Title: Resurgence, uniform WKB and complex instantonsÂ

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Abstract: The theory of resurgence connects perturbative and non-perturbative physics. Focusing on certain one-dimensional quantum mechanical systems with degenerate harmonic minima, I will explain how the resurgent trans-series expansions for the low lying energy eigenvalues follow from the exact quantization condition via the uniform WKB approach. In the opposite spectral region (with high lying eigenvalues), in contrast to the divergent asymptotic expansions expressed as trans-series, the relevant expansions are convergent. However, due to the poles in the expansion coefficients, they contain non-perturbative contributions which can be identified with complex instantons. I will demonstrate that in each spectral region there are striking relation between perturbative and non-perturbative expansions even though the nature of these expansions are very different. Notably, the quantum mechanical examples that I will discuss encode the vacua of certain supersymmetric gauge theories in their spectra.

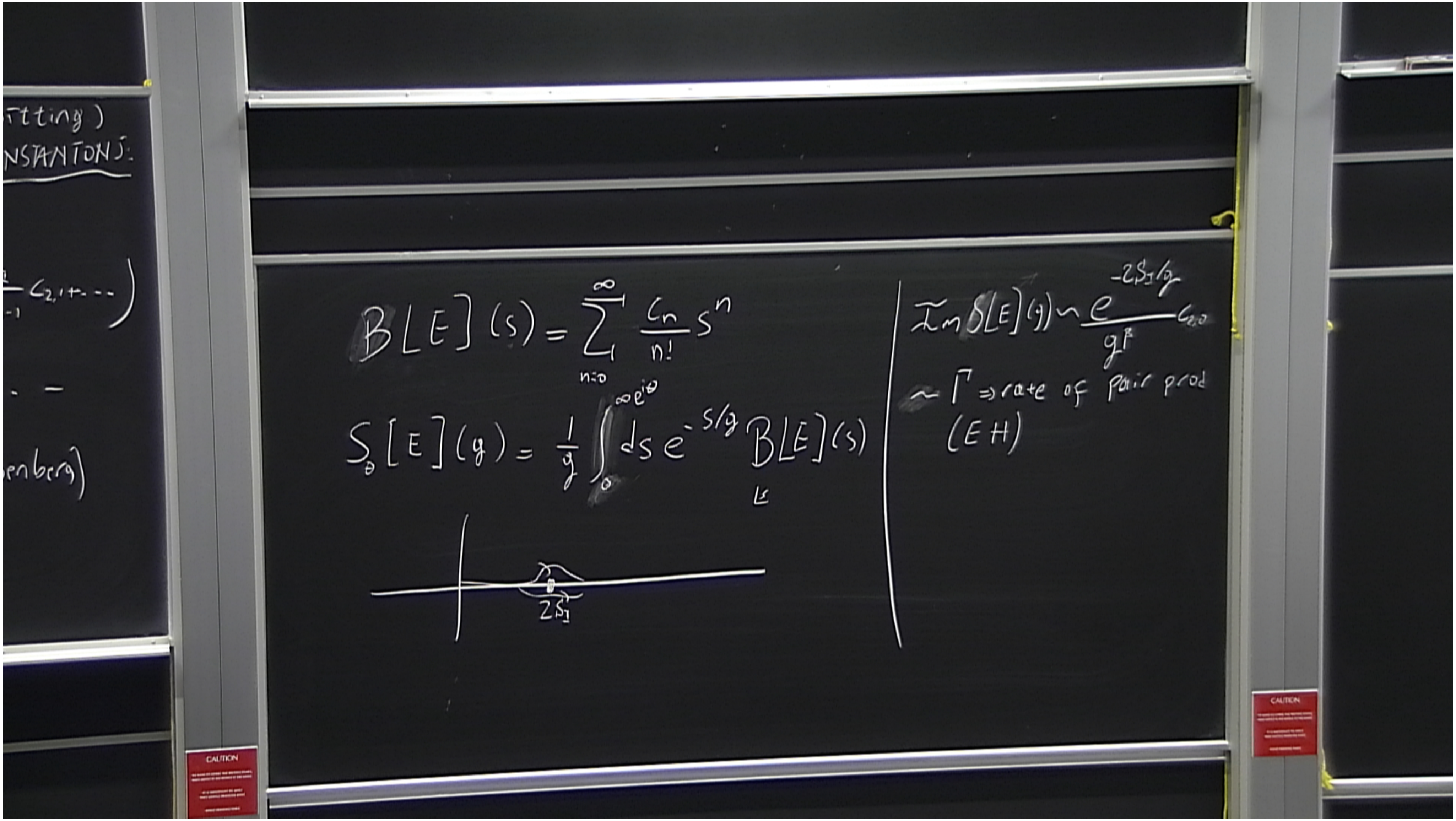
RESURGENCE, UNIFORM WKB & COMPLEX INSTANTONS (time permitting)

$$E(g) \sim \sum_{n=0}^{\infty} c_n g^n \quad ; \quad c_n \sim \frac{\Gamma(n+\beta)}{(2S_I)^{n+\beta}}\left(C_{1,0} + \frac{2S_I}{n+\beta-1} C_{1,1} + \dots\right)$$

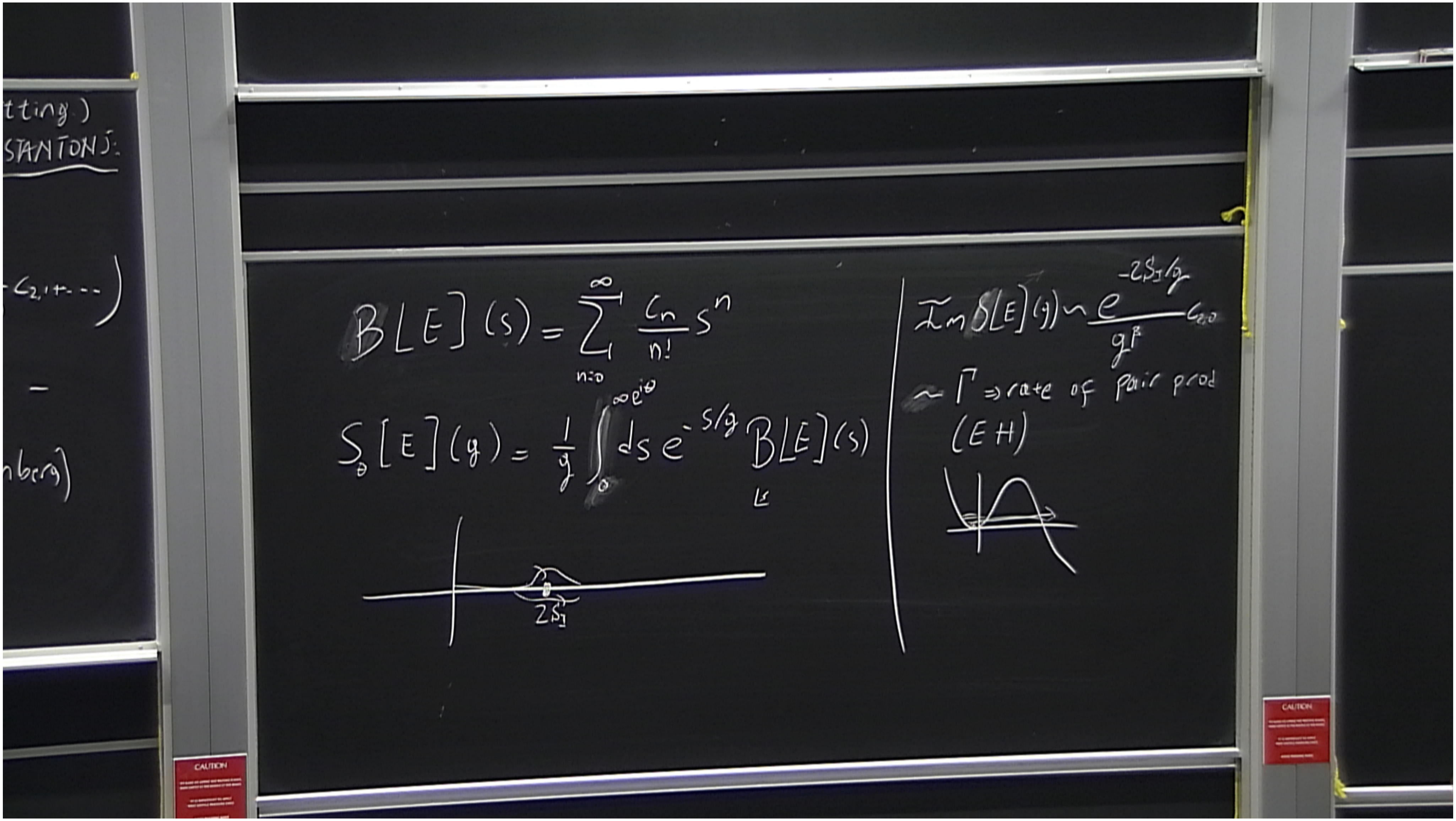
• QM, QFT, Strings, Matrix models, hydrodynamics, ... —

ex: $$c_n \sim \frac{(-1)^{n+1}}{8}\,\frac{\Gamma(2n+1)}{\pi^{2n+4}}\,\zeta(2n+4) \quad \text{(Euler Heisenberg)}$$

only $\vec{B}$



tting)
NSTANTONS:
enberg)
$B[E](s) = \sum_{n=0}^{\infty} \frac{c_n}{n!} s^n$
$S_\theta[E](g) = \frac{1}{g} \int_0^{\infty e^{i\theta}} ds\, e^{-s/g} B[E](s)$
$2S_I$
$\mathrm{Im}\, S[E](g) \sim \frac{e^{-2S_I/g}}{g^\beta} c_{2,0}$
$\sim \Gamma \Rightarrow$ rate of pair prod
(EH)
CAUTION
CAUTION



tting)
STANTONS:
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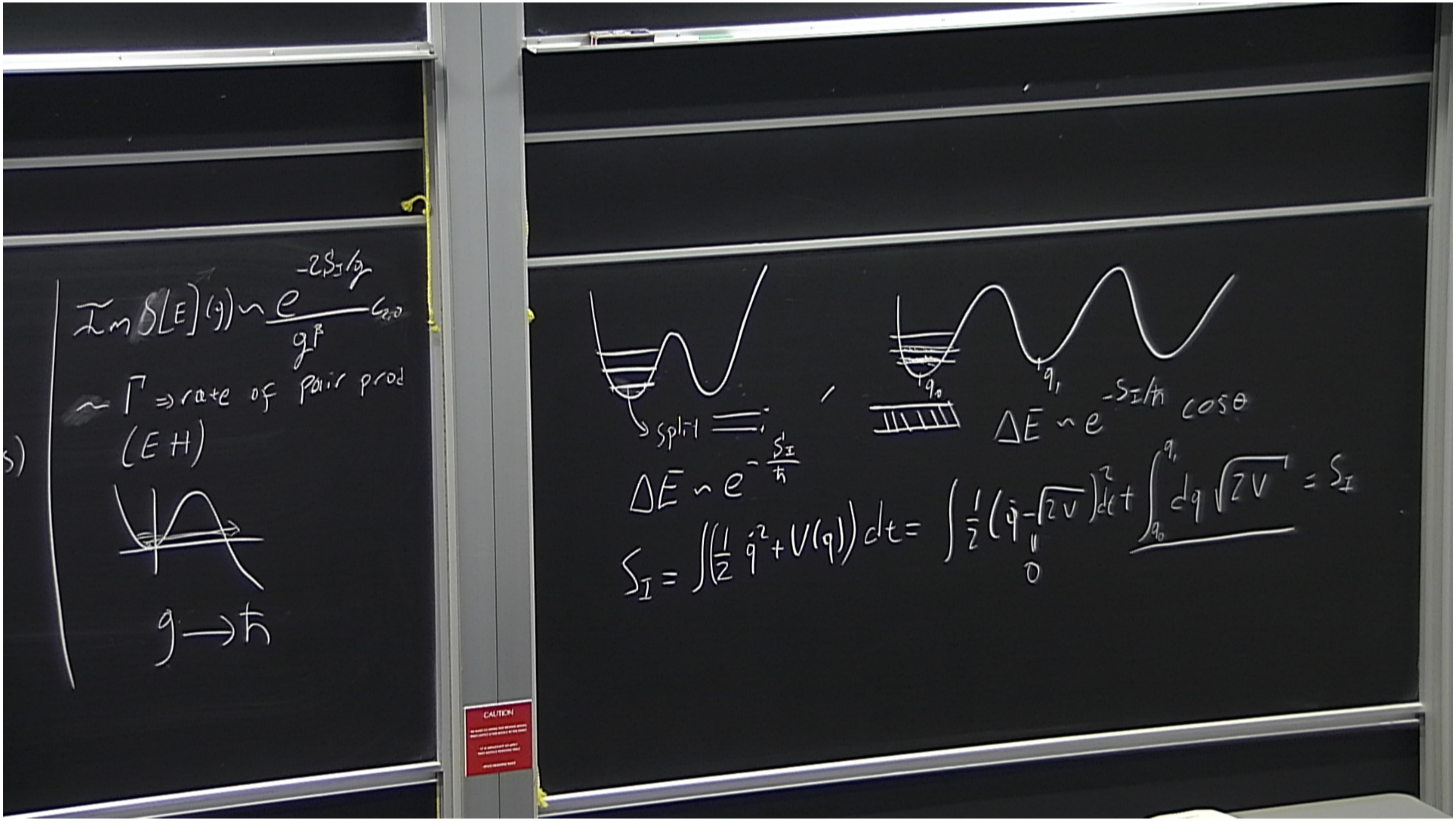
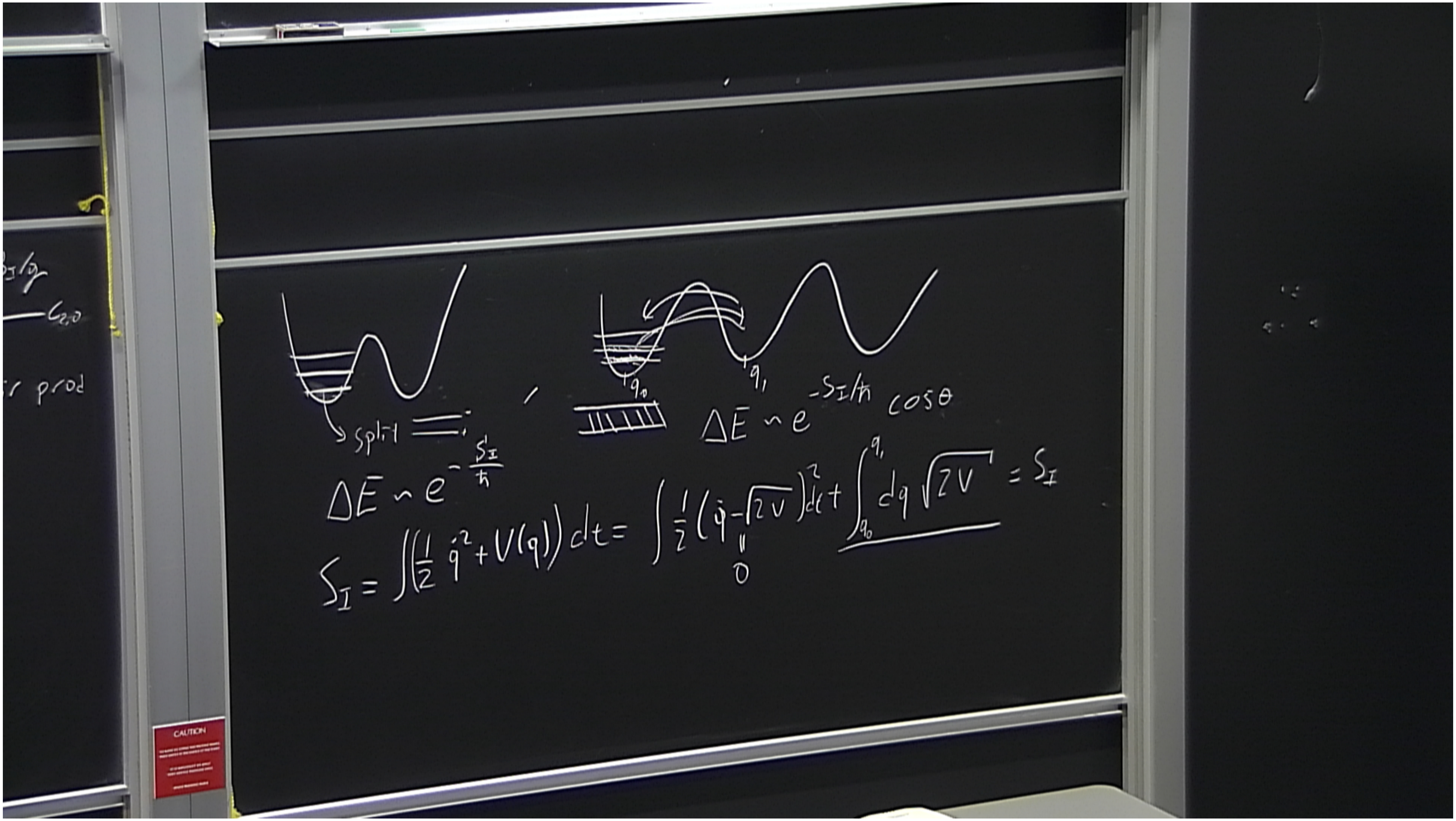


$\mathrm{Im}\, \delta[E](g) \sim \frac{e^{-2S_I/g}}{g^\beta}$
$\sim \Gamma \Rightarrow$ rate of pair prod
(E H)
$g \to \hbar$
split
q_0
q_1
$\Delta E \sim e^{-\frac{S_I'}{\hbar}}$
$\Delta E \sim e^{-S_I/\hbar} \cos\theta$
$S_I = \int \left(\frac{1}{2}\dot{q}^2 + V(q)\right) dt = \int \frac{1}{2}\left(\dot{q} - \sqrt{2V}\right)^2 dt + \int_{q_0}^{q_1} dq \sqrt{2V} = S_I$


split
$\Delta E \sim e^{-\frac{S_I}{\hbar}}$
q_0
q_1
$\Delta E \sim e^{-S_I/\hbar} \cos\theta$
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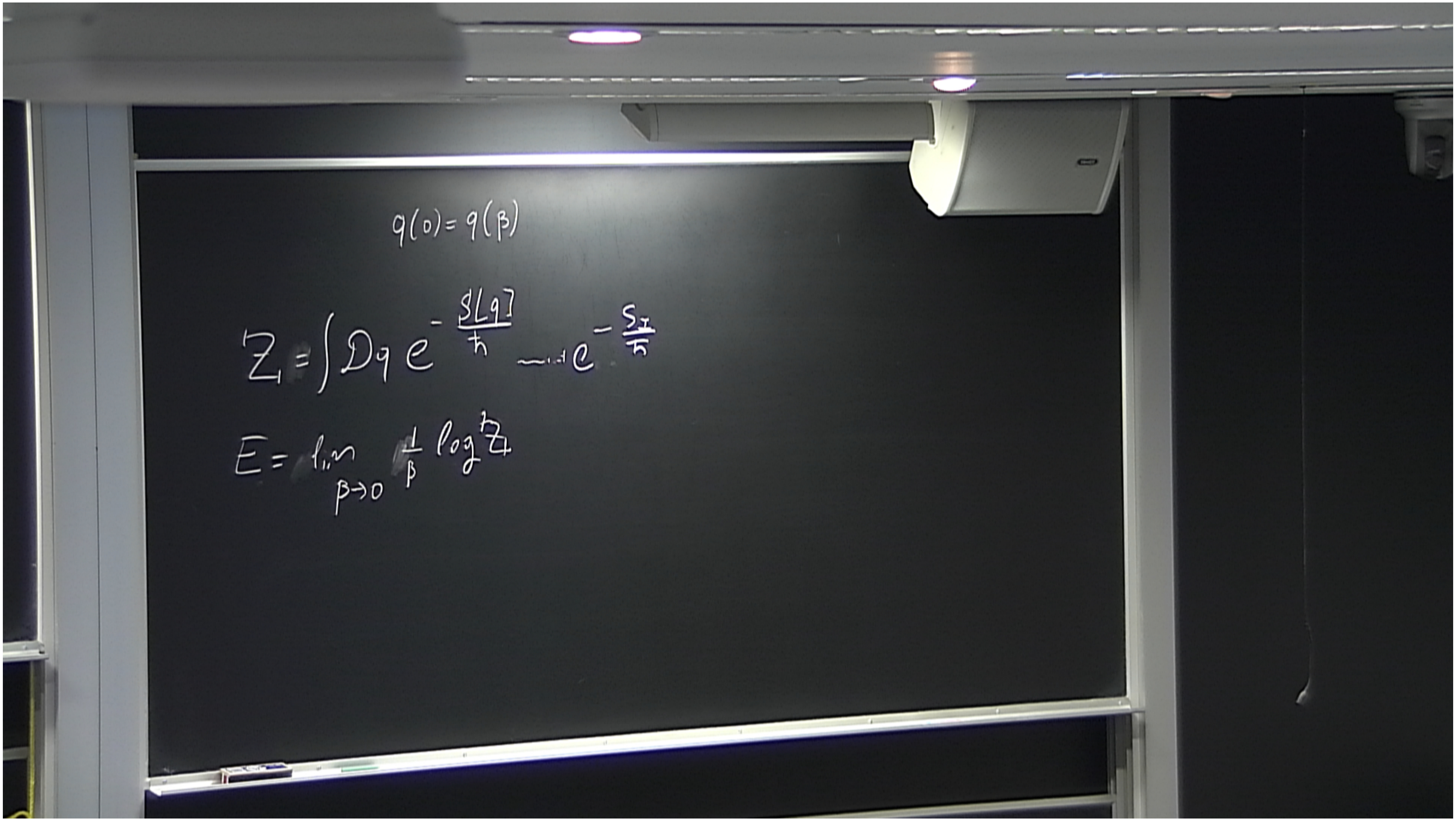
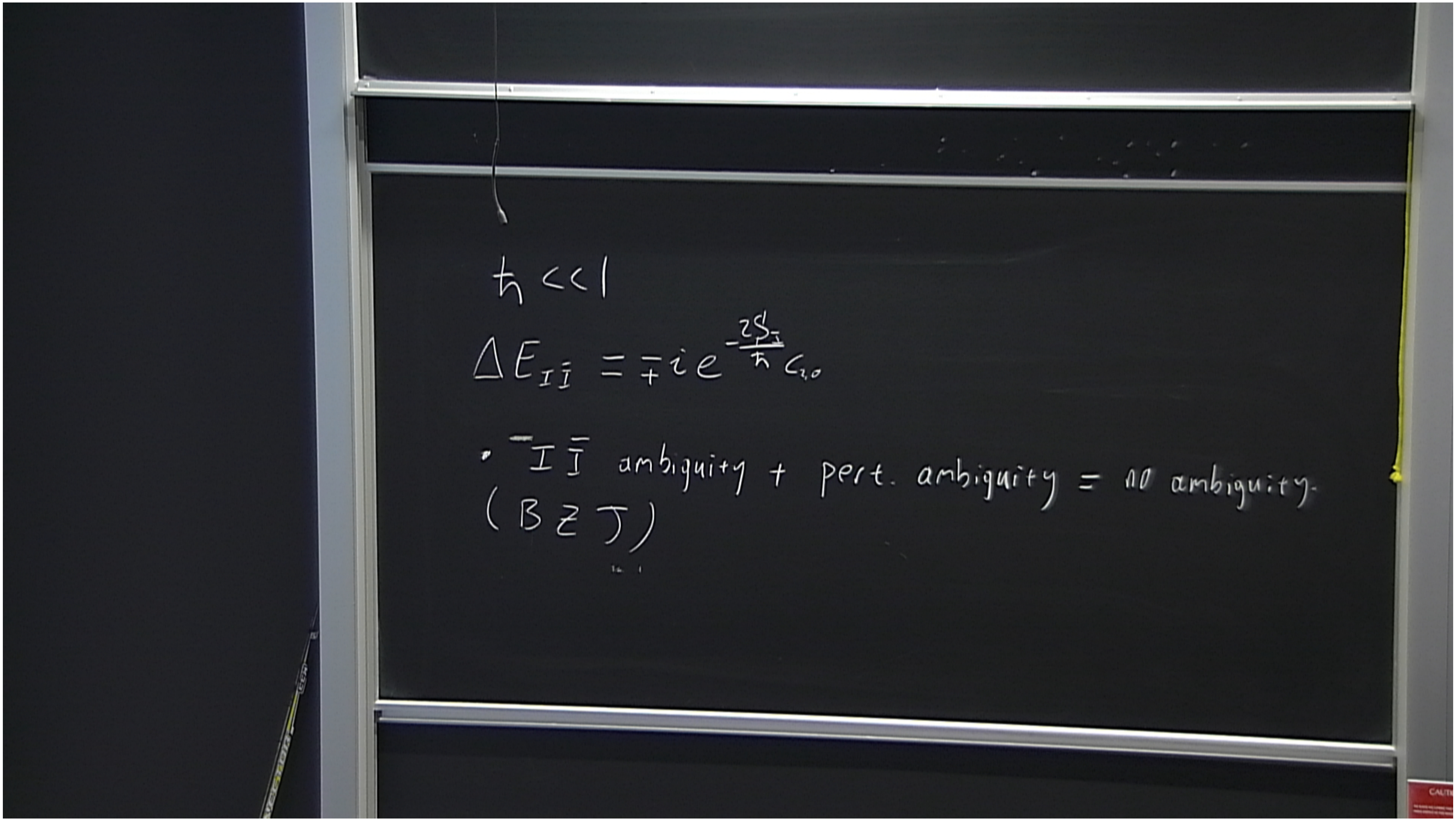
0
$C_{2,0}$
r prod


$q(0) = q(\beta)$
$Z = \int \mathcal{D}q \, e^{-\frac{S[q]}{\hbar}} \sim e^{-\frac{S_T}{\hbar}}$
$E = \lim_{\beta \to 0} -\frac{1}{\beta} \log Z$


$\hbar << 1$
$\Delta E_{I\bar{I}} = \mp i e^{-\frac{2S_I}{\hbar}} c_{2,0}$
• $I\bar{I}$ ambiguity + pert. ambiguity = no ambiguity.
(BZJ)

$\hbar \ll 1$

$$\Delta E_{I\bar{I}} = \mp i\, e^{-\frac{2S_I}{\hbar}} C_{1,0}$$

- $I\bar{I}$ ambiguity + pert. ambiguity = no ambiguity. (BZJ)
- I + $I I \bar{I}$ = no ambiguity
  $\vdots$

$$E = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k-1} \sum_{m=0}^{\infty} c^{(m)}_{n,k,\ell}\, \hbar^{n} e^{-kS_I/\hbar} \log^{\ell}\left(\frac{1}{\hbar}\right) e^{im\theta}$$

ity.

$$E = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k-1} \sum_{m=0}^{\infty} c^{(m)}_{n,k,l} \, \hbar^{n} \, e^{-kS_I/\hbar} \, \log^{l}\left(\frac{1}{\hbar}\right) e^{im\theta}$$

"topological charge"

$\rightarrow$ Bloch angle

perturbative fluct.

"instanton fugacity"

"q.z.m"

"trans-monomials"

"topological charge"

$$E = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k-1}\sum_{m=0}^{\infty} c^{(m)}_{n,k,l}\, \hbar^{n}\, e^{-kS_I/\hbar}\, \log^{l}\!\left(\frac{1}{\hbar}\right) e^{im\theta}$$

θ → Bloch angle

perturbative fluct. — "instanton fugacity" — "q.z.m" — "trans-monomials"

Stringent constraints on $c^{(m)}_{n,l,k}$'s $\Rightarrow$ "resurgence"

- $\hbar N \ll 1$ (validity), $\hbar N \gg 1$ has a very different expansion. ($N \to \infty$)

II) WEAK COUPLING & UNIFORM WKB:

$V = \cos z$ $\qquad -\frac{\hbar^2}{2}\psi'' + V\psi = u\psi$

$$\psi = \frac{D_\nu\left(\sqrt{\frac{2}{\hbar}}\, f(z)\right)}{\sqrt{f'(z)}} \rightarrow \text{parabolic cylinder} \Rightarrow D_\nu''(x) + \left(\nu + \frac{1}{2} - \frac{x^2}{4}\right) D_\nu(x) = 0$$

$$(V-u) - 2f^2 f'^2 - 2\hbar(\nu + 1/2) f'^2 + \frac{\hbar^2}{4}\sqrt{f'}\left(\frac{f''}{(f')^{3/2}}\right)' = 0$$

• I + III = no ambiguity

$D_\nu(x)=0$

$F \sim f_0 + \hbar f_1 + \dots$

$u \sim \underset{(=1)}{u_0} + \hbar u_1 + \dots$

i) Local analysis

$\Rightarrow$ solve for f_i & u_i recursively

$$u_\nu(\hbar) \sim \hbar(\nu+1/2) + \sum_{n=2}^{\infty}{}' \hbar^n P_n(\nu+1/2)$$

$\to$ polynomials of $\nu+1/2$ of order n

$\nu = N \Rightarrow$ perturbative exp.

ii) "Global" analysis

$$\psi(\pi) = e^{i\theta}\psi(-\pi) \quad \text{(boundary cond.)}$$

$$\arg \hbar = \pm\varepsilon$$

$$\mathcal{D}_\nu(x) \sim x^\nu e^{-\frac{x^2}{4}} F_-(x) + e^{\pm i\pi\nu}\frac{\sqrt{2\pi}}{\Gamma(-\nu)} x^{-1-\nu} e^{\frac{x^2}{4}} F_+(x)$$

$$F_-(x) \sim \sum_{k=0}^{\infty} \frac{(-\nu/2)_k (1/2-\nu/2)_k}{k!}\left(-\frac{2}{x^2}\right)^k$$

$$F_+(x) \sim \quad \ldots \quad \left(\frac{2}{x^2}\right)^k$$

plug the ansatz into (*). After all the dust settles

$$\left(\frac{32}{\hbar}\right)^{-B}\frac{e^{A/2}}{\Gamma(\frac{1}{2}-B)} + e^{\pm i\pi B}\left(\frac{32}{\hbar}\right)^{B}\frac{e^{-A/2}}{\Gamma(\frac{1}{2}+B)} = \sqrt{\frac{2}{\pi}}\cos\theta$$

$B = \nu + 1/2$

$$\frac{A}{2} = \frac{1}{\hbar}\int_{-\pi}^{\pi}\sqrt{2(V-u_0)}\,dz + \hbar(\dots) = \frac{S_I}{\hbar} + \hbar(\dots) + \hbar^2(\dots)$$

plug the ansatz into (*). After all the dust settles

$$\left(\frac{32}{\hbar}\right)^{-B}\frac{e^{A/2}}{\Gamma(\frac{1}{2}-B)} + e^{\pm i\pi B}\left(\frac{32}{\hbar}\right)^{B}\frac{e^{-A/2}}{\Gamma(\frac{1}{2}+B)} = \sqrt{\frac{2}{\pi}}\cos\theta$$

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- leading inst. order (pert.)

  $\nu = N$

- 1-instanton. $\nu = N + \delta\nu$, $\delta\nu \sim e^{-S/\hbar}\left(1+\hbar(\dots)\right)\cos\theta$

$$u_\nu(\hbar) \simeq \underbrace{u_N(\hbar)}_{\text{pert.}} + \underbrace{\delta\nu \frac{\partial u}{\partial N}}_{\text{1-instanton}}$$

$$\Delta u \sim e^{-\frac{S_I}{\hbar}} \cos\theta$$

just settles

$\sqrt{\frac{2}{\pi}}\cos\theta$

$\hbar^2(\dots)$

- leading inst. order (pert.)

  $\nu = N$

- 1-instanton. $\nu = N + \delta\nu$, $\delta\nu \sim e^{-S/\hbar}(1+\hbar(\dots))\cos\theta$

$$u_\nu(\hbar) \simeq \underbrace{u^0_N(\hbar)}_{\text{pert.}} + \underbrace{\delta\nu \frac{\partial u}{\partial N}}_{\text{1-instanton}}$$

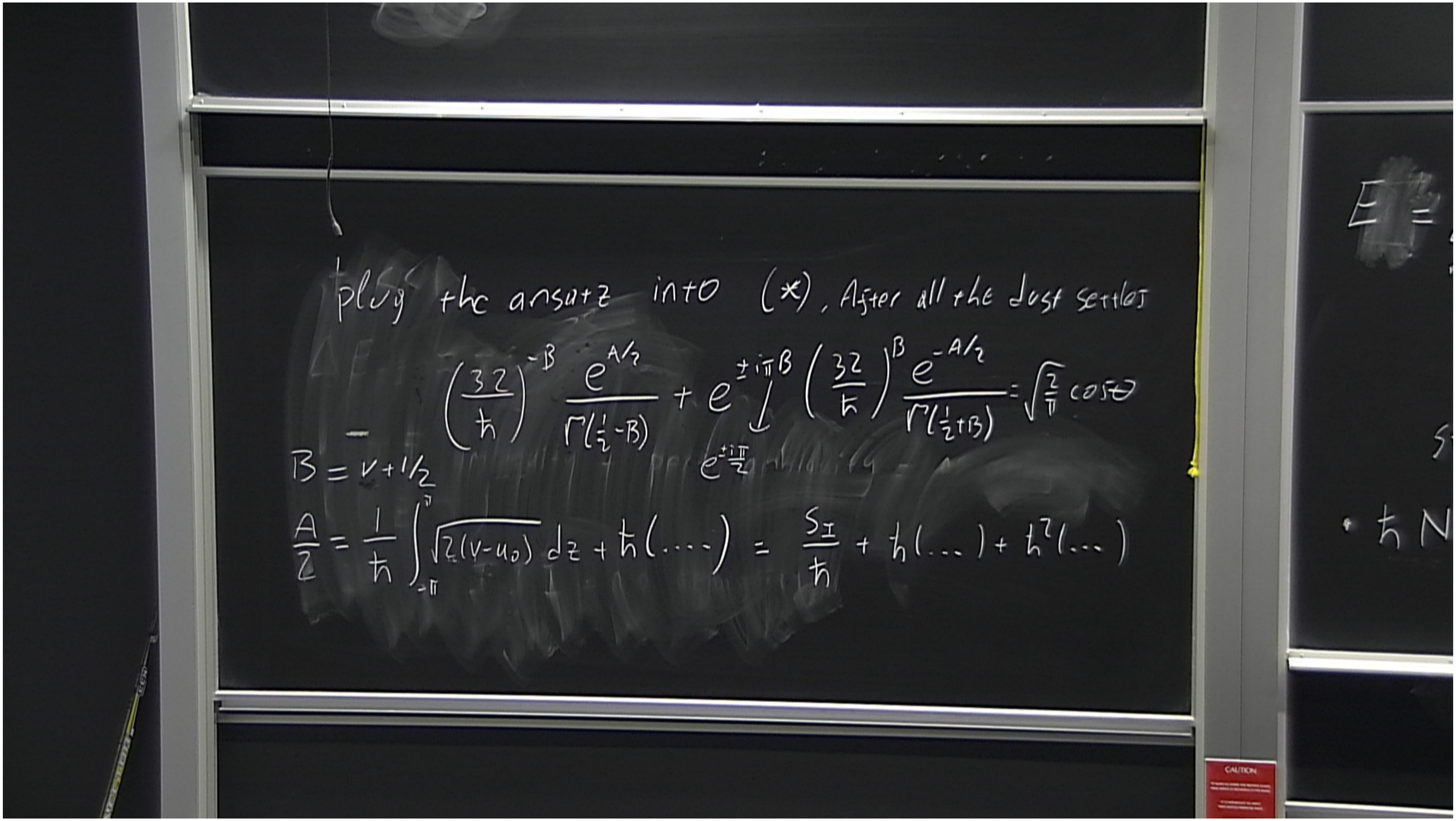
$$u^0_B \sim \hbar B + \sum_{n=2}^{\infty} \hbar^n P_n(B)$$

$$\Delta u \sim e^{-\frac{S_I}{\hbar}} \cos\theta$$

• 2-instanton

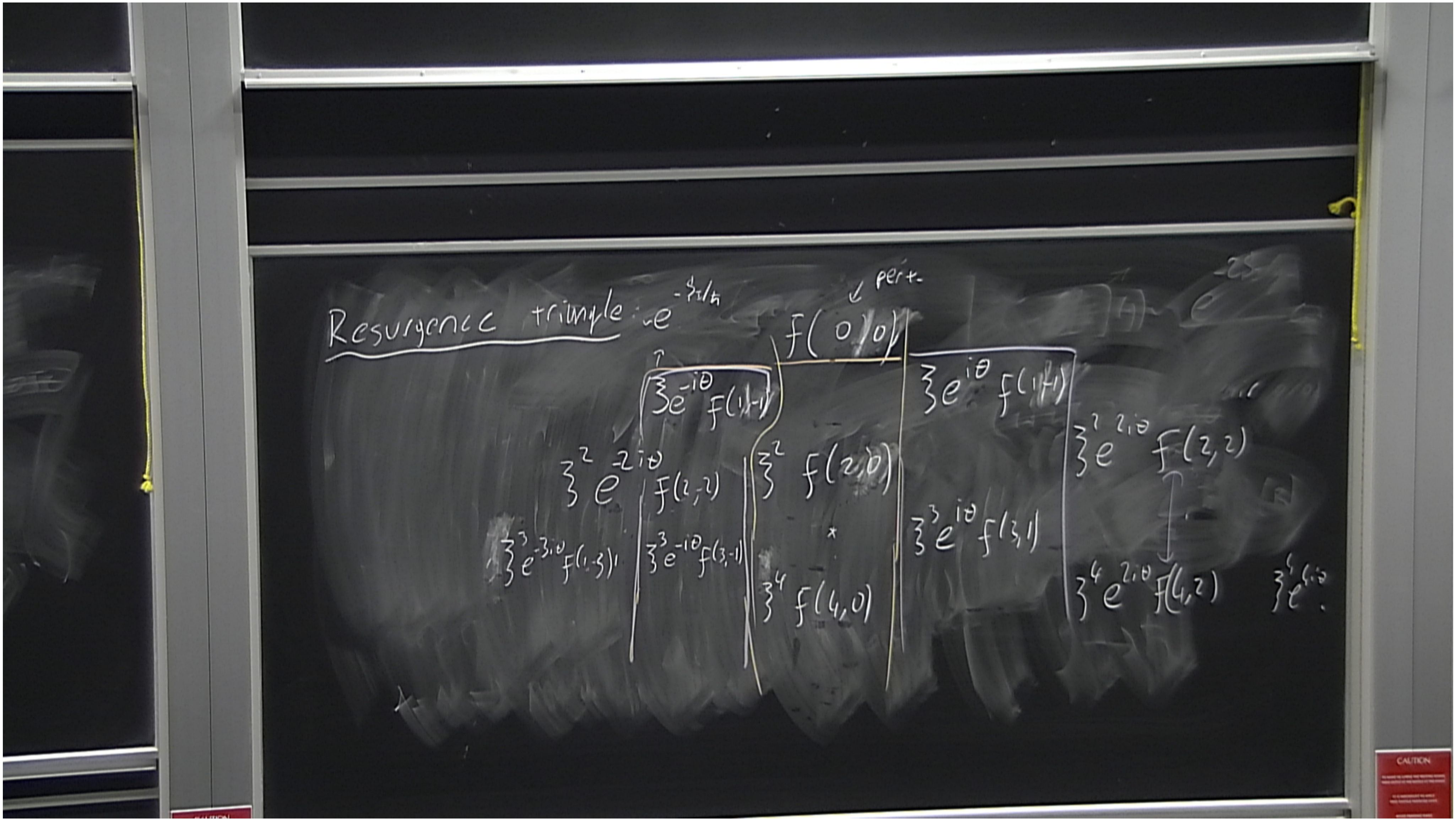
$$\nu = N + \delta\nu \qquad \delta\nu \hookrightarrow e^{-S_I/\hbar}(\ldots) + e^{-2S_I/\hbar}(\ldots)$$

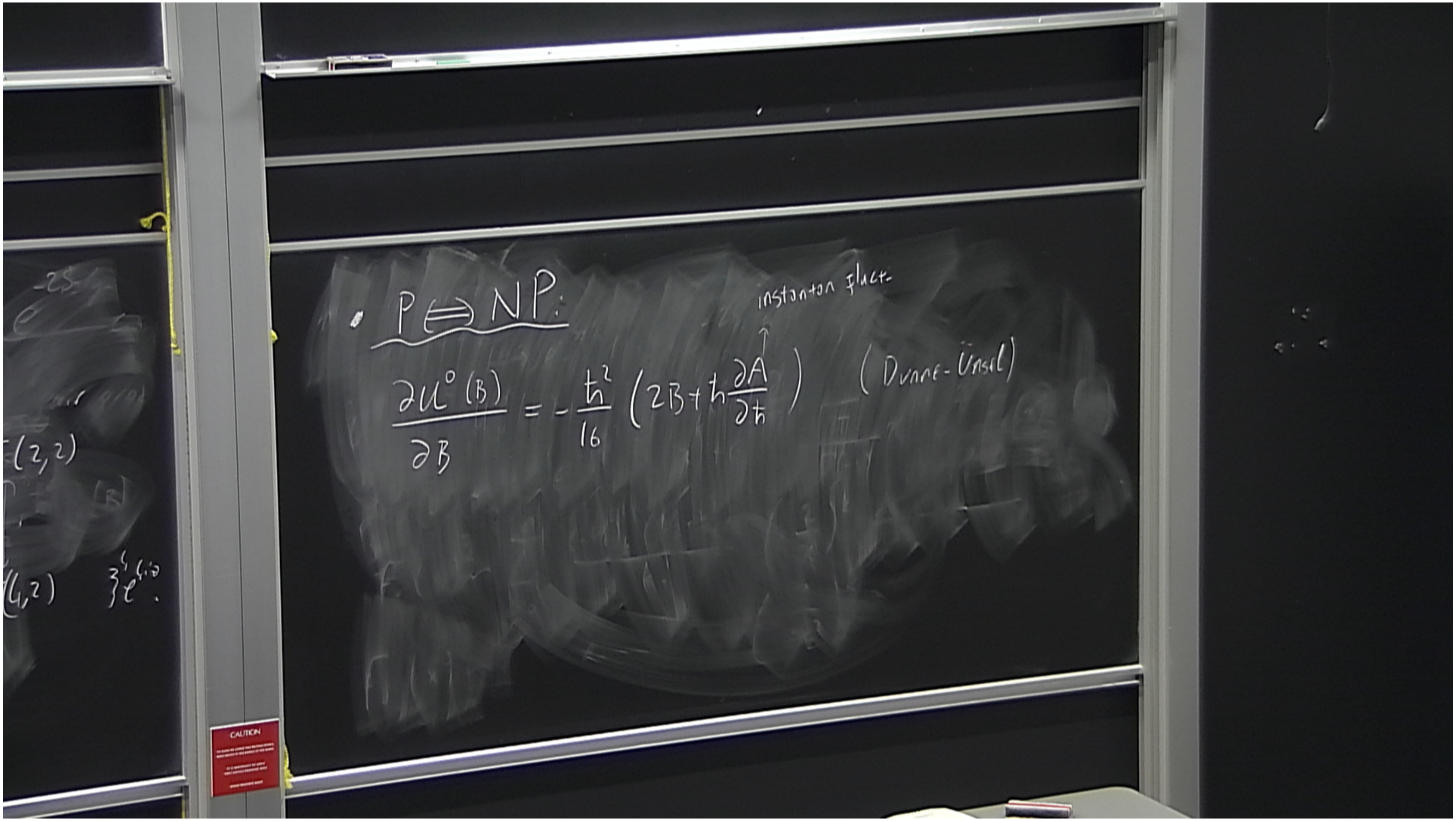
$$u(\hbar) \sim u^0_N(\hbar) + \delta\nu \frac{\partial u^0}{\partial N} + \frac{1}{2}(\delta\nu)^2 \frac{\partial^2 u^0}{\partial N^2}$$

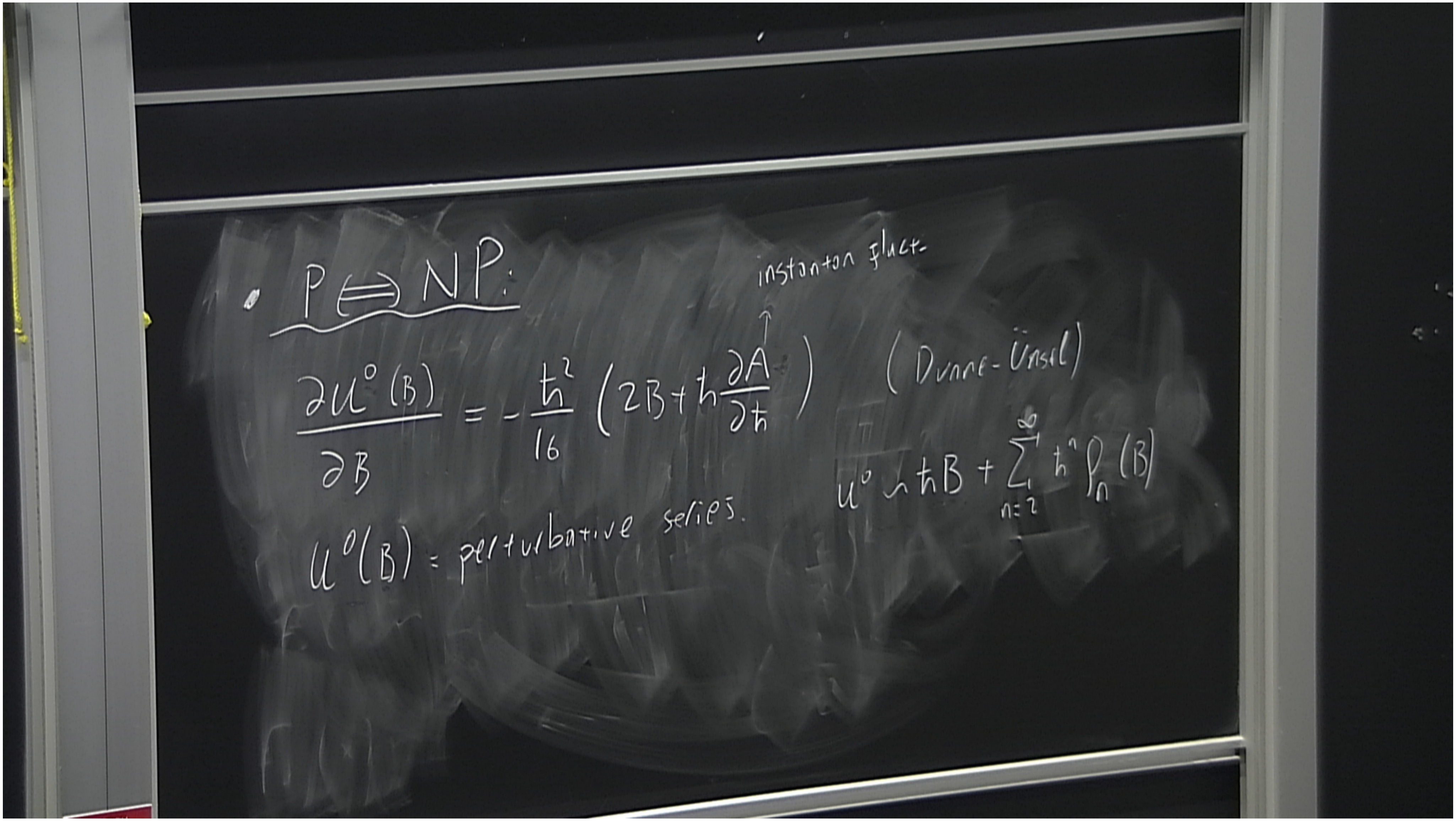

plug the ansatz into (*), After all the dust settles
$\left(\frac{32}{\hbar}\right)^{-B}\frac{e^{A/2}}{\Gamma(\frac{1}{2}-B)} + e^{\pm i\pi B}\left(\frac{32}{\hbar}\right)^{B}\frac{e^{-A/2}}{\Gamma(\frac{1}{2}+B)} = \sqrt{\frac{2}{\pi}}\cos\theta$
$e^{\pm i\frac{\pi}{2}}$
$B = \nu + 1/2$
$\frac{A}{2} = \frac{1}{\hbar}\int_{-\pi}^{\pi}\sqrt{2(v-u_0)}\,dz + \hbar(\ldots) = \frac{S_I}{\hbar} + \hbar(\ldots) + \hbar^2(\ldots)$

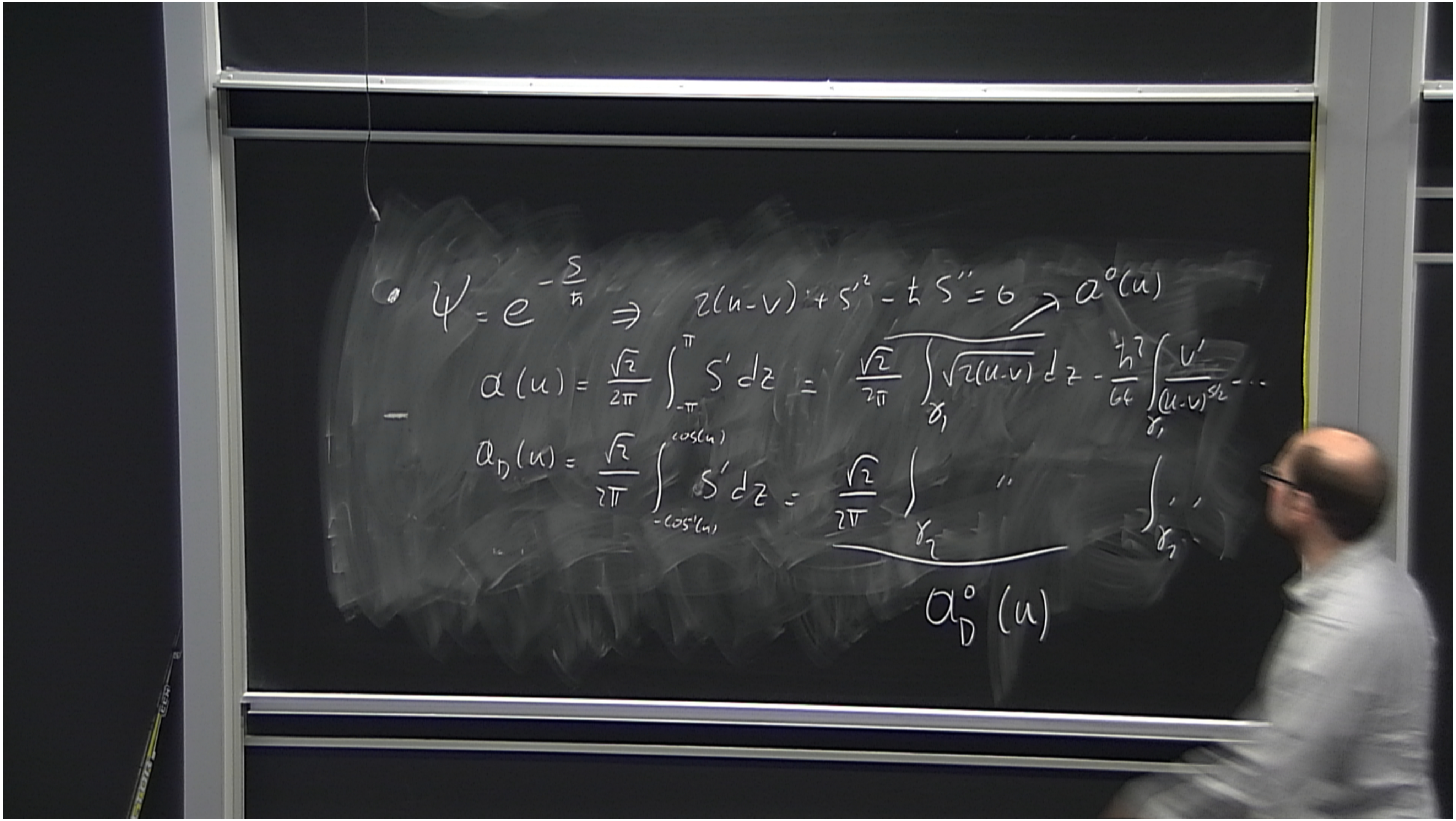
$$C_n \sim \frac{n!}{(2S_I)^n}\left(1 - \frac{5}{2}\frac{1}{n} - \frac{13}{8}\frac{1}{n(n-1)} - \cdots\right)$$

$$\mathrm{Im}\, U_0^{(2)} \sim e^{-\frac{2S_I}{\hbar}}\left(1 - \frac{5}{2}\left(\frac{\hbar}{2S_I}\right)^2 - \frac{13}{8}\left(\frac{\hbar}{2S_I}\right)^4 - \cdots\right)$$


Resurgence triangle: $e^{-S_I/\hbar}$
pert.
$f(0,0)$
$\xi e^{-i\theta} f(1,-1)$
$\xi e^{i\theta} f(1,1)$
$\xi^2 e^{-2i\theta} f(2,-2)$
$\xi^2 f(2,0)$
$\xi^2 e^{2i\theta} f(2,2)$
$\xi^3 e^{-3i\theta} f(1,-3)$
$\xi^3 e^{-i\theta} f(3,-1)$
$\xi^3 e^{i\theta} f(3,1)$
$\xi^4 f(4,0)$
$\xi^4 e^{2i\theta} f(4,2)$


P ⟺ NP:
instanton fluct.
$\frac{\partial u^0(B)}{\partial B} = -\frac{\hbar^2}{16}\left(2B + \hbar\frac{\partial A}{\partial \hbar}\right)$
(Dunne-Ünsal)

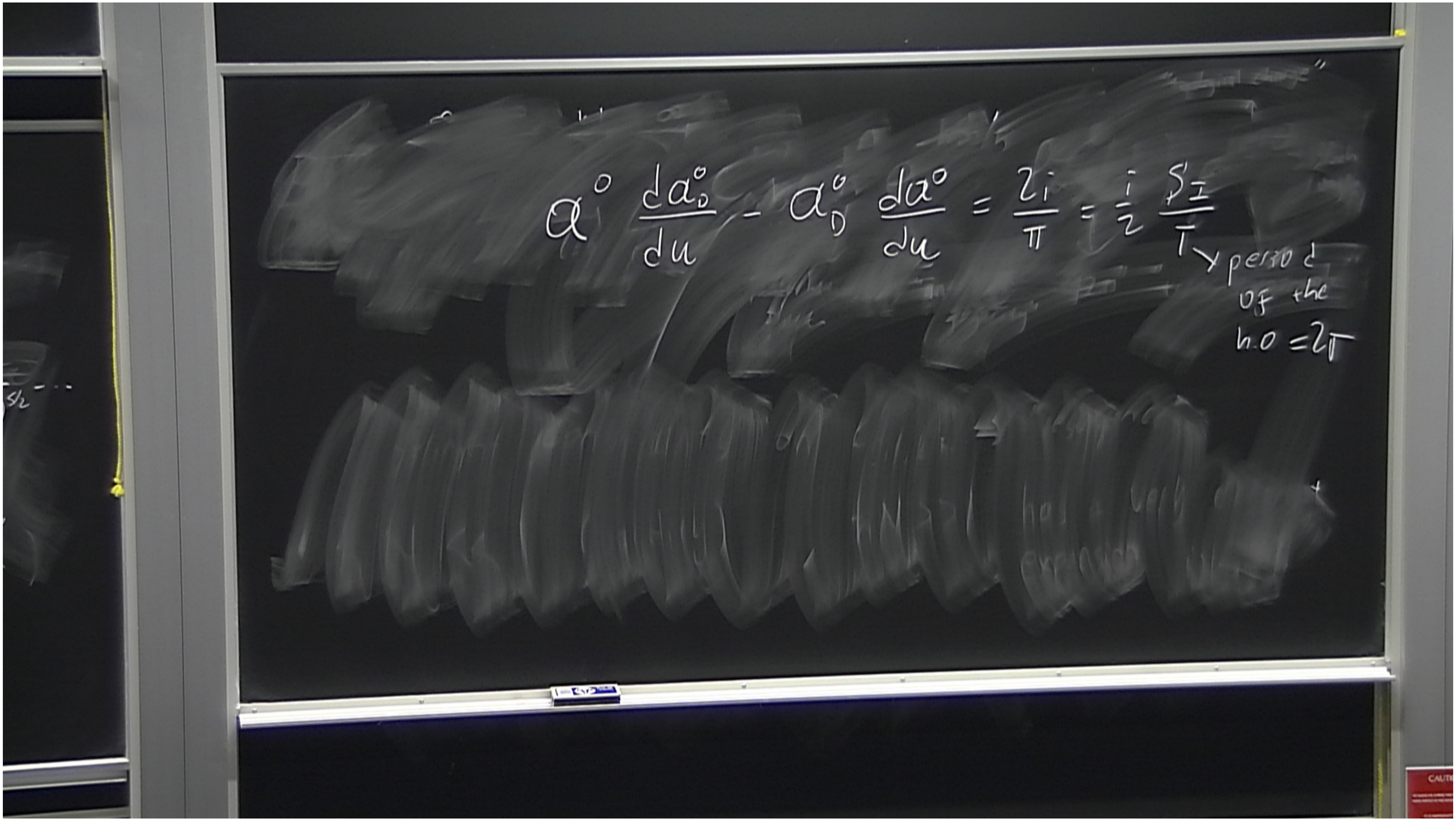

P ⟺ NP:
instanton fluct.
$\frac{\partial u^0(B)}{\partial B} = -\frac{\hbar^2}{16}\left(2B + \hbar\frac{\partial A}{\partial \hbar}\right)$
(Dunne-Ünsal)
$u^0 \sim \hbar B + \sum_{n=2}^{\infty} \hbar^n \rho_n(B)$
$u^0(B)$ = perturbative series.


$\Psi = e^{-\frac{S}{\hbar}} \Rightarrow 2(u-V) + S'^2 - \hbar S'' = 0 \rightarrow a^0(u)$
$a(u) = \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} S' dz = \frac{\sqrt{2}}{2\pi} \int_{\gamma_1} \sqrt{2(u-V)}\, dz - \frac{\hbar^2}{64} \int_{\gamma_1} \frac{V'}{(u-V)^{5/2}} - \dots$
$a_D(u) = \frac{\sqrt{2}}{2\pi} \int_{-\cos^{-1}(u)}^{\cos(u)} S' dz = \frac{\sqrt{2}}{2\pi} \int_{\gamma_2} \text{''} \int_{\gamma_2} \dots$
$a_D^0(u)$

$$\bullet \quad \psi = e^{-\frac{S}{\hbar}} \Rightarrow 2(u-V) + S'^2 - \hbar S'' = 0 \quad a^0(u)$$

$$a(u) = \frac{\sqrt{2}}{2\pi}\int_{-\pi}^{\pi} S'\,dz = \overbrace{\frac{\sqrt{2}}{2\pi}\int_{\gamma_1}\sqrt{2(u-V)}\,dz}^{a^0(u)} - \frac{\hbar^2}{64}\int_{\gamma_1}\frac{V'}{(u-V)^{5/2}} \cdots$$

$$a_D(u) = \frac{\sqrt{2}}{2\pi}\int_{-\cos^{-1}(u)}^{\cos(u)} S'\,dz = \underbrace{\frac{\sqrt{2}}{2\pi}\int_{\gamma_2} \;''}_{a_D^0(u)} \quad \int_{\gamma_2} \cdots$$


$a^0 \frac{da_D^0}{du} - a_D^0 \frac{da^0}{du} = \frac{2i}{\pi} = \frac{i}{2}\frac{S_I}{T}$
period of the h.o $= 2\pi$

leading order in $\hbar$ $\leftarrow$ $a^0 \dfrac{da_D^0}{du} - a_D^0 \dfrac{da^0}{du} = \dfrac{2i}{\pi} = \dfrac{i}{2}\dfrac{S_I}{T}$

T period of the h.o $= 2\pi$

$$\left(a - \hbar\frac{\partial a}{\partial \hbar}\right)\frac{\partial a_D}{\partial u} - \left(a_D - \hbar\frac{\partial a_D}{\partial \hbar}\right)\frac{\partial a}{\partial u} = \frac{2i}{\pi}$$

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$$\left(a - \hbar\frac{\partial a}{\partial \hbar}\right)\frac{\partial a_D}{\partial u} - \left(a_D - \hbar\frac{\partial a_D}{\partial \hbar}\right)\frac{\partial a}{\partial u} = \frac{2i}{\pi}$$

(G.B. Dunne)