Title: Black Holes, Thermodynamics, and Lagrangians

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Abstract:

Black Holes, Thermodynamics, and Lagrangians

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Lagrangians

If you had asked me 25 years ago, I would have said that Lagrangians in classical field theory were mainly useful as nmemonic devices for remembering field equations: It is easier to remember a Lagrangian (e.g., $L = \sqrt{-g}R$) than to remember the field equations (e.g., $G_{ab} = 0$). Thus, it was my view that for applications to classical physics, you can dispense with the Lagrangian if you've already memorized the field equations.

Some people seem to believe that Lagrangians are somehow necessary and/or sufficient to have well posed field equations. This is not true.

However, I was wrong about Lagrangians being useful classically only for mnemonic purposes. Lagrangians provide classical field theories—particularly those with local symmetries—with vitally important auxiliary structure. This additional structure is crucial to understanding the thermodynamic properties of classical black holes.

Diffeomorphism Covariant Lagrangians

Consider a diffeomorphism covariant Lagrangian $L(g, \psi)$ for a metric g_{ab} and arbitrary matter fields $\psi^{a_1 \cdots}{}_{b_1 \ldots}$ in n spacetime dimensions. Diffeomorphism covariance means if $\delta g = \pounds_X g$ and $\delta\psi = \pounds_X \psi$, then $\delta L = \pounds_X L$. It implies that L can be written as a function of g, the Riemann curvature of g, covariant derivatives of the Riemann curvature, and ψ and its covariant derivatives. Let $\phi \equiv (g, \psi)$ denote the collection of dynamical variables.

i) View the Lagrangian as an n-form rather than a scalar density. ("d" commutes with pullbacks and it is generally much more straightforward to apply Stokes' theorem than Gauss' law.)

Example: Einstein-Hilbert Lagrangian for vacuum general relativity:

$$L_{a_1 \dots a_n} = \frac{1}{16\pi} R \, \epsilon_{a_1 \dots a_n}$$

ii) Don't put the Lagrangian under an integral sign so as to pretend that you are calculating an action. (The action integral over spacetime generally doesn't converge anyway. People usually put in a boundary and spend time and effort computing the boundary term. It is much easier to work with the local Lagrangian.)

First variation:

$$\delta L = E \cdot \delta\phi + d\theta(\phi, \delta\phi)$$

People normally throw away the "boundary term" θ (or choose a boundary term in the action so as to cancel θ) and keep the Euler-Lagrange equations of motion $E = 0$. For our purposes, the more useful thing to do is: iii) Throw away the equations of motion and keep the boundary term θ.

Example: For the Einstein-Hilbert Lagrangian, we obtain

$$\theta_{a_1 \ldots a_{n-1}} = \frac{1}{16\pi} g^{ac} g^{bd} (\nabla_d \delta g_{bc} - \nabla_c \delta g_{bd}) \epsilon_{c a_1 \ldots a_{n-1}}$$

Symplectic current ($(n-1)$-form):

$$\omega(\phi; \delta_1 \phi, \delta_2 \phi) = \delta_1 \theta(\phi; \delta_2 \phi) - \delta_2 \theta(\phi; \delta_1 \phi) \, .$$

Then $d\omega = 0$ if $\delta_1 \phi$ and $\delta_2 \phi$ satisfy the linearized equations of motion, $\delta_1 E = \delta_2 E = 0$. The symplectic form is obtained by integrating ω over a Cauchy surface Σ:

$$W_\Sigma(\phi; \delta_1\phi, \delta_2\phi) \equiv \int_\Sigma \omega(\phi; \delta_1\phi, \delta_2\phi)$$

Example: For the Einstein-Hilbert Lagrangian, we obtain

$$W_\Sigma(g; \delta_1 g, \delta_2 g) = -\frac{1}{32\pi}\int_\Sigma (\delta_1 h_{ab}\delta_2 p^{ab} - \delta_2 h_{ab}\delta_1 p^{ab}),$$

with

$$p^{ab} \equiv h^{1/2}(K^{ab} - h^{ab}K).$$

Hamiltonians and Hamilton's Equations

Let X be a vector field on spacetime defining a notion of "time translations." If you know the Hamiltonian H_X, you have broken up the phase space variables into "p" and "q" and you are trying to write the equations of motion $E = 0$ as time evolution equations, then it is quite useful to use the usual form of Hamilton's of motion:

$$\pounds_X q = \delta H_X / \delta p \,, \qquad \pounds_X p = -\delta H_X / \delta q$$

But if you know the equations of motion (e.g., from a Lagrangian), you haven't broken up phase space into "p" and "q," variables, and you are trying to figure out if a Hamiltonian, H_X, exists and, if so, what it is, then:

iv) A much more useful form of Hamilton's equations of motion is:

$$\delta H_X = W_\Sigma(\phi; \delta\phi, \pounds_X \phi)$$

for all $\delta\phi$ if and only if ϕ satisfies $E = 0$.

Noether Current and Charge

Diffeomorphism covariance of L means that for any vector field X, the field variation $\delta\phi = \pounds_X \phi$ produces the Lagrangian variation $\delta L = \pounds_X L(\phi)$. This implies that the Noether current

$$\mathcal{J}_X \equiv \theta(g, \pounds_X g) - X \cdot L$$

is conserved, $d\mathcal{J}_X = 0$, whenever $E = 0$. Since this holds for all X, this further implies that $\mathcal{J}_X$ takes the form

$$\mathcal{J}_X = X \cdot C + dQ_X \, .$$

where $C = 0$ whenever the equations of motion hold (and C satisfies a Bianchi identity, so the equations

$C = 0$ are naturally viewed as "constraints." Q_X is called the Noether charge. For the Einstein-Hilbert Lagrangian, Q_X takes the form

$$(Q_X)_{a_1 \ldots a_{n-2}} = -\frac{1}{16\pi} \nabla_b X_c \epsilon^{bc}{}_{a_1 \ldots a_{n-2}}$$

Fundamental Variational Identity

Taking the first variation of the two formulas for $\mathcal{J}_X$, we obtain

$$\begin{aligned}\omega(\phi; \delta\phi, \pounds_X \phi) &= X \cdot [E(g) \cdot \delta\phi] + X \cdot \delta C \\ &\quad + d\left[\delta Q_X(\phi) - X \cdot \theta(\phi; \delta\phi)\right]\end{aligned}$$

Comparing with Hamilton's equations of motion, we obtain

$$\delta H_X = \int_\Sigma X \cdot \delta C + \int_{\partial\Sigma} [\delta Q_X(\phi) - X \cdot \theta(\phi; \delta\phi)]$$

Thus, the Hamiltonian of a diffeomorphism covariant theory always takes the the form of a volume integral of a "pure constraint" term plus a surface term. In the asymptotically flat case, the ADM conserved quantities are defined as the boundary term from infinity:

$$\delta H_X = \int_\infty [\delta Q_X(\phi) - X \cdot \theta(\phi; \delta\phi)]$$

The First Law of Black Hole Mechanics

For a stationary black hole, choose X to be the horizon Killing field

$$K^a = t^a + \sum \Omega_i \phi_i^a$$

Integration of the fundamental identity yields:

$$0 = \delta M - \sum_i \Omega_i \delta J_i - \int_B \delta Q_K \, .$$

Can show further that

$$\delta Q_K = \frac{\kappa}{2\pi} \delta S$$

where

$$S = -2\pi \int_B E_R^{abcd} n_{ab} n_{cd}$$

with $E_R^{abcd} \equiv \delta L / \delta R_{abcd}$. One may therefore identify S as the entropy of the black hole.

Canonical Energy

Define the canonical energy of a perturbation $\delta\phi$ of a stationary black hole by

$$\mathcal{E}_K \equiv W_\Sigma \left(\phi; \delta\phi, \pounds_K \delta\phi\right)$$

[Need to impose gauge conditions at B to make $\mathcal{E}_K$ gauge invariant.] If $\mathcal{E}_K$ has a positive flux at infinity and through the black hole horizon (as is the case for axisymmetric perturbations in general relativity), then positivity of $\mathcal{E}_K$ for perturbations with $\delta M = \delta J_i = 0$ is necessary and sufficient for linear dynamic stability. But the second variation of our fundamental identity yields

$$\mathcal{E}_K = \delta^2 M - \sum_i \Omega_i \delta^2 J_i - \frac{\kappa}{2\pi}\delta^2 S\,.$$

which shows that positivity of $\mathcal{E}_K$ is necessary and sufficient for local thermodynamic stability (maximum of S at fixed M, J_i). Thus, for black holes in general relativity, dynamic and thermodynamic stability are equivalent.

Conclusion

It is well known that there is a deep connection between black holes and thermodynamics. The Lagrangian structure of classical general relativity plays a deep role in this relationship.