

Title: Scattering Amplitudes and Riemann Surfaces

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Abstract: In 2003 Witten introduced twistor string theory as a novel description of the scattering matrix of the maximally supersymmetric Yang-Mills theory in four dimensions. In these lectures I will give an introduction to the developments that have led to new formulations, also based on Riemann surfaces, of a large variety of theories, with and without supersymmetry, in arbitrary space-time dimensions.

$$m(\alpha|\beta) = \int_{\mathcal{L}} \left(\prod_{\alpha \neq \beta} \frac{dG_\alpha}{E_\alpha} \right) |ijk|^2 \frac{1}{(G_{\alpha(1)} - G_{\alpha(2)}) \cdots (G_{\alpha(n)} - G_{\alpha(1)})} \times \frac{1}{(G_{\beta(1)} - G_{\beta(2)}) \cdots (G_{\beta(n)} - G_{\beta(1)})}$$

Integrand is theory dependent.

Claim: This object is universal
for theories of massless particles.

$$\mathcal{L} = \bigcup_{G \in \mathbb{Z}_n} \mathcal{L}_G \int_{\mathcal{L}} d\mu_n$$

- * Scalar
- * Gluons (Yang Mills)
- * Gravitons (Einstein's GR)

Lecture II

Summary: $\mathcal{L} = \frac{1}{2} \partial_\mu \varphi^{\alpha\tilde{\alpha}} \partial^\mu \varphi_{\alpha\tilde{\alpha}} - \frac{1}{2} \underbrace{m^2}_{m=0} (\varphi^{\alpha\tilde{\alpha}})^2 - \frac{g}{3!} f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \varphi^{\alpha\tilde{a}} \varphi^{\beta\tilde{b}} \varphi^{\gamma\tilde{c}}$

Massless particles.

$$A(k_1^\mu, \dots, k_n^\mu) = A(k_1^\mu, \dots, k_n^\mu) \int^D \left(\sum_{a=1}^n k_a^\mu \right) \quad \varphi^{\alpha\tilde{\alpha}}: \underbrace{\mathbb{R}^{1,D-1}}_{\text{Space-Time}} \rightarrow \mathbb{R}$$

Flavor group $U(N) \times U(\tilde{N})$ $\uparrow \hat{f}$ struct. const. of the Space-Time
corresp. Lie Algebras.

$$A(k_1^\mu, \dots, k_n^\mu) = \sum_{\alpha, \beta \in S_n / \mathbb{Z}_n} \text{tr}(T^{\alpha_{(1)}} \dots T^{\alpha_{(n)}}) \text{tr}(\tilde{T}^{\tilde{\alpha}_{(1)}} \dots \tilde{T}^{\tilde{\alpha}_{(n)}}) m(\alpha | \beta)$$

Permutation $\alpha: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$

$\alpha(1)$

$$\alpha = (\alpha(1), \alpha(2), \dots, \alpha(n))$$

Functions

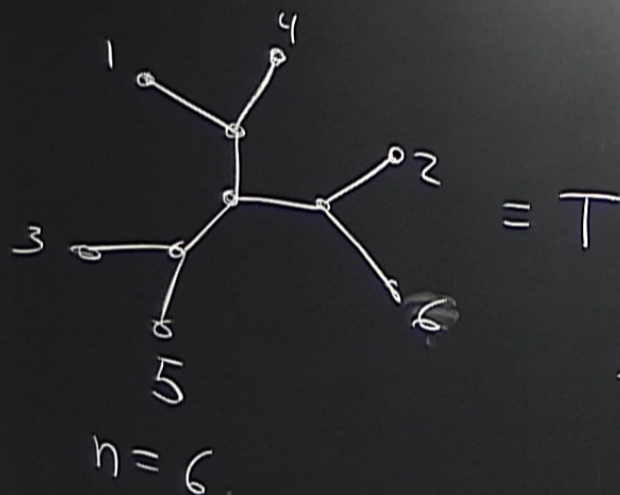
$$m(\alpha | \beta) = \prod (k_a k_b)$$

$\rightarrow SO(1, D-1)$
Lorentz invariant

$$\rightarrow \equiv k_a^\mu k_b^\nu \eta_{\mu\nu}$$

Construction of $m(\alpha|\beta)$:

- 1) Connected tree graphs with n vertices of degree 1. (Labels)
 $n-2$ " " " 3.



$$E: \{\text{Graphs}\} \rightarrow \left\{ \begin{array}{l} \text{Rational} \\ \text{Funct of } k_1, k_2 \end{array} \right\}$$

$$E(T) = \text{Function.}$$

Set of edges:

$$E_{(T)}^{\text{int}} = \left\{ \begin{array}{l} \text{edges that connect two degree 3} \\ \text{vertices} \end{array} \right\}$$

Every $e \in E^{int}(T)$

$$L_e \cup R_e = \{1, 2, \dots, n\}$$

Note: $\sum_{a,b \in L_e} k_a \cdot k_b = \sum_{a,b \in R_e} k_a \cdot k_b$

$\Downarrow$
 P_e^2

Follows from $\sum_{a=1}^n k_a^2 = 0$

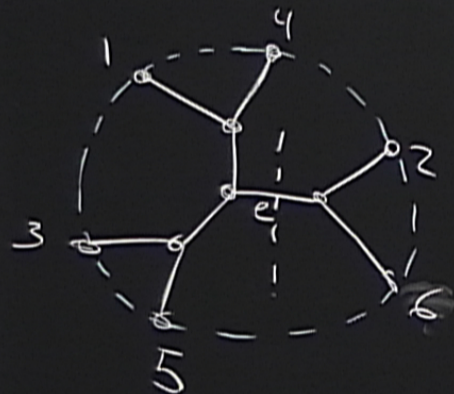
need $k_a^2 \cdot k_{a_j} = 0$

$$E(T) = \prod_{e \in E^{int}} \frac{1}{P_e^2}$$

Note that T has many planar embeddings.
Each emb. induces an ordering of labels $\{1, 2, \dots, n\}$
 $(1, 4, 2, 6, 5, 3)$

Construction of $m(\alpha|\beta)$:

- 1) Connected tree graphs with n vertices of degree 1. (Labels)
 $n-2$ " " " 3.



$= T$

$n=6$

$$C : \{\text{Graphs}\} \rightarrow \left\{ \begin{array}{l} \text{Rational} \\ \text{Funct of } k_1, k_2 \end{array} \right\}$$

$$C(T) = \text{Function.}$$

Set of edges:

$$E_{(T)}^{\text{int}} = \left\{ \text{edges that connect two degree 3 vertices} \right\}$$

$$C(T) = \prod_{p \in T} \frac{1}{p_c^2}$$

Note that T has many planar embeddings.
Each emb. induces an ordering of labels $\{1, 2, \dots, n\}$
 $(1, 4, 2, 6, 5, 3) = \alpha$

We say that T is consistent with α if
it has a planar embed. that gives rise to α .

Def: $T(\alpha) = \left\{ \text{Set of all tree graphs that are consistent with } \alpha \right\}$

$$m(\alpha|\beta) = \sum_{T \in \Gamma(\alpha) \cap \Gamma(\beta)} \mathcal{E}(T)$$

QFT way of
computing $m(\alpha|\beta)$

$$T(\alpha_1, \dots, \alpha_n) = \sum_{\alpha, \beta \in S_n / \mathbb{Z}_n} \text{Tr}(T^{\alpha_1} \dots T^{\alpha_n}) \text{Tr}(T^{\alpha} \dots T^{\beta}) m(\alpha | \beta)$$

Examples: $n=4$

$$T((1234)) = \left\{ \begin{array}{c} \text{Diagram 1: A central node with four edges labeled 1, 2, 3, 4. Edges 1 and 2 are on the left, 3 and 4 on the right.} \\ \text{Diagram 2: A central node with four edges labeled 1, 2, 3, 4. Edges 1 and 3 are on the left, 2 and 4 on the right.} \end{array} \right\}$$

$$T((1243)) = \left\{ \begin{array}{c} \text{Diagram 1: A central node with four edges labeled 1, 2, 3, 4. Edges 1 and 2 are on the left, 3 and 4 on the right.} \\ \text{Diagram 2: A central node with four edges labeled 1, 2, 3, 4. Edges 1 and 4 are on the left, 2 and 3 on the right.} \end{array} \right\}$$

$$m(\alpha|\beta) = \sum_{T \in \mathcal{T}(\alpha) \cap \mathcal{T}(\beta)} \mathcal{E}(T)$$

QFT way of
computing $m(\alpha|\beta)$

$$m(1234|1243) = \frac{1}{2k_1 k_2}.$$

$$m(\alpha|\beta) = \sum_{T \in \mathcal{T}(\alpha) \cap \mathcal{T}(\beta)} \mathcal{E}(T)$$

QFT way of
computing $m(\alpha|\beta)$

$$m(1234|1243) = \frac{1}{2k_1 k_2}$$

$$m(1234|1234) = \frac{1}{2k_1 k_2} + \frac{1}{2k_1 k_3}$$

Exercise: $|\mathcal{T}((12345))| = 5$

$$m(12345|14253) = ?$$

Moduli Computation of $m(\alpha/\beta)$

Moduli space of genus 0 curves with n marked points.

$$\mathcal{M}_{0,n} = \left((\mathbb{CP}^1)^n \setminus \Delta \right) / \text{PSL}(2, \mathbb{C})$$

$$\Delta = \bigcup_{i \neq j} \left\{ (G_1, G_2, \dots, G_n) \in (\mathbb{CP}^1)^n \mid G_i = G_j \right\}$$

$$(1) - 2k_1 \cdot k_2$$

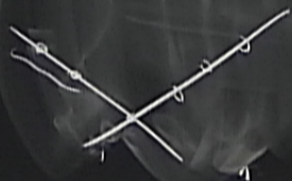
Moduli Computation of $m(\alpha/\beta)$

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$\partial \overline{\mathcal{M}}_{0,n}$



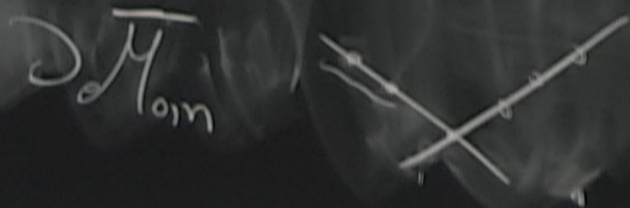
(1) $2k_1 \cdot k_2$

$\mathcal{M}_{0,n}$ Computation of $m(\alpha/\beta)$

Moduli space of genus 0 curves with n marked points.

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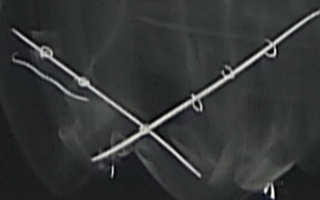
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$\partial \overline{\mathcal{M}}_{0,n}$



2) Real valued function on $\mathcal{M}_n$

Start with $\Delta = \sum_{1 \leq a \leq b \leq n} S_{ab} \ln |G_a - G_b|$

Impose that Δ be invariant under $\text{PSL}(2, \mathbb{C})$ transf.

$$G \rightarrow \frac{AG+B}{CG+D} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{PSL}(2, \mathbb{C})$$

$$S = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}_{n \times n}$$

$$S_{ab} = S_{ba}$$

$$S_{aa} = 0$$

2) Real valued function on M_{nn}

Start with $\Delta = \sum_{1 \leq a \leq b \leq n} S_{ab} \ln |G_a - G_b|$

Impose that Δ be invariant under

$PSL(2, \mathbb{C})$ transf.

$$G \rightarrow \frac{AG+B}{CG+D}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in PSL(2, \mathbb{C})$$

$$\Leftrightarrow \sum_{b=1}^n S_{ab} = 0 \quad \forall a$$

$$S = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}_{n \times n}$$

$$S_{ab} = S_{ba}$$

$$S_{aa} = 0$$



All rows add up to zero.

3) Critical points of Δ . Use $\text{PSL}(2, \mathbb{R})$ to fix $\{\sigma_i^*, \sigma_j^*, \sigma_k^*\}$

$$\frac{\partial \Delta}{\partial \sigma_a} = 0 \quad a \neq \{i, j, k\}$$

$$E_a = \sum_{\substack{b=1 \\ b \neq a}}^n \frac{S_{ab}}{\sigma_a - \sigma_b}$$

$$E_a = 0 \quad a \neq \{i, j, k\}$$

4) For generic S_{ab} 's, the critical points are non-degenerate.

$$\Phi_{ab} = \frac{\partial^2 \Delta}{\partial \sigma_a \partial \sigma_b}$$

$$\det \Phi_{(n-3) \times (n-3)}^{[ijk]} \neq 0 \quad \text{at the critical points}$$

3) Critical points of Δ . Use $PSL(2, \mathbb{R})$ to fix $\{\sigma_i^*, \sigma_j^*, \sigma_r^*\}$

$$\frac{\partial \Delta}{\partial \sigma_a} = 0 \quad a \neq \{i, j, r\}$$

$$E_a = \sum_{\substack{b=1 \\ b \neq a}}^n \frac{S_{ab}}{\sigma_a - \sigma_b}$$

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4) For generic S_{ab} 's, the critical points are non-degenerate.

$$\overline{\Phi}_{ab} = \frac{\partial^2 \Delta}{\partial \sigma_a \partial \sigma_b}$$

$$\det \overline{\Phi}_{(n-3) \times (n-3)}^{[ijk]} \neq 0 \quad \text{at the critical points}$$

5) Fact: $\frac{\det \overline{\Phi}_{[ijk]}^{[ijk]}}{|ijk|^2}$

is independent of the choice.

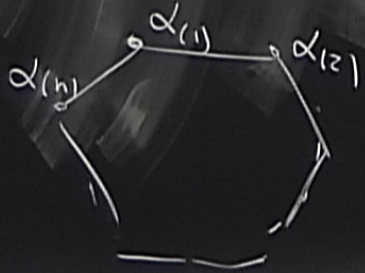
$$|ijk| = \det \begin{pmatrix} 1 & 1 & 1 \\ \sigma_i & \sigma_j & \sigma_r \\ \sigma_i^2 & \sigma_j^2 & \sigma_r^2 \end{pmatrix}$$

c) Let $\mathcal{Z} = \{ \Gamma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in (\mathbb{CP}^1)^n ; \Gamma \text{ is a critical point of } \Delta \}$

Claim :

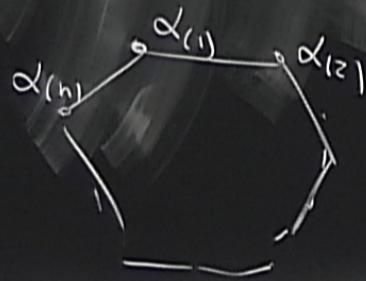
$$m(\alpha|\beta) = \sum_{\Gamma \in \mathcal{Z}_\Delta} \frac{1}{d\Gamma|d\Phi} \times \frac{1}{(\sigma_{\alpha(1)} - \sigma_{\alpha(2)})(\sigma_{\alpha(2)} - \sigma_{\alpha(3)}) \dots (\sigma_{\alpha(n)} - \sigma_{\alpha(1)})} \times$$

$$\frac{1}{(\sigma_{\beta(1)} - \sigma_{\beta(2)})(\sigma_{\beta(2)} - \sigma_{\beta(3)}) \dots (\sigma_{\beta(n)} - \sigma_{\beta(1)})}$$



Claim :

$$m(\alpha|\beta) = \sum_{\Gamma \in \mathbb{Z}_A} \frac{1}{d\Gamma|\Phi} \times \frac{1}{(\Gamma_{\alpha(1)} - \Gamma_{\alpha(2)})(\Gamma_{\alpha(2)} - \Gamma_{\alpha(3)}) \dots (\Gamma_{\alpha(n)} - \Gamma_{\alpha(1)})} \times$$



if

$$S_{ab} = 2 K_a K_b$$

$$\frac{1}{(\Gamma_{\beta(1)} - \Gamma_{\beta(2)})(\Gamma_{\beta(2)} - \Gamma_{\beta(3)}) \dots (\Gamma_{\beta(n)} - \Gamma_{\beta(1)})}$$

Main Computation of $m(\alpha|\beta)$

$$m(12 \dots n | 12 \dots n) = \text{Sum over all planar trees} \Rightarrow 2^{n-1} \frac{(2n-5)!!}{(n-1)!}$$

$$= \# \text{ of critical points of } \Delta \text{ is } (n-3)!$$

Residue Formula for $m(\alpha|\beta)$

$$F(T) = \frac{1}{2k_1 k_2}$$

Review the def. of multidimens. residues.
Given a map $f: \mathbb{C}^m \rightarrow \mathbb{C}^m$
 $\uparrow$
 rational

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 and a rational fund. $h: \mathbb{C}^m \rightarrow \mathbb{C}$. $z = (z_1, \dots, z_m)$

The residue associated with $\omega = h(z) \frac{dz_1}{f_1} \wedge \frac{dz_2}{f_2} \wedge \dots \wedge \frac{dz_m}{f_m}$

$$\text{Res}_{[z]} \omega = \frac{1}{(2\pi i)^m} \int \omega$$

c a zero of f $\mathcal{C} = \{ |f_i| = c \} = T^m$

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The residue associated with $\omega = h(z) \frac{dz_1}{f_1} \wedge \frac{dz_2}{f_2} \wedge \dots \wedge \frac{dz_m}{f_m}$

$$\text{Res}_{[z^*]} \omega = \frac{1}{(2\pi i)^m} \int \omega = \frac{h(z^*)}{\det \left(\frac{\partial(f_1, \dots, f_m)}{\partial(z_1, \dots, z_m)} \right) \Big|_{z=z^*}}$$

z^* = zero of f $\mathcal{C} = \{ |f_i| = \epsilon \} = T^m$

$$m(\alpha|\beta) = \left(\int_{\mathcal{L}} \prod_{\alpha \neq i, j, k} \frac{dG_\alpha}{E_\alpha} |ijk|^2 \right) \frac{1}{(G_{\alpha(1)} - G_{\alpha(2)}) \cdots (G_{\alpha(n)} - G_{\alpha(1)})} \times \frac{1}{(G_{\beta(1)} - G_{\beta(2)}) \cdots (G_{\beta(n)} - G_{\beta(1)})}$$

$$\mathcal{L} = \bigcup_{G \in \mathbb{Z}_p} \mathcal{L}_G$$