Title: Quasilocal mass in general relativity

Date: May 01, 2015 04:15 PM

URL: http://pirsa.org/15050138

Abstract:

Quasi-local Mass in General Relativity

Shing-Tung Yau

Harvard University

For the 60th birthday of Gary Horowtiz

U. C. Santa Barbara, May. 1, 2015

This talk is based on joint work with Po-Ning Chen and Mu-Tao Wang.

As is well known, it is not possible to find mass density of gravity in general relativity.

The mass density would have to be first derivative of the metric tensor which is zero in suitable chosen coordinate at a point.

But we still desire to measure the total mass in a space like region bounded by a closed surface.

The mass due to gravity should be computable from the intrinsic and the extrinsic geometry of the surface.

It has been important question to find the right definition.

Penrose gave a talk on this question in my seminar in the Institute for Advances Study in 1979, the year before Gary and Andy came to be postdocts.

The quantity is called quasilocal mass.

Penrose listed it as the first major problem in his list of open problems.

Many people, including Penrose, Hawking-Horowitz, Brown-York and others worked on this problem and various definitions were given.

I thought about this problem and attempted to look at it from point of view of mathematician.

I list properties that the definition should satisfy :

1. It should be nonnegative and zero for any closed surfaces in flat Minkowski spacetime

2. It should converge to the familiar ADM mass for asymptotically flat spacetime if we have a sequence of coordinate spheres that approaches the spatial infinity of an asymptotic flat slice.

3. It should convergent to the Bondi mass when the spheres divergent to the cut at null infinity

4. It should be equivalent to the standard Komar mass in a stationary spacetime .

It turns out that this is not so easy to find such a definition.

In the time symmetric case, my former student Robert Bartnik proposed a definition which satisfies the above properties. But his definition does not allow him to give an effective calculation of the mass.

About 15 years ago, I was interested in how to formulate a criterion for existence of black hole, that Kip Thorne called hoop conjecture.

The statement says that if the quasi-local mass of a closed surface is greater than certain multiple of the diameter of the surface, then the closed surface will collapse to a black hole. (perhaps the length of shortest closed geodesic is a better quantity than diameter)

Hence a good definition of quasi-local mass is needed.

I was visiting Hong Kong at that time and I lectured on related materials.

Luen-Fai Tam told me that he can prove that the total mean curvature of a round sphere that bounds a three dimensional manifold with positive scalar curvature must be smaller than the total mean curvature of the same sphere when it is in Euclidean space

He later generalized the statement with Shi to sphere that is not necessary round.

When I came back to Harvard, Melissa Liu and I generalized this statement of Shi-Tam to spheres that are the boundary of a three dimensional space like hyper surface in a spacetime which satisfies the dominant energy condition.

The total mean curvature is replaced by the total integral of the spacetime length of the mean curvature vector. This is a quantity independent of the choice of the three manifolds that the surface may bound. The difference between this quantity and the corresponding quantity of the isometric embedding of the surface into Euclidean space is positive.

I thought this should be the quasi local mass of the surface .

Then I found out that in the time symmetric case, this was in fact derived to be the quasi local mass by Brown-York and Hawking-Horowitz based on Hamiltonian formulation. (Their definition actually depend on the choice of the three manifold that the surface bounds)

The definition of Liu-Yau is quite good as it satisfies most of the properties mentioned above

However the mass so defined is too positive and may not be trivial for surfaces in the Minkowski spacetime. This is also true for the mass of Brown-York and Hawking-Horowitz.

Hence Mu-Tao Wang and I changed the definition and considered isometric embedding of the two dimensional surface into a Minkowski spacetime.

Such embedding are not unique and we have to optimize the quantities among all embeddings.

To be precise, the Wang-Yau definition of the quasi local mass can be defined in the following way:
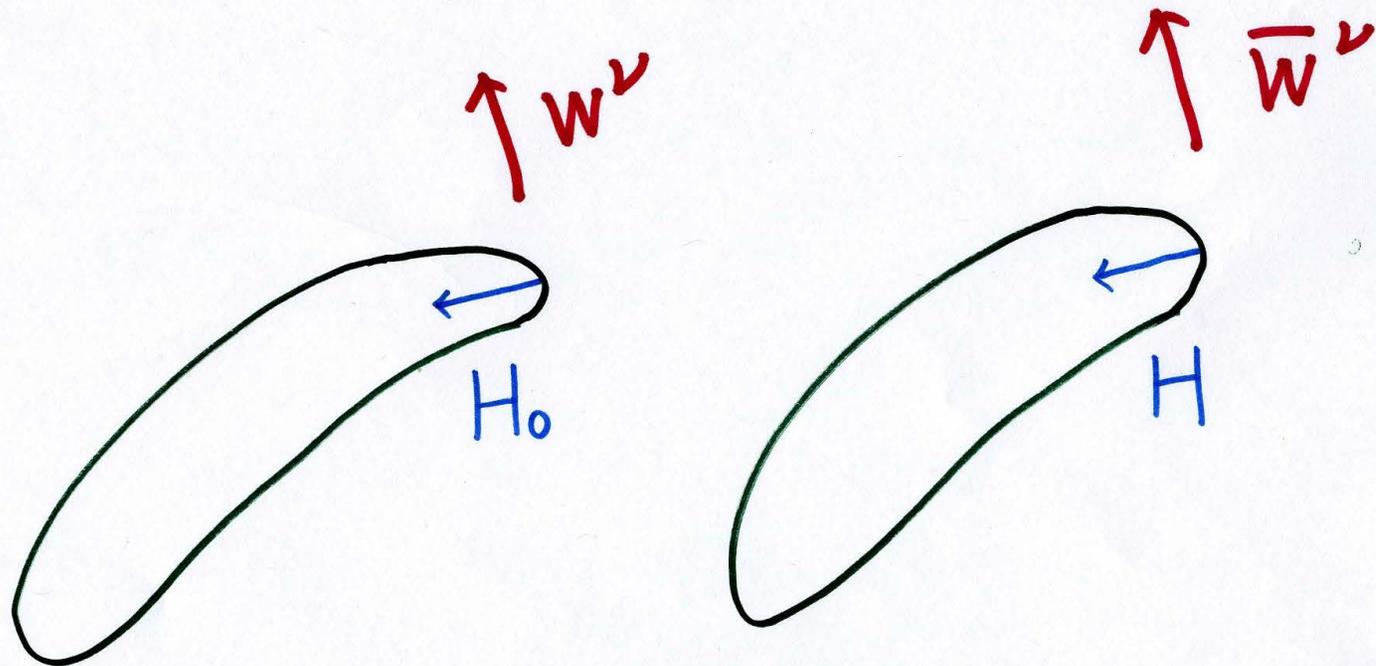
Given a surface S, we assume that its mean curvature vector is spacelike. We embed S isometrically into $\mathbb{R}^{3,1}$.

Given any constant unit future time-like vector w (observer) in $\mathbb{R}^{3,1}$, we can define a future directed time-like vector field $\overline{w}$ along S by requiring

$$\langle H_0, w \rangle = \langle H, \overline{w} \rangle$$

where H_0 is the mean curvature vector of S in $\mathbb{R}^{3,1}$
   and H is the mean curvature vector of S in spacetime.

$$\langle H_0, W \rangle = \langle H, \bar{W} \rangle$$

$$W^\nu = N n^\nu + N^\nu \qquad \bar{W}^\nu = N \bar{n}^\nu + N^\nu$$

Note that given any surface S in $\mathbb{R}^{3,1}$ and a constant future time-like unit vector w^ν, there exists a canonical gauge n^μ (future time-like unit normal along S) such that

$$\int_S N^2 K_0 + N^\mu (p_0)_{\mu\nu} r^\nu$$

is equal to the total mean curvature of $\hat{S}$, the projection of S onto the orthogonal complement of w^μ.

In the expression, we write $w^\mu = N\, n^\mu + N^\mu$ alone the surface S. r^μ is the space-like unit normal orthogonal to n^μ, and p_0 is the second fundamental form calculated by the three surface defined by S and r^μ.

$\uparrow W^\nu$

$\mathbb{R}^{3.1}$

n^ν

$\mathbb{R}^3$
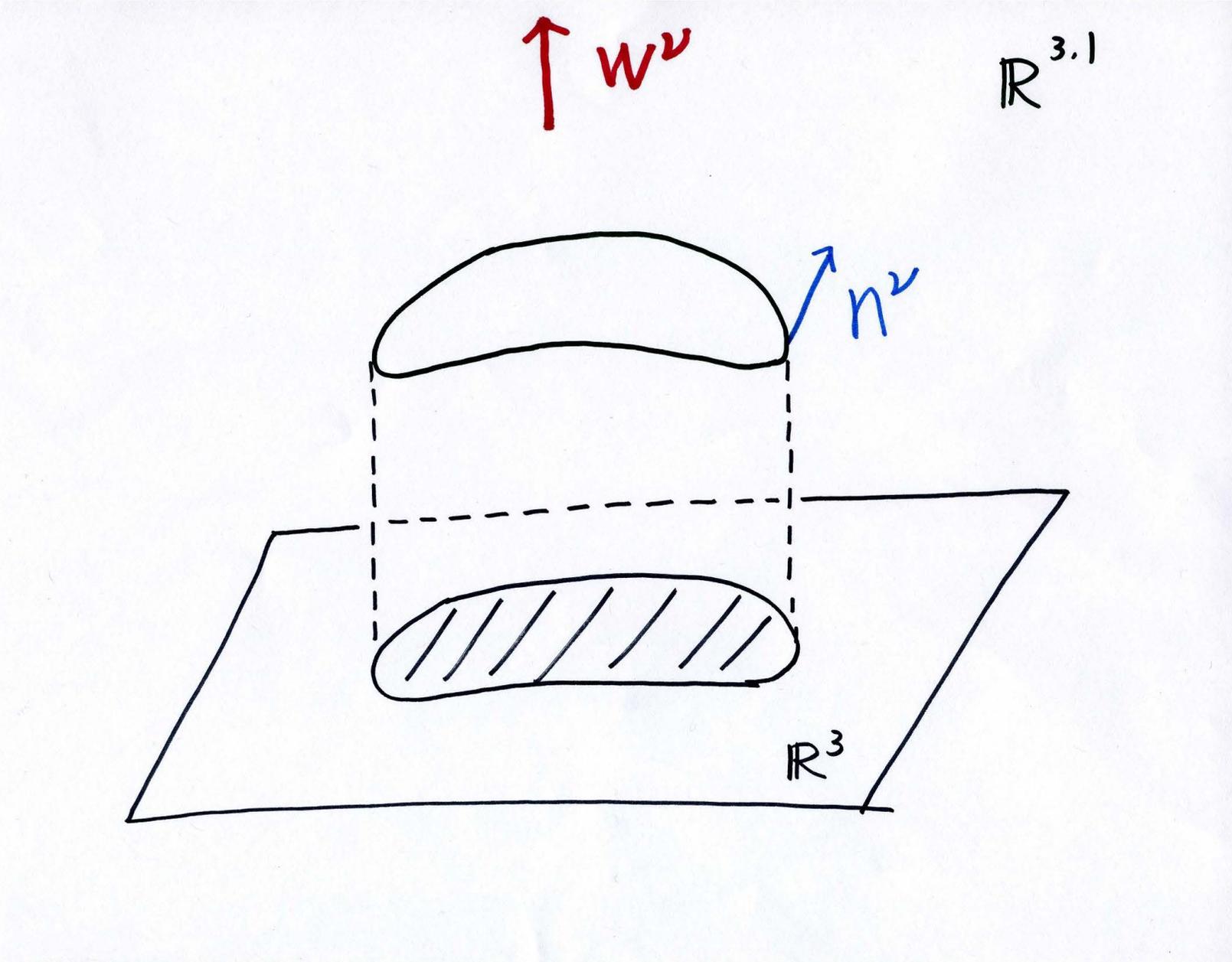
From the matching condition and the correspondence $(w^\mu, n^\mu) \to (\overline{w}^\mu, \overline{n}^\mu)$, we can define a similar quantity from the data in spacetime

$$\int_S N\,^2\overline{K} + N^\mu(\overline{p})_{\mu\nu}\overline{r}^\nu \ .$$

We write $E(w)$ to be

$$8\pi E(w) = \int_S N\,^2\overline{K} + N^\mu(\overline{p})_{\mu\nu}\overline{r}^\nu - \int_S N\,^2 K_0 + N^\mu(p_0)_{\mu\nu}r^\nu$$

and define the quasi-local mass to be

$$\inf E(w)$$

where the infimum is taken among all isometric embeddings into $\mathbb{R}^{3,1}$ and timelike unit constant vector $w \in \mathbb{R}^{3,1}$.

The Euler-Lagrange equation (called the optimal embedding equation) for minimizing $E(w)$ is

$$div_S\left(\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}\cosh\theta|H| - \nabla\theta - V\right)$$

$$- (\widehat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd})\frac{\nabla_b\nabla_a\tau}{\sqrt{1+|\nabla\tau|^2}} = 0$$

where $\sinh\theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}$, V is the tangent vector on Σ that is dual to the connection one-form $\langle\nabla^N_{(\cdot)}\frac{J}{|H|}, \frac{H}{|H|}\rangle$ and $\hat{\sigma}$, $\widehat{H}$ and $\hat{h}$ are the induced metric, mean curvature and second fundamental form of $\widehat{S}$ in $\mathbb{R}^3$.

In general, the above equation should have an unique solution τ. We prove that $E(w)$ is non-negative among admissible isometric embedding into Minkowski space.

In summary, given a closed space-like 2-surface in spacetime whose mean curvature vector is space-like, we associate an energy-momentum four-vector to it that depends only on the first fundamental form, the mean curvature vector and the connection of the normal bundle with the properties

1. It is Lorentzian invariant;

2. It is trivial for surfaces sitting in Minkowski spacetime and future time-like for surfaces in spacetime which satisfies the local energy condition.

Our quasi-local mass also satisfies the following important properties:

3. When we consider a sequence of spheres on an asymptotically flat space-like hypersurface, in the limit, the quasi-local mass (energy-momentum) is the same as the well-understood ADM mass (energy-momentum);

4. When we take the limit along a null cone, we obtain the Bondi mass(energy-momentum).

5. When we take the limit approaching a point along null geodesics, we recover the energy-momentum tensor of matter density when matter is present, and the Bel-Robinson tensor in vacuum.

These properties of the quasi-local mass is likely to characterize the definition of quasi-local mass, i.e. any quasi-local mass that satisfies all the above five properties may be equivalent to the one that we have defined.

Strictly speaking, we associate each closed surface not a four-vector, but a function defined on the light cone of the Minkowski spacetime. Note that if this function is linear, the function can be identified as a four-vector.

It is a remarkable fact that for the sequence of spheres converging to spatial infinity, this function becomes linear, and the four-vector is defined and is the ADM four-vector that is commonly used in asymptotically flat spacetime. For a sequence of spheres converging to null infinity in Bondi coordinate, the four vector is the Bondi-Sachs four-vector.

It is a delicate problem to compute the limit of our quasi-local mass at null infinity and spatial infinity. The main difficulties are the following:

(i) The function associated to a closed surface is non-linear in general;

(ii) One has to solve the Euler-Lagrange equation for energy minimization.

For (i), the following observation is useful:

For a family of surfaces Σ_r and a family of isometric embeddings X_r of Σ_r into $\mathbb{R}^{3,1}$, the limit of quasi-local mass is a linear function under the following general assumption that the mean curvature vectors are comparable in the sense

$$\lim_{r \to \infty} \frac{|H_0|}{|H|} = 1$$

where H is the the spacelike mean curvature vector of Σ_r in N and H_0 is that in the image of X_r in $\mathbb{R}^{3,1}$.

Under the comparable assumption of mean curvature, the limit of our quasi-local mass with respect to a constant future time-like vector $T_0 \in \mathbb{R}^{3,1}$ is given by

$$\lim_{r \to \infty} \frac{1}{8\pi} \int_{\Sigma_r} \left[-\langle T_0, \frac{J_0}{|H_0|} \rangle (|H_0| - |H|) \right.$$
$$\left. - \langle \nabla^{\mathbb{R}^{3,1}}_{\nabla \tau} \frac{J_0}{|H_0|}, \frac{H_0}{|H_0|} \rangle + \langle \nabla^{N}_{\nabla \tau} \frac{J}{|H|}, \frac{H}{|H|} \rangle \right] d\Sigma_r$$

where $\tau = -\langle T_0, X_r \rangle$ is the time function with respect to T_0.

This expression is linear in T_0 and defines an energy-momentum four-vector at infinity.

At the spatial infinity of an asymptotically flat spacetime, the limit of our quasi-local mass is

$$\lim_{r \to \infty} \frac{1}{8\pi} \int_{\Sigma_r} \left(|H_0| - |H| \right) d\Sigma_r = M_{ADM}$$

$$\lim_{r \to \infty} \frac{1}{8\pi} \int_{\Sigma_r} \langle \nabla^N_{-\nabla X_i} \frac{J}{|H|}, \frac{H}{|H|} \rangle d\Sigma_r = P_i$$

where $\begin{pmatrix} M \\ P_i \end{pmatrix}$ is the ADM energy-momentum four-vector, assuming the embeddings X_r into $\mathbb{R}^3$ inside $\mathbb{R}^{3,1}$.

At the null infinity, the limit of quasi-local mass was found by Chen-Wang-Yau to recover the Bondi-Sachs energy-momentum four-vector.

On a null cone $w = c$ as r goes to infinity, the limit of the quasi-local mass is

$$\lim_{r \to \infty} \frac{1}{8\pi} \int_{\Sigma_r} (|H_0| - |H|) d\Sigma_r = \frac{1}{8\pi} \int_{S^2} 2m \, dS^2$$

$$\lim_{r \to \infty} \frac{1}{8\pi} \int_{\Sigma_r} \langle \nabla^N_{-\nabla X_i} \frac{J}{|H|}, \frac{H}{|H|} \rangle d\Sigma_r = \frac{1}{8\pi} \int_{S^2} 2m X_i \, dS^2$$

where $(X_1, X_2, X_3) = (\sin\theta \sin\phi, \sin\theta \cos\phi, \cos\theta)$.

The following two properties are important for solving the Euler-Lagrange equation for energy minimization:

(a) The limit of quasi-local mass is stable under $O(1)$ perturbation of the embedding;

(b) The four-vector obtained is equivariant with respect to Lorentzian transformations acting on X_r.

We observe that momentum is an obstruction to solving the Euler-Lagrange equation near a boosted totally geodesics slice in $\mathbb{R}^{3,1}$. Using (b), we find a solution by boosting the isometric embedding according to the energy-momentum at infinity. Then the limit of quasi-local mass is computed using (a) and (b).

In evaluating the small sphere limit of the quasilocal energy, we pick a point p in spacetime and consider C_p the future light cone generated by future null geodesics from p. For any future directed timelike vector e_0 at p, we define the affine parameter r along C_p with respect to e_0. Let S_r be the level set of the affine parameter r on C_p.

We solve the optimal isometric equation and find a family of isometric embedding X_r of S_r which locally minimizes the quasi-local energy.

With respect to X_r, the quasilocal energy is again linearized and is equal to

$$\frac{4\pi}{3} r^3 T(e_0, \cdot) + O(r^4)$$

which is the expected limit.

In the vacuum case, i.e. $T = 0$, the limit is non-linear with the linear term equal to

$$\frac{1}{90} r^5 Q(e_0, e_0, e_0, \cdot) + O(r^6)$$

with an additional positive correction term in quadratic expression of the Weyl curvature.

The linear part consists of the Bel-Robinson tensor and is precisely the small-sphere limit of the Hawking mass which was computed by Horowitz and Schmidt.

The Bel-Robinson tensor satisfies conservation law and is an important tool in studying the dynamics of Einstein's equation, such as the stability of the Minkowski space (Christodoulou-Klainerman) and the formation of trapped surface in vacuum (Christodoulou)

Po-Ning Chen joined in the research about four years ago and we can now defined quasilocal angular momentum and center of gravity

We define quasi-local conserved quantities in general relativity by using the optimal isometric embedding to transplant Killing fields in the Minkowski spacetime back to the 2-surface a physical spacetime.

To each optimal isometric embedding, a dual element of the Lie algebra of the Lorentz group is assigned. Quasi-local angular momentum and quasi-local center of mass correspond to pairing this element with rotation Killing fields and boost Killing fields, respectively.

Consider the following quasi-local energy density ρ

$$\rho = \frac{\sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}}{\sqrt{1+|\nabla\tau|^2}}$$

and momentum density j

$$j = \rho\nabla\tau - \nabla[\sinh^{-1}(\frac{\rho\Delta\tau}{|H_0||H|})] - \alpha_{H_0} + \alpha_H.$$

The optimal embedding equation takes a simple form:

$$div(j) = 0.$$

The quasi-local conserved quantity of Σ with respect to an optimal isometric embedding (X, T_0) and a Killing field K is

$$E(\Sigma, X, T_0, K) = \frac{(-1)}{8\pi} \int_\Sigma \left[\langle K, T_0 \rangle \rho + j(K^\top) \right] d\Sigma.$$

Suppose $T_0 = A(\frac{\partial}{\partial X^0})$ for a Lorentz transformation A.

The quasi-local conserved quantities corresponding to $A(X^i \frac{\partial}{\partial X^j} - X^j \frac{\partial}{\partial X^i})$ are called the quasi-local angular momentum and the ones corresponding to $A(X^i \frac{\partial}{\partial X^0} + X^0 \frac{\partial}{\partial X^i})$ are called the quasi-local center of mass integrals.

The quasi-local angular momentum and center of mass satisfy the following important properties

1 The quasi-local angular momentum and center of mass vanish for any surfaces in the Minkowski space

2 They obey classical transformation laws under the action of the Poincaré group.

3 For axially symmetric spacetime, the quasi-local angular momentum reduces to the Komar definition.

We further justify these definitions by considering their limits as the total angular momentum J^i and the total center of mass C^i of an isolated system. They satisfies the following important properties:

1. All total conserved quantities vanish on any spacelike hypersurface in the Minkowski spacetime, regardless of the asymptotic behavior.

2. The new total angular momentum and total center of mass are always finite on any vacuum asymptotically flat initial data set of order one.

3. Under the vacuum Einstein evolution of initial data sets, the total angular momentum is conserved and the total center of mass obeys the dynamical formula $\partial_t C^i(t) = \frac{p^i}{p^0}$ where p^ν is the ADM energy-momentum four vector.

We recently compute the quasi-local mass of "spheres of unit size" at null infinity to capture the information of gravitational radiation.

The set-up (following Chandrasekhar) is a gravitational perturbation of the Schwarzschild spacetime which is governed by the Regge-Wheeler equation.

We take a sphere of a fixed areal radius and push it all the way to null infinity. The limit of the geometric data is still that of a standard configuration and thus the optimal embedding equation can be solved in a similar manner.

Let me discuss the result of axial perturbation in more detail.

We consider a metric perturbation of the form

$$-(1 - \frac{2m}{r})dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta (d\phi - q_2 dr - q_3 d\theta)^2$$

The linearized vacuum Einstein equation is solved by a separation of variable Ansatz in which q_2 and q_3 are explicitly given by the Teukolsky function and the Legendre function.
In particular,

$$q_3 = \sin(\sigma t) \frac{C(\theta)}{\sin^3 \theta} \frac{(r^2 - 2mr)}{\sigma^2 r^4} \frac{d}{dr}(rZ^{(-)})$$

for a solution of frequency σ.

After the change of variable $r_* = r + 2m \ln(\frac{r}{2m} - 1)$, $Z^{(-)}$ satisfies the Regge-Wheeler equation:

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right)Z^{(-)} = V^{(-)}Z^{(-)},$$

where

$$V^{(-)} = \frac{r^2 - 2mr}{r^5}[(\mu^2 + 2)r - 6m],$$

and μ is the separation of variable constant.

On the Schwarzschild spacetime

$$-(1 - \frac{2m}{r})dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$
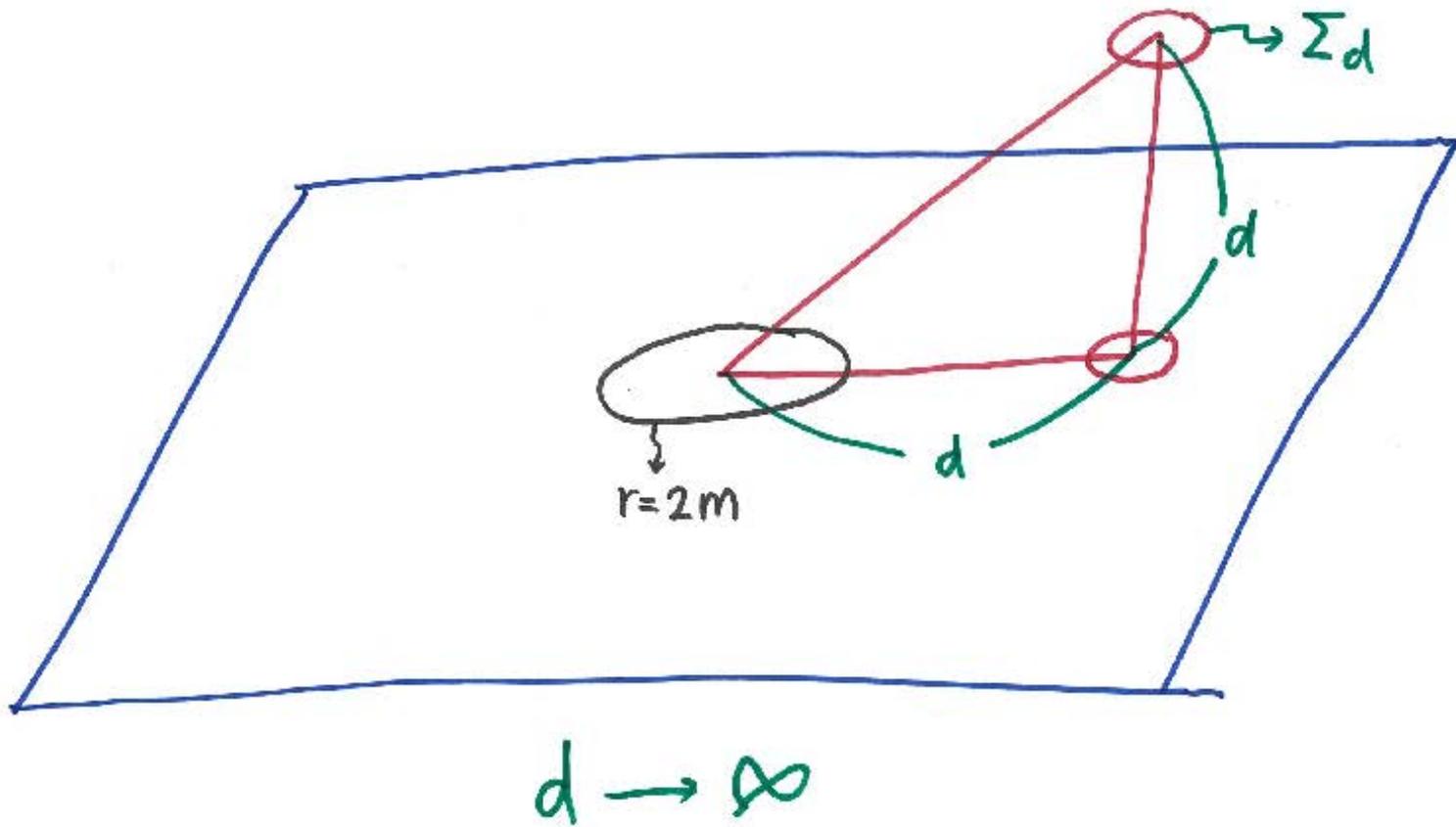
we consider an asymptotically flat Cartesian coordinate system (t, y_1, y_2, y_3) with $y_1 = r\sin\theta\sin\phi$, $y_2 = r\sin\theta\cos\phi$, $y_3 = r\cos\theta$. Given $(d_1, d_2, d_3) \in \mathbb{R}^3$ with $d^2 = \sum_{i=1}^{3} d_i^2$, consider the 2-surface

$$\Sigma_d = \{(d, y_1, y_2, y_3) : \sum_{i=1}^{3}(y_i - d_i)^2 = 1\}.$$

We compute the quasi-local mass of Σ_d as $d \to \infty$.

Σ_d

d

d

$r = 2m$

$d \longrightarrow \infty$

Denote

$$A(r) = \frac{(r^2 - 2mr)}{\sigma^2 r^3} \frac{d}{dr}(rZ^{(-)}),$$

the linearized optimal embedding equation of Σ_d is reduced to two linear elliptic equations on the unit 2-sphere S^2:

$$\Delta(\Delta + 2)\tau = [-A''(1 - Z_1^2) + 6A'Z_1 + 12A]Z_2 Z_3$$
$$(\Delta + 2)N = (A'' - 2A'Z_1 + 4A)Z_2 Z_3$$

where τ and N are the respective time and radial components of the solution, and Z_1, Z_2, Z_3 are the three standard first eigenfunctions of S^2. A' and A'' are derivatives with respect to r and r^2 is substituted by $r^2 = d^2 + 2Z_1 + 1$ in the above equations.

The quasi-local mass of Σ_d with respect to the above optimal isometric embedding is then

$$\frac{1}{d^2} \frac{C^2(\theta)}{\sin^6 \theta} \{\sin^2(\sigma d) E_1 + \sigma^2 \cos^2(\sigma d) E_2\} + O(\frac{1}{d^3})$$

where

$$E_1 = \int_{S^2} (1/2) \left[A^2 Z_2^2 (7 Z_3^2 + 1) + 2 A A' Z_1 Z_3^2 (3 Z_2^2 - 1) - N(\Delta + 2) N \right]$$

$$E_2 = \int_{S^2} \left[A^2 Z_2^2 Z_3^2 - \tau \Delta (\Delta + 2) \tau \right]$$

In fact, the quasi-local mass density ρ of Σ_d can be computed at the pointwise level. Up to an $O(\frac{1}{d^3})$ term

$$\rho = (K - \frac{1}{4}|H|^2)$$

$$- \frac{(|H| - 2)^2}{4} + \frac{1}{d^2}\{\frac{1}{2}|\nabla^2 N|^2 + ((\Delta + 2)N)^2 - \frac{1}{4}(\Delta N)^2$$

$$- \frac{1}{4}(\Delta \tau)^2 + \frac{1}{2}[\nabla^a \nabla^b (\tau_a \tau_b) - |\nabla \tau|^2 - \Delta|\nabla \tau|^2]\}$$

where K is the Gauss curvature of Σ_d.

The first line, which integrates to zero, is of the order of $\frac{1}{d}$ and is exactly the mass aspect function of the Hawking mass. The $\frac{1}{d^2}$ term of the quasi local mass $\int_{\Sigma_d} \rho \, d\mu_{\Sigma_d}$ has contributions from the second and third lines (of the order of $\frac{1}{d^2}$), the $\frac{1}{d^2}$ term of the first line, and the $\frac{1}{d}$ term of the area element $d\mu_{\Sigma_d}$. The above integral formula is obtained after performing integrations by parts and applying the optimal embedding equation several times. Generalizations to gravitational perturbations of the Kerr spacetime are in progress.