Title: How to (path) integrate by differentiating

Date: Jun 11, 2015 02:30 PM

URL: http://pirsa.org/15060010

Abstract: <p>Path integrals are at the heart of quantum field theory. In spite of their covariance and seeming simplicity, they are hard to define and evaluate. In contrast, functional differentiation, as it is used, for example, in variational problems, is relatively straightforward. This has motivated the development of new techniques that allow one to express functional integration in terms of functional differentiation. In fact, the new techniques allow one to express integrals in general through differentiation. These techniques therefore add to the general toolbox for integration and integral transforms such as the Fourier and Laplace transforms. Here, we review some of these results, we give simpler proofs and we add new results, for example, on expressing the Laplace transform and its inverse in terms of derivatives, results that may be of use in quantum field theory, e.g., in the context of heat traces.</p>

How to (path) integrate by differentiating

New tools for QFT and gravity

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The problem

- Integration is hard, harder than differentiation.

- Path integrals (which are also functional Fourier transforms)

$$Z[J] \;=\; \int e^{iS[\phi]+i\int J\phi\ d^nx} D[\phi]$$

  are harder than functional derivatives.

- If only integration could be expressed in terms of differentiation! **Or can it?**

Overview

- Main message?
  - New, convenient methods for integration and integral transforms such as Fourier and Laplace, using only derivatives.
- Advantages?
  - Often quicker, simpler.
  - Handles distributions well.
  - For cases that are too hard, offers new perturbative approaches.
- Applications to QFT
  - Expresses functional integrations and functional transforms in terms of functional differentiation.
  - Offers new perturbative approaches.

New representations of integration:

$$\int_a^b f(x)\, dx = \lim_{\epsilon \to 0} f(\partial_\epsilon)\, \frac{e^{\epsilon b} - e^{\epsilon a}}{\epsilon}$$

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{\epsilon \to 0^+} \left(f(\partial_\epsilon) + f(-\partial_\epsilon)\right)\, \frac{1}{\epsilon}$$

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi\, \delta(i\partial_x)\, f(x)$$

and:

Integration: $$\int_{-\infty}^{\infty} f(x)\, dx \;=\; \lim_{x\to 0} 2\pi \; f(-i\partial_x)\; \delta(x)$$

Fourier: $$\mathcal{F}[f](x) \;=\; \sqrt{2\pi}\; f(-i\partial_x)\; \delta(x)$$

Laplace: $$\mathcal{L}[f](x) \;=\; f(-\partial_x)\; \frac{1}{x}$$

Inverse Laplace: $$\mathcal{L}^{-1}[f](x) \;=\; f(\partial_x)\; \delta(x)$$

and:

Integration: $$\int_{-\infty}^{\infty} f(x)\, dx \;=\; \lim_{x\to 0} 2\pi\; f(-i\partial_x)\;\delta(x)$$

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Inverse Laplace: $$\mathcal{L}^{-1}[f](x) \;=\; f(\partial_x)\;\delta(x)$$

Examples for integration

Recall:

$$\int_{-\infty}^{\infty} f(x)\; dx = \lim_{x\to 0} 2\pi\; f(-i\partial_x)\;\delta(x)$$

For example:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\; dx &= 2\pi \lim_{x\to 0} \frac{1}{2i}\left(e^{\partial_x} - e^{-\partial_x}\right)\frac{1}{-i\partial_x}\;\delta(x) \\
&= \pi \lim_{x\to 0}\left(e^{\partial_x} - e^{-\partial_x}\right)(\Theta(x) + c) \\
&= \pi \lim_{x\to 0}\left(\Theta(x+1) + c - \Theta(x-1) - c\right)
\end{aligned}$$

Examples for integration

Similarly, one quickly obtains, e.g.,

$$\int_{-\infty}^{\infty} \frac{\sin^5(x)}{x}\, dx = 3\pi/8$$

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\, dx = \pi$$

$$\int_{-\infty}^{\infty} \frac{(1-\cos(tx))}{x^2}\, dx = \pi|t|$$

$$\int_{-\infty}^{\infty} x^2 \cos(x)\, e^{-x^2}\, dx = \sqrt{\pi}e^{-1/4}/4$$

(exercise)

Examples for Fourier

Now how much harder is Fourier?

Fourier transforming is even easier than integrating!

Examples for Fourier

Recall the new methods for integration and Fourier:

Integration: $$\int_{-\infty}^{\infty} f(x)\, dx \;=\; \lim_{x \to 0} 2\pi\, f(-i\partial_x)\, \delta(x) \quad \checkmark$$

Fourier: $$\mathcal{F}[f](x) \;=\; \sqrt{2\pi}\, f(-i\partial_x)\, \delta(x)$$

Therefore:
Obtain Fourier transform simply by *not* taking the limit (and by dividing by $\sqrt{2\pi}$).

Examples for Fourier

For example, for $f(x) = \sin(x)/x$, recall:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx &= 2\pi \lim_{x\to 0} \frac{1}{2i}\left(e^{\partial_x} - e^{-\partial_x}\right)\frac{1}{-i\partial_x}\,\delta(x) \\
&= \pi \lim_{x\to 0}\left(e^{\partial_x} - e^{-\partial_x}\right)\left(\Theta(x) + c'\right) \\
&= \pi \lim_{x\to 0}\left(\Theta(x+1) - \Theta(x-1)\right) \\
&= \pi
\end{aligned}$$

By not taking the limit and by dividing by $\sqrt{2\pi}$, we obtain immediately:

$$\mathcal{F}[f](x) \;=\; \sqrt{\pi/2}\;\;\left(\Theta(x+1) - \Theta(x-1)\right)$$

Examples for Fourier

Consider Fourier transform of plane waves $f(x) = e^{ixy}$.

According to our method,

$$\mathcal{F}[f](x) = \sqrt{2\pi}\, f(-i\partial_x)\, \delta(x)$$

we obtain:

$$\begin{aligned}\mathcal{F}[f](x) &= \sqrt{2\pi}\, e^{y\partial_x}\, \delta(x) \\ &= \sqrt{2\pi}\, \delta(x+y)\end{aligned}$$

And the plane waves from a basis of the function space.

Examples for Laplace

Recall:

Integration: $$\int_{-\infty}^{\infty} f(x)\, dx \;=\; \lim_{x\to 0} 2\pi\, f(-i\partial_x)\, \delta(x)$$

Fourier: $$\mathcal{F}[f](x) \;=\; \sqrt{2\pi}\, f(-i\partial_x)\, \delta(x)$$

Laplace: $$\mathcal{L}[f](x) \;=\; f(-\partial_x)\, \frac{1}{x}$$

Inverse Laplace: $$\mathcal{L}^{-1}[f](x) \;=\; f(\partial_x)\, \delta(x)$$

Examples for Laplace

If we apply the new Laplace transform method

$$\mathcal{L}[f](x) \;=\; f(-\partial_x)\,\frac{1}{x}$$

to monomials $f(x) = x^n$ we obtain:

$$L[f](x) = (-\partial_x)^n\,\frac{1}{x} \;=\; \frac{n!}{x^{n+1}}$$

And the monomials form a basis in the function space.

Example for inverse Laplace

Consider a heat kernel trace:

$$h(t) = \sum_n e^{-\lambda_n t}$$

Given $h(t)$, the spectrum $\{\lambda_n\}$ is known to be recoverable via inverse Laplace transform.

Why? Using the new inverse Laplace transform method, namely

$$\mathcal{L}^{-1}[f](x) \;=\; f(\partial_x)\,\delta(x)$$

this is easy to see:

$$\mathcal{L}^{-1}[h](\lambda) \;=\; h(\partial_\lambda)\,\delta(\lambda)$$

Example for inverse Laplace

Consider a heat kernel trace:

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Why? Using the new inverse Laplace transform method, namely

$$\mathcal{L}^{-1}[f](x) \;=\; f(\partial_x)\,\delta(x)$$

this is easy to see:

$$\begin{aligned}\mathcal{L}^{-1}[h](\lambda) \;&=\; h(\partial_\lambda)\,\delta(\lambda)\\ &=\; \sum_n e^{-\lambda_n \partial_\lambda}\,\delta(\lambda) = \sum_n \delta(\lambda - \lambda_n)\end{aligned}$$

Associated perturbative expansions

We want to apply the new methods

$$\text{Integration:} \quad \int_{-\infty}^{\infty} f(x)\, dx \;=\; \lim_{x\to 0} 2\pi\, f(-i\partial_x)\,\delta(x)$$

$$\text{Fourier:} \quad \mathcal{F}[f](x) \;=\; \sqrt{2\pi}\, f(-i\partial_x)\,\delta(x)$$

$$\text{Laplace:} \quad \mathcal{L}[f](x) \;=\; f(-\partial_x)\,\frac{1}{x}$$

$$\text{Inverse Laplace:} \quad \mathcal{L}^{-1}[f](x) \;=\; f(\partial_x)\,\delta(x)$$

to QFT.

Associated perturbative expansions

But what if in

$$Z[J] \;=\; \int e^{iS[\phi]+i\int J\phi\ d^nx} D[\phi]$$

the action $S[\phi]$ is not suitable to solve the integral or Fourier (or Laplace) transform with our new methods exactly?

And that's the norm of course!

Associated perturbative expansions

- On the basic level, what if $f(x)$ is too complicated, e.g., for:

$$\mathcal{F}[f](x) \;=\; \sqrt{2\pi}\; f(-i\partial_x)\;\delta(x)$$

- Opportunity: Use any regularizations of δ such as or

$$\delta_\sigma(x) = (2\pi\sigma)^{-1/2} e^{-x^2/2\sigma}$$

to obtain, e.g.:

$$\mathcal{F}[f](x)\; dx \approx \sqrt{2\pi}\; f(-i\partial_x)\; \delta_\sigma(x)$$

- Obtain weak & strong coupling expansions and others...

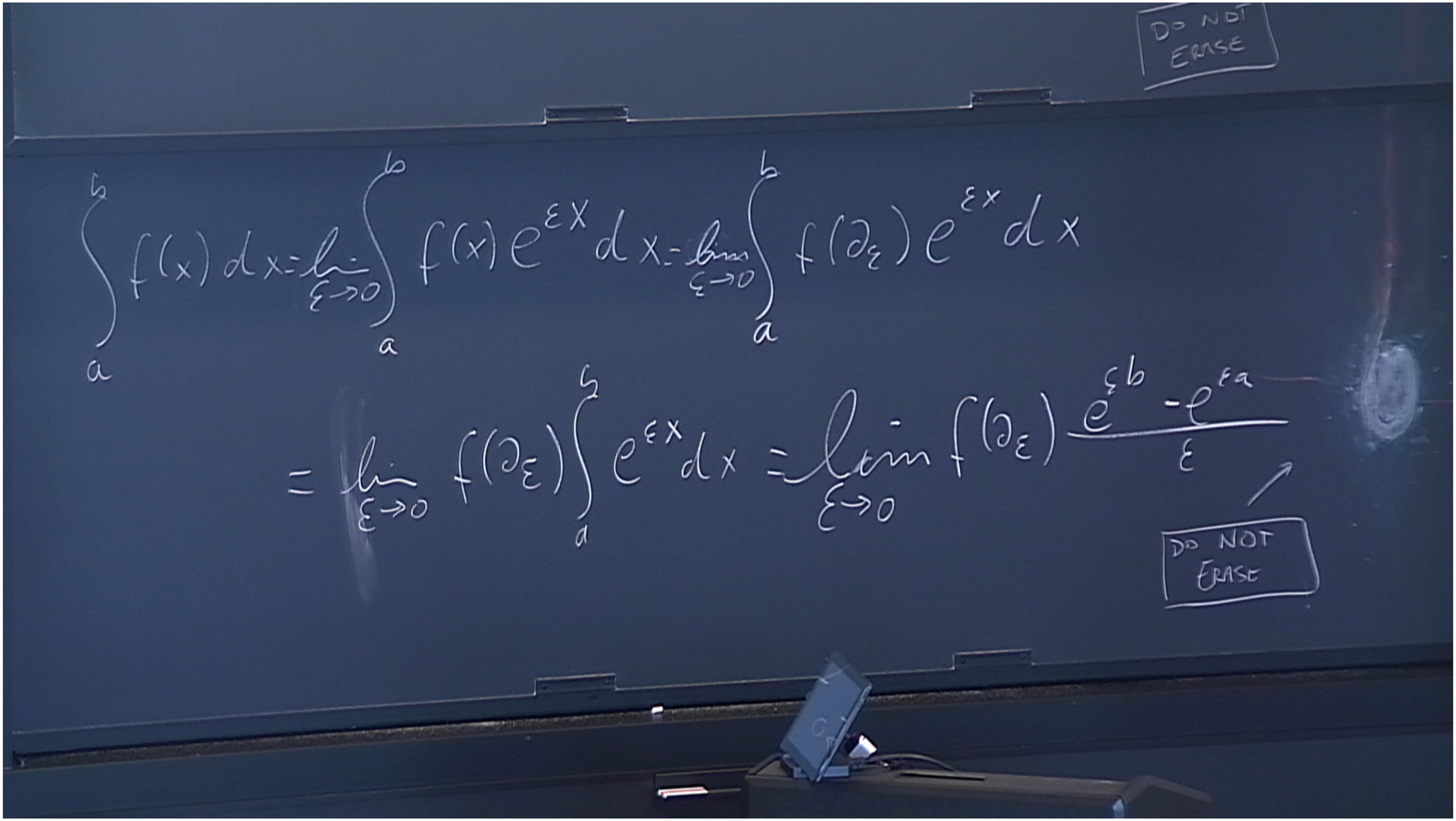
Outlook

- What is the full size of the space of functions and distributions to which these methods apply?

- Relation to Stoke's theorem?

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

- Relation to fermionic integration, a unifying formalism?



DO NOT ERASE
$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0}\int_a^b f(x)\,e^{\varepsilon x}\,dx = \lim_{\varepsilon\to 0}\int_a^b f(\partial_\varepsilon)\,e^{\varepsilon x}\,dx$$
$$= \lim_{\varepsilon\to 0} f(\partial_\varepsilon)\int_a^b e^{\varepsilon x}\,dx = \lim_{\varepsilon\to 0} f(\partial_\varepsilon)\,\frac{e^{\varepsilon b}-e^{\varepsilon a}}{\varepsilon}$$
DO NOT ERASE

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0} f(\partial_\varepsilon)\,\frac{e^{\varepsilon b}-e^{\varepsilon a}}{\varepsilon}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{\varepsilon\to 0^+}\Big(f(\partial_\varepsilon)+f(-\partial_\varepsilon)\Big)\,\frac{1}{\varepsilon}$$

DO NOT ERASE

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi\, f(-i\partial_x)\,\delta(x)\Big|_{x=0}$$

$$= \lim_{\varepsilon\to 0} f(\partial_\varepsilon)\int_a^b e^{\varepsilon x}\,dx = \lim_{\varepsilon\to 0} f(\partial_\varepsilon)\,\frac{e^{\varepsilon b}-e^{\varepsilon a}}{\varepsilon}$$

DO NOT ERASE

$$f'(x) = \lim_{\varepsilon \to 0} \left(f(x+\varepsilon) - f(x) \right) \frac{1}{\varepsilon}$$

$$\int_{-\infty}^{+\infty} f(x)\, dx = \lim_{\varepsilon \to 0^+} \left(f(\partial_\varepsilon) + f(-\partial_\varepsilon) \right) \frac{1}{\varepsilon}$$