Title: What discrete states have a continuum limit?

Date: May 14, 2015 11:50 AM

URL: http://pirsa.org/15050075

Abstract: Renormalization to low energies is widely used in condensed matter theory to reveal the low energy degrees of freedom of a system, or in high energy physics to cure divergence problems. Here we ask which states can be seen as the result of such a renormalization procedure, that is, which states can â€œrenormalized to high energies". Intuitively, the continuum limit is the limit of this "renormalization" procedure. We consider three definitions of continuum limit and characterise which states satisfy either one in the context of Matrix Product States.

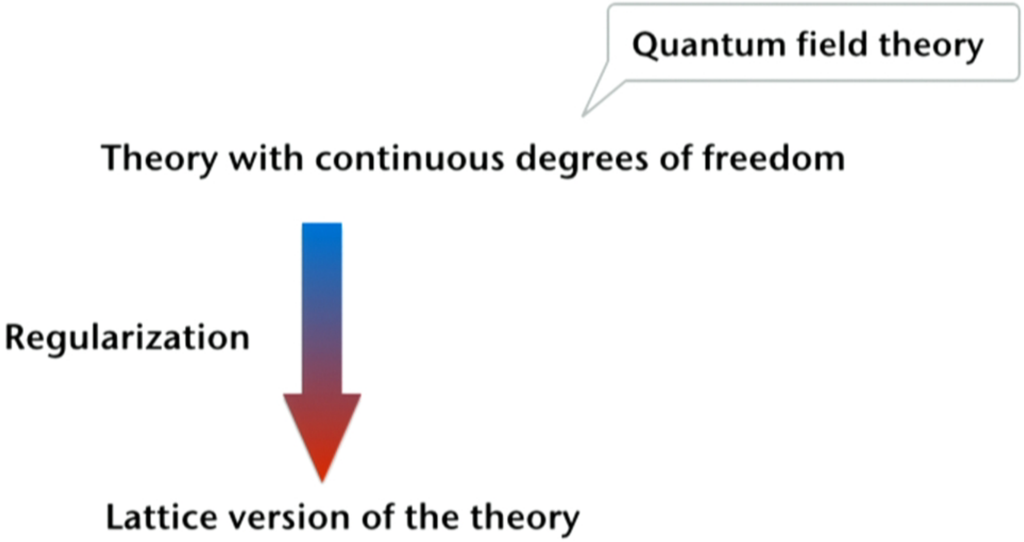
Joint work with N. Schuch, D. Perez-Garcia and I. Cirac.

Which discrete states have a continuum limit?

Gemma De las Cuevas (MPQ Garching, Germany)
Norbert Schuch (Aachen, Germany)
David Pérez-García (Madrid, Spain)
J. Ignacio Cirac (MPQ Garching, Germany)

work in progress

Information theoretic foundations for Physics, PI, May 14, 2015

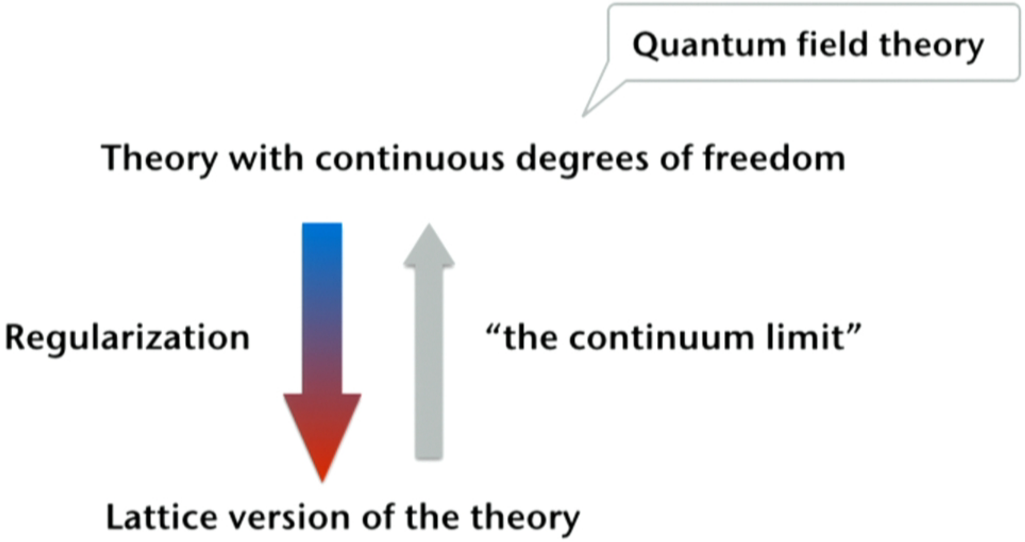


Quantum field theory

Theory with continuous degrees of freedom

Regularization

Lattice version of the theory



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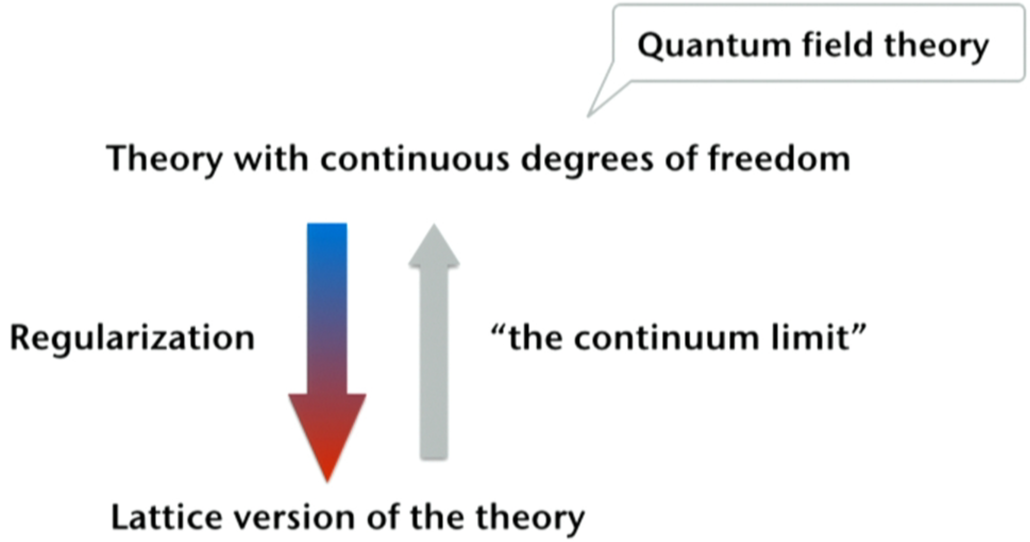

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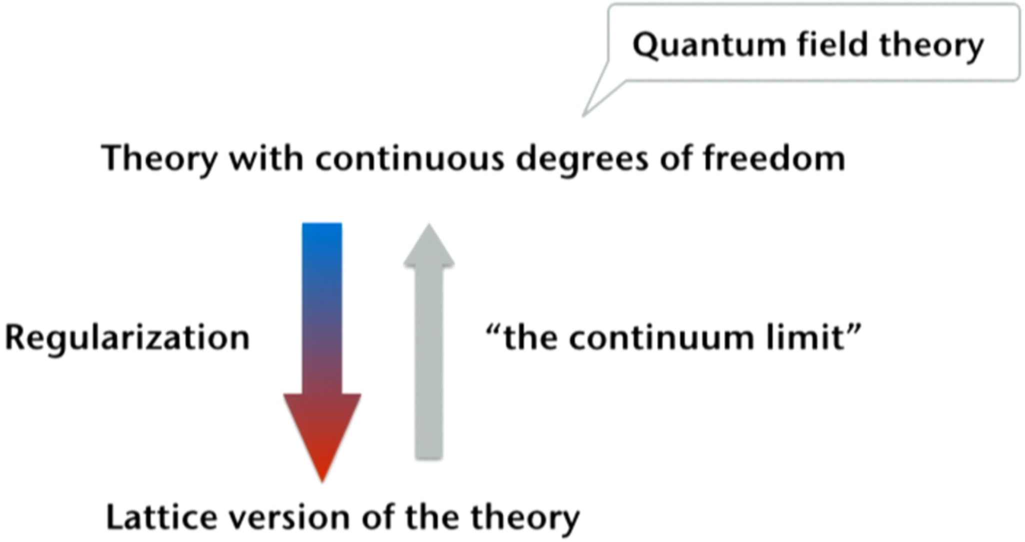

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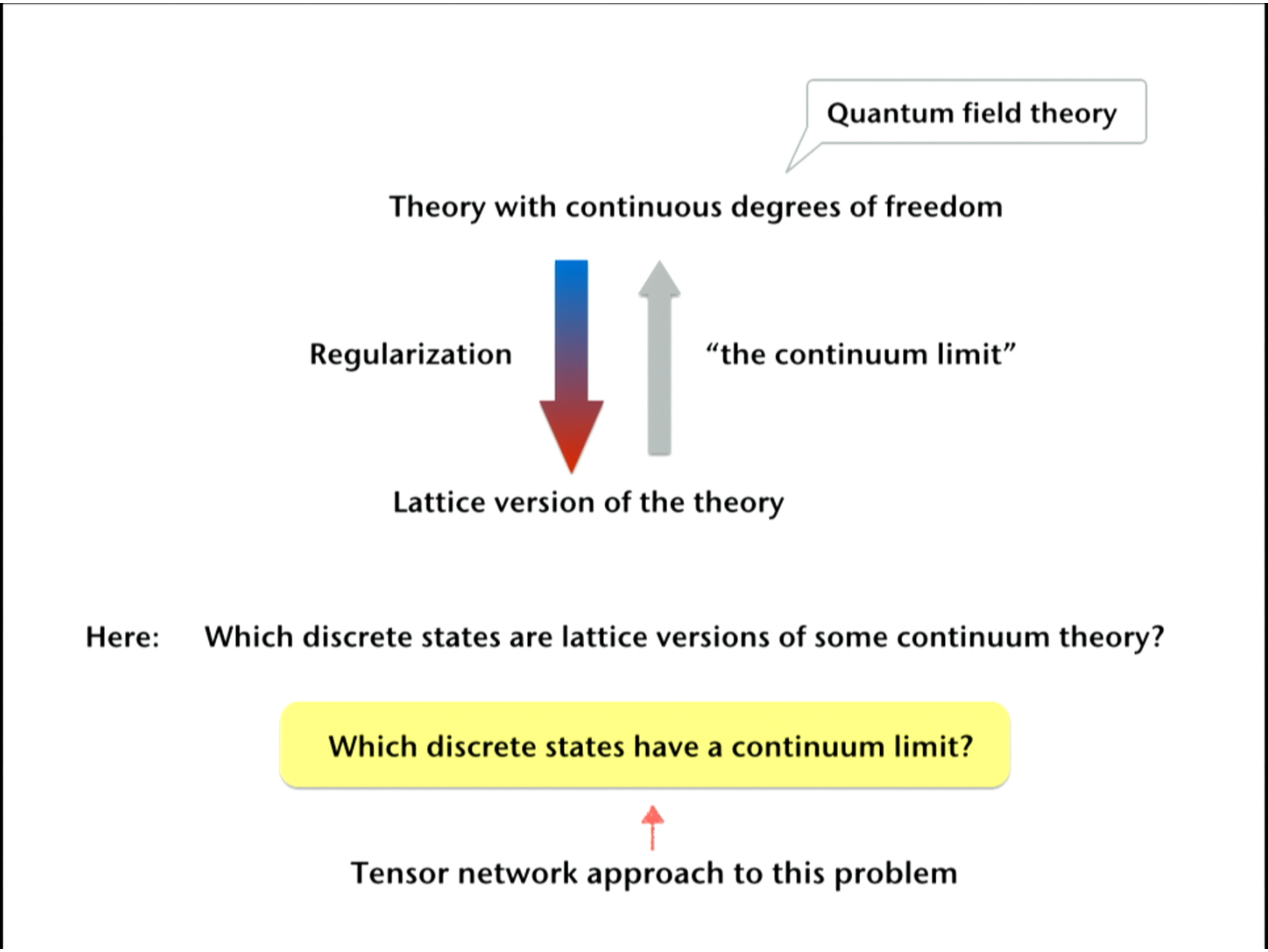

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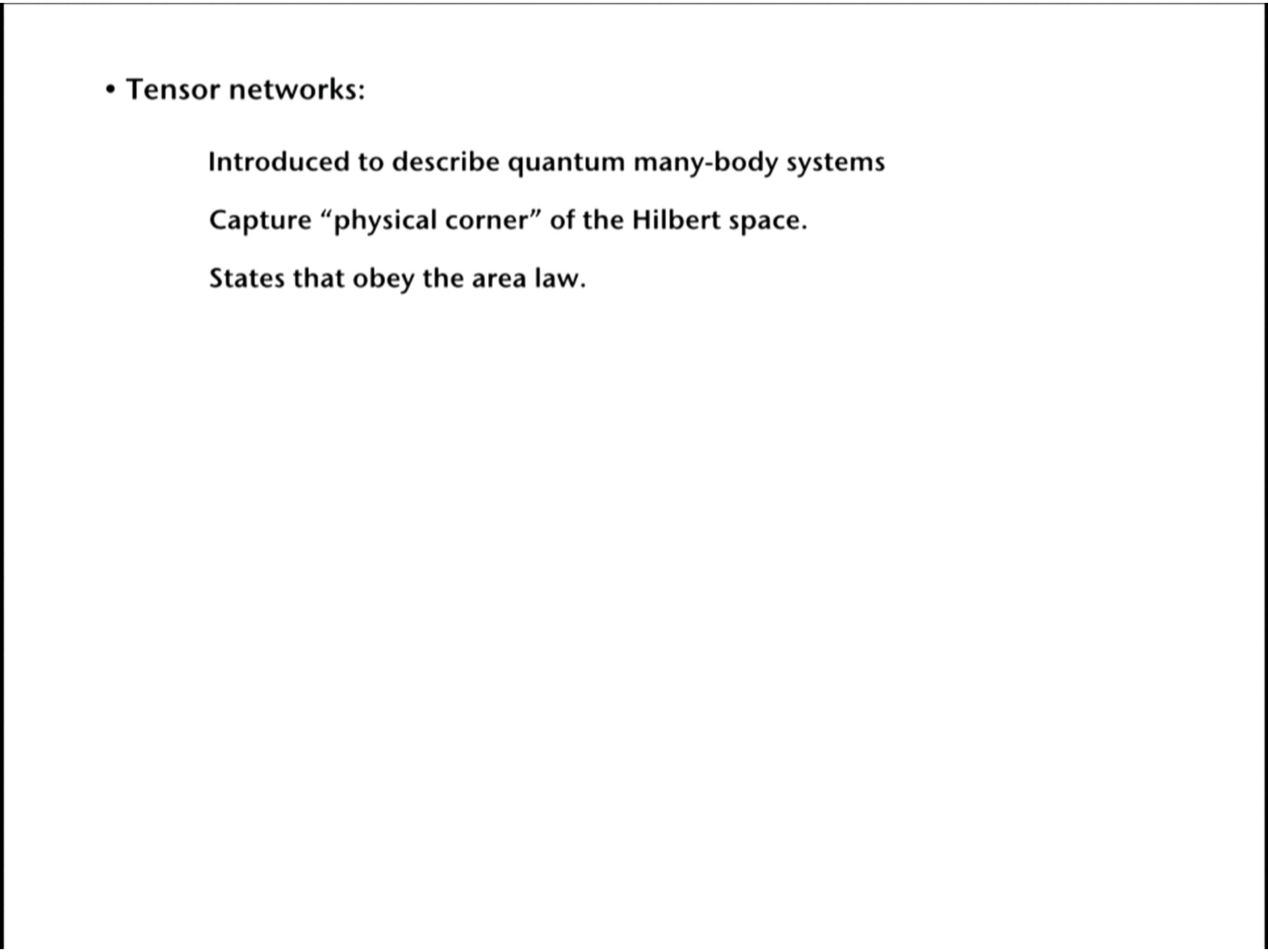
Tensor network approach to this problem



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  Introduced to describe quantum many-body systems

  Capture “physical corner” of the Hilbert space.

  States that obey the area law.

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This work In 1D, translational invariant, particular RG transformations

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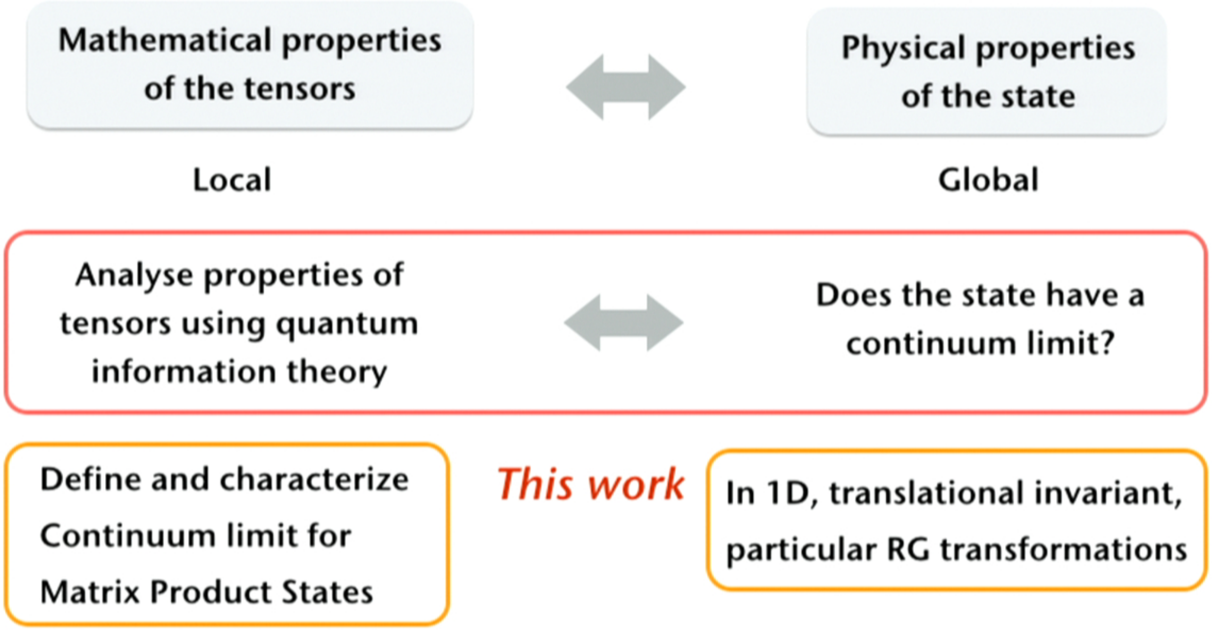

Mathematical properties of the tensors ⟷ Physical properties of the state

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This work

Define and characterize Continuum limit for Matrix Product States

In 1D, translational invariant, particular RG transformations

Outline

- The setting

    RG transformations for MPS

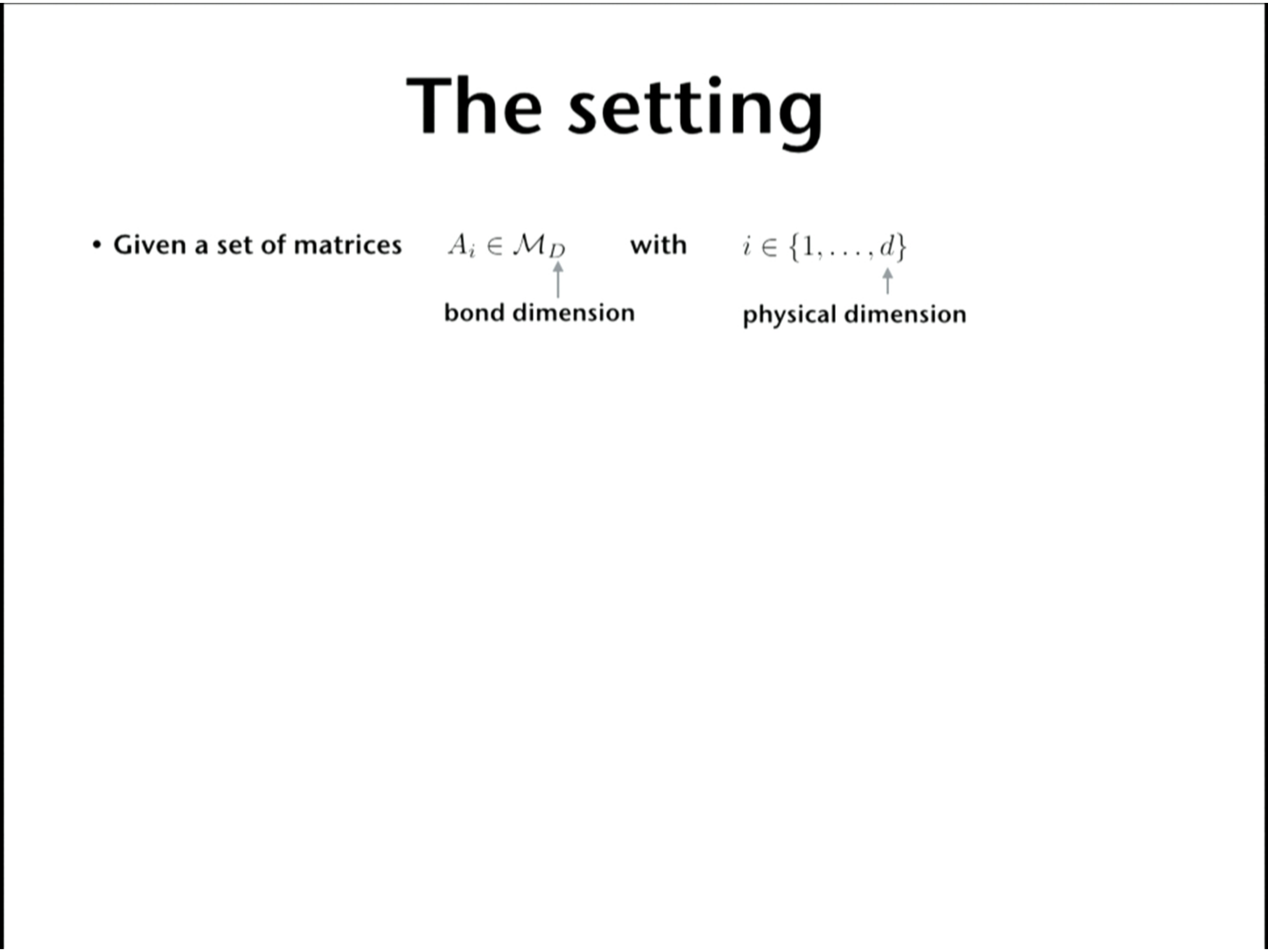
    RG transformations in terms of Quantum channels

- Continuum limit 1
- Continuum limit 2
- Conclusions & Outlook

The setting

We focus on the case

- In 1 spatial dimension
- Translational invariant
- with Periodic Boundary Conditions



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$$|\psi_N\rangle = c_N \sum_{i_1 \ldots i_N = 1}^{d} \mathrm{Tr}(A_{i_1} A_{i_2} \ldots A_{i_N}) |i_1 \ldots i_N\rangle$$

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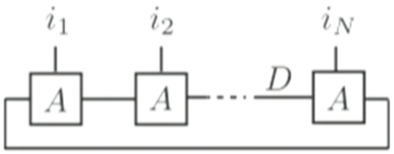

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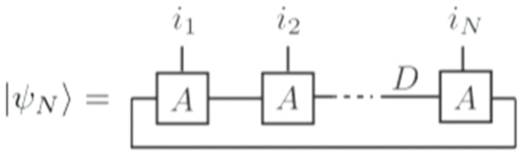

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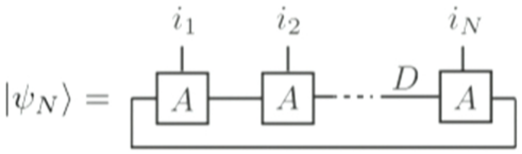

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Matrix Product State (MPS)

The setting

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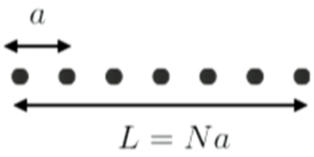
describes a spin chain



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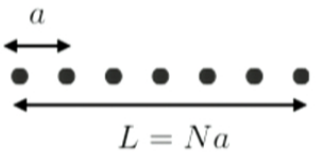

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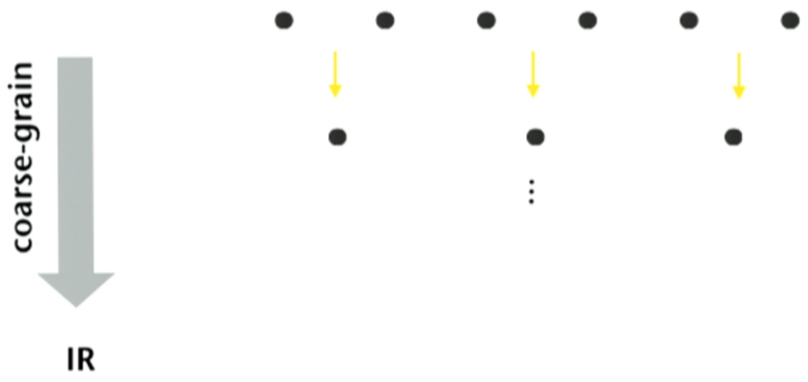
fixed lattice spacing *a*

- **Usually the continuum limit means** $$\left(\begin{array}{l} L \quad \text{fixed} \\ a \to 0 \\ N \to \infty \end{array} \right. \qquad L = Na$$

Here it will mean $a \to 0$ **but N is not fixed.**

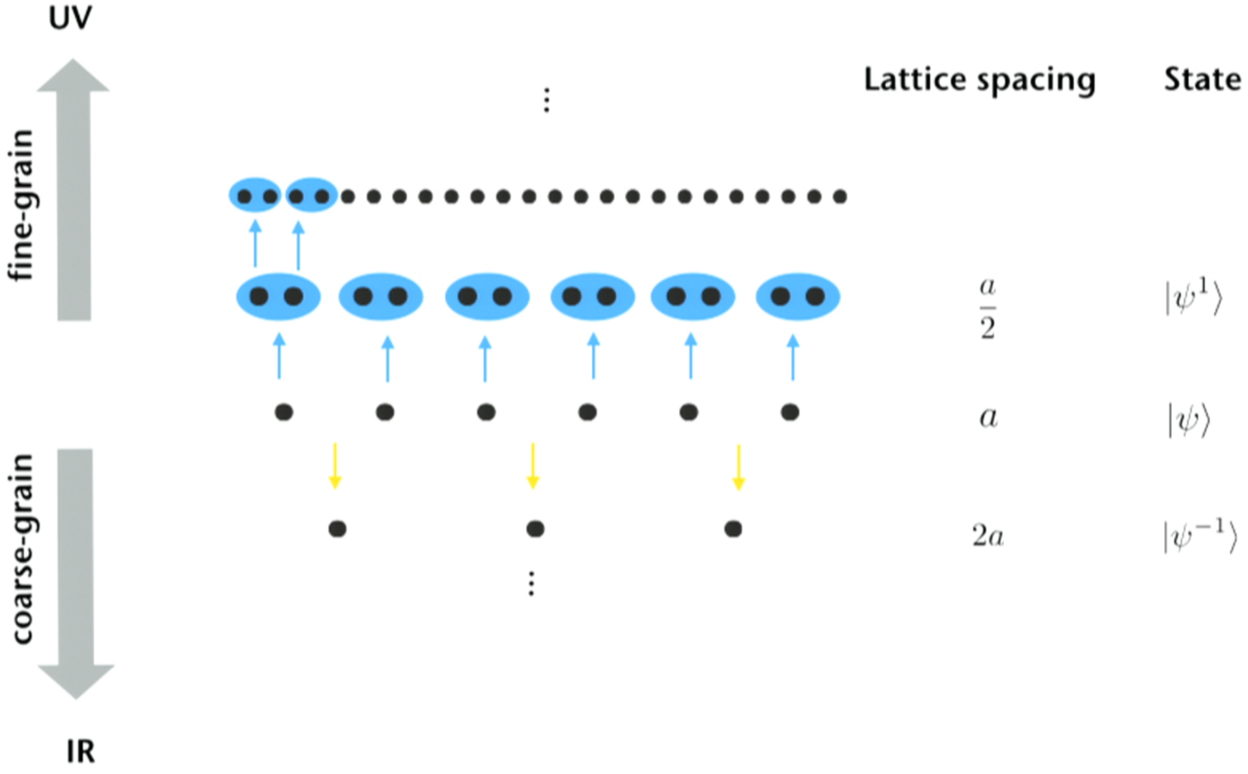
RG transformations for MPS

RG in real space



Lattice spacing	State
a	$\|\psi\rangle$
$2a$	$\|\psi^{-1}\rangle$

RG in real space



Coarse-grain

- **Definition:**

$|\psi^{-1}\rangle$ is the coarse-grained version of $|\psi\rangle$

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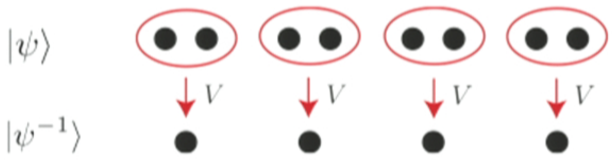

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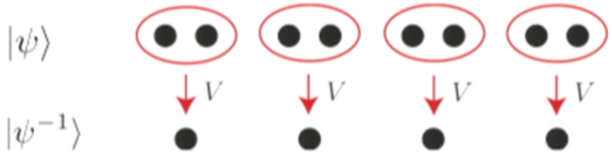

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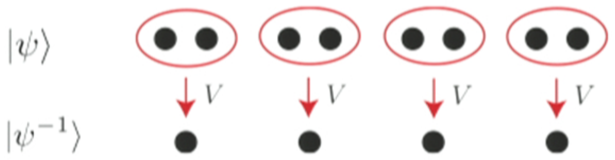

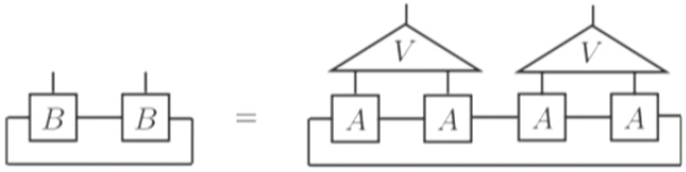

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- **Remarks:**

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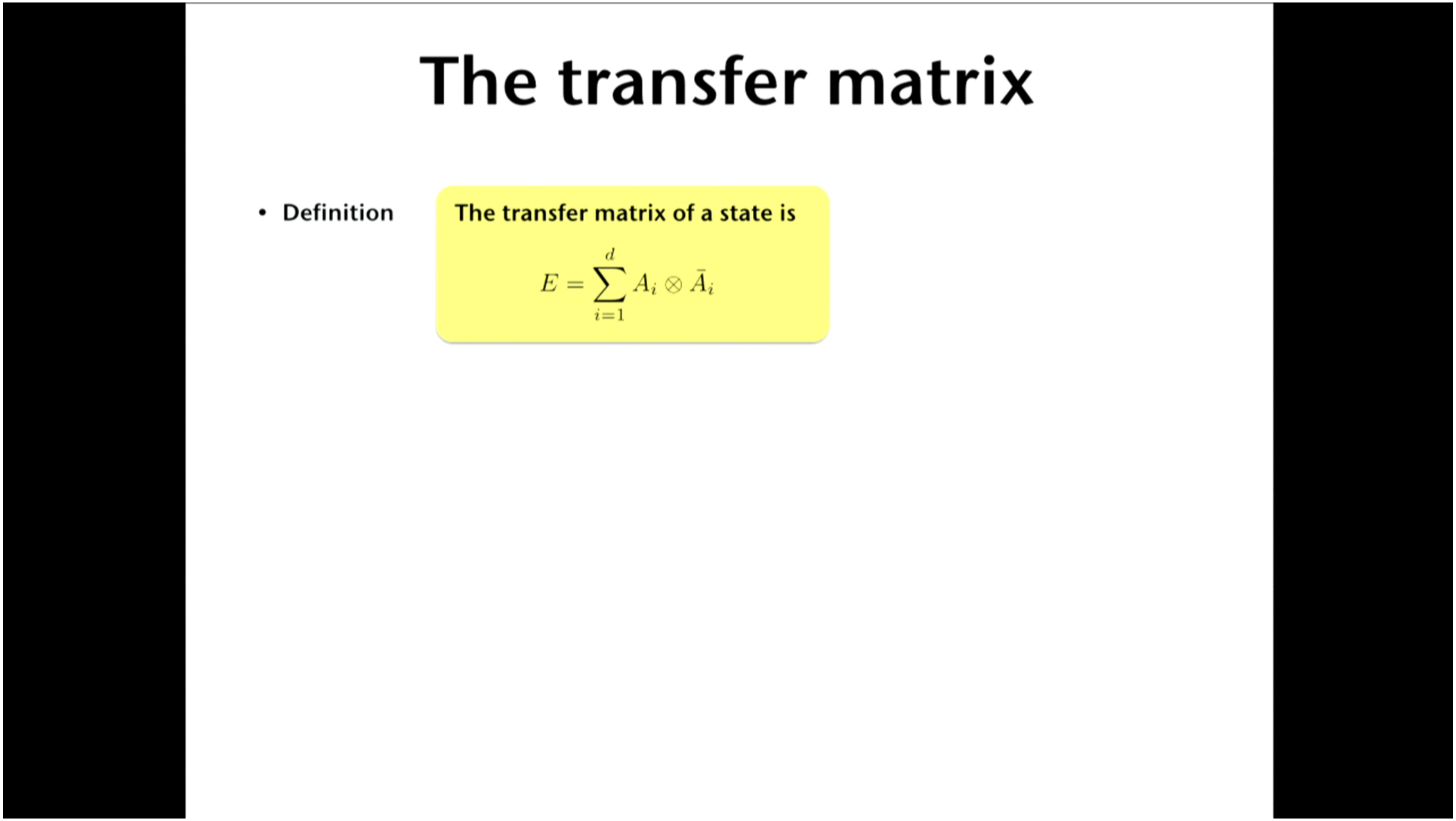
  - Because of the structure of MPS, same bond dim., same physical dim.



The transfer matrix

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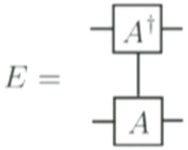
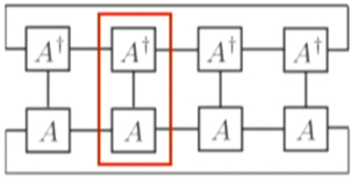


E.g. the norm of the state $\langle \psi_N | \psi_N \rangle = |c_N|^2 \, \mathrm{Tr}(E^N)$



- **Facts:**

The transfer matrix is a completely positive map.

It can be made trace preserving by a choice of the gauge.

Coarse-graining

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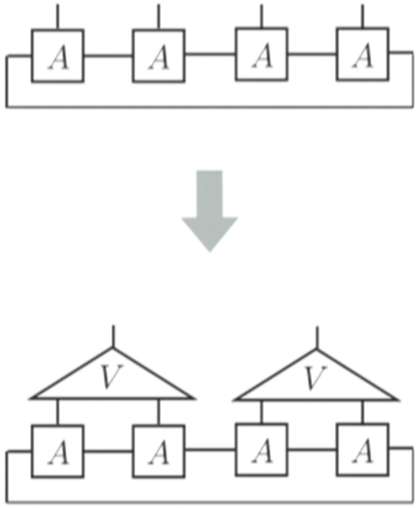
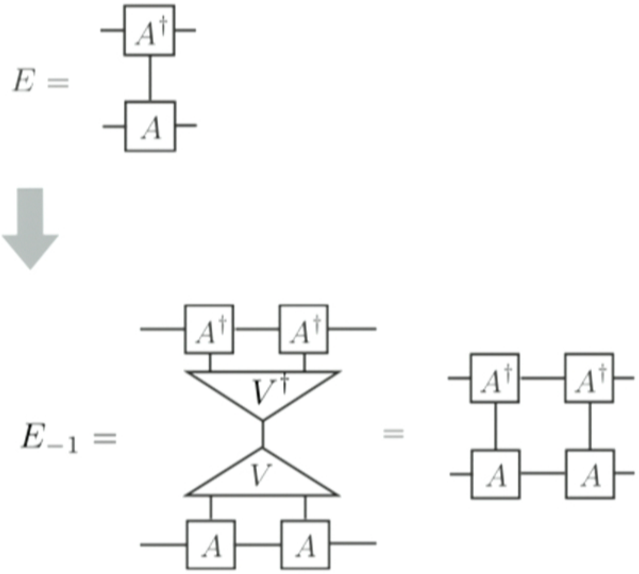


Transfer matrix:



Coarse-graining

- Observation: Coarse-graining a state corresponds to taking the square of its transfer matrix.

- Note:

The square of a quantum channel is always a valid quantum channel. ⟺ Every state can be coarse-grained.

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- **Proposition:**

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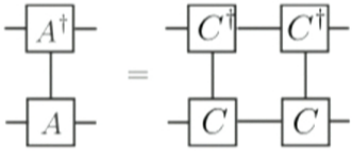

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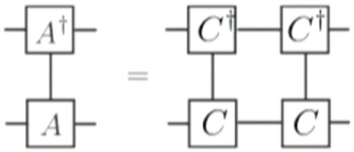

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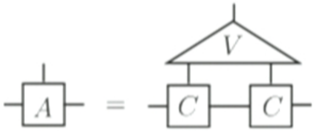

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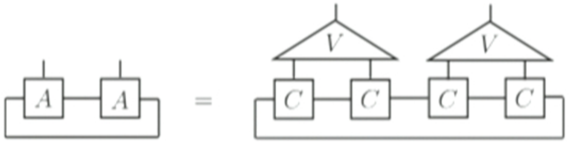

Fine-graining

- **Proposition:**

A state can be fine-grained once if and only if its transfer matrix has a square root which is a valid transfer matrix.

- **Observe:**

Not every quantum channel is divisible into two quantum channels.

⟺

Not every state can be fine-grained

Continuum limit 1

- **Definition:**

  A state has a Continuum limit 1

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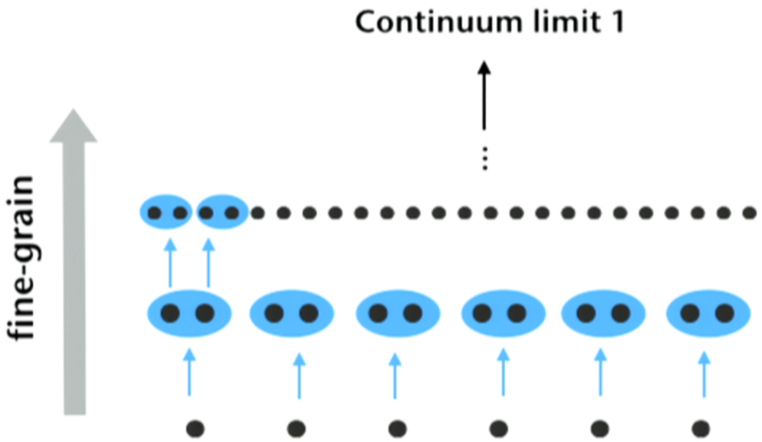
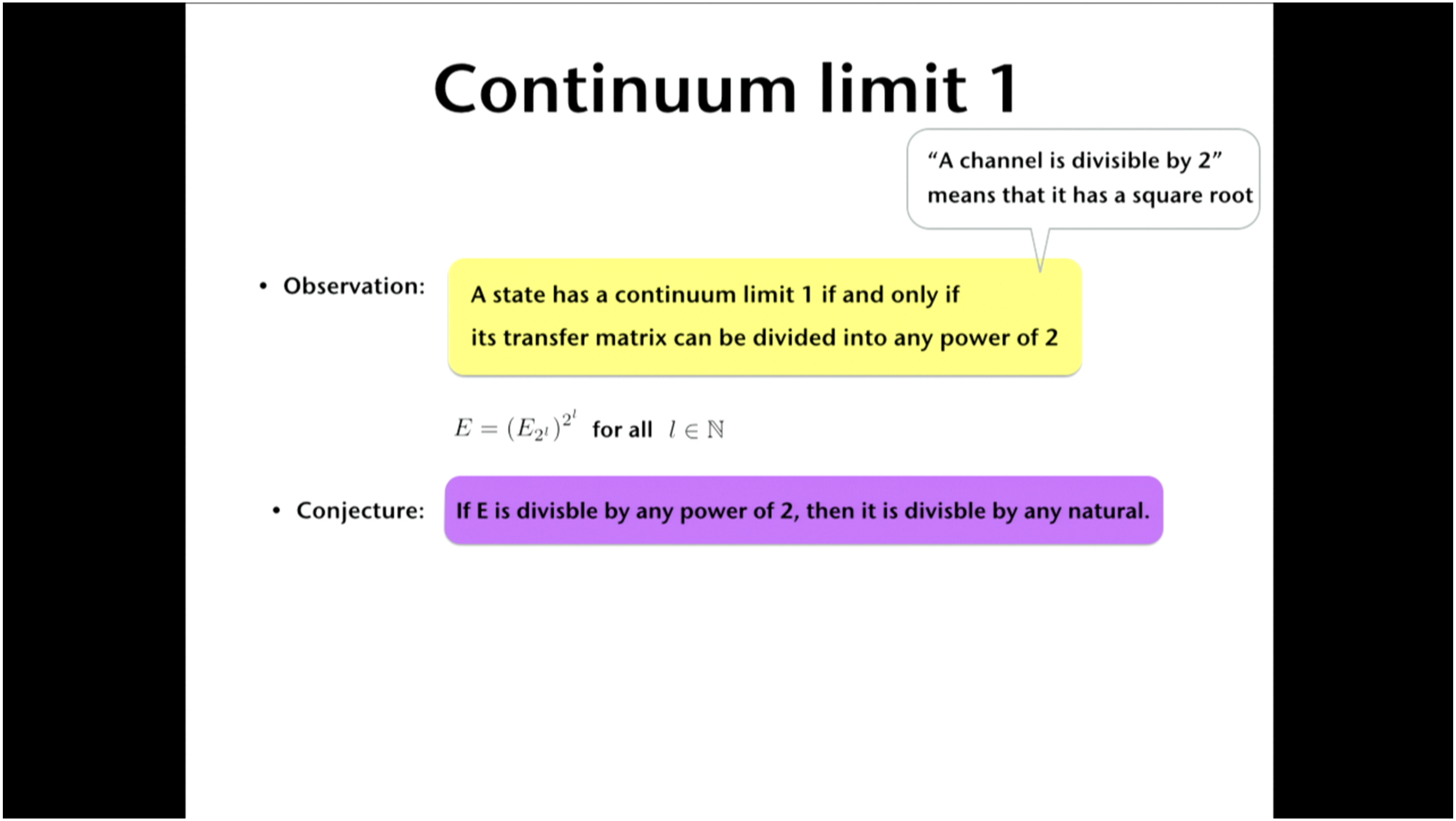


Continuum limit 1

- **Observation:**

A state has a continuum limit 1 if and only if its transfer matrix can be divided into any power of 2



Continuum limit 1

"A channel is divisible by 2"
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$E = (E_{2^l})^{2^l}$ for all $l \in \mathbb{N}$

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Infinitely divisible channel

From now on, I will assume it is true.

Continuum limit 1

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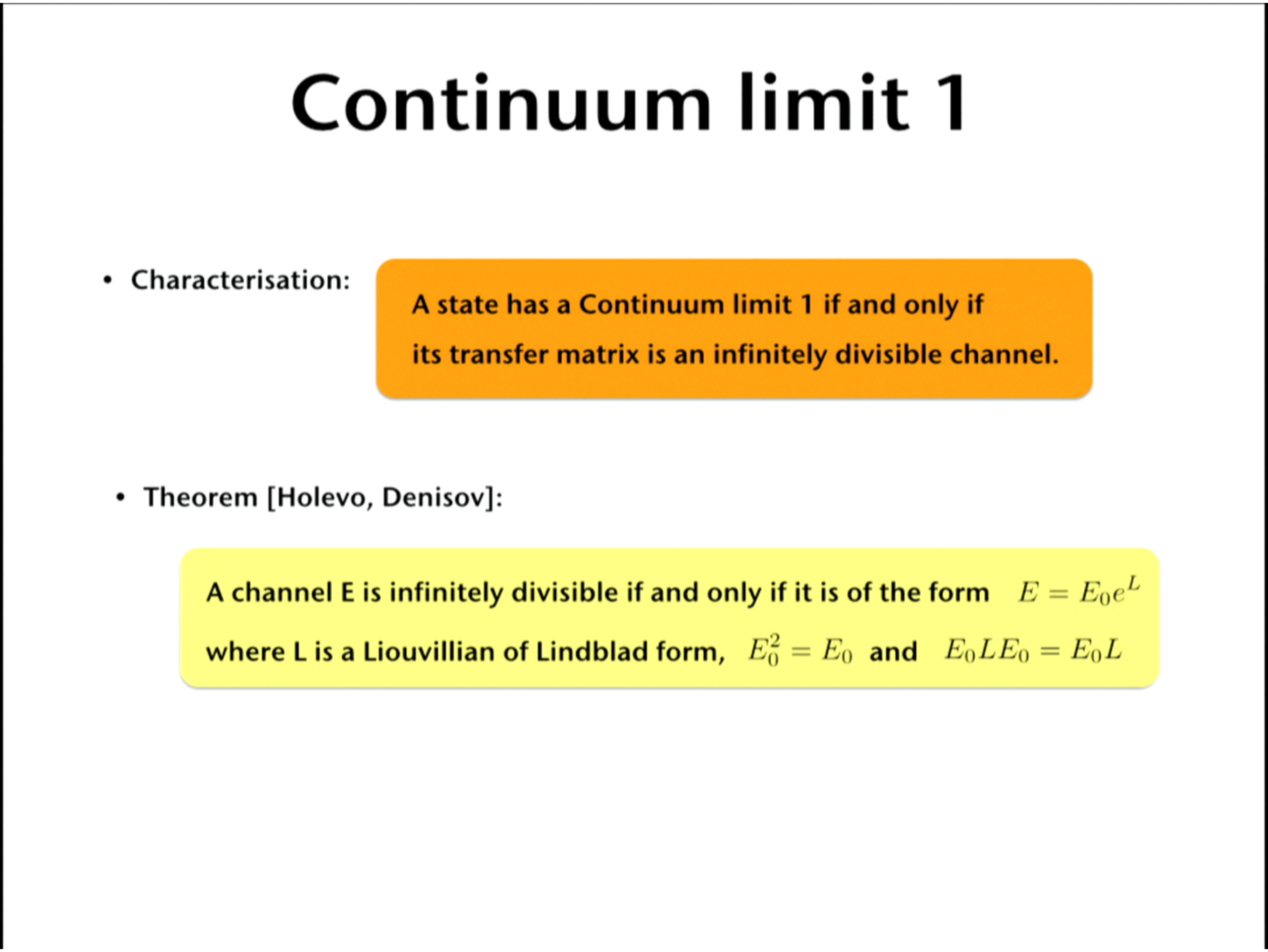
A state has a Continuum limit 1 if and only if its transfer matrix is an infinitely divisible channel.



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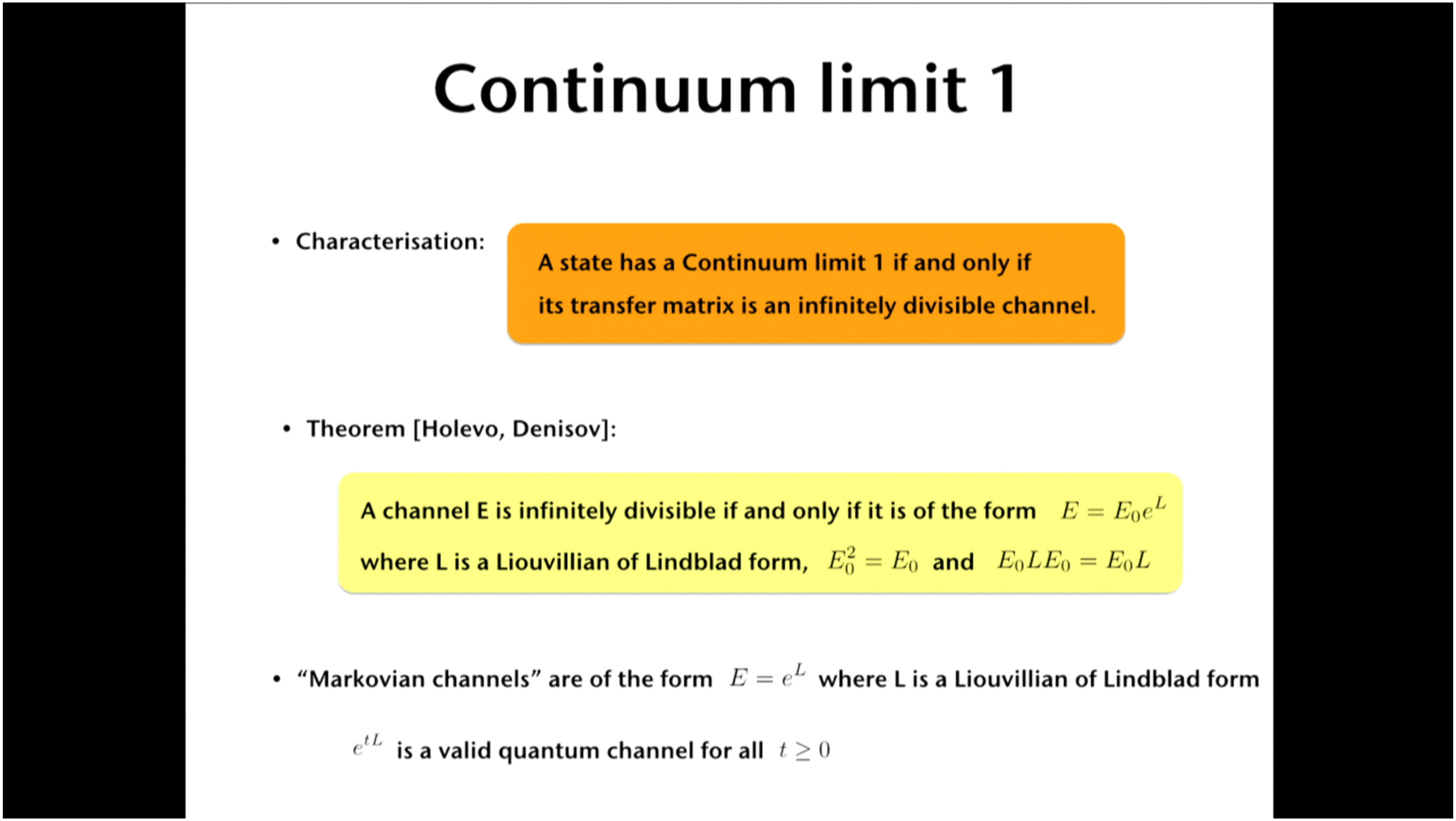
- **Theorem [Holevo, Denisov]:**

A channel E is infinitely divisible if and only if it is of the form $E = E_0 e^L$ **where L is a Liouvillian of Lindblad form,** $E_0^2 = E_0$ **and** $E_0 L E_0 = E_0 L$



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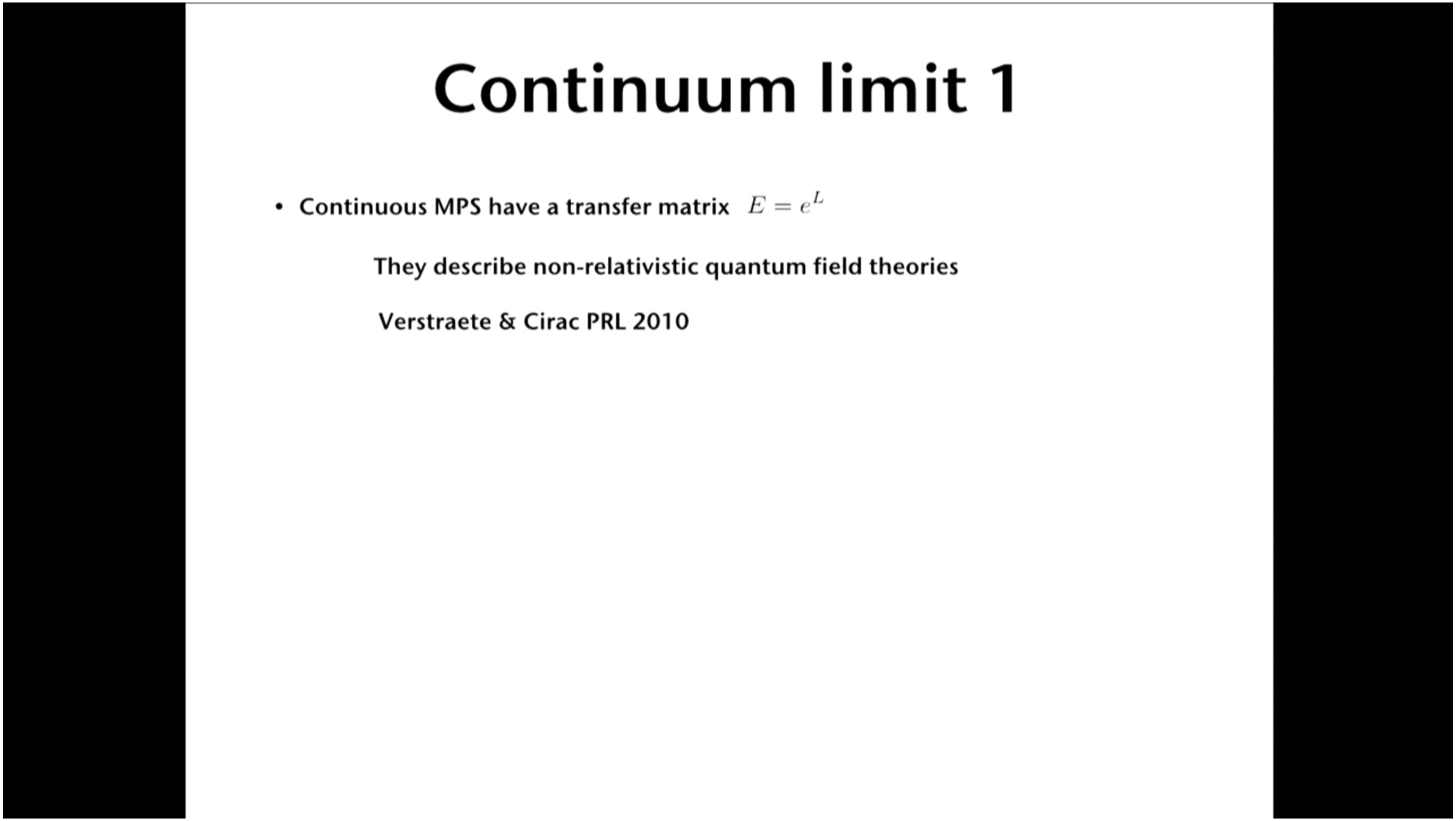
- “Markovian channels” are of the form $E = e^L$ where L is a Liouvillian of Lindblad form

e^{tL} is a valid quantum channel for all $t \geq 0$

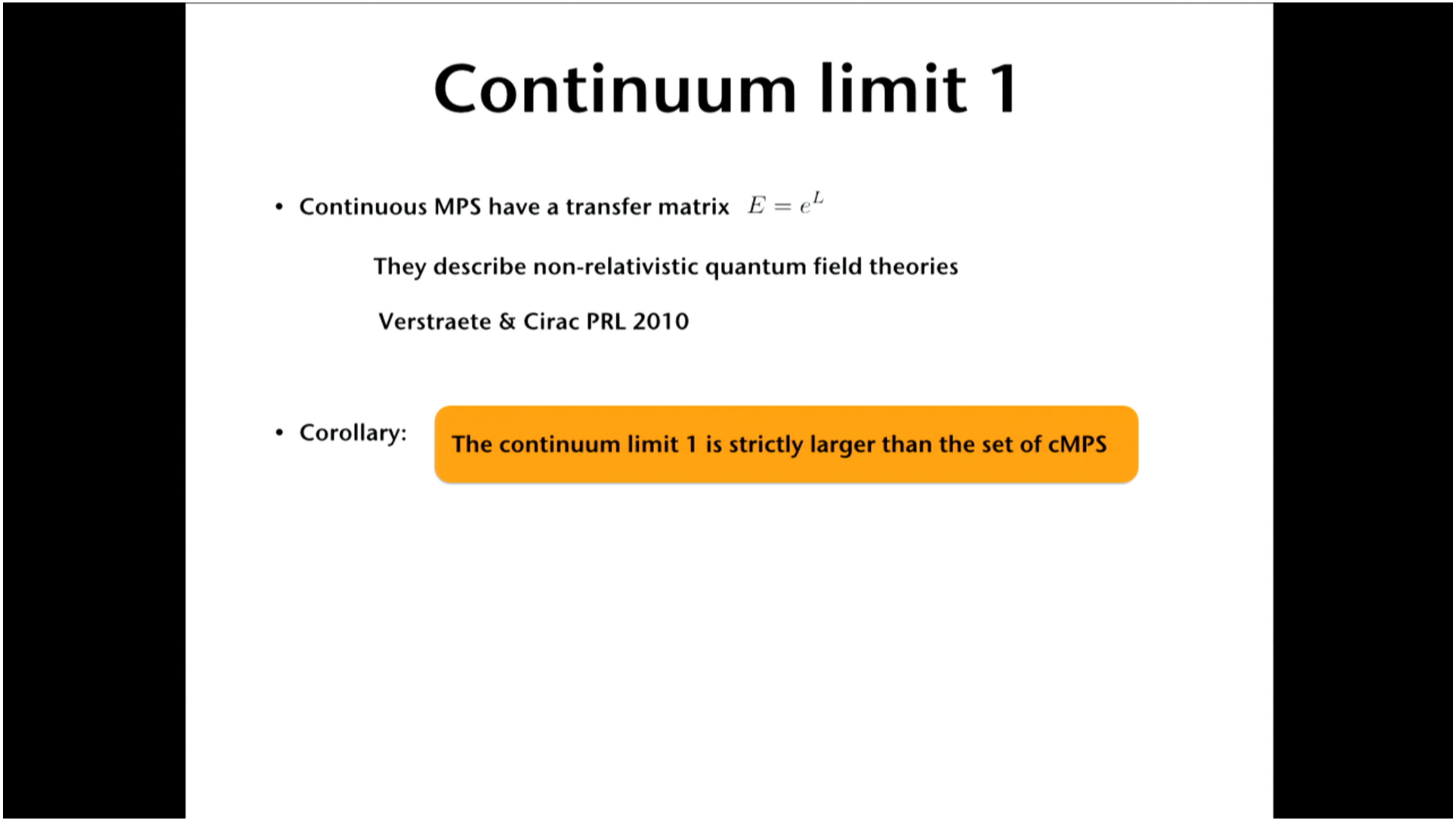


Continuum limit 1

- Continuous MPS have a transfer matrix $E = e^{L}$

They describe non-relativistic quantum field theories

Verstraete & Cirac PRL 2010



Continuum limit 1

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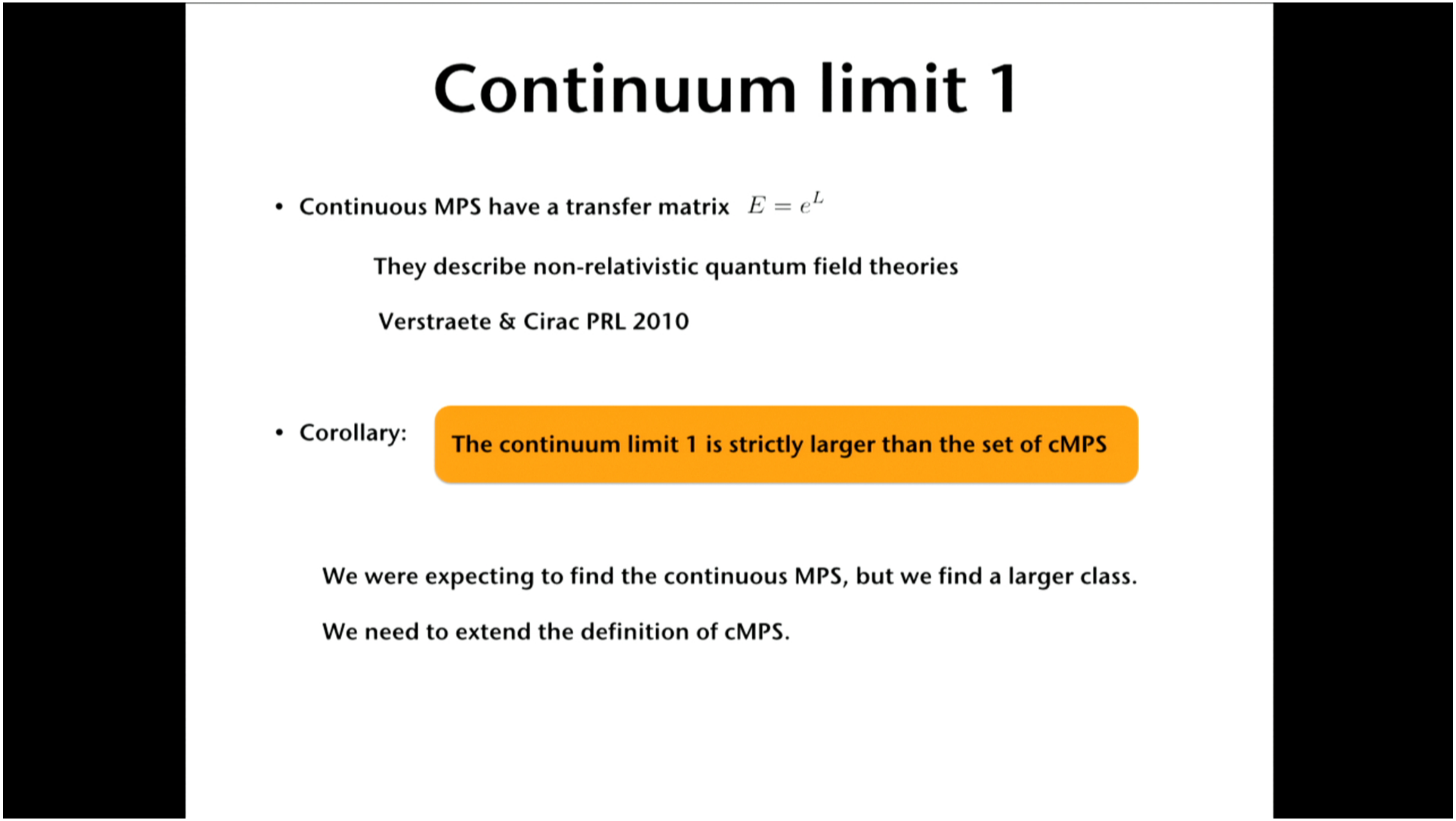
  Verstraete & Cirac PRL 2010

- Corollary: **The continuum limit 1 is strictly larger than the set of cMPS**



Continuum limit 1

- Continuous MPS have a transfer matrix $E = e^L$

  They describe non-relativistic quantum field theories

  Verstraete & Cirac PRL 2010

- Corollary: **The continuum limit 1 is strictly larger than the set of cMPS**

We were expecting to find the continuous MPS, but we find a larger class.

We need to extend the definition of cMPS.

Continuum limit 2

- **Definition:** A state has a Continuum limit 2 if it has a Continuum limit 1 after a finite number of coarse-graining steps

Continuum limit 2

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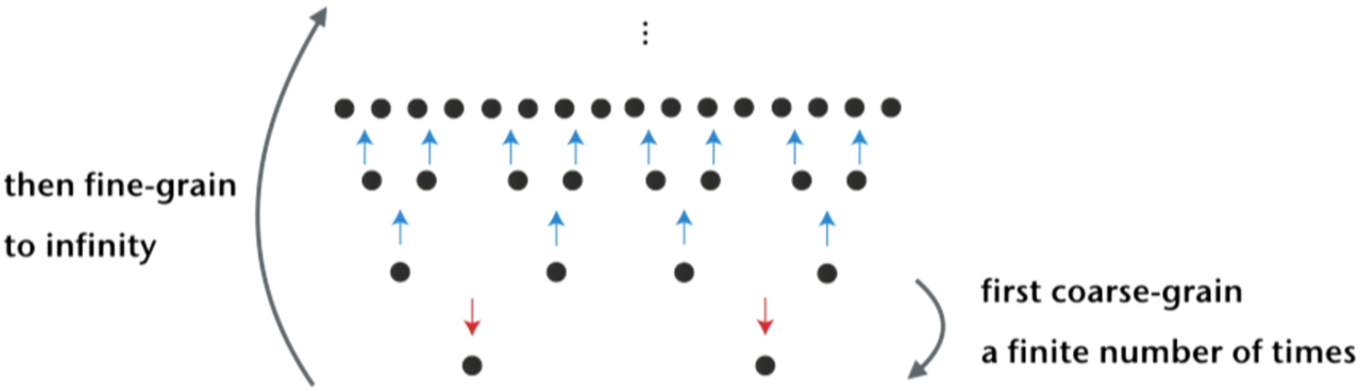
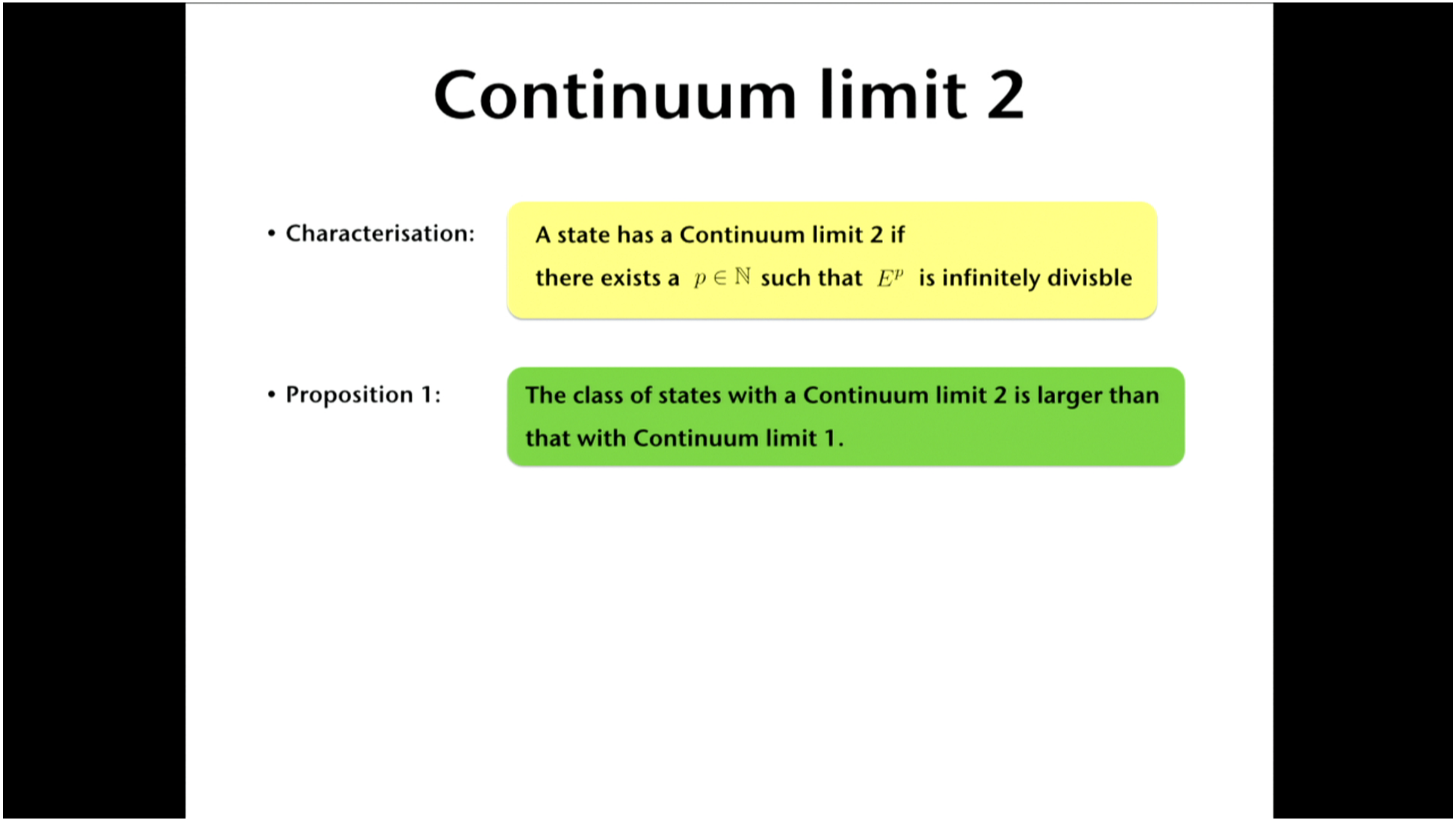


Continuum limit 2

- **Characterisation:**

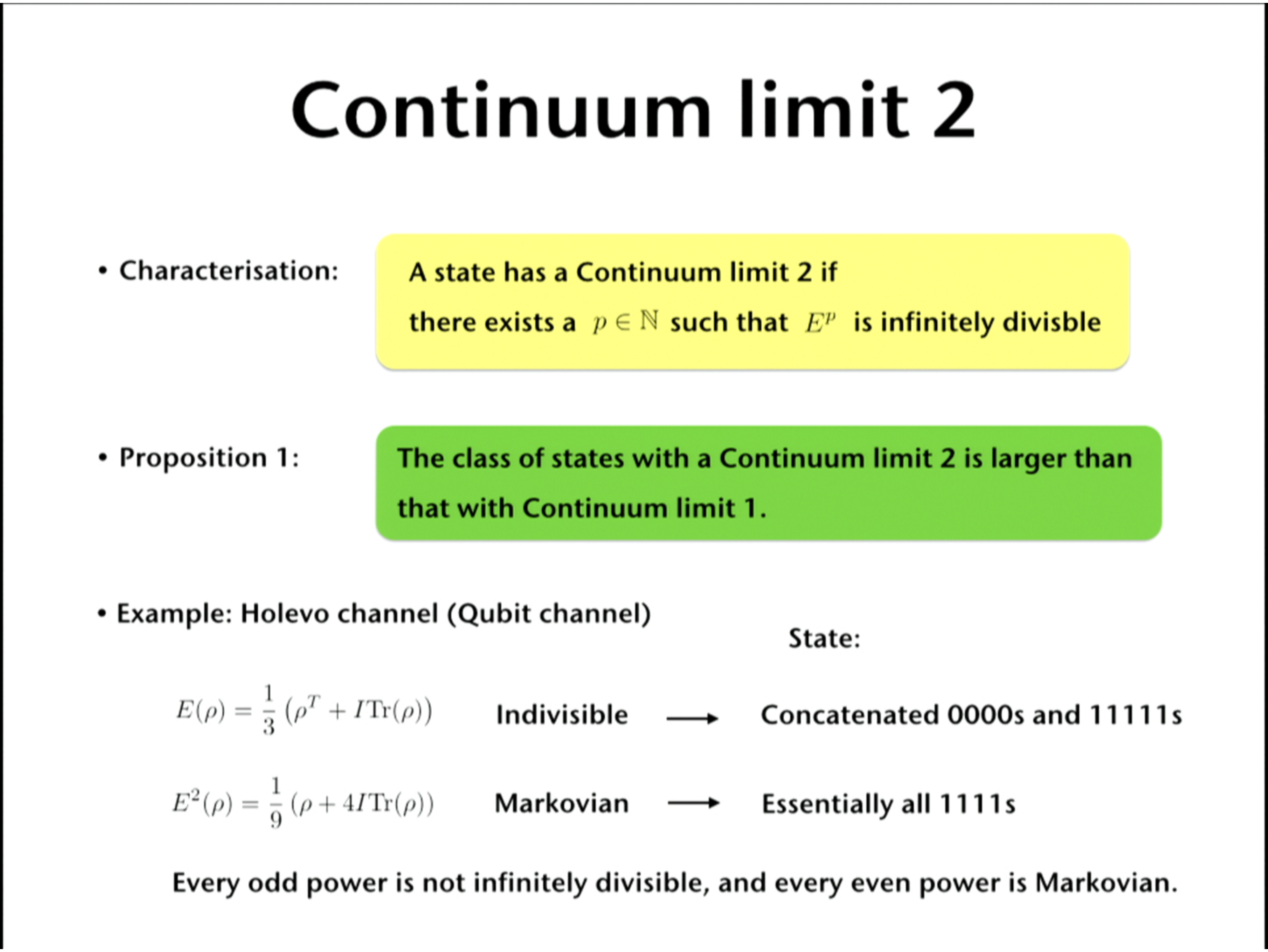
A state has a Continuum limit 2 if there exists a $p \in \mathbb{N}$ such that E^p is infinitely divisble



Continuum limit 2

- **Characterisation:** A state has a Continuum limit 2 if there exists a $p \in \mathbb{N}$ such that E^p is infinitely divisble

- **Proposition 1:** The class of states with a Continuum limit 2 is larger than that with Continuum limit 1.



Continuum limit 2

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- **Proposition 1:** The class of states with a Continuum limit 2 is larger than that with Continuum limit 1.

- **Example: Holevo channel (Qubit channel)**

			State:
$E(\rho) = \frac{1}{3}\left(\rho^T + I\mathrm{Tr}(\rho)\right)$	Indivisible	⟶	Concatenated 0000s and 1111s
$E^2(\rho) = \frac{1}{9}\left(\rho + 4I\mathrm{Tr}(\rho)\right)$	Markovian	⟶	Essentially all 1111s

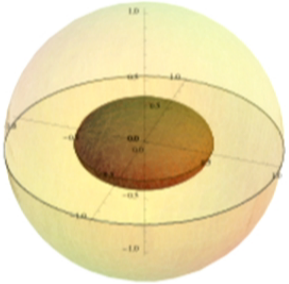
Every odd power is not infinitely divisible, and every even power is Markovian.

Continuum limit 2

- **Proposition 2:** **Not all states have a Continuum limit 2.**

- **Example: Pancake channel** $E = \mathrm{diag}(1, a, a, a^2/2)$ **in the Pauli basis**

Image in the Bloch sphere:



Wolf & Cirac , CMP 2008

Continuum limit 2

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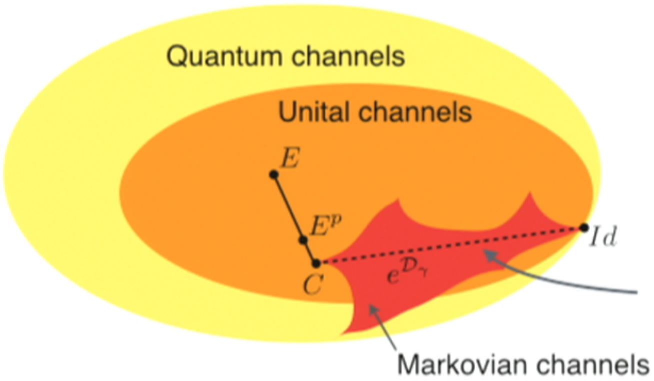
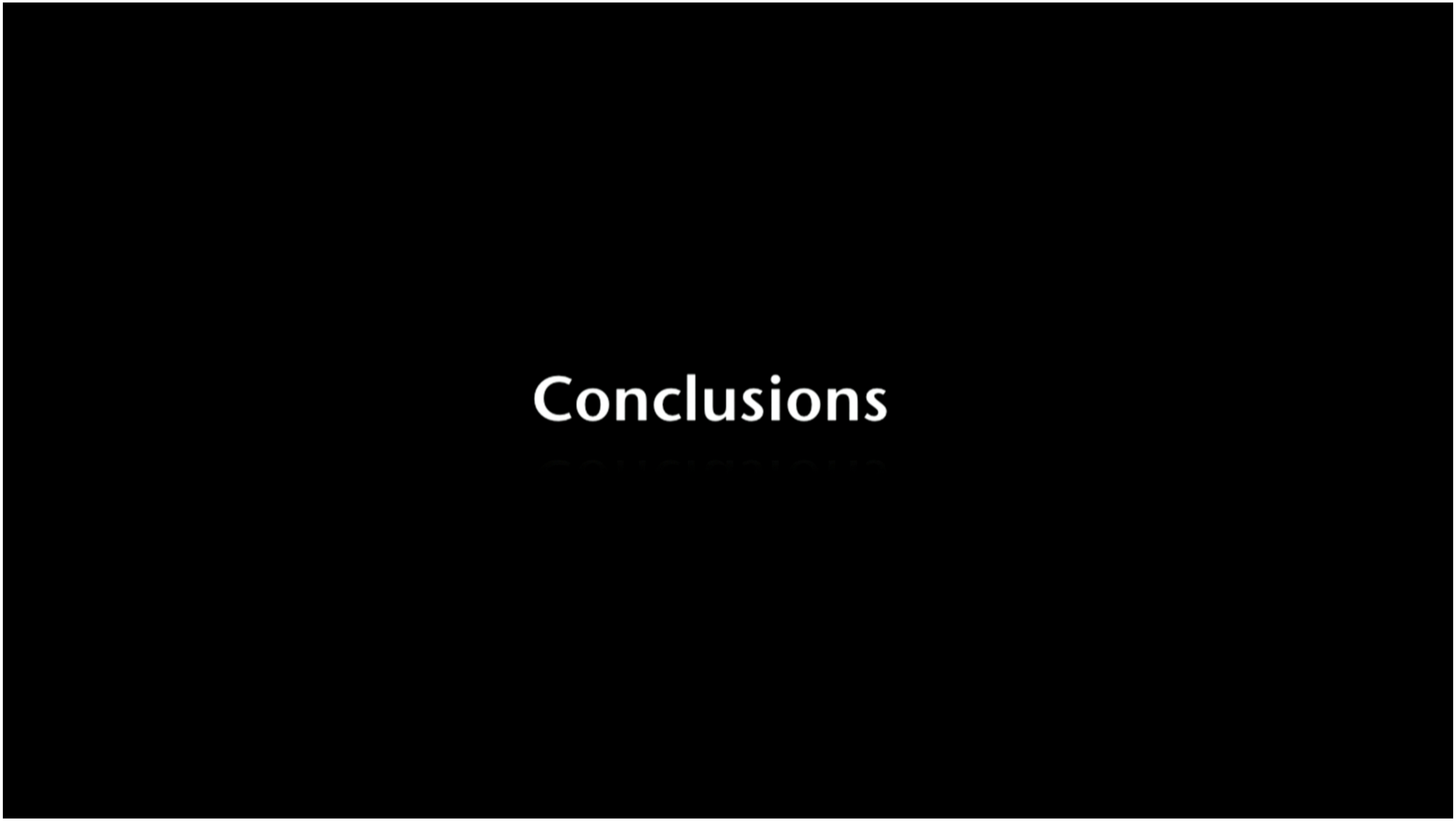
Note: arbitrarily close to the closure of Markovian channels



depolarizing channel

with various strengths

Conclusions



Conclusions

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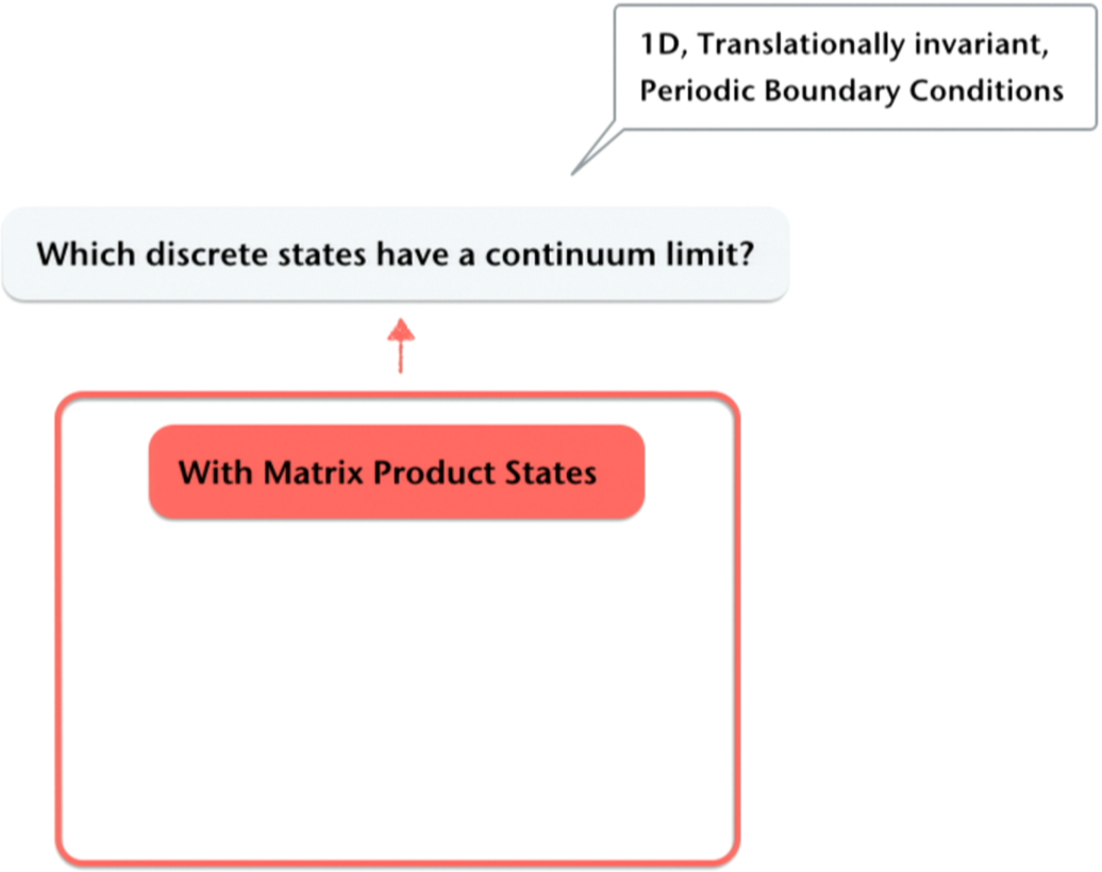
1D, Translationally invariant,
Periodic Boundary Conditions

Which discrete states have a continuum limit?

Conclusions



1D, Translationally invariant,
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Which discrete states have a continuum limit?

With Matrix Product States

Conclusions

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With Matrix Product States

- Transfer matrix of a state is a quantum channel.

Conclusions

1D, Translationally invariant,
Periodic Boundary Conditions

Which discrete states have a continuum limit?

↑

With Matrix Product States

- Transfer matrix of a state is a quantum channel.
- A state can be fine-grained once iff its transfer matrix can be divided into two transfer matrices.

Conclusions

- Continuum limit 1: the limit of the fine-graining process

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All states whose transfer matrix is an infinitely divisble channel

(assuming conjecture)

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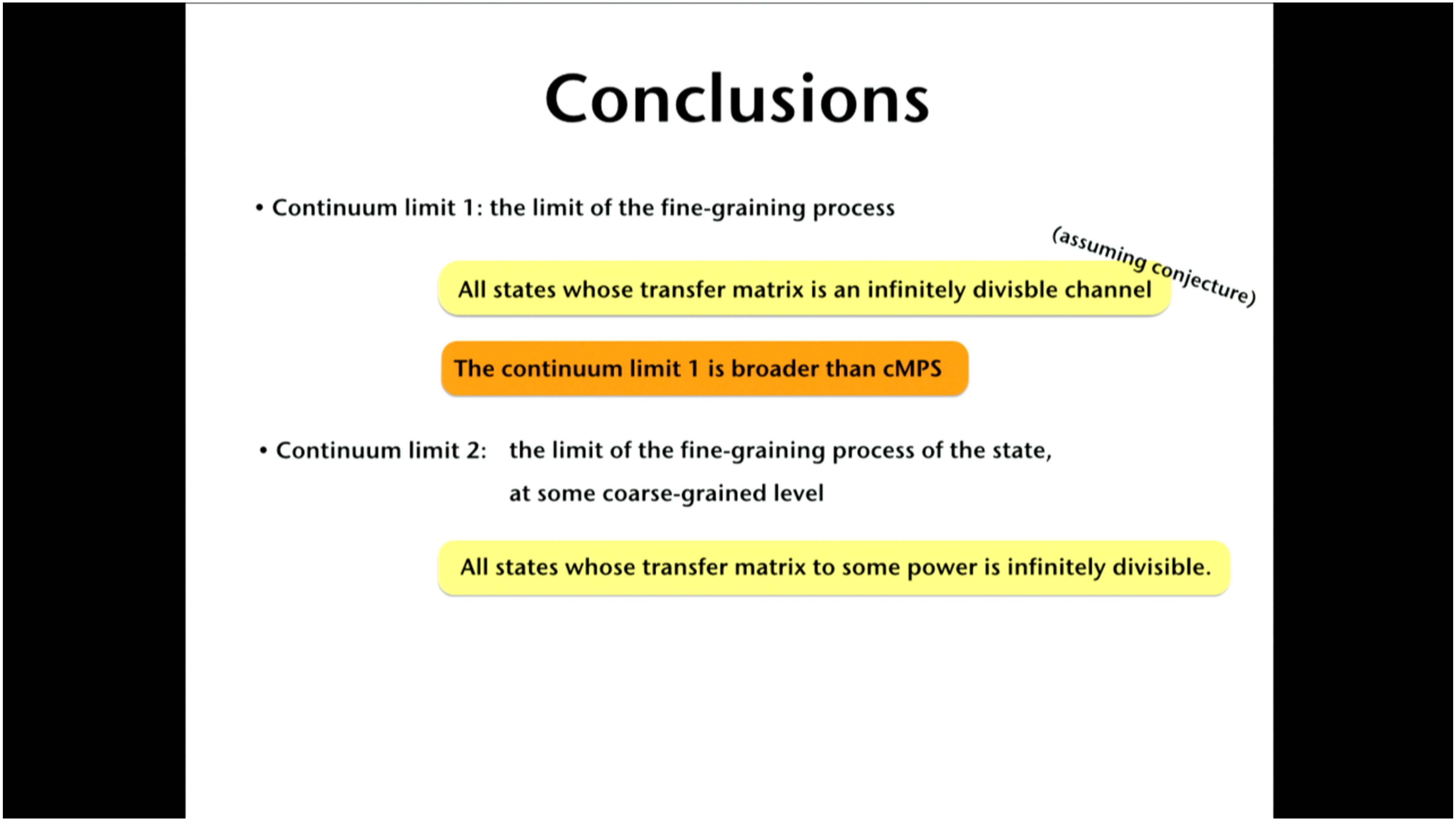
- Continuum limit 2: the limit of the fine-graining process of the state, at some coarse-grained level



Conclusions

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Conclusions

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(assuming conjecture)

The continuum limit 1 is broader than cMPS

- Continuum limit 2: the limit of the fine-graining process of the state, at some coarse-grained level

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Larger class than those with continuum limit 1.

Outlook

(after completion of this work)

Continuum limit 3: based on expectation values of observables

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(after completion of this work)

Continuum limit 3:
based on expectation
values of observables

Many natural generalisations of this work:

- Non-translational invariant
- Boundary conditions
- Other RG schemes: MERA?
- Non-exact RG transformations
- States in more spatial dimensions

Compare with approaches by Osborne / Beny /
Brockt, Haegeman, Jennings, Osborne, Verstraete

Continuum limit 2

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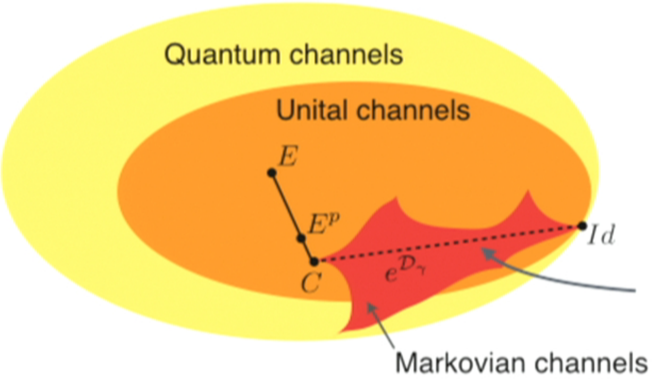

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