Title: Entropy, majorization, and thermodynamics in general probabilistic theories.

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Abstract: Much progress has recently been made on the fine-grained thermodynamics and statistical mechanics of microscopic physical systems, by conceiving of thermodynamics as a resource theory: one which governs which transitions between states are possible using specified "thermodynamic" (e.g. adiabatic or isothermal) means. In this talk we lay some groundwork for investigating thermodynamics in generalized probabilistic theories. We describe simple, but fairly strong, postulates: unique spectrality, projectivity, and symmetry of transition probabilities, that imply that a system has a well-behaved convex analogue of the spectrum, and show that the spectrum of a state majorizes the outcome probabilities of any fine-grained measurement, allowing the operationally defined measurement entropy (and Schur-concave analogues) to be calculated from the spectrum. These are implied by, but probably weaker than, Axioms 1 (weak spectrality) and 2 (strong symmetry) of a recent characterization of Jordan-algebraic and quantum systems by Barnum, Mueller, and Ududec. It is an open question whether theories beyond the Jordan-algebraic ones satisfy them. We describe how part of von Neumann's argument that spectral entropy is a good candidate for thermodynamic entropy generalizes to systems satisfying our postulates, and discuss whether its assumptions are reasonable there, suggesting that the extendibility of certain processes to reversible ones is crucial. We will discuss further postulates and results that might suffice to obtain, in this more general setting, a thermodynamical resource theory similar to the one that is emerging for quantum theory.

(Joint work with Jon Barrett, Marius Krumm, Matt Leifer, Markus Mueller.)

Entropy, majorization and thermodynamics in general probabilistic systems

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Perimeter Institute, May 11, 2015

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Collaborators: Markus Mueller (Western; Heidelberg; PI); Cozmin Ududec (Invenia Technical Computing; PI; Waterloo), Jon Barrett (Oxford), Marius Krumm (Heidelberg)

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Probabilistic Theories

Theory: Set of systems
System: Specified by bounded convex sets of allowed states, allowed measurements, allowed dynamics compatible with each measurement outcome. (Could view as a category.)
Composite systems: Rules for combining systems to get a composite system, e.g. tensor product in QM. (Could view as making it a symmetric monoidal category)
Remark: Framework (e.g. convexity, monoidality...) justified operationally. Very weakly constraining.

Background: characterization of quantum systems

HB, Markus Müller, Cozmin Ududec

1. **Weak Spectrality**: every state is in convex hull of a set of perfectly distinguishable pure (i.e. extremal) states
2. **Strong Symmetry**: Every set of perfectly distinguishable pure states transforms to any other such set of the same size **reversibly**.
3. **No irreducibly three-slit** (or more) **interference.**
4. **Energy observability**: Systems have nontrivial continuously parametrized reversible dynamics. Generators of one-parameter continuous subgroups ("Hamiltonians") are associated with nontrivial conserved observables.

•$1-4 \Longrightarrow$ standard quantum system (over $\mathbb{C}$)

•$1-3 \Longrightarrow$ irreducible Jordan algebraic systems, and classical.

•$1-2 \Longrightarrow$ "projective" (filters onto faces), self-dual systems

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State spaces and measurements

Normalized states of system A: Convex compact set Ω_A of dimension $d-1$, embedded in $A \simeq \mathbb{R}^d$ as the base of a regular **cone** A_+ of unnormalized states (nonnegative multiples of Ω_A).
Measurement outcomes: linear functionals $A \to \mathbb{R}$ called **effects** whose values on states in Ω_A are in $[0,1]$.
Unit effect u_A has $u_A(\Omega_A) = 1$.
Measurements: Indexed sets of effects e_i with $\sum_i e_i = u_A$ (or continuous analogues).
Effects generate the **dual cone** A^*_+, of functionals nonnegative on A_+.
Sometimes we may wish to restrict measurement outcomes to a (regular) subcone, call it $A^\#_+$, of A^*_+. If no restriction, system **saturated**.
(A_+ is **regular**: closed, generating, convex, pointed. It makes A an **ordered linear space** (inequalities can be added and multiplied by positive scalars), with order $a \geq b := a - b \in A_+$.)

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Inner products, internal representation of the dual and self-duality

In a *real* vector space A an inner product $(\_,\_)$ is equivalent to a linear isomorphism $A \to A^*$. $y \in A$ corresponds to the functional $x \mapsto (y,x)$. GPT theories often represented this way (Hardy, Barrett...).

- **Internal dual** of A_+ relative to inner product: $A_+^{*int} := \{y \in A : \forall x \in A_+ (y,x) \geq 0\}$. Isomorphic to A_+^*).
- If there *exists* an inner product relative to which $A_+^{*int} = A_+$, A is called **self-dual**.
- Self-duality is stronger than A_+ affinely isomorphic to A_+^*! (examples)
- related to time reversal

Examples

Classical: A is the space of n-tuples of real numbers; $u(x) = \sum_{i=1}^{n} x_i$. So Ω_A is the probability simplex, A_+ the positive (i.e.nonnegative) orthant $x : x_i \geq 0, i \in 1, ..., n$

Quantum: $A = \mathscr{B}_h(\mathbf{H})$ = self-adjoint operators on complex (f.d.) Hilbert space $\mathbf{H}$; $u_A(X) = \mathrm{Tr}(X)$. Then Ω_A = density operators. A_+ = positive semidefinite operators.

Squit (or P/Rbit): Ω_A a square, A_+ a four-faced polyhedral cone in $\mathbb{R}^3$.

Inner-product representations: $\langle X, Y \rangle = \mathrm{tr}\, XY$ (Quantum) $\langle x, y \rangle = \sum_i x_i y_i$ (Classical)

Quantum and classical cones are self-dual! Squit cone is not, but is isomorphic to dual.

Jordan Algebraic Systems

- Pascual Jordan, (Z. Phys, 1932 or 1933):
  - **Jordan algebra:** abstracts properties of Hermitian operators.
  - Symmetric product $\bullet$ abstracts $A \bullet B = \frac{1}{2}(AB + BA)$.
  - Jordan identity: $a \bullet (b \bullet a^2) = (a \bullet b) \bullet a^2$.
  - **Formally real JA:** $a^2 + b^2 = 0 \implies a = b = 0$. Makes the cone of squares a candidate for unnormalized state space.
- Jordan, von Neumann, Wigner (Ann. Math., **35**, 29-34 (1934)): irreducible f.d. formally real Jordan algebras are:
  - quantum systems (self-adjoint matrices) over $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$;
  - systems whose state space is a ball (aka "spin factors");
  - 3×3 Hermitian octonionic matrices ("exceptional" JA).
- f.d. homogeneous self-dual cones are precisely the cones of squares in f.d. formally real Jordan algebras. (Koecher 1958, Vinberg 1960)

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Faces of convex sets

Face of convex C: subset S such that if $x \in S$ & $x = \sum_i \lambda_i y_i$, where $y_i \in C$, $\lambda_i > 0$, $\sum_i \lambda_i = 1$, then $y_i \in S$.

Exposed face: intersection of C with a supporting hyperplane. Classical, quantum, squit examples.

For effects e, $F_e^0 := \{x \in \Omega) : e(x) = 0\}$ and $F_e^1 := \{x \in \Omega : e(x) = 1\}$ are exposed faces of Ω.

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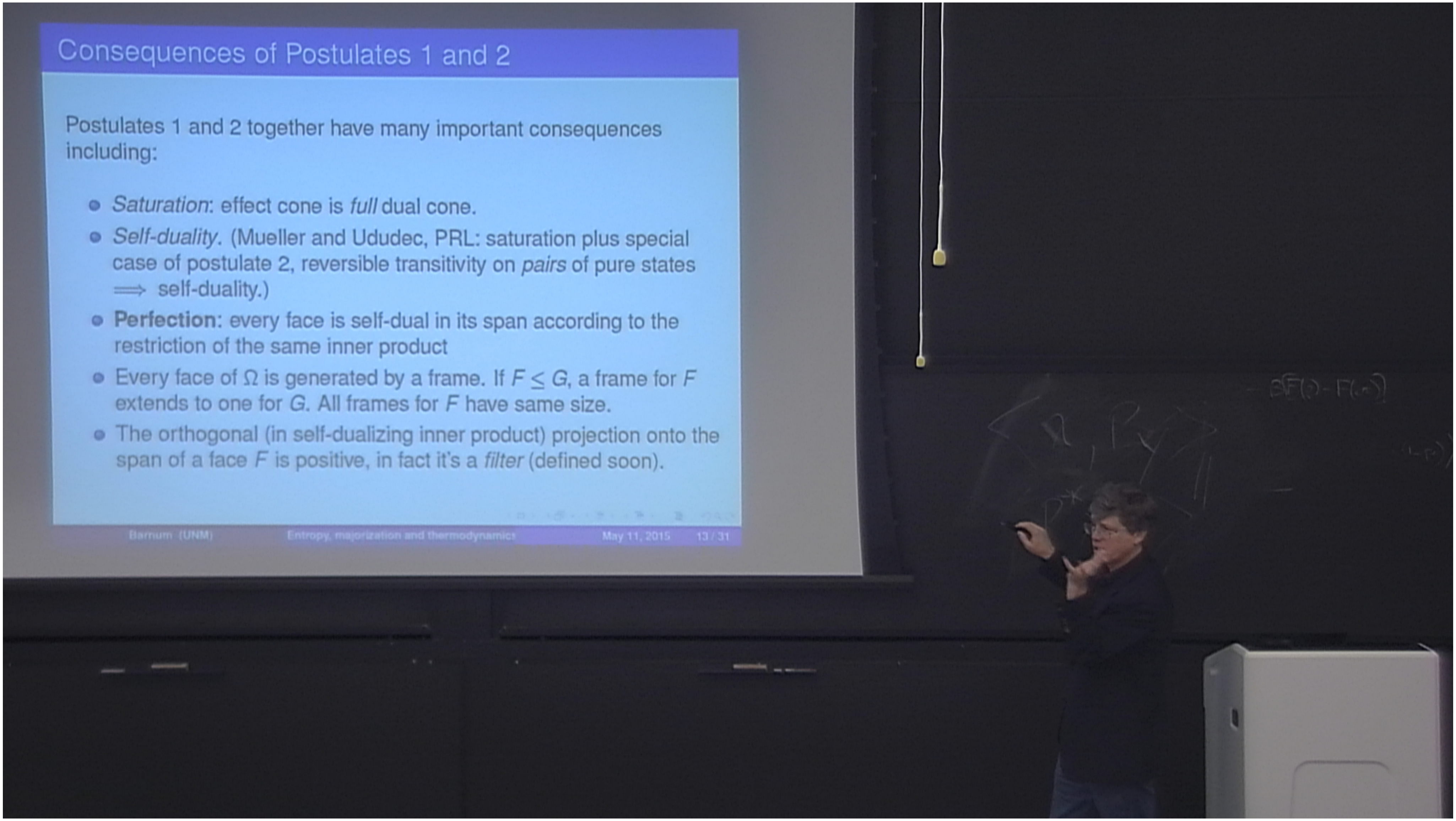
Filters

Convex abstraction of QM's Projection Postulate (Lüders version): $\rho \mapsto Q\rho Q$ where Q is the orthogonal projector onto a subspace of Hilbert space $\mathscr{H}$.
Helpful in abstracting interference.

Filter := Normalized positive linear map $P : A \to A$: $P^2 = P$, with P and P^* both complemented.
Complemented means $\exists$ filter P' such that $\mathrm{im}\, P \cap A_+ = \ker P' \cap A_+$.
Normalized means $\forall \omega \in \Omega \;\; u(P\omega) \leq 1$.

- Dual of Alfsen and Shultz' notion of **compression**.
- Filters are **neutral**: $u(P\omega) = u(\omega) \implies P\omega = \omega$.
- Ω called **projective** if every face is the positive part of the image of a filter.


Consequences of Postulates 1 and 2
Postulates 1 and 2 together have many important consequences including:
Saturation: effect cone is full dual cone.
Self-duality. (Mueller and Ududec, PRL: saturation plus special case of postulate 2, reversible transitivity on pairs of pure states ⟹ self-duality.)
Perfection: every face is self-dual in its span according to the restriction of the same inner product
Every face of Ω is generated by a frame. If F ≤ G, a frame for F extends to one for G. All frames for F have same size.
The orthogonal (in self-dualizing inner product) projection onto the span of a face F is positive, in fact it's a filter (defined soon).
Barnum (UNM)
Entropy, majorization and thermodynamics
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The lattice of faces

- **Lattice**: partially ordered set such that every pair of elements has a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.
- The faces of any convex set, ordered by set inclusion, form a lattice.
- **Complemented lattice**: bounded lattice in which every element x has a **complement**: x' such that $x \vee x' = 1$, $x \wedge x' = 0$. (Remark: x' not necessarily unique.)
- **orthocomplemented** if equipped with an order-reversing complementation: $x \leq y \implies x' \geq y'$. (Remark: still not necessarily unique.)
- Orthocomplemented lattices satisfy DeMorgan's laws.

Orthomodularity

- **Orthomodularity**: $F \leq G \implies G = F \vee (G \wedge F')$. (draw)
- For projective systems, define $F' := \mathrm{im}\,_+P'_F$. Then $'$ is an orthocomplementation, and the face lattice is orthomodular. (Alfsen & Shultz)
- OMLs are "Quantum logics"
- OML's are precisely those orthocomplemented lattices that are determined by their Boolean subalgebras.
- Closely related to **Principle of Consistent Exclusivity** (A. Cabello, S. Severini, A. Winter, `arxiv 1010.2163`):
  If a set of sharp outcomes e_i are pairwise jointly measurable, their probabilities sum to 1 or less in *any* state.
  Limit on noncontextuality.

Multi-slit interference I

To adapt Rafael Sorkin's k-th order interference to our framework, need k-slit experiments.

k-slit mask: Set of filters $P_1, ..., P_k$ onto distinguishable faces. Define $P_J := \bigvee_{i \in J} P_i$. (Notation: $P_{ij...n} = P_i \vee P_j \vee \cdots \vee P_n$.)

In QM: maps $\rho \mapsto Q_i \rho Q_i$, where Q_i are projectors onto orthogonal subspaces S_i of $\mathscr{H}$.

- **2nd-order interference** if for some 2-slit mask,

$$P_1 + P_2 \neq P_{12}. \tag{1}$$

- **3rd-order interference** if for some 3-slit mask,

$$P_{12} + P_{13} + P_{23} - P_1 - P_2 - P_3 \neq P_{123}. \tag{2}$$

(Zero in quantum theory; easy to check at Hilbert space/pure-state level.)

Multi-slit interference II

k-th order interference if for some mask $M = \{P_1, .., P_k\}$,

$$\sum_{r=1}^{k-1} (-1)^{r-1} \sum_{|J|=k-r} P_J \neq P_M \,. \tag{3}$$

• Equivalently $F_M = \mathrm{lin} \cup_{|J|=k-1} F_J$ (no "k**-th order coherence**"). (Ududec, Barnum, Emerson, *Found. Phys.* **46**: 396-405 (2011). (`arxiv: 0909.4787`) for $k = 3$, in prep. arbitrary k (& CU thesis).)

Components of a state in $F_M \setminus \mathrm{lin} \cup_{|J|=k-1} F_J$ are k-th order "coherences". In QM: off-block-diagonal density matrix elements.

• No k-th order $\implies$ no $k+1$-st order.

Symmetry of transition probabilities

- Given projectivity, for each atomic projective unit $p = P^* u$ (P an **atomic** (:= minimal nonzero) filter) the face $P\Omega$ contains a single pure state, call it $\hat{p}$.

$p \mapsto \hat{p}$ is 1:1 from atomic projective units onto extremal points of Ω (pure states).

- **Symmetry of transition probabilities**: for atomic projective units a, b, $a(\hat{b}) = b(\hat{a})$.

A self-dual projective cone has symmetry of transition probabilities.

Theorem (Araki 1980; we rediscovered...)

Projectivity $\implies$ (**STP** $\equiv$ **Perfection**).

Characterizing Jordan algebraic systems

Theorem (Adaptation of Alfsen & Shultz, Thm 9.3.3)

Let a finite-dimensional system satisfy

(a) **Projectivity**: *there is a filter onto each face*

(b) **Symmetry of Transition Probabilities**, *and*

(c) **Filters Preserve Purity**: *if ω is a pure state, then $P\omega$ is a nonnegative multiple of a pure state.*

Then Ω is the state space of a formally real Jordan algebra.

Theorem (Barnum, Müller, Ududec)

(**Weak Spectrality** & **Strong Symmetry**) $\implies$ **Projectivity** & **STP**;
WS & **SS** & **No Higher Interference** $\implies$ **Filters Preserve Purity**.
Jordan algebraic system thus obtained must be either irreducible or classical. (All such satisfy **WS**, **SS**, **No HOI**.*)*

Energy Observability

Definition

Let A, Ω be a system with a group of reversible transformations $\mathscr{G}_A$ having non-trivial Lie algebra $\mathfrak{g}_A$.
Energy observable assignment: injective linear map $\phi : \mathfrak{g}_A \to A^*$ such that

- $\phi(X) \circ X = 0$ for all $X \in \mathfrak{g}_A$
- $u_A \notin \mathrm{ran}(\phi)$.

We say that "energy is an observable" in A if $\mathfrak{g}_A \neq \{0\}$ and if there exists an energy observable assignment.

Theorem (Barnum, Müller, Ududec)

A finite dimensional system satisfies **Weak Spectrality**, **Strong Symmetry**, **No Higher Interference**, *and* **Energy Observability** *iff it is a standard quantum system (over a complex Hilbert space).*

Covering Law

An alternative to **Filters Preserve Purity** in Alfsen-Shultz theorem is the **Covering Law** for face lattice:

Definition

Element b of lattice **covers** element a if $a \leq b$ and there is nothing between them. **Atom**: covers 0.

Covering law: For every element F and atom a, either $F \vee a = a$ or $F \vee a$ covers a. (draw)

Thermodynamics and the geometry of systems

Thermo/stat mech phenomena are a natural arena for information-related principles to play a role in physics.

Thermodynamic protocols (e.g. for moving between nonequilibrium states using adiabatic and isothermal processes at cost governed by $E - TS$ in some limit) tend to involve

- *Spectra* (provided by **Unique Spectrality**),
- Plenty of reversible transformations (provided by **Strong Symmetry**),
- Possibly, measurements using *filters* (Maxwell's demon?) (provided by **Projectivity**).
- Association of reversible evolution with conserved energy, cf. **Energy observability**.
- Maybe don't need **No HOI** , or **Filters Preserve Purity**?

Initial results relevant to thermo

(HB, Jonathan Barrett, Markus Mueller, Marius Krumm)
(QPL 2015, submitted, and M. Krumm, masters' thesis Heidelberg)

Definition

Unique Spectrality: every state has a decomposition into perfectly distinguishable pure states and all such decompositions use the same probabilities.

Stronger than Weak Spectrality (example), but implied by Weak Spectrality and Strong Symmetry.

Definition

For $x, y \in \mathbb{R}^n$, $x \prec y$, x **is majorized by** y, means that $\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}$ for $k = 1, \ldots, n-1$, and $\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow}$.

Spectral measurement probabilities majorize

A measurement $\{e_i\}$ is **fine-grained** if e_i are on extremal rays of A_+^*.

Theorem (H. Barnum, J. Barrett, M. Müller, M. Krumm)

Let a system satisfy Unique Spectrality, Symmetry of Transition Probabilities, and Projectivity. (Equivalently, Perfection and Unique Spectrality.) Then for any state ω and fine-grained measurement $e_1, \ldots, e_n$, the vector $\mathbf{p} = [e_1(\omega), \ldots, e_n(\omega)]$ is majorized by the vector of probabilities of outcomes for a spectral measurement on ω.

Corollary

Let $\omega' = \int_K d\mu(T) T_\mu(\rho)$, where $d\mu(T)$ is a normalized measure on the compact group K of reversible transformations. Then $\omega \preceq \omega'$.

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *Schur-concave* if for every $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{v}$ majorizes $\mathbf{w}$ implies $f(\mathbf{v}) \leq f(\mathbf{w})$.

Entropy-like; mixing-monotone.

Proposition

Every concave symmetric function is Schur-concave.

Definition (Measurement, preparation, spectral "entropies")

Let χ be a Schur-concave function. Define
$\chi^{meas}(\omega) = \min_{fine-grained\ measurements} \chi([e_1(\omega), ..., e_{\#outcomes}(\omega)])$.
$\chi^{meas}(\omega) =$ minimum over convex decompositions of $\omega = \sum_i p_i \omega_i$) of ω into pure states, of $\chi(\mathbf{p})$.
$\chi^{spec}(\omega) := \chi(spec(\omega))$.

Rényi entropies

Definition (Rényi entropies)

$$H_\alpha(\mathbf{p}) := \frac{1}{1-\alpha} \log\left(\sum_i p_i^\alpha\right)$$

for $\alpha \in (0,1) \cup (1,\infty)$.

$$H_0(\mathbf{p}) := \lim_{\alpha\to 0} H_\alpha(\mathbf{p}) = -\log|\mathrm{supp}\ \mathbf{p}|.$$

$$H_1(\mathbf{p}) = \lim_{\alpha\to 1} H_\alpha(\mathbf{p}) = H(\mathbf{p}).$$

$$H_\infty(\mathbf{p}) = \lim_{\alpha\to\infty} H_\alpha(\mathbf{p}) = -\log\max_j p_j.$$

Concave, Schur-concave.

Proposition (Corollary of "spectral probabilities majorize".)

In a **perfect** *system (equivalently one with* **spectrality**, **projectivity**, *and* **STP**), *any concave and Schur-concave function of finegrained measurement outcome probabilities is minimized by the spectral measurement.*

So Rényi measurement entropy = spectral Rényi entropy.

Proposition

Assume **Weak Spectrality**, **Strong Symmetry**. *Then* $H_2^{prep} = H_2^{meas}$. *("Collision entropies".)*

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Proposition

Assume **Weak Spectrality**, **Strong Symmetry**. *If* $H_0^{prep} = H_0^{meas}$ *then* **No Higher Interference** *holds (and vice versa). (So systems are Jordan-algebraic.)*

Because $H_0^{prep} = H_0^{meas}$ is basically the covering law given the background assumptions.

Could enable some purification axiom that implies $H_0^{prep} = H_0^{meas}$ via steering (e.g. locally tomographic purification with identical marginals) to imply Jordan-algebraic systems.

Further observations:

Filters allow for **emergent classicality**: generalized *decoherence* onto classical subsets of the state space: $\omega \mapsto P_1\omega + P_2\omega + \cdots + P_n\rho$, P_i filters.

Open question: the operator projecting out higher-order interference is a projector. Is it positive? If so, **higher-order decoherence** possible. Could make HOI more plausible as potential trans-quantum physics.

Filters might be useful in **information-processing** protocols like computation, data compression ("project onto typical subspace"), coding.

Postulate 4 recalls Noether's theorem. Relation to a moment map?

von Neumann argument for spectral entropy may work. Does it require dilating certain transformations to reversible ones?

Failure of local tomography— a problem for extensivity?