Title: Some combinatorial comments on amplitudes

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Abstract: I will begin with the perspective that the perturbative expansion is an augmented generating function and then discuss some of the results which follow from this perspective.

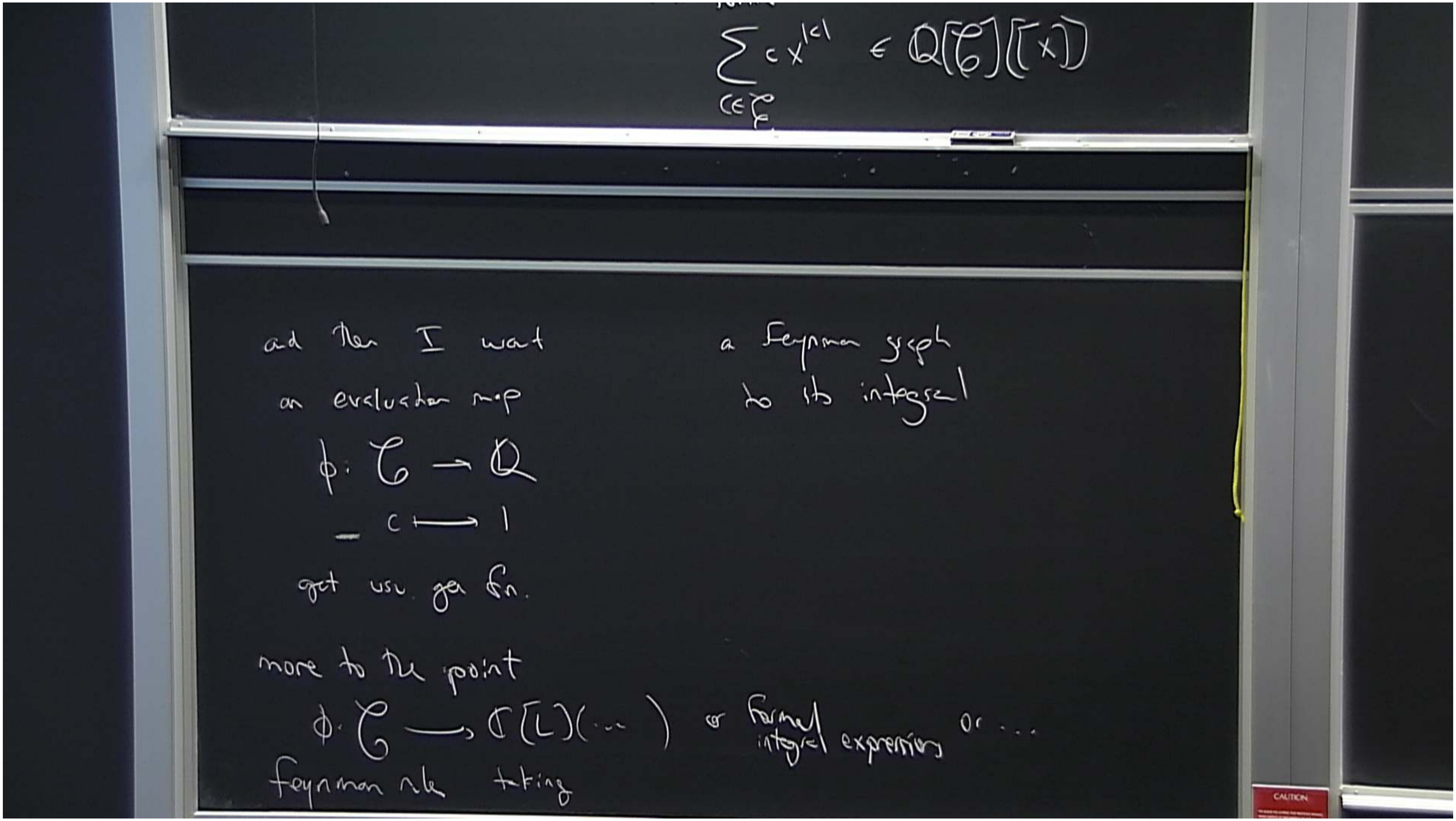
...gmented
generating functions
and combinatorial
Dyson-Schwinger
equations.

I like excuses to sum
over graphs

Precisely
$\mathcal{C}$ comb class

and form

$$\sum_{c \in \mathcal{C}} c\, x^{|c|} \in \mathbb{Q}[\mathcal{C}][[x]]$$



$\sum_{c \in \mathcal{G}} c\, x^{|c|} \in \mathbb{Q}[\mathcal{G}][[x]]$
and then I want
an evaluation map
$\phi: \mathcal{G} \to \mathbb{Q}$
$c \mapsto 1$
get usu. gen fn.
a Feynman graph
to its integral
more to the point
$\phi: \mathcal{G} \to \mathbb{C}[L](\cdots)$ or formal integral expressions or ...
feynman rule taking

How to specify an interesting comb class with a focus on rooted trees.

I want rooted trees
with no plane structure
no valence restrictions.
call this $\tilde{\mathcal{T}}$

Define $B_+ : \mathbb{Q}[\mathcal{T}] \to \mathbb{Q}[\mathcal{T}]$

on a forest = monomial in rooted trees

$B_+(t_1 \dots t_n) =$ $t_1 \cdots t_n$

interesting comb class

with a focus on rooted

trees.

I want rooted trees

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call this $\mathcal{T}$

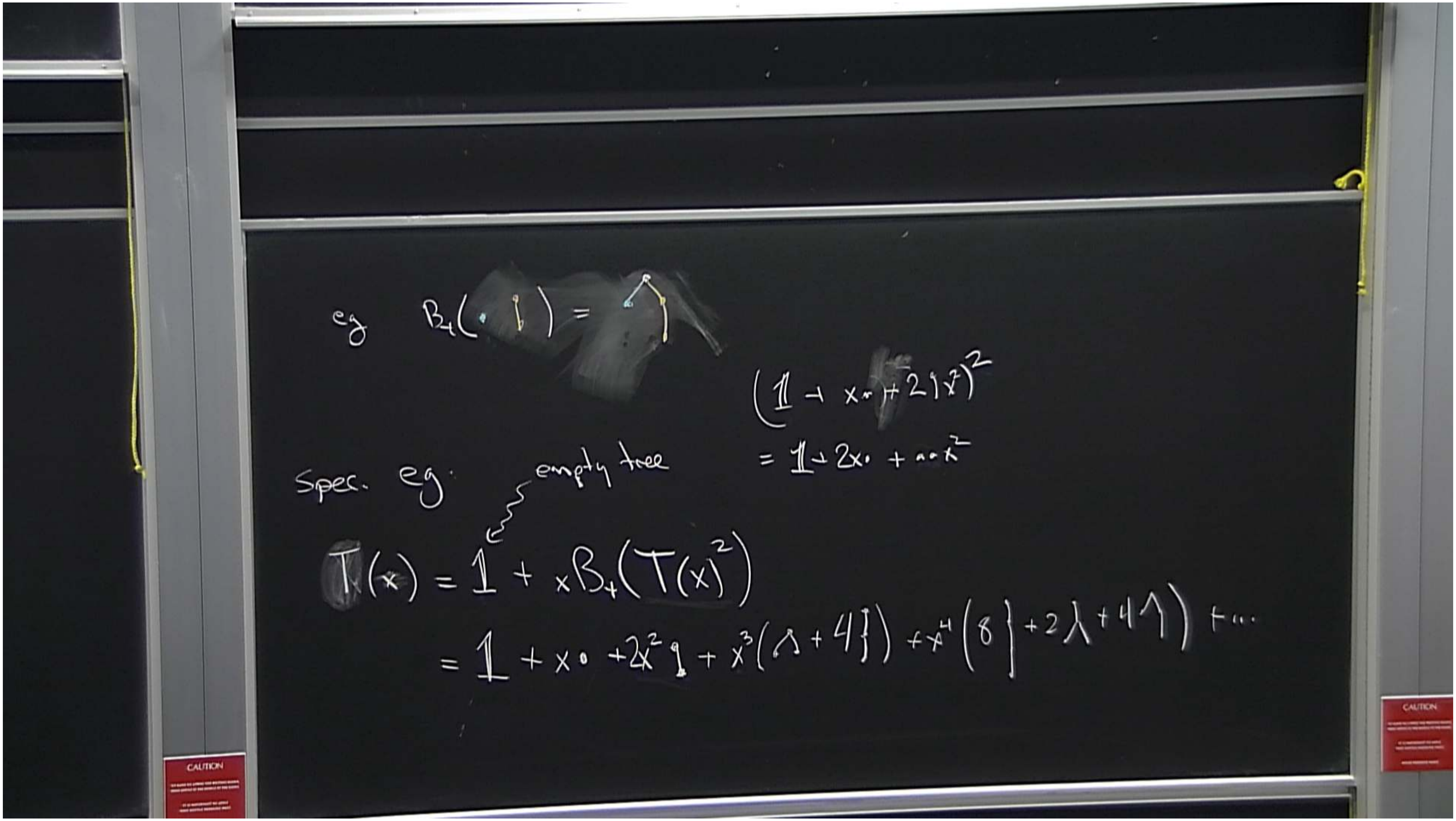
$\mathbb{Q}[\mathcal{T}] \to \mathbb{Q}[\mathcal{T}]$

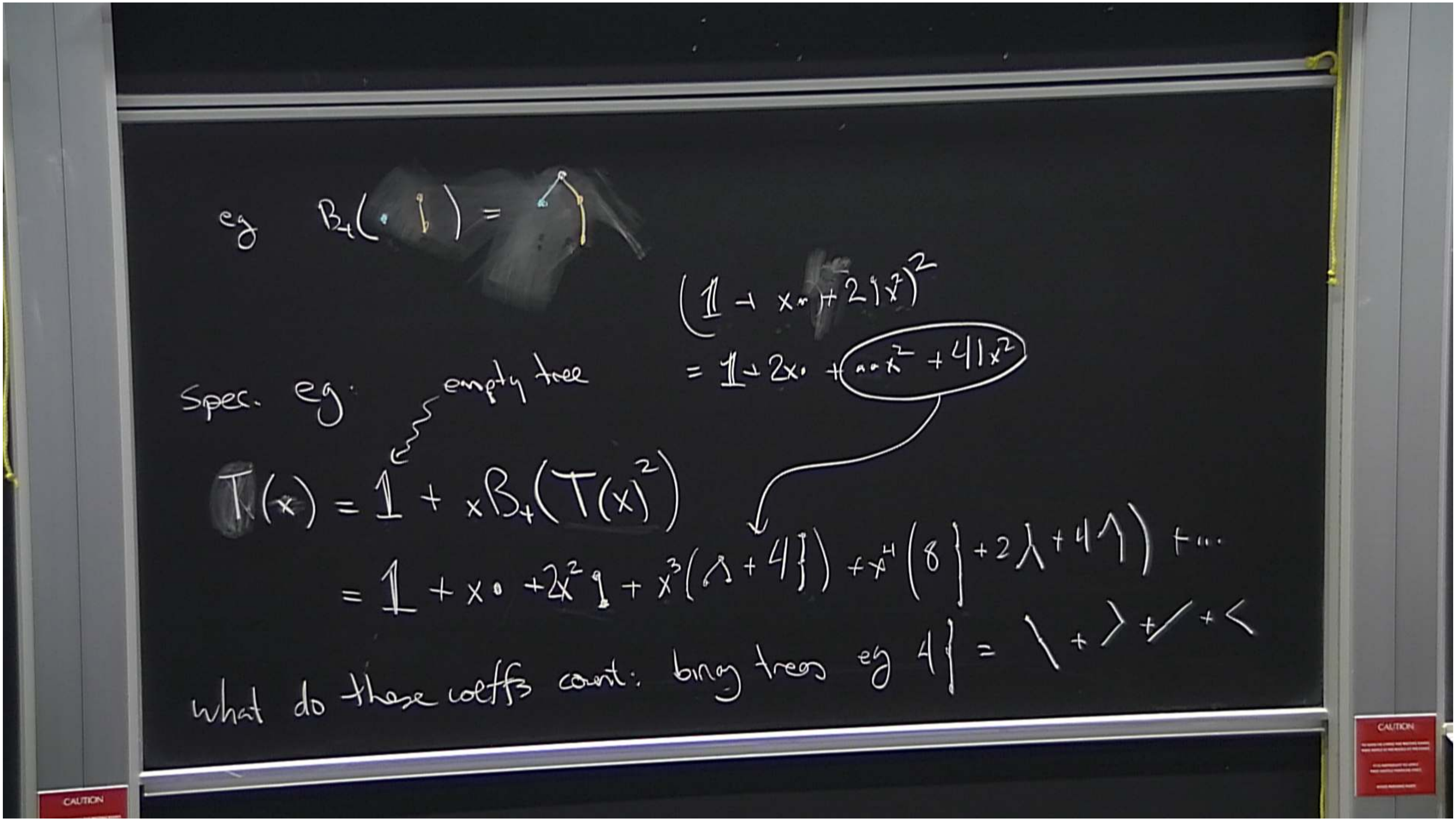
vs a forest = monomial in rooted trees

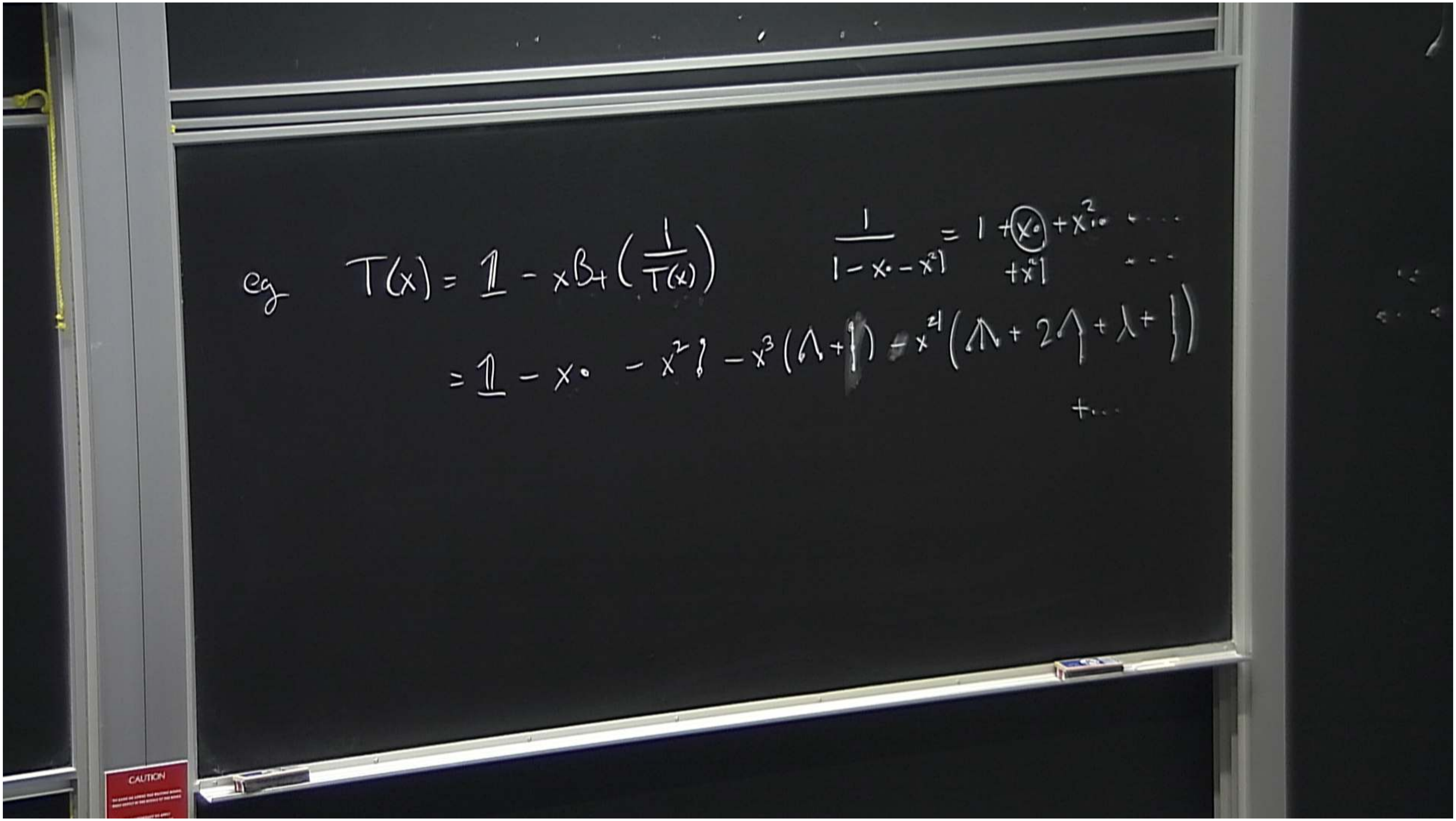
$B_+(t_1 \dots t_n) =$ $t_1 \cdots t_n$

use B_+ to build

interesting classes


eg $B_+($
$(1 + x + 2x^2)^2$
$= 1 + 2x + x^2$
Spec. eg:
empty tree
$T(x) = 1 + x B_+(T(x)^2)$
$= 1 + x + 2x^2 + x^3(+ 4) + x^4(8 + 2 + 4) + \dots$


eg $B_+(\ \ ) =$
$(1 + x + 2x^2)^2$
$= 1 + 2x + x^2 + 4x^2$
Spec. eg:
empty tree
$T(x) = 1 + xB_+(T(x)^2)$
$= 1 + x + 2x^2 + x^3(\ + 4\ ) + x^4(8\ + 2\ + 4\ ) + \dots$
what do these coeffs count: binary trees eg


eg T(x) = 1 - xB+(1/T(x))
+...

no valence restrictions.

call those $\tilde{T}$

eg $B_+(\cdot\ |) =$

$(1 + x\cdot + 2|x^2)^2$

$= 1 + 2x\cdot + \cdot\cdot x^2 + 4|x^2$

Spec. eg: empty tree

$T(x) = 1 + xB_+(T(x)^2)$

$= 1 + x\cdot + 2x^2| + x^3(\wedge + 4|) + x^4(8| + 2\lambda + 4\wedge) + \dots$

what do these coeffs count: binary trees eg $4| = \backslash + \rangle + / + \langle$

eg $T(x) = \mathbb{1} - x B_+\left(\frac{1}{T(x)}\right)$ $\qquad \frac{1}{1 - x\bullet - x^2} = 1 + x\bullet + x^2 \cdots$

$+ x^2 \cdots$

$= \mathbb{1} - x\bullet - x^2 - x^3(+) - x^4(+ 2 + +)$

$+ \cdots$

Call these kinds of things

Combinatorial Dyson-Schwinger equations.

$) + \cdots$

$+ / + <$

Combinatorial DSEs

have alg. gen. fns

as solution.

Apply Feynman rules ϕ

- to the Comb DSE
  get Analytic DSE
- to the solution get
  Greens functions.

② Tree hook weights

(joint with Brad Jones)

A simple case of the preceding ϕ just gives numbers (leading log of more realistic cases)

The enum. comb community studied this indep

specifically

$$\phi(t) = \prod_{v \in t} B_{|t_v|}$$

where t_v is the subtree rooted at v

a Feynman graph

A simple case of ... specifically

preceeding ϕ just

$$\phi(t) = \prod_{v \in t} B_{|t_v|}$$

gives numbers (leading log of more realistic cases)

where t_v is the subtree rooted at v.

eg tree factorial

$$t! = \prod_{v \in t} |t_v|$$

eg  $! = 4 \cdot 3 \cdot 1 \cdot 1 = 12$

Then the hook length series ("Greens function") is

$$F_{T,B}(x) = \sum_{t \in T} \phi(t)\, x^{|t|}$$

Hook weight formulas – nice if $F_{T,B}$ has a closed form or interpretation.

call this ")

...ated

...ction")

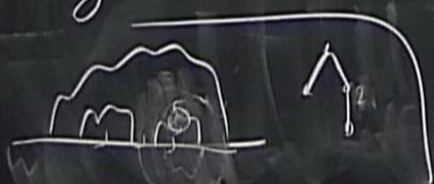
...t) $x^{|t|}$

$F_{T,B}$

So us
nice comb. DSEs
and nice Feynman rules
with nice Greens functions

$$T(x) = \mathbb{1} - x B_+\left(\frac{1}{T(x)^2}\right)$$

eg



$$\phi(t) = \frac{L^{|t|}}{t!}$$

then $G(x,L) = (1-3xL)^{\frac{1}{3}}$

then $G(x, L) = ($

③ More physical DSEs

Here is something closer to what you might think of as a Dyson-Schwinger eqn.

(Broadhurst - Kreimer)

$$G(\alpha, L) = 1 - \left(\frac{\alpha}{q^2} \int d^4k \, \frac{k\cdot q}{|k|^2 G\left(\alpha, \log\frac{|k|^2}{\mu^2}\right) |k+q|^2} - \cdots \right|_{q^2=\mu^2} \right)$$

then $G(x, L) = \ldots$

(3) More physical DSEs in Yukawa theory

inserted into itself in all ways

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eg 

$T(x) = \mathbb{1} - x B_+\left(\frac{1}{T(x)^2}\right)$

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ee rooted

function")

$\phi(t)\, x^{|t|}$

T

of $F_{T,B}$

tahler.

③ More physical DSEs in Yukawa theory

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$L = \log q^2/\mu^2$

eg

Call

eg $T(x) = \mathbb{1} - xB_+\left(\frac{1}{T(x)}\right)$

Now do the following to

1. expand $G(\alpha, L) = 1 - \sum_{k\geq 1} \gamma_k(\alpha) L^k$

   plug in and expand geom. series

2. swap $\sum, \int$
3. use $\frac{\partial^i}{\partial(\rho)^i} y^{-\rho}\Big|_{\rho=0} = (-1)^i \log^i y$
4. swap $\int, \frac{\partial}{\partial\rho}$
5. undo 1

get

② Swap $\langle \, , \, \rangle$

③ use $\frac{\partial^i}{\partial(-\rho)^i} y^{-\rho}\Big|_{\rho=0} = (-1)^i \log^i y$

get $G(\alpha, L) = 1 - \alpha\, G\left(\alpha, \frac{\partial}{\partial(-\rho)}\right)^{-1} \left(e^{-L\rho} - 1\right) F(\rho)\Big|_{\rho=0}$

② Swap $\Sigma, \int$

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integral for

with $\int$ regularized by ρ.

(2) swap $\sum, \int$

(3) use $\frac{\partial^i}{\partial(-\rho)^i} y^{-\rho}\Big|_{\rho=0} = (-1)^i \log^i y$

get $$G(\alpha, L) = 1 - \alpha \underbrace{G\left(\alpha, \frac{\partial}{\partial(-\rho)}\right)^{-1}}_{\text{series in } \alpha,\ \frac{\partial}{\partial\rho}} \underbrace{(e^{-L\rho} - 1) F(\rho)}_{\text{series } L,\rho}\Big|_{\rho=0}$$

integral for

Slightly more generally

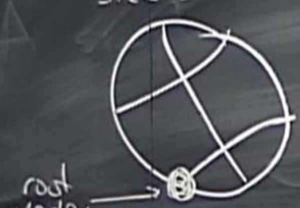
$$G(\alpha, L) = 1 - \sum_{k \geq 1} \alpha^k G\left(\alpha, \frac{\partial}{\partial(-\rho)}\right)^{1-sk} (e^{-L\rho} - 1) F_k(\rho)\Big|_{\rho=0}$$

where $F_k(\rho) = \sum_{i \geq 0} f_{k,i} \rho^{i-1}$

with $\int$ regularized by ρ.

where the sum runs
our rooted connected
chord diagrams

size 3



$|C| = \#$ chords

need recursive chord order

root chord is 1
remove root chord, get
connected components $C_1, C_2, \ldots, C_k$
order C_1 next then C_2 etc.

eg

and in general Markus Hihn

then $G(x, L) = (1 - \ldots xL)$

$F(\rho) = f_0 \rho^{-1} + f_1 + f_2 \rho + \cdots$

say a chord is terminal if everything it crosses occurs before it in the recursive chord order

let $t_1 < t_2 < \ldots < t_\ell$ be the terminal chords

then $b(C) = t_1$

$f_C = f_{t_\ell - t_{\ell-1}} \cdots f_{t_3 - t_2} f_{t_2 - t_1} f_0^{|C| - \ell}$

$\ldots C_k$