Title: Resurgent analysis and its applications to the Witten Laplacian

Date: May 30, 2015 11:00 AM

URL: http://pirsa.org/15050069

Abstract: The first lecture will be devoted to the review of the classical theory of the Witten Laplacian, the second -- to the concepts of resurgent analysis. The third -- to applications of the resurgent analysis to the Witten Laplacian. Time permitting, we will touch upon some foundational questions of resurgent analysis.

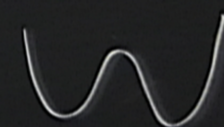
## Microlocal properties of sheaves & complex WKB.

$$(-\hbar^2 \partial_x^2 + V(x))\,\psi(x,\hbar) = 0. \qquad 1/\hbar \leftrightarrow s$$

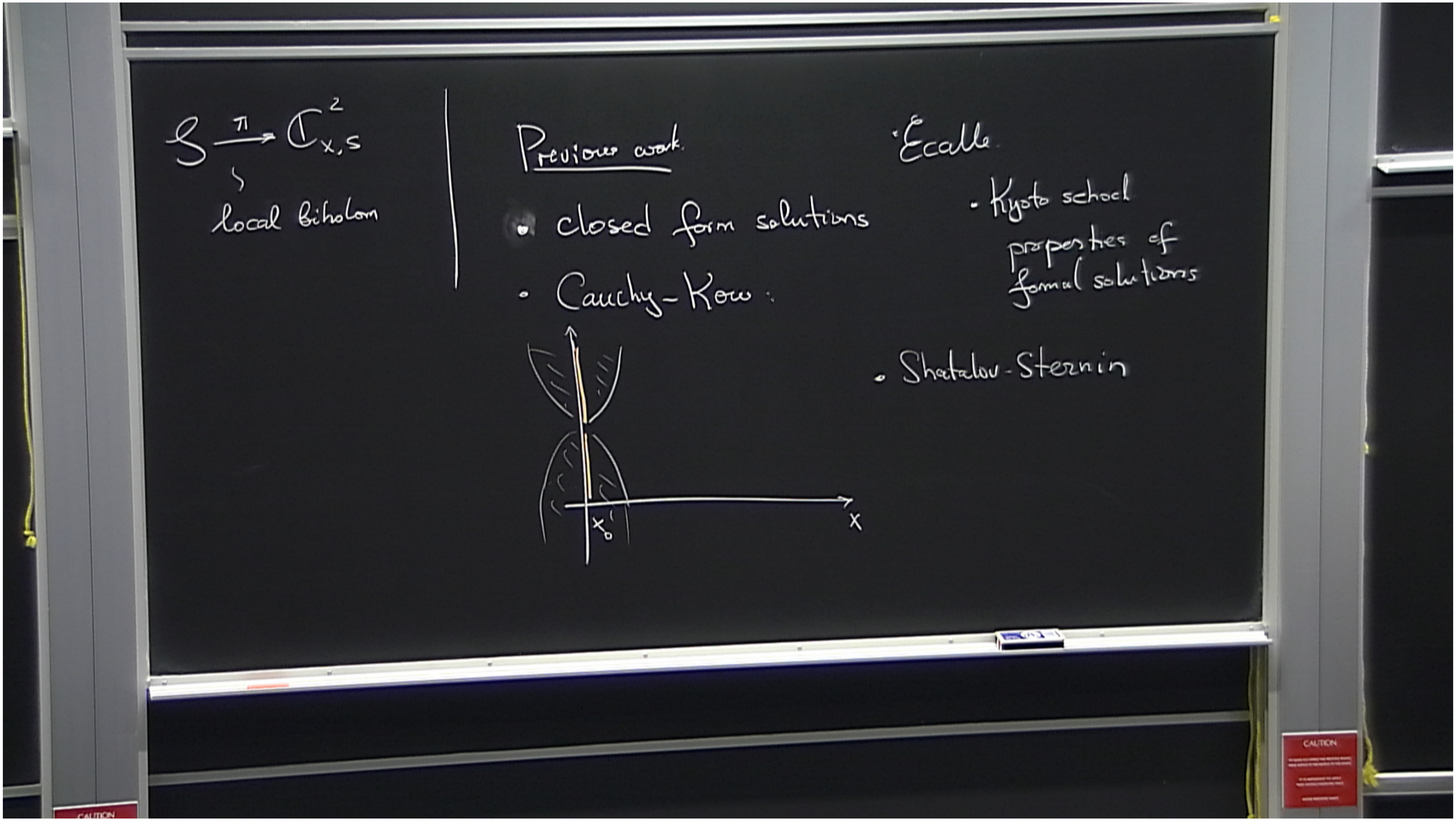
$$(-\partial_x^2 + V(x)\partial_s^2)\,\Psi(x,s) = 0.$$

## Microlocal properties of sheaves & complex WKB.

$$\left(-\hbar^2 \partial_x^2 + V(x)\right)\psi(x,\hbar) = 0. \qquad 1/\hbar \leftrightarrow s$$



$$\begin{cases} \left(-\partial_x^2 + V(x)\partial_s^2\right)\Psi(x,s) = 0. \\ x_0 \quad V(x_0) \neq 0. \qquad \Psi(x,s)\big|_{x_0} = \psi_0(s), \ \dfrac{\partial}{\partial x}\Psi(x,s)\Big|_{x_0=1} = \psi_1(s) \end{cases}$$


$S \xrightarrow{\pi} \mathbb{C}^2_{x,s}$
local biholom
Previous work
• closed form solutions
• Cauchy-Kow
$x_0$
$x$
Écalle
• Kyoto school
properties of formal solutions
• Shatalov-Sternin

$$f' = A(x)\, f$$

$$f(x) = c_0 + \int_{x_0}^{x} A(t)\, f(t)\, dt$$

$$f(x) = c_0 + \int_{x_0}^{x} A(t)\, c_0\, dt + \dots + \underbrace{\int_0^{x} dx_1\, A(x_1) \int_0^{x_1} A(x_2)\, dx_2 + \dots}_{n\text{-fold.}}$$

$$f = A(x) f$$

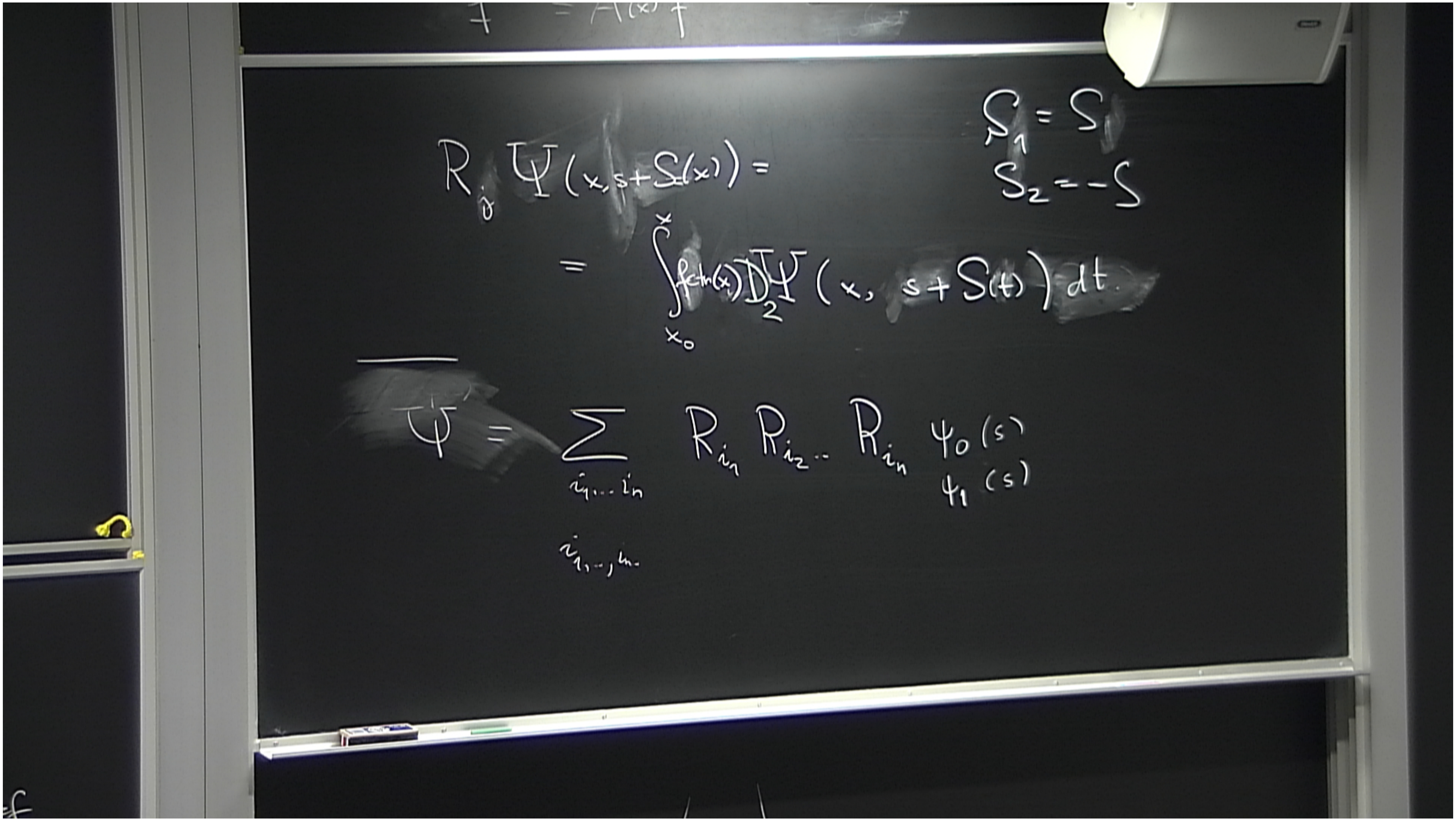
$$S_1 = S$$
$$S_2 = -S$$

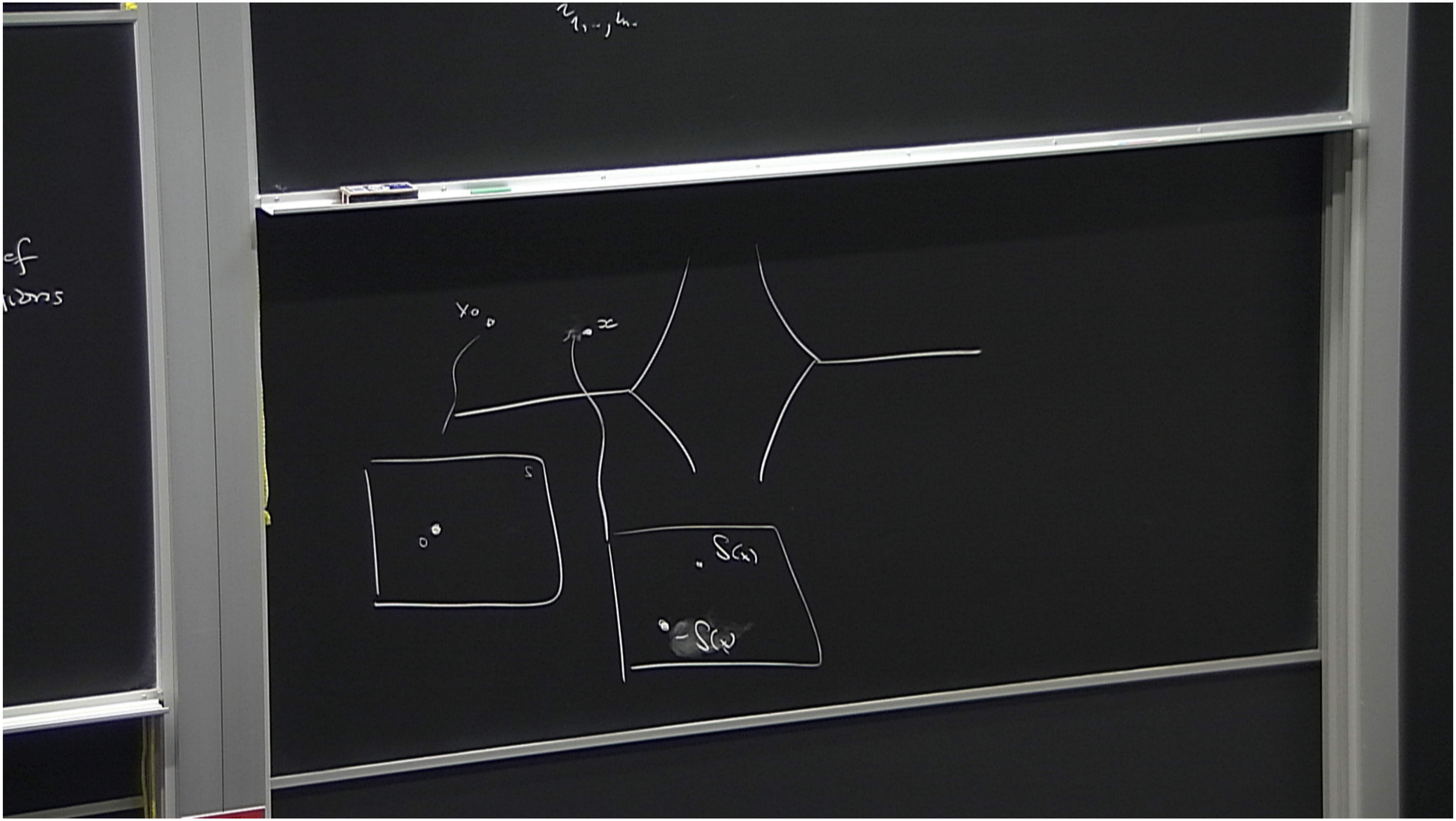
$$R_j \Psi(x, s + S(x)) =$$

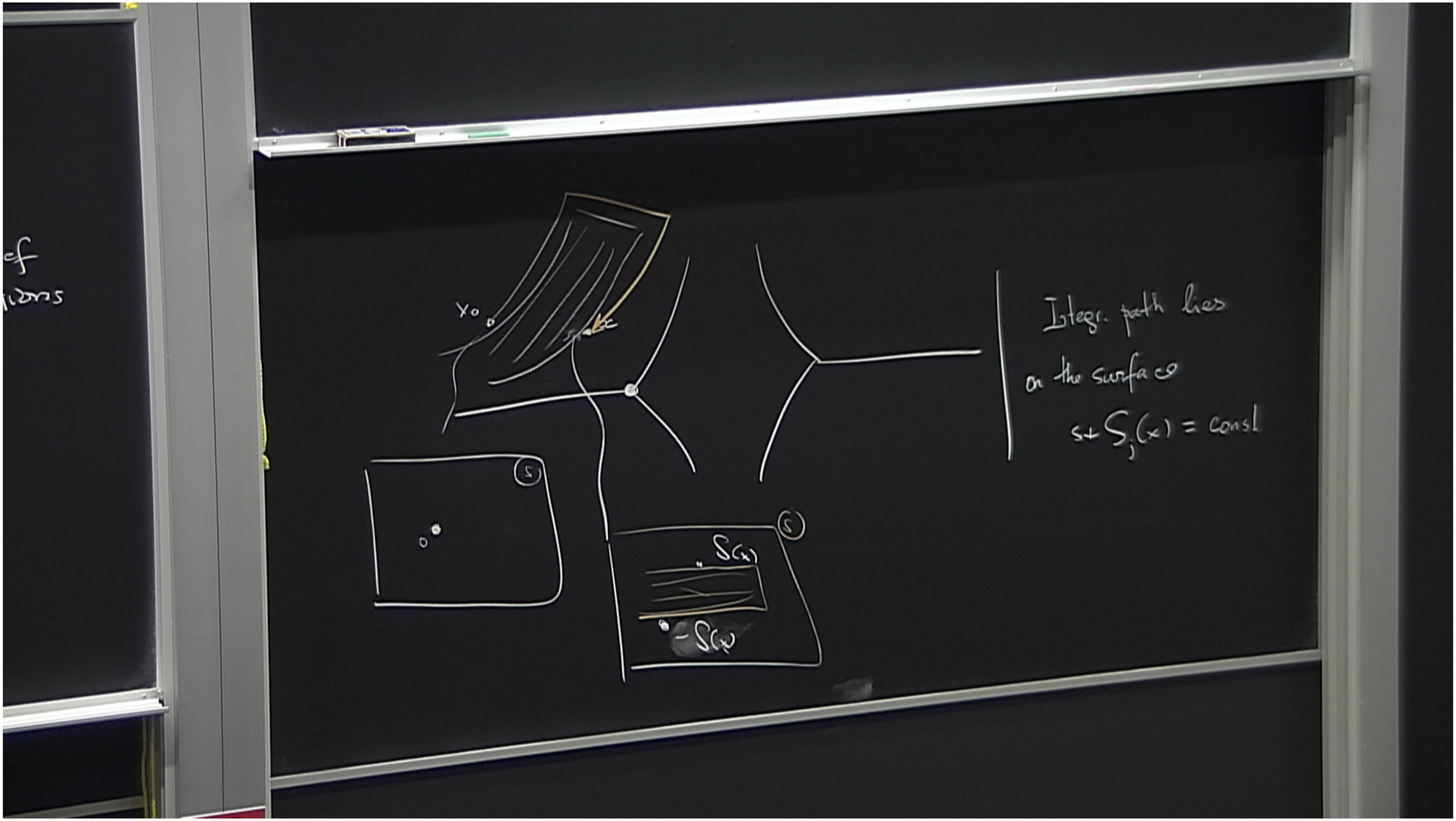
$$= \int_{x_0}^{x} f(t-h(x)) D_2 \Psi(x, s + S(t))\, dt.$$

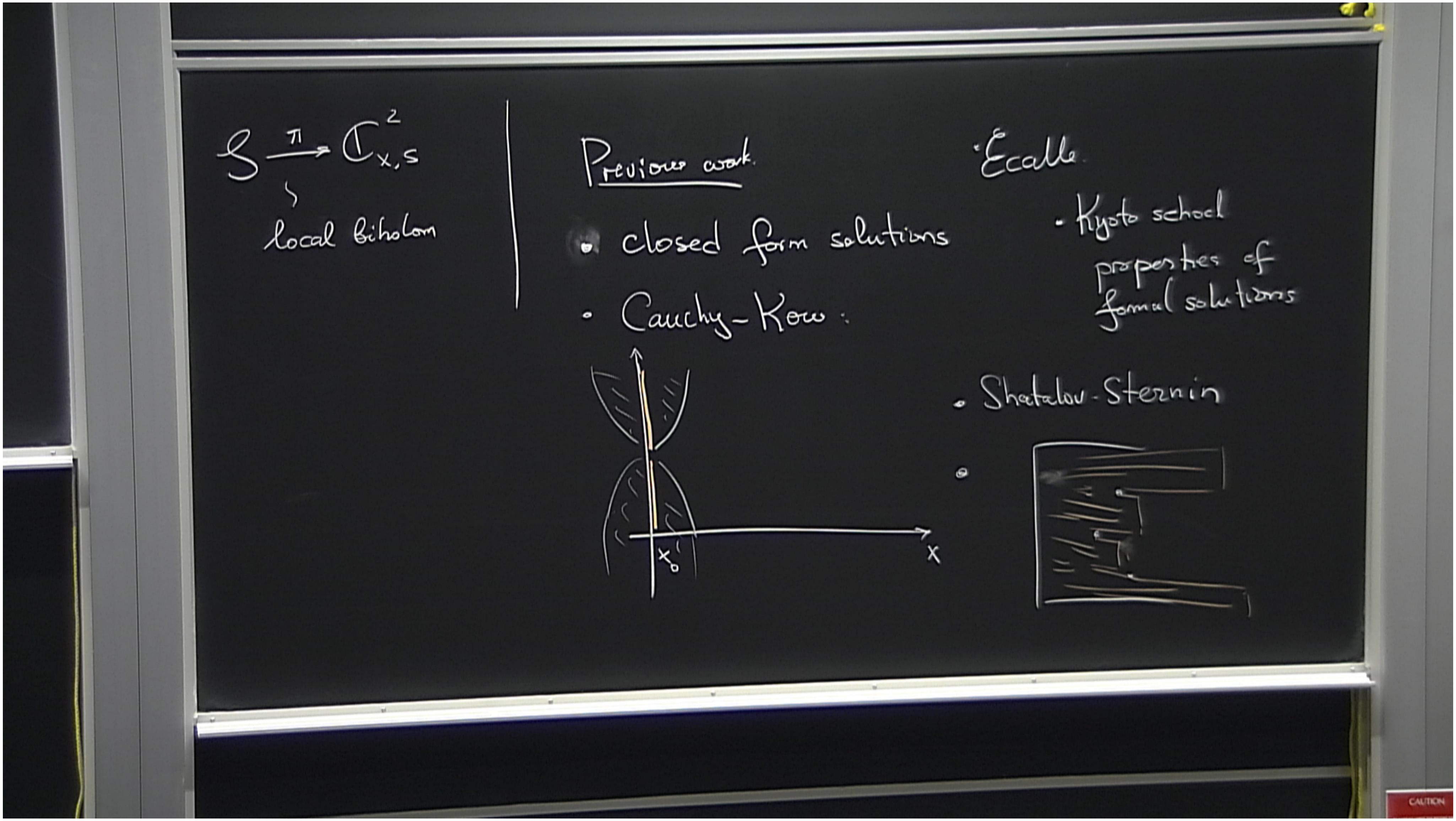
$$\Psi = \sum_{i_1 \ldots i_n} R_{i_1} R_{i_2} \ldots R_{i_n} \psi_0(s)$$
$$\psi_1(s)$$

$$i_1, \ldots, i_n$$


R_j Ψ(x, s+S(x)) =
S_1 = S
S_2 = −S
= ∫_{x_0}^{x} D_2 Ψ(x, s+S(t)) dt
Ψ = Σ_{i_1...i_n} R_{i_1} R_{i_2} .. R_{i_n} ψ_0(s)
ψ_1(s)
i_1, .., i_n


x₀
x
S(x)
-S(x)


x0
S(x)
-S(x)
Integr. path lies
on the surface
s + Sj(x) = const

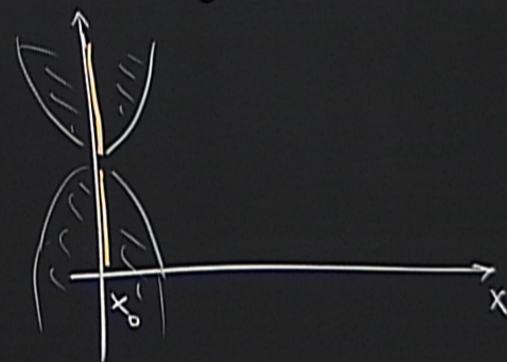

S →π C²x,s
local bihol
Previous work.
closed form solutions
Cauchy-Kow:
x₀
X
Écalle.
Kyoto school
properties of
formal solutions
Shatalov-Sternin

$$\mathcal{S} \xrightarrow{\pi} \mathbb{C}^2_{x,s}$$

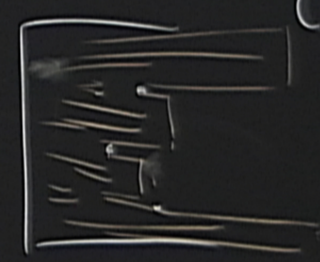
local biholom

## Previous work.

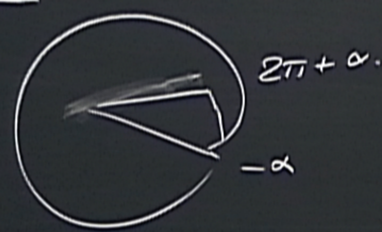
- closed form solutions
- Cauchy-Kow :




- Écalle.
- Kyoto school
  properties of formal solutions
- Shatalov-Sternin
- C. Simpson



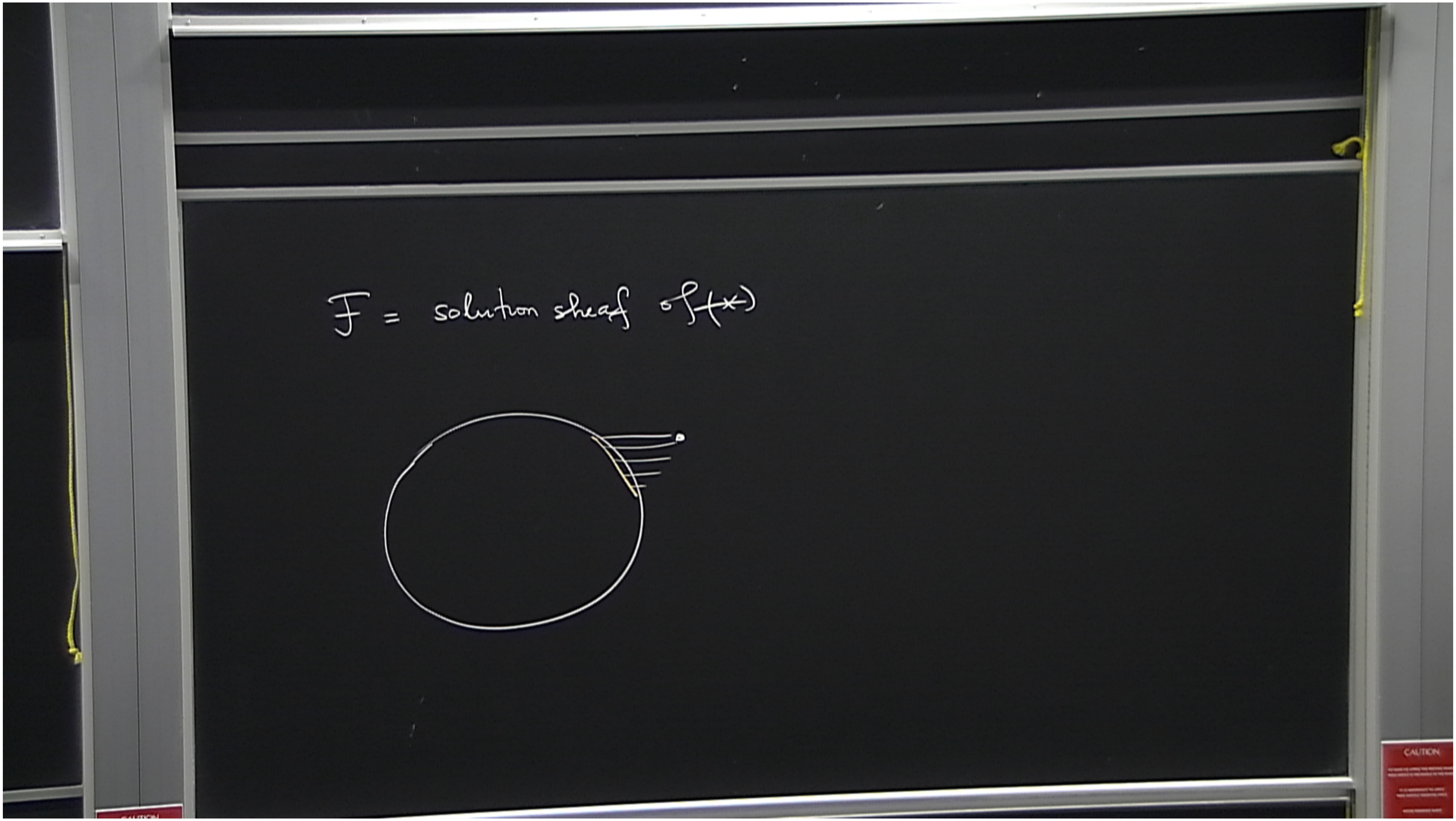
$$\mathcal{S}(x) = \int_{x_0}^{x} \sqrt{V(x)}\,dx \quad \text{defined on } \left(\mathbb{C}_x \setminus V^{-1}(x)\right)^{\sim}$$

$S_\alpha =$ 


$$S_\alpha = \{-\alpha < \operatorname{Im} z < \alpha + 2\pi\} \xrightarrow[\exp]{} \mathbb{C}_s$$

$$\psi_0(s),\ \psi_1(s) \in \mathcal{O}(S_\alpha)$$

$$\left(-\hbar^2 \partial_x^2 + V(x)\right)\psi(x,\hbar) = 0 \qquad 1/\hbar \leftrightarrow s$$


$\mathcal{F}$ = solution sheaf of $f(x)$

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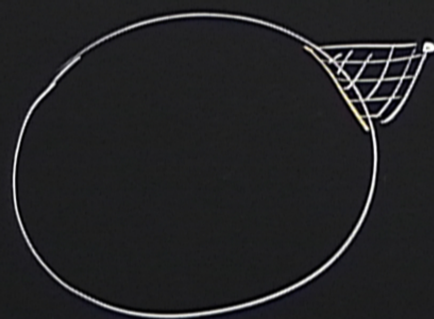
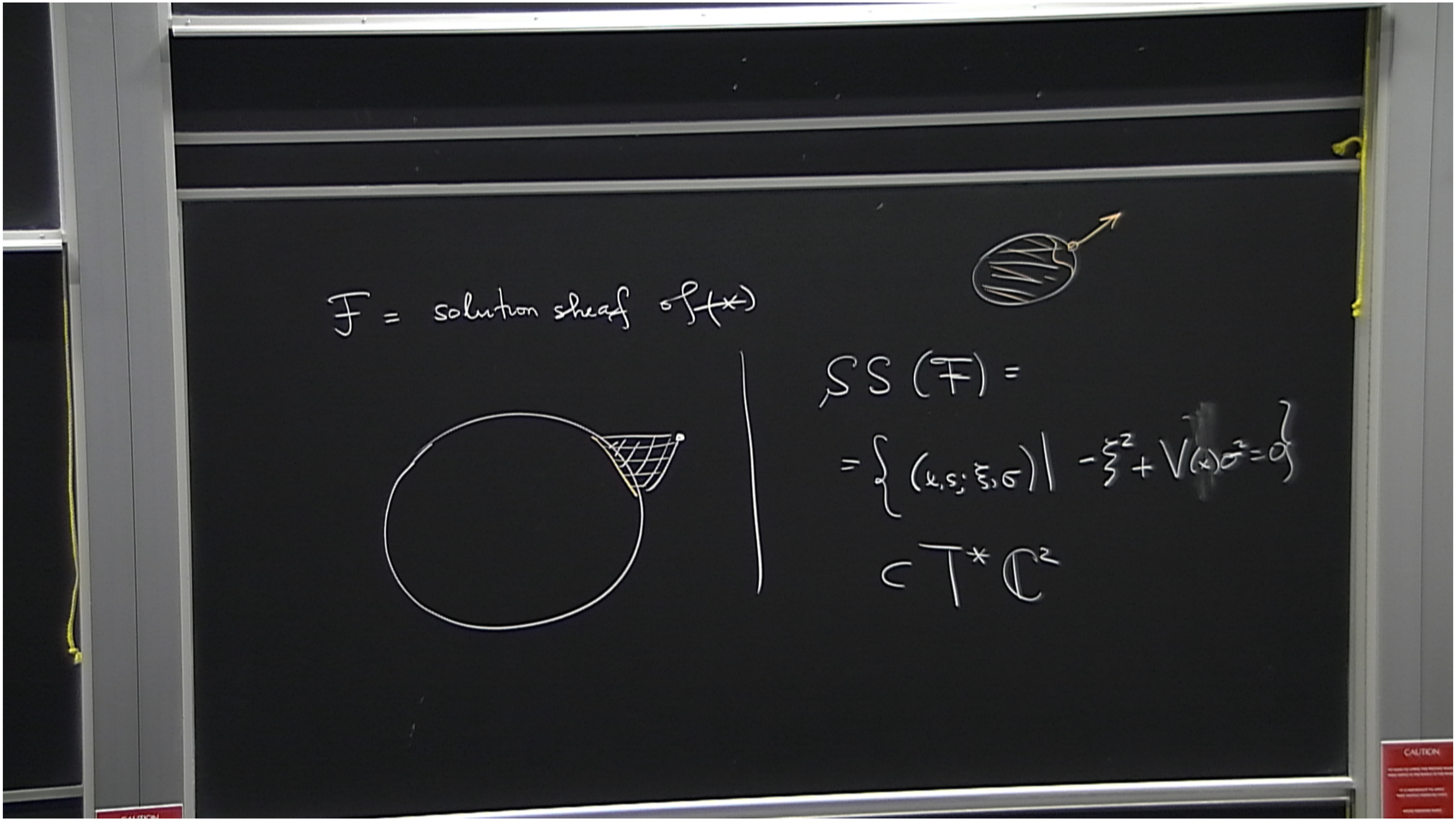


$$SS(\mathcal{F}) = \left\{ (x, s; \xi, \sigma) \,\middle|\, -\xi^2 + V(x)\sigma^2 = 0 \right\} \subset T^*\mathbb{C}^2$$


$\mathcal{F}$ = solution sheaf of
$SS(\mathcal{F}) =$
$= \{ (x,s;\xi,\sigma) \mid -\xi^2 + V(x)\sigma^2 = 0 \}$
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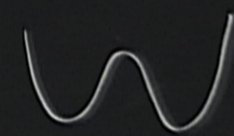
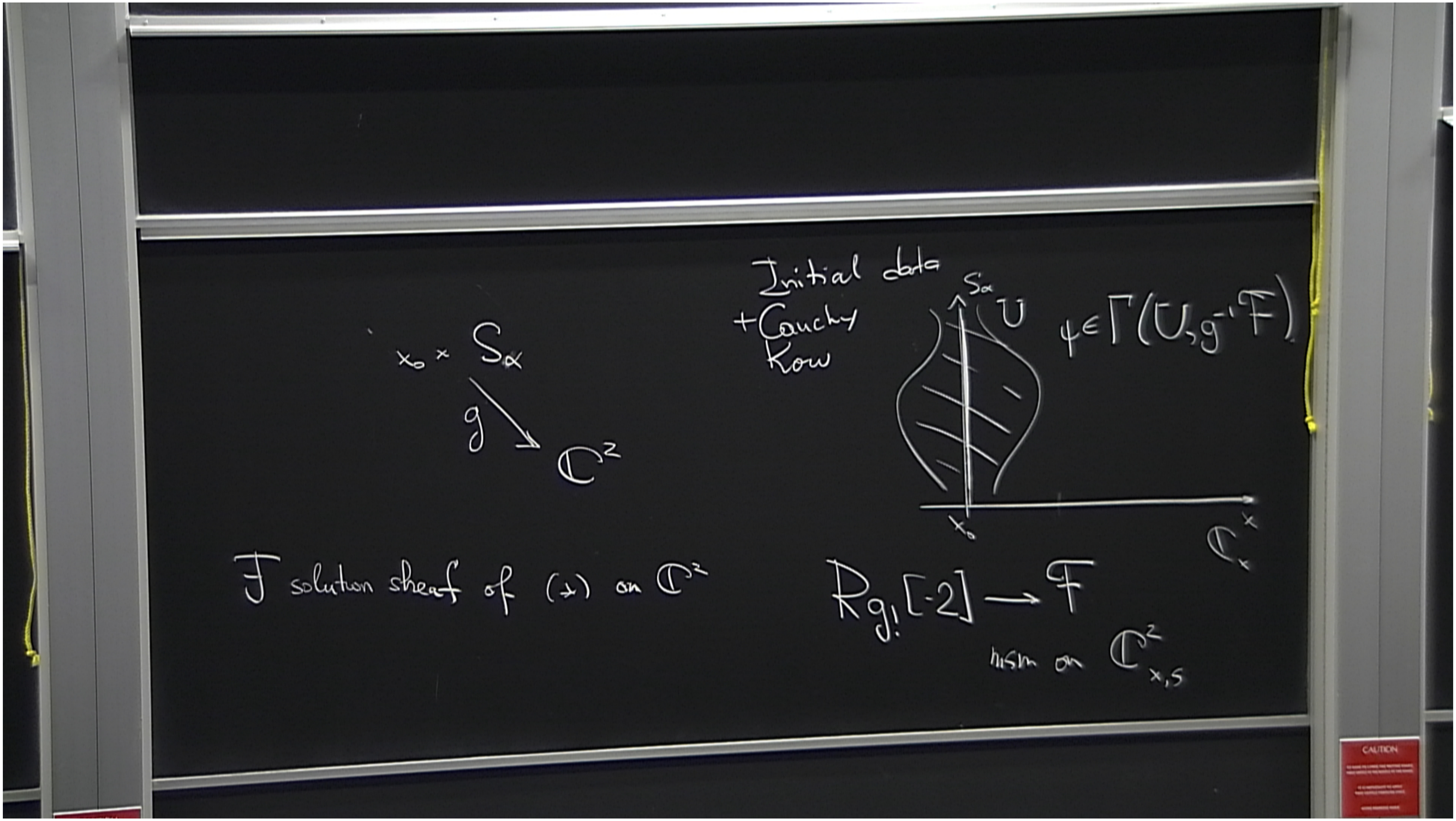
## Microlocal properties of sheaves & complex WKB.

$$\left(-h^2 \partial_x^2 + V(x) + hV_1(x) + \dots\right)\psi(x,h) = 0. \qquad 1/h \leftrightarrow s$$

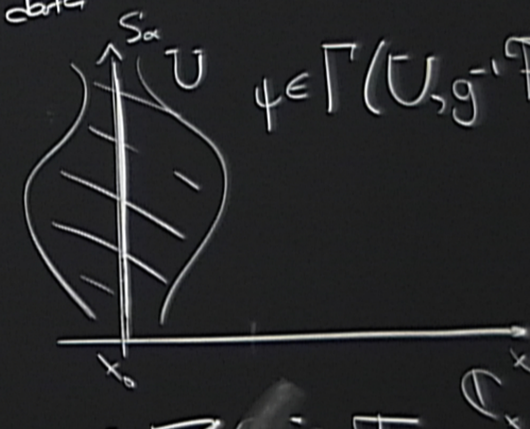


$$\begin{cases} \left(-\partial_x^2 + V(x)\partial_s^2\right)\Psi(x,s) = 0. \qquad (*) \\ x_0 \quad V(x_0) \neq 0. \qquad \Psi(x,s)\big|_{x_0} = \psi_0(s),\ \dfrac{\partial}{\partial x}\Psi(x,s)\Big|_{x_0=1} = \psi_1(s) \end{cases}$$


Initial data
+ Cauchy
Kow
$x_0 \times S_\alpha$
$g$
$\mathbb{C}^2$
$S_\alpha$
$U$
$\psi \in \Gamma(U, g^{-1}\mathcal{F})$
$x_0$
$\mathbb{C}_x$
$\mathcal{F}$ solution sheaf of $(*)$ on $\mathbb{C}^2$
$Rg_![-2] \longrightarrow \mathcal{F}$
ism on $\mathbb{C}^2_{x,s}$

$x_0 \times S_\alpha \xrightarrow{g} \mathbb{C}^2$

Initial data
+ Cauchy
Kow




$\psi \in \Gamma(U, g^{-1}\mathcal{F})$

$\mathcal{F}$ solution sheaf of $(*)$ on $\mathbb{C}^2$

$Rg_![-2]\,\mathbb{Z}_{S_\alpha} \to \mathcal{F}.$

ism on $\mathbb{C}^2_{x,s}$

$S_\alpha = \ldots \quad \exp$

$\psi_0(s),\ \psi_1(s) \in \mathcal{O}(S_\alpha)$

<u>Thm</u>. There is a distinguished triangle ("semiorthog. decomp'n)

$$R g_! \mathbb{Z}_{S_\alpha}[-2] \longrightarrow \Phi \longrightarrow \delta \xrightarrow{+1}$$

$$\Phi \in \mathcal{C}, \qquad \delta \in {}^{\perp}\mathcal{C}$$

$\mathcal{C}$ = the full subcat of $D^b(X \times \mathbb{C})$ with S.S. $\subset$

$\subset \{-\xi^2 + V(x)\sigma^2 = 0\}$

**Thm.** There is a distinguished triangle ("semiorthog. decomp'n)

$$\begin{tikzcd} R g_! \mathbb{Z}_{S_\alpha}[-2] \arrow[r] \arrow[dr, "\text{int.}"'] & \Phi \arrow[r] \arrow[d, dashed, "\exists !"'] & \delta \arrow[r, "+1"] & {} \\ & \mathcal{F} & & \end{tikzcd}$$

$\Phi \in \mathcal{C}$, $\quad \delta \in {}^{\perp}\mathcal{C}$

$\mathcal{C}$ = the full subcat of $D^b(X \times \mathbb{C})$ with s.s. $\subset$

$\subset \{-\xi^2 + V(x)\sigma^2 = 0\}$

$$\Phi_0 = H^0(\Phi)$$

$$\mathcal{S} = \text{étale}(\Phi_0).$$

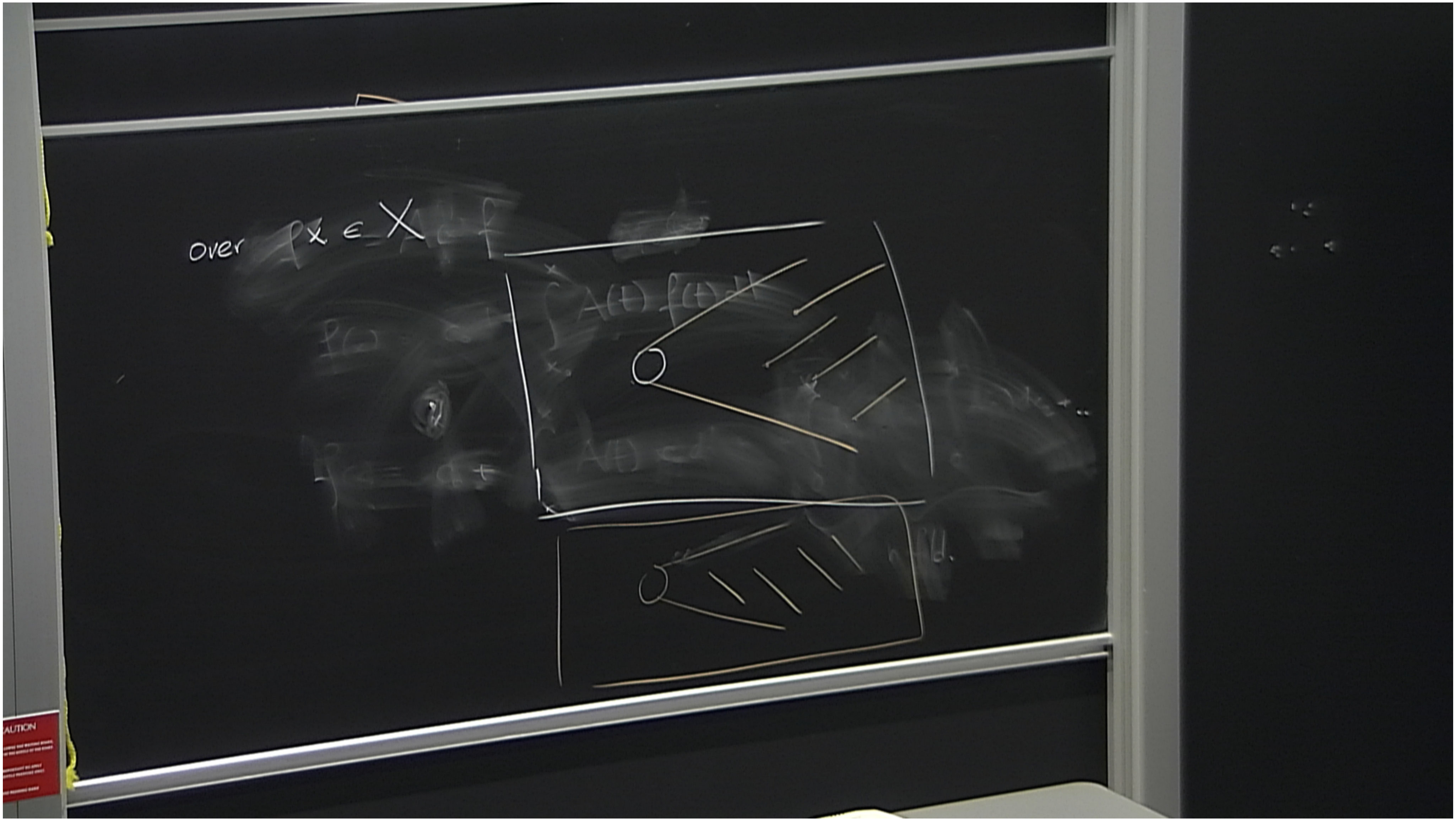
$$\begin{array}{c} \mathcal{S} \\ \downarrow \\ X \times \mathbb{C} \end{array}$$

$$\Phi_0 = H^0(\Phi)$$

$$\begin{array}{c} \mathcal{S} = \text{étale}(\Phi_0). \\ \pi \big\downarrow \\ X \times \mathbb{C} \end{array}$$

$\pi^{-1}\Phi_0$ has a tautological section

$\pi^{-1}u$( tautol. section of $\pi^{-1}\Phi$ ) is a sectin of $\pi^{-1}F$.


Over