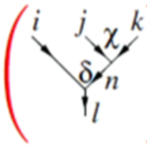
F-move: $\Phi\Big($

$\Big) = \sum_{n=0}^{N} \sum_{\chi=1}^{N_{kjn}} \sum_{\delta=1}^{N_{nil}} F^{ijk;m\alpha\beta}_{l;n\chi\delta} \Phi\Big($

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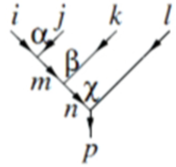
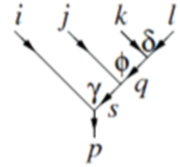
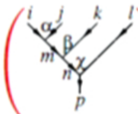
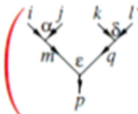
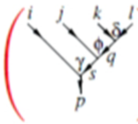
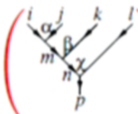
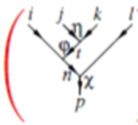
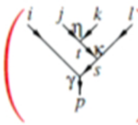
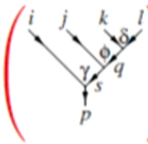
Consistent conditions for $F^{ijk;m\alpha\beta}_{l;n\chi\delta}$: the pentagon identity

 can be trans. to  through two different paths:

$\Phi\Big($  $\Big) = \sum_{q,\delta,\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon}\,\Phi\Big($  $\Big) = \sum_{q,\delta,\epsilon;s,\phi,\gamma} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma}\,\Phi\Big($  $\Big)$,

$\Phi\Big($  $\Big) = \sum_{t,\eta,\varphi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi}\,\Phi\Big($  $\Big) = \sum_{t,\eta,\varphi;s,\kappa,\gamma} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma}\,\Phi\Big($  $\Big)$

$= \sum_{t,\eta,\kappa;\varphi;s,\kappa,\gamma;q,\delta,\phi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi}\,\Phi\Big($  $\Big)$.

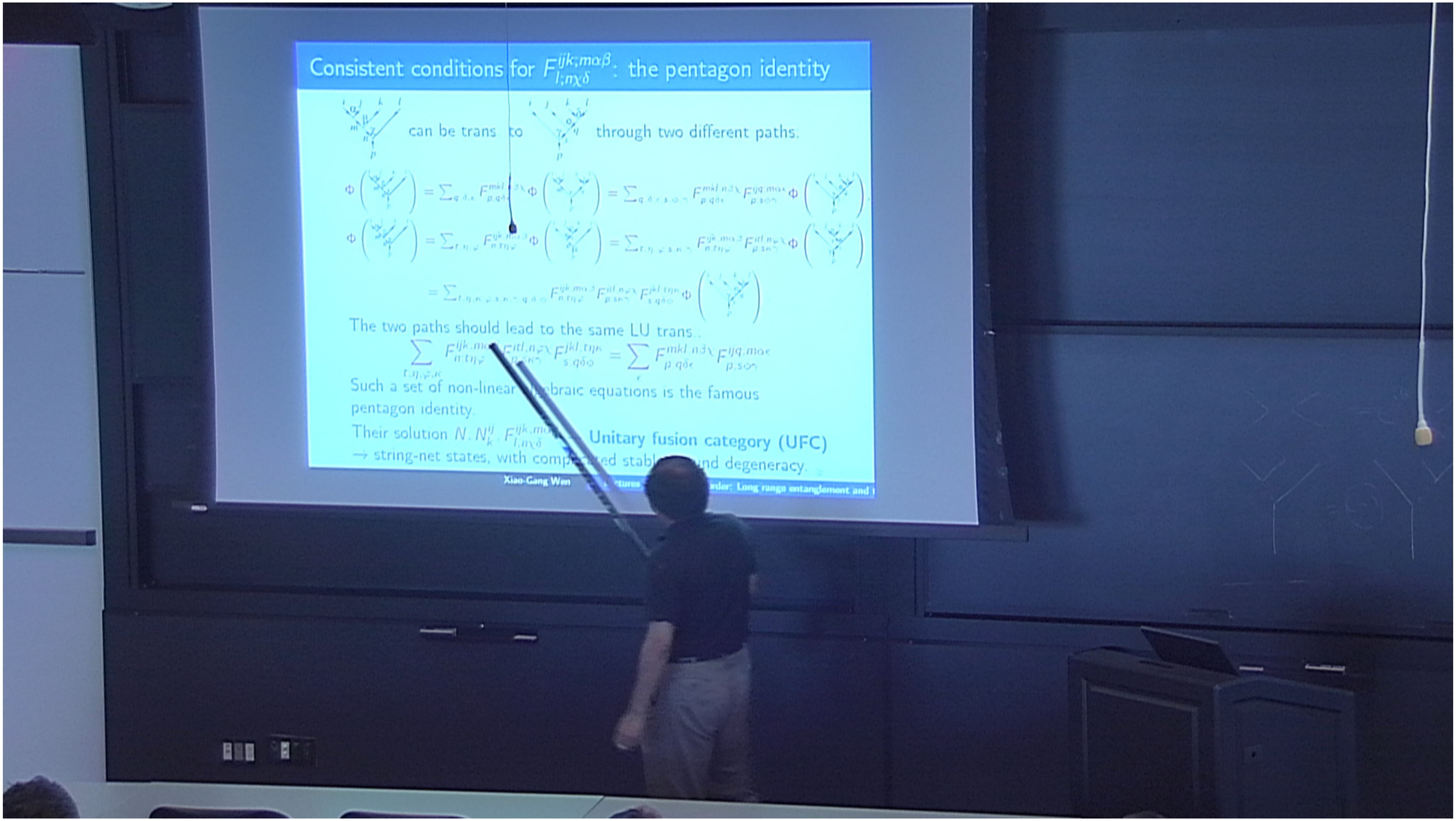
The two paths should lead to the same LU trans.:

$$\sum_{t,\eta,\varphi,\kappa} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi} = \sum_{\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma}$$

Such a set of non-linear algebraic equations is the famous pentagon identity.

Their solution $N, N^{ij}_k, F^{ijk;m\alpha\beta}_{l;n\chi\delta}$ → **Unitary fusion category (UFC)**

→ string-net states, with complicated stable ground degeneracy.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Unitary fusion category (UFC)
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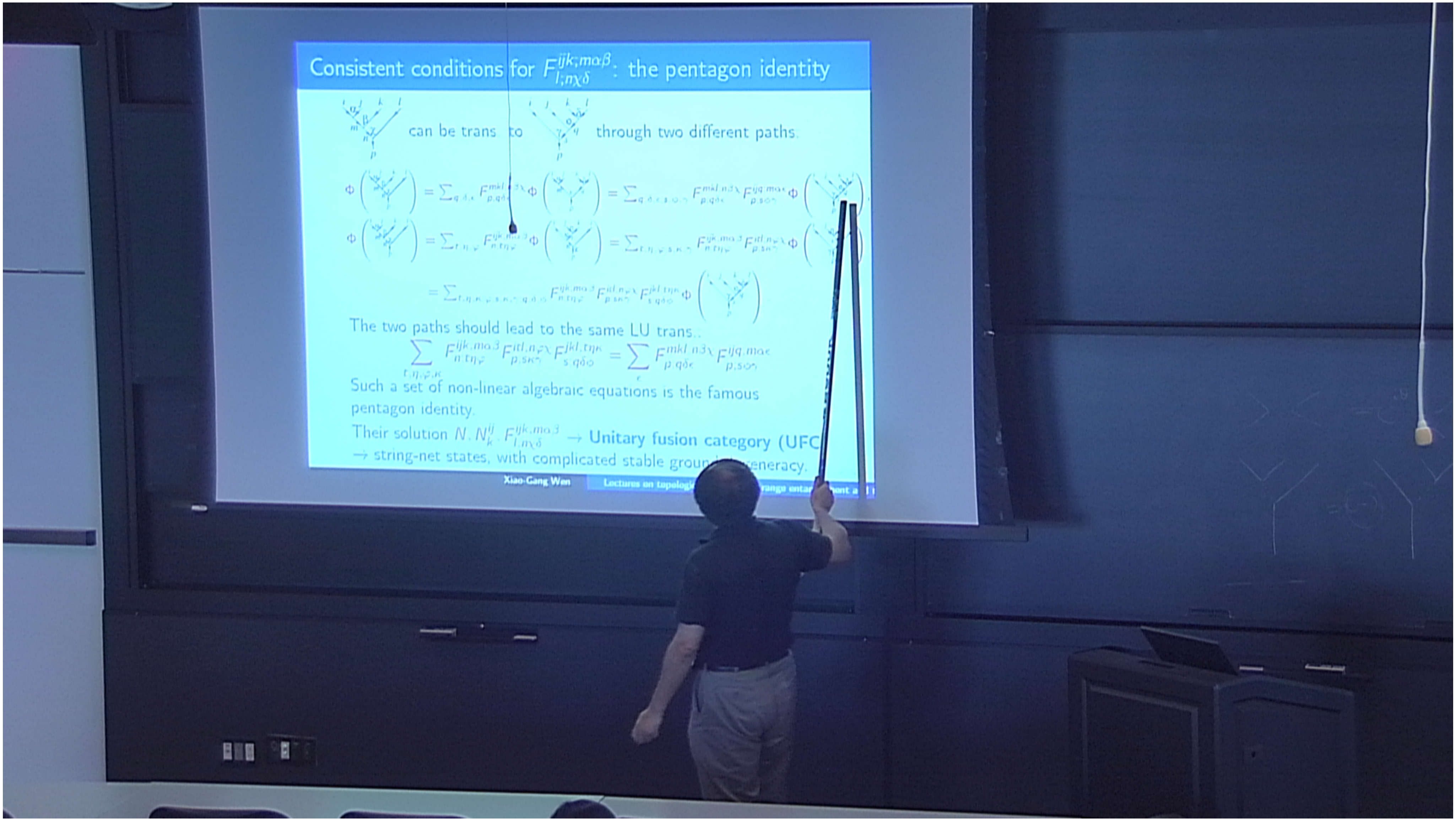
Xiao-Gang Wen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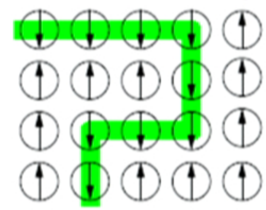
Example of Topologically ordered states

To make topological order, we need to sum over many different product states, but we should not sum over everything.

$\sum_{\text{all spin config.}} |\uparrow\downarrow ..\rangle = |\rightarrow\rightarrow ..\rangle$

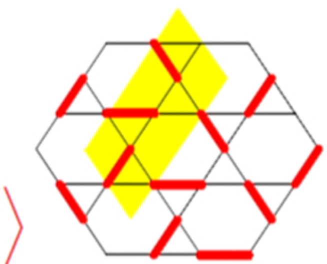


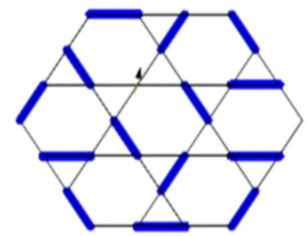
- *sum* over a subset of spin config.:

$|\Phi^{Z_2}_{\text{loops}}\rangle = \sum$ 

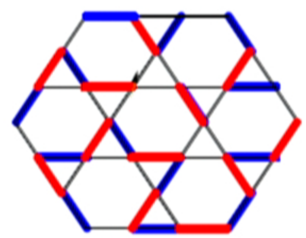
$|\Phi^{DS}_{\text{loops}}\rangle = \sum (-)^{\#\text{ of loops}}$ 

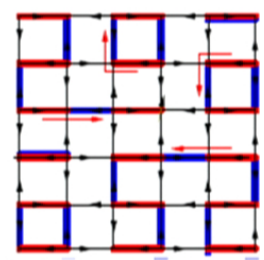
$|\Phi^{\theta}_{\text{loops}}\rangle = \sum (e^{i\theta})^{\#\text{ of loops}}$ 

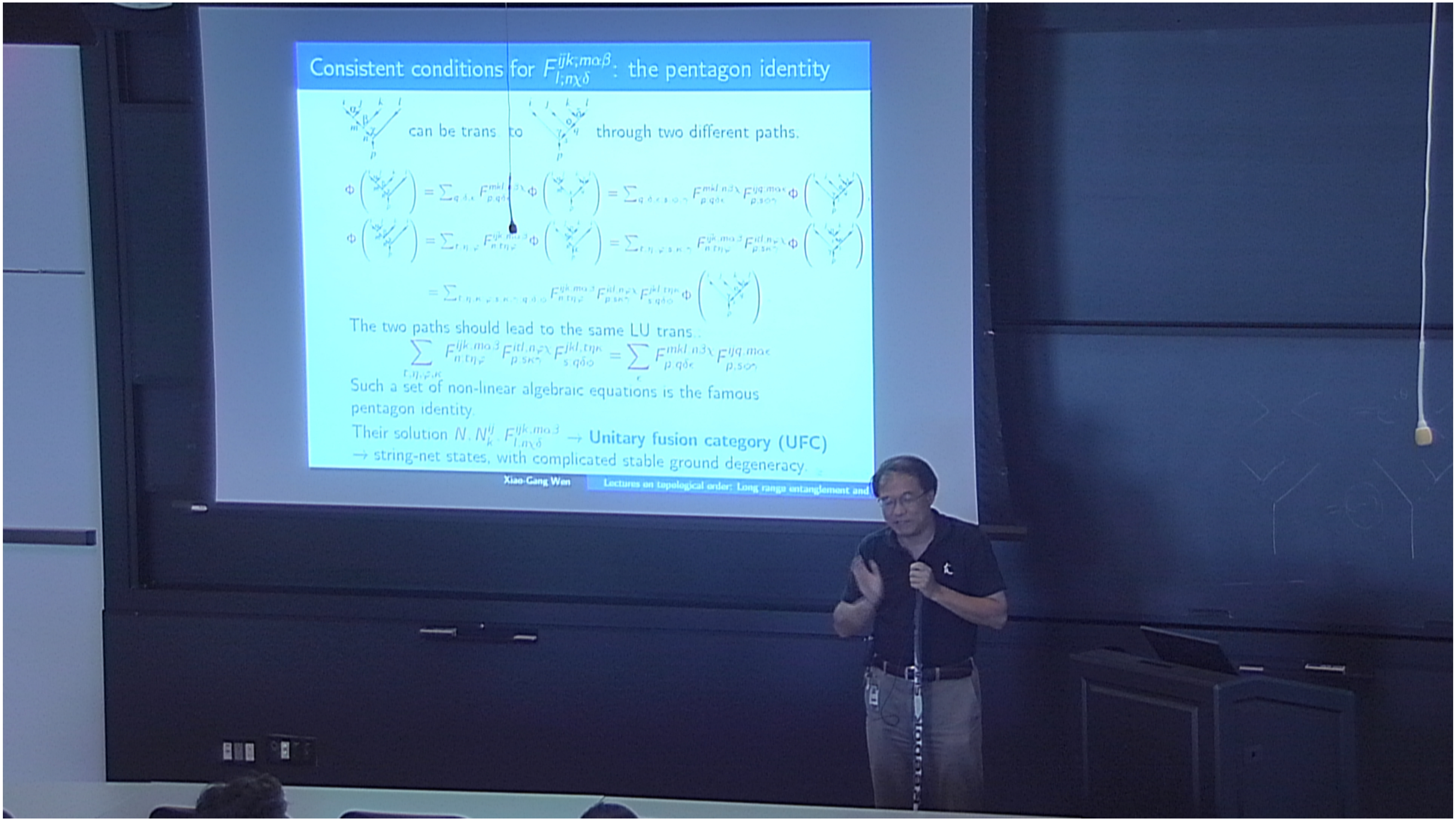




- Can the above wavefunction be the ground states of local Hamiltonians?



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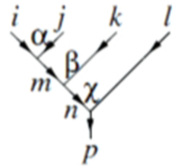
Lectures on topological order: Long range entanglement and

Consistent conditions for $F^{ijk;m\alpha\beta}_{l;n\chi\delta}$: the pentagon identity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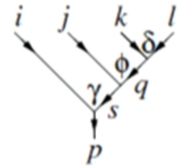
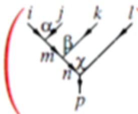
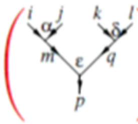
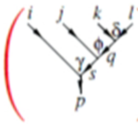
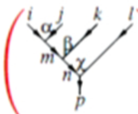
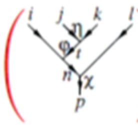
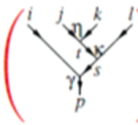
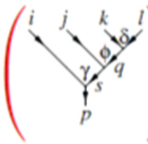

through two different paths:

Φ () $= \sum_{q,\delta,\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} \Phi$ () $= \sum_{q,\delta,\epsilon;s,\phi,\gamma} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma} \Phi$ (),

Φ () $= \sum_{t,\eta,\varphi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} \Phi$ () $= \sum_{t,\eta,\varphi;s,\kappa,\gamma} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} \Phi$ ()

$= \sum_{t,\eta,\kappa;\varphi;s,\kappa,\gamma;q,\delta,\phi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi} \Phi$ ().

The two paths should lead to the same LU trans.:

$$\sum_{t,\eta,\varphi,\kappa} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi} = \sum_{\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma}$$

Such a set of non-linear algebraic equations is the famous pentagon identity.

Their solution $N, N^{ij}_k, F^{ijk;m\alpha\beta}_{l;n\chi\delta}$ → **Unitary fusion category (UFC)**

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