Title: Feynman integrals and motives in configuration spaces

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Abstract: This talk will cover aspects of Feynman integral computations in configuration spaces, and some related mathematical problems, and the occurrence of motives and periods, focusing on joint work with Ozgur Ceyhan.

Question:

When are Feynman integrals periods of mixed Tate motives? (multiple zeta values: extensive example collection Broadhurst–Kreimer)

- Two methods of computing Feynman integrals: momentum space or configuration space (Fourier transform)

$$G_m^{\mathbb{R}}(x_s - x_t) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} dp \, \frac{e^{ip\cdot(x_s - x_t)}}{p^2 + m^2 + i\epsilon}$$

- focus here on *configuration space* picture
- Note: Fourier transform not an algebraic operation (a priori not obvious why it should preserve the nature of motives and periods)

Periods and motives: $\int_\sigma \omega$ numbers obtained integrating an algebraic differential form over a cycle defined by algebraic equations
Constraints on numbers obtained as periods from the motive of the variety!

- Periods of mixed Tate motives over $\mathbb{Z}$ are combinations of Multiple Zeta Values (coeffs. rationals and powers of $2\pi i$)

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{n_1 > n_2 > \cdots > n_r \geq 1} n_1^{-k_1} n_2^{-k_2} \cdots n_r^{-k_r}$$

- Francis Brown, *Mixed Tate motives over* $\mathbb{Z}$, Annals of Math 2012, arXiv:1102.1312

Feynman integrals and periods in momentum space: MZVs frequently occur, but known non-mixed-Tate cases (Brown–Doryn, Schnetz)

General setting: scalar perturbative QFTs

$$S(\phi) = \int \mathscr{L}(\phi) d^D x = S_0(\phi) + S_{int}(\phi)$$

in D dimensions, with Lagrangian density

$$\mathscr{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathscr{L}_{int}(\phi)$$

Perturbative expansion: Feynman rules and Feynman diagrams

$$S_{eff}(\phi) = S_0(\phi) + \sum_{\Gamma} \frac{U(\Gamma, \phi)}{\#\mathrm{Aut}(\Gamma)} \quad \text{(1PI graphs)}$$

Amplitudes $U(\Gamma)$ for fixed external edges of the graph are integral (generally divergent) on:

- momenta associated to internal edges of the graph with momentum conservation rules at vertices
- configurations associated to vertices of the graph with divergences where coordinates collide (diagonals)

• Configuration space: wonderful compactifications of graph configuration spaces; mixed Tate motives; Feynman amplitude and Laplacian Green functions; explicit results using Gegenbauer polynomial expansion; pullback to wonderful compactification, cohomologous to algebraic form with logarithmic poles; deformation and renormalization.

Main geometric setting:

1. $X \supset \mathbb{A}^D$ algebraic variety
2. $\mathrm{Conf}_\Gamma(X) = X^{V_\Gamma} \setminus \cup_e \Delta_e$ configuration space
3. $\overline{\mathrm{Conf}}_\Gamma(X)$ "wonderful compactification"

Two different problems: (1) real case, (2) complexified case

Feynman amplitude in configuration space (dim $D = 2\lambda + 2$)

Version N.1: real case

$$\omega_\Gamma = \prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} \bigwedge_{v \in V_\Gamma} dx_v$$

defines a $\mathscr{C}^\infty$-differential form on X^{V_Γ} with singularities along diagonals $x_{s(e)} = x_{t(e)}$

- not algebraic form
- not closed form
- chain of integration:

$$\sigma_\Gamma = X(\mathbb{R})^{V_\Gamma}$$

Version N.2: complexification

$Z = X \times X$ with projection $p : Z \to X$, $p : z = (x, y) \mapsto x$

$$\omega_\Gamma^{(Z)} = \prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2D-2}} \bigwedge_{v \in V_\Gamma} dx_v \wedge d\bar{x}_v$$

where $\|x_{s(e)} - x_{t(e)}\| = \|p(z)_{s(e)} - p(z)_{t(e)}\|$

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- closed form
- chain of integration:

$$\sigma^{(Z,y)} = X^{V_\Gamma}(\mathbb{C}) = X^{V_\Gamma} \times \{y = (y_v)\} \subset Z^{V_\Gamma} = X^{V_\Gamma} \times X^{V_\Gamma}$$

for a fixed $y = (y_v \mid v \in V_\Gamma)$

The real case and the physical amplitude:

Momentum space: momentum variables k_e with $e \in E_\Gamma$

$\Gamma(\phi)$ built from edge-propagators

$$\frac{1}{(m^2 + \|k_e\|^2)}$$

Configuration space: position variables x_v with $v \in V_\Gamma$

$\Gamma(\phi)$ built from propagators:

$$G_{0,\mathbb{R}}(x_{s(e)} - x_{t(e)}) = \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}}, \quad \text{where} \quad D = 2\lambda + 2$$

or massive

$$G_{m,\mathbb{R}}(x_{s(e)} - x_{t(e)}) = \frac{m^\lambda}{(2\pi)^{(\lambda+1)}} \|x_{s(e)} - x_{t(e)}\|^{-\lambda} \mathscr{K}_\lambda(m\|x_{s(e)} - x_{t(e)}\|)$$

with $\mathscr{K}_\nu(z)$ modified Bessel function

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Dual pictures:

- $G_{0,\mathbb{R}}(x_{s(e)} - x_{t(e)})$ Green function of Laplacian; $G_{m,\mathbb{R}}(x_{s(e)} - x_{t(e)})$ fundamental solution of Helmholtz equation $(\Delta + m^2)G = \delta$
- Fourier transform: (test functions $\varphi \in \mathscr{S}(\mathbb{R}^D)$)

$$\widehat{(G_{0,\mathbb{R}} \star \varphi)}(k) = \frac{4\pi^{D/2}}{\Gamma(\lambda)} \frac{1}{\|k\|^2} \widehat{\varphi}(k)$$

$$\widehat{(G_{m,\mathbb{R},} \star \varphi)}(k) = \frac{1}{(m^2 + \|k\|^2)} \widehat{\varphi}(k)$$

complexification not directly computing the physical amplitude, but a closely related problem with interestingly different structure!

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Algebraic versus $\mathscr{C}^\infty$-formulation:

- Algebraic formulation: extend from real to complex variables using a quadratic form instead of the Euclidean norm
- used in momentum space Feynman amplitudes ($n = \#E_\Gamma$)

$$U(\Gamma) = \int \frac{\delta(\sum_{i=1}^{n} \epsilon_{v,i} k_i + \sum_{j=1}^{N} \epsilon_{v,j} p_j)}{q_1 \cdots q_n} \, d^D k_1 \cdots d^D k_n$$

quadratic form

$$q_e(k_e) = \sum_{j=1}^{D} k_{e,j}^2 + m^2$$

- can use also in configuration space Feynman amplitude (massless; $m = \# V_\Gamma$)

$$U(\Gamma) = \int \frac{1}{Q_1 \cdots Q_n} d^D x_{v_1} \cdots d^D x_{v_m}$$

$$Q_e(x_{s(e)}, x_{t(e)}) = \sum_{j=1}^{D} (x_{s(e),j} - x_{t(e),j})^2$$

- Advantages: *get an algebraic differential form*
- Disadvantages: *singular on a hypersurface* (whose motive is difficult to control)
- situation similar to the momentum space approach with "graph hypersurfaces"...

We follow a different approach with $\mathscr{C}^\infty$-forms

- Analytic formulation: extend from real to complex variables using the Euclidean norm

$$\omega_\Gamma = \prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} \bigwedge_{v \in V_\Gamma} dx_v$$

over chain of integration $\sigma_\Gamma = \mathbb{R}^{\# V_\Gamma}$

- Advantages: *Singular on diagonals* (motive will be easy to control)
- Disadvantages: *not an algebraic differential form* (only smooth)

Different methods:

- Version N.1 real case: explicit computation of regularized integral

$$\int_{\sigma_\Gamma} \omega_\Gamma$$

- use expansion of Green function in Gegenbauer polynomials: explicit occurrence of multiple zeta values
- isolate one term in the motive (canonically described in representation theoretic terms) that remains mixed Tate even when the overall Feynman motive is not mixed Tate

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- Version N.2 complexification: cohomological method
- pullback $\omega_\Gamma^{(Z)}$ to wonderful compactification of configuration space
- cohomologous to algebraic form with log poles
- regularize to separate poles from chain of integration
- in this complexified case motive is mixed Tate

(not necessarily over $\mathbb{Z}$ or $\mathbb{Q}$)

Version N.1: Explicit computations of Feynman amplitudes (real case):

Step 1: explicit chains in X^{V_Γ}

- Acyclic orientations: Γ no looping edges, $\Omega(\Gamma)$ set of acyclic orientations; Stanley: $(-1)^{V_\Gamma} P_\Gamma(-1)$ acyclic orientations where $P_\Gamma(t)$ chromatic polynomial
- orientation $\mathbf{o} \in \Omega(\Gamma) \Rightarrow$ partial ordering of vertices $w \geq_{\mathbf{o}} v$
- chain with boundary $\partial \Sigma_{\mathbf{o}} \subset \cup_{e \in E_\Gamma} \Delta_e$

$$\Sigma_{\mathbf{o}} := \{(x_v) \in X^{V_\Gamma}(\mathbb{R}) \ : \ r_w \geq r_v \text{ whenever } w \geq_{\mathbf{o}} v\}$$

middle dimensional relative homology class

$$[\Sigma_{\mathbf{o}}] \in H_{|V_\Gamma|}(X^{V_\Gamma}, \cup_{e \in E_\Gamma} \Delta_e)$$

- $\Sigma_{\mathbf{o}} \setminus \cup_v \{r_v = 0\}$ bundle fiber $(S^{D-1})^{V_\Gamma}$ base

$$\overline{\Sigma}_{\mathbf{o}} = \{(r_v) \in (\mathbb{R}^*_+)^{V_\Gamma} \ : \ r_w \geq r_v \text{ whenever } w \geq_{\mathbf{o}} v\}$$

Step 2: Gegenbauer polynomials

- Generating function and orthogonality ($|t| < 1$ and $\lambda > -1/2$)

$$\frac{1}{(1-2tx+t^2)^\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) t^n$$

$$\int_{-1}^{1} C_n^{(\lambda)}(x) C_m^{(\lambda)}(x) \, (1-x^2)^{\lambda-1/2} dx = \delta_{n,m} \frac{\pi 2^{1-2\lambda} \Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma(\lambda)^2}$$

- $D = 2\lambda + 2$ Newton potential expansion in Gegenbauer polynomials:

$$\frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} = \frac{1}{\rho_e^{2\lambda}(1 + (\frac{r_e}{\rho_e})^2 - 2\frac{r_e}{\rho_e}\omega_{s(e)} \cdot \omega_{t(e)})^\lambda}$$

$$= \rho_e^{-2\lambda} \sum_{n=0}^{\infty} (\frac{r_e}{\rho_e})^n C_n^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}),$$

with $\rho_e = \max\{\|x_{s(e)}\|, \|x_{t(e)}\|\}$ and $r_e = \min\{\|x_{s(e)}\|, \|x_{t(e)}\|\}$ and with $\omega_v \in S^{D-1}$

Gegenbauer polynomials and spherical harmonics

- orthonormal basis $\{Y_j\}$ of the Hilbert space $\mathscr{H}_n(S^{D-1})$ of spherical harmonics on S^{D-1} of degree n
- zonal spherical harmonics

$$Z^{(n)}_{\omega_1}(\omega_2) = \sum_{j=1}^{\dim \mathscr{H}_n(S^{D-1})} Y_j(\omega_1)\overline{Y_j(\omega_2)}$$

- Gegenbauer polynomials

$$C_n^{(\lambda)}(\omega_1 \cdot \omega_2) = c_{D,n}\, Z^{(n)}_{\omega_1}(\omega_2)$$

$$c_{D,n} = \frac{Vol(S^{D-1})\,(D-2)}{2n+D-2}$$

- dimension $D = 4$

$$\dim \mathscr{H}_n(S^3) = \binom{n+3}{n} - \binom{n+1}{n-2} = (n+1)^2$$

Step 3: angular and radial integrals

- on chain of integration $\sigma_\Gamma = X(\mathbb{R})^{V_\Gamma}$ Feynman integral becomes (Version N.1)

$$\sum_{\mathbf{o}\in\Omega(\Gamma)} m_{\mathbf{o}} \int_{\Sigma_{\mathbf{o}}} \prod_{e\in E_\Gamma} r_{t_{\mathbf{o}}(e)}^{-2\lambda} \left(\sum_n \left(\frac{r_{s_{\mathbf{o}}(e)}}{r_{t_{\mathbf{o}}(e)}}\right)^n C_n^{(\lambda)}(\omega_{s_{\mathbf{o}}(e)} \cdot \omega_{t_{\mathbf{o}}(e)}) \right) dV$$

with positive integers $m_{\mathbf{o}}$ (multiplicities) and volume form $dV = \prod_v d^D x_v = \prod_v r_v^{D-1} dr_v \, d\omega_v$

- angular integrals:

$$\mathscr{A}_{(n_e)_{e\in E_\Gamma}} = \int_{(S^{D-1})^{V_\Gamma}} \prod_e C_{n_e}^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}) \prod_v d\omega_v$$

- radial integrals:

$$\sum_{\mathbf{o}\in\Omega(\Gamma)} m_{\mathbf{o}} \int_{\bar{\Sigma}_{\mathbf{o}}} \prod_{e\in E_\Gamma} \mathscr{F}(r_{s_{\mathbf{o}}(e)}, r_{t_{\mathbf{o}}(e)}) \prod_v r_v^{D-1} dr_v$$

$$\mathscr{F}(r_{s_{\mathbf{o}}(e)}, r_{t_{\mathbf{o}}(e)}) = r_{t_{\mathbf{o}}(e)}^{-2\lambda} \sum_{n_e} \mathscr{A}_{n_e} \left(\frac{r_{s_{\mathbf{o}}(e)}}{r_{t_{\mathbf{o}}(e)}}\right)^{n_e}$$

Example: polygons and polylogarithms

- Γ polygon with k edges, $D = 2\lambda + 2$:

$$\mathscr{A}_n = \left(\frac{\lambda 2\pi^{\lambda+1}}{\Gamma(\lambda+1)(n+\lambda)} \right)^k \cdot \dim \mathscr{H}_n(S^{2\lambda+1})$$

$\mathscr{H}_n(S^{2\lambda+1})$ space of harmonic functions deg n on $S^{2\lambda+1}$
(Gegenbauer polynomial and zonal spherical harmonics)

- when $D = 4$, Feynman amplitude:

$$(2\pi^2)^k \sum_{\mathbf{o}} m_{\mathbf{o}} \int_{\bar{\Sigma}_{\mathbf{o}}} \mathrm{Li}_{k-2}\Big(\prod_i \frac{r_{w_i}^2}{r_{v_i}^2}\Big) \prod_v r_v \, dr_v$$

polylogarithm functions

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

vertices v_i, w_i sources and tails of oriented paths of $\mathbf{o}$

Problem: computations intractable very quickly for larger graphs!

- Can reduce to trivalent vertices and use triple integrals of harmonic functions: Gaunt coefficients $\langle Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2} Y^{(n_3)}_{\ell_3} \rangle_D$ Racah's factorization in terms of *isoscalar factors*

$$\langle Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2}, Y^{(n_3)}_{\ell_3} \rangle_D = \begin{pmatrix} n_1 & n_2 & n_3 \\ n'_1 & n'_2 & n'_3 \end{pmatrix}_{D:D-1} \langle Y^{(n'_1)}_{\ell'_1}, Y^{(n'_2)}_{\ell'_2}, Y^{(n'_3)}_{\ell'_3} \rangle_{D-1}$$

$\ell_i = (n'_i, \ell'_i)$ with $n'_i = m_{D-2,i}$ and $\ell'_i = (m_{D-3,i}, \ldots, m_{1,i})$

- There are general explicit (but complicated) expressions for the isoscalar factors

- Focus on term with $\ell_i = 0$: this $(\underline{n}, \underline{0})$-term is the "deepest" term, canonically defined $SO(D)$-representation theoretically

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- For $D = 4$ "leading term" involving multiple series related to MVZs:

Mordell–Tornheim multiple series:

$$\zeta_{MT,k}(s_1,\ldots,s_k;s_{k+1}) = \sum_{(n_1,\ldots,n_k)\in\mathscr{R}_P^{(k)}} n_1^{-s_1}\cdots n_k^{-s_k}(n_1+\cdots+n_k)^{-s_{k+1}}$$

$$\mathscr{R} = \mathscr{R}_P^{(k)} := \{(n_1,\ldots,n_k)\,|\,n_i > 0,\ \ i = 1,\ldots,k\}$$

Apostol–Vu multiple series:

$$\zeta_{AV,k}(s_1,\ldots,s_k;s_{k+1}) = \sum_{(n_1,\ldots,n_k)\in\mathscr{R}_{MP}^{(k)}} n_1^{-s_1}\cdots n_k^{-s_k}(n_1+\cdots+n_k)^{-s_{k+1}}$$

$$\mathscr{R} = \mathscr{R}_{MP}^{(k)} := \{(n_1,\ldots,n_k)\,|\,n_k > \cdots > n_2 > n_1 > 0\}$$

- known these expressible in terms of multiple zeta values

Version N.1 Conclusion:

- by this method can see explicit integrals leading to multiple zeta values arising from the deepest $(\underline{n}, \underline{0})$-term: representation theoretically identifies a part of the motive that remains mixed-Tate
- ... computations become easily extremely complicated even for simple graphs!
- expect occurrences of non-mixed-Tate periods from other terms in the isoscalar factors, beyond the $(\underline{n}, \underline{0})$ term, for sufficiently large graphs!

Version N.2: Graph configuration spaces

X a smooth projective algebraic variety that contains a dense $\mathbb{A}^D$: for instance $X = \mathbb{P}^D$, with D spacetime dimension.

Feynman amplitude ω_Γ on X^{V_Γ}

Singularities of Feynman amplitude along diagonals

$$\Delta_e = \{(x_v)_{v \in V_\Gamma} \mid x_{v_1} = x_{v_2} \text{ for } \partial_\Gamma(e) = \{v_1, v_2\}\}$$

Graph configuration space:

$$Conf_\Gamma(X) = X^{V_\Gamma} \setminus \bigcup_{e \in E_\Gamma} \Delta_e$$

Goal N.1: compactify $Conf_\Gamma(X)$ to a smooth projective algebraic variety $\overline{Conf}_\Gamma(X)$ so that

$$\overline{Conf}_\Gamma(X) \setminus Conf_\Gamma(X)$$

is a normal crossings divisor

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Variants: Version N.2 of configuration space for amplitude $\omega_\Gamma^{(Z)}$

$$F(X,\Gamma) = Z^{V_\Gamma} \setminus \bigcup_{e\in E_\Gamma} \Delta_e^{(Z)} \cong (X\times X)^{V_\Gamma} \setminus \bigcup_{e\in E_\Gamma} \Delta_e^{(Z)}$$

with diagonals

$$\Delta_e^{(Z)} \cong \{(z_v \mid v\in V_\Gamma) \in Z^{V_\Gamma} \mid p(z_{s(e)}) = p(z_{t(e)})\}$$

Relation to previous:

$$F(X,\Gamma) \simeq \mathrm{Conf}_\Gamma(X) \times X^{V_\Gamma}$$

$$\Delta_e^{(Z)} \cong \Delta_e \times X^{V_\Gamma}$$

Compactify to $\overline{F(X,\Gamma)}$ in same way

Blowup construction of wonderful compactifications

- Connected induced subgraphs: $\mathbb{SG}_k(\Gamma) = \{\gamma \in \mathbb{SG}(\Gamma) \mid |V_\gamma| = k\}$ and polydiagonals $\hat{\Delta}_\gamma = \{x_{s(e)} = x_{t(e)} \ : \ e \in E_\gamma\}$ (same as diagonal Δ_γ if γ connected)
- Arrangement of subvarieties in X^{V_Γ}: $\mathscr{S}_\Gamma$ polydiagonals of disjoint unions of connected induced subgraphs
- Building set for the arrangement:

$$\mathscr{G}_\Gamma = \{\Delta_\gamma \ : \ \gamma \text{ induced, biconnected } \}$$

$$Conf_\Gamma(X) = X^{V_\Gamma} \setminus \cup_{\gamma \subset \mathscr{G}_\Gamma} \Delta_\gamma$$

- Start with $Y_0 = X^{V_\Gamma}$; obtain Y_k from Y_{k-1} by blowup along the iterated *dominant transforms* (=proper transform or inverse image of exceptional divisor) of

$$\cup_{\gamma \in \mathscr{G}_{n-k+1,\Gamma}} \Delta_\gamma$$

with $\mathscr{G}_{k,\Gamma} = \mathscr{G}_\Gamma \cap \mathbb{SG}_k(\Gamma)$ then

$$Y_{n-1} = \overline{Conf}_\Gamma(X)$$

Boundary structure

- $\mathscr{G}_\Gamma$-nests: sets of biconnected induced subgraphs with $\gamma \cap \gamma' = \emptyset$ or $\gamma \cap \gamma' = \{v\}$ single vertex or $\gamma \subseteq \gamma'$ or $\gamma' \subseteq \gamma$

$$\overline{Conf}_\Gamma(X) \smallsetminus Conf_\Gamma(X) = \bigcup_{\Delta_\gamma \in \mathscr{G}_\Gamma} D_\gamma$$

- divisors D_γ (iterated dominant transform of Δ_γ) with

$$D_{\gamma_1} \cap \cdots \cap D_{\gamma_\ell} \neq \emptyset \Leftrightarrow \{\gamma_1, \ldots, \gamma_\ell\} \text{ is a } \mathscr{G}_\Gamma\text{-nest}$$

and transverse intersections

- strata parameterized by forests of nested subgraphs (as in Fulton–MacPherson case)
- case of $\overline{F(X,\Gamma)}$ completely analogous

Motives of configuration spaces – Key ingredient: Blowup formulae

- For mixed motives (Voevodsky category):

$$\mathfrak{m}(\mathrm{Bl}_V(Y)) \cong \mathfrak{m}(Y) \oplus \bigoplus_{k=1}^{\mathrm{codim}_Y(V)-1} \mathfrak{m}(V)(k)[2k]$$

- For Grothendieck classes Bittner relation

$$[\mathrm{Bl}_V(Y)] = [Y] - [V] + [E] = [Y] + [V]([\mathbb{P}^{\mathrm{codim}_Y(V)-1}] - 1)$$

exceptional divisor E

- Conclusion: the motive of $\overline{Conf}_\Gamma(X)$ and of $\overline{F(X,\Gamma)}$ is mixed Tate if X is mixed Tate.

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Voevodsky motive: (quasi-projective smooth X)

$$\mathfrak{m}(\overline{Conf}_\Gamma(X)) = \mathfrak{m}(X)^{V_\Gamma} \oplus \bigoplus_{\mathcal{N}\in\mathcal{G}_\Gamma\text{-nests},\mu\in M_{\mathcal{N}}} \mathfrak{m}(X)^{V_{\Gamma/\delta_{\mathcal{N}}(\Gamma)}}(\|\mu\|)[2\|\mu\|]$$

where $M_{\mathcal{N}} := \{(\mu_\gamma)_{\Delta_\gamma\in\mathcal{G}_\Gamma} : 1 \leq \mu_\gamma \leq r_\gamma - 1,\ \mu_\gamma \in \mathbb{Z}\}$ with $r_\gamma = r_{\gamma,\mathcal{N}} := \dim(\cap_{\gamma'\in\mathcal{N}:\gamma'\subset\gamma}\Delta_{\gamma'}) - \dim\Delta_\gamma$ and $\|\mu\| := \sum_{\Delta_\gamma\in\mathcal{G}_\Gamma}\mu_\gamma$

$$\Gamma/\delta_{\mathcal{N}}(\Gamma) = \Gamma//(\gamma_1\cup\cdots\cup\gamma_r)$$

for $\mathcal{N} = \{\gamma_1,\ldots,\gamma_r\}$

Class in the Grothendieck ring:

$$[\overline{Conf}_\Gamma(X)] = [X]^{V_\Gamma} + \sum_{\mathcal{N}\in\mathcal{G}_\Gamma\text{-nests}} [X]^{V_{\Gamma/\delta_{\mathcal{N}}(\Gamma)}} \sum_{\mu\in M_{\mathcal{N}}} \mathbb{L}^{\|\mu\|}$$

Chow motive: (smooth projective X): from result of Li Li on wonderful compactifications of arrangements of subvarieties

Pullback and forms with logarithmic poles

- $\pi_\gamma^*(\omega_\Gamma^{(Z)})$ pullback to iterated blowup $\overline{F(X,\Gamma)}$ of Z^{V_Γ} along dominant transforms of $\Delta_\gamma^{(Z)}$ of biconnected induced subgraphs
- Divergence locus union of divisors (dominant transforms of $\Delta_\gamma^{(Z)}$)

$$\bigcup_{\Delta_\gamma^{(Z)} \in \mathscr{G}_\Gamma} D_\gamma^{(Z)}$$

- Chain of integration $\tilde{\sigma}_\Gamma^{(Z,y)} = \overline{\mathrm{Conf}}_\Gamma(X) \times \{y\} \subset \overline{F(X,\Gamma)}$ intersects divergence locus in

$$\mathscr{D}_\Gamma = \bigcup_{\Delta_\gamma^{(Z)} \in \mathscr{G}_\Gamma} D_\gamma \times \{y\} \subset \overline{\mathrm{Conf}}_\Gamma(X) \times \{y\}$$

- pullback $\pi_\gamma^*(\omega_\Gamma^{(Z)})$ on $\tilde{\sigma}_\Gamma^{(Z,y)}$ smooth closed form on

$$\overline{\mathrm{Conf}}_\Gamma(X) \smallsetminus \left(\bigcup_{\gamma \in \mathscr{G}_\Gamma} D_\gamma \right)$$

Pullback and forms with logarithmic poles

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Smooth and algebraic forms

- de Rham cohomology of a smooth quasi-projective varieties computed using algebraic differential forms (Grothendieck)
- if complement of normal crossings divisor can use forms with log poles (Deligne)

$$H^*(\mathscr{U}) \simeq \mathbb{H}^*(\mathscr{X}, \Omega^*_{\mathscr{X}}(\log(\mathscr{D})))$$

- $\mathscr{X}$ smooth projective variety $\dim_{\mathbb{C}} m$; $\mathscr{D}$ normal crossings divisor; $\mathscr{U} = \mathscr{X} \setminus \mathscr{D}$; ω smooth closed differential form $\deg m$ on $\mathscr{U}$;
$\Rightarrow \exists$ algebraic differential form η log poles along $\mathscr{D}$, with
$[\eta] = [\omega] \in H^m_{dR}(\mathscr{U})$
- Conclusion: $\exists$ algebraic form $\eta_\Gamma^{(Z)}$ with log poles along union of D_γ, cohomologous to $\pi_\gamma^*(\omega_\Gamma^{(Z)})$ on $\tilde{\sigma}_\Gamma^{(Z,y)}$
- Warning: motive over $\mathbb{Q}$, but algebraic form maybe larger field!

Regularization problem

• $\eta_\Gamma^{(z)}$ algebraic differential form; $\tilde{\sigma}_\Gamma^{(z,y)}$ algebraic cycle: Feynman integral becomes

$$\int_{\tilde{\sigma}_\Gamma^{(z,y)} \setminus \mathscr{D}_\Gamma} \eta_\Gamma^{(z)}$$

would be a period... but divergent!! (because of intersection $\mathscr{D}_\Gamma$ of chain with divisors)

• need a regularization procedure: separate chain of integration from divergence locus

Three different regularization methods

1. Principal value current regularization and iterated Poincaré residues
2. Deformation to the normal cone
3. Algebraic renormalization via Hopf algebras and Rota–Baxter algebras

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Current regularization

- Regularized Feynman amplitude:

$$\langle PV(\eta_\Gamma^{(Z)}), \varphi\rangle = \lim_{\lambda\to 0} \int_{\tilde{\sigma}_\Gamma^{(Z,y)}} |f_n|^{2\lambda_n} \cdots |f_1|^{2\lambda_1} \eta_\Gamma^{(Z,y)} \ \varphi$$

where φ test functions; $n = n_\Gamma = \#\mathscr{G}_\Gamma$; and f_k equation of $D_{\gamma_k}^{(Z)}$

- Ambiguities of regularization:

$$\tilde{\sigma}_{\Gamma,\mathscr{N},\epsilon}^{(Z,y)} := \tilde{\sigma}_\Gamma^{(Z,y)} \cap T_{\mathscr{N},\epsilon}(f) \cap N_{\mathscr{N},\epsilon}(f)$$

$$T_{\mathscr{N},\epsilon}(f) = \{|f_k| = \epsilon_k,\ k = 1, \ldots, r\}$$

$$N_{\mathscr{N},\epsilon}(f) = \{|f_k| > \epsilon,\ k = r+1, \ldots, n\}$$

n graphs in $\mathscr{G}_\Gamma$ ordered so that first r in the nest $\mathscr{N}$

$$\lim_{\epsilon\to 0} \int_{\tilde{\sigma}_{\Gamma,\mathscr{N},\epsilon}^{(Z,y)}} \varphi\, \eta_\Gamma^{(Z,y)}$$

has a residue (iterated Poincaré residue) supported on

$$V_{\mathscr{N}}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)}$$

Iterated Poincaré residue

$$\int_{\Sigma_{\mathcal{N}}} \mathscr{R}_{\mathcal{N}}(\eta_\Gamma) = \frac{1}{(2\pi i)^r} \int_{\mathscr{L}_{\mathcal{N}}(\Sigma_{\mathcal{N}})} \eta_\Gamma$$

$(2D|V_\Gamma| - r)$-cycle $\Sigma_{\mathcal{N}}$ in $V_{\mathcal{N}}^{(Z)}$; iterated Leray coboundary $\mathscr{L}_{\mathcal{N}}(\Sigma_{\mathcal{N}})$ in $\overline{F(X,\Gamma)}$ is a T^r-torus bundle over $\Sigma_{\mathcal{N}}$

- If the variety X is a mixed Tate motive, these residues are all periods of mixed Tate motives
- On intersections of chain of integration and divergence loci

$$\langle \mathscr{R}_{\mathcal{N}}(\eta_\Gamma), V_{\mathcal{N}} \rangle = \int_{V_{\mathcal{N}} \times \{y\}} \mathscr{R}_{\mathcal{N}}(\eta_\Gamma)$$

Iterated Poincaré residue

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Deformation to the normal cone

- extend integral

$$\int_{\tilde{\sigma}_\Gamma^{(Z,y)}} \pi_\Gamma^*(\omega_\Gamma^{(Z)})$$

to a larger ambient deformation space where can separate $\tilde{\sigma}_\Gamma^{(Z,y)}$ from the divergence locus

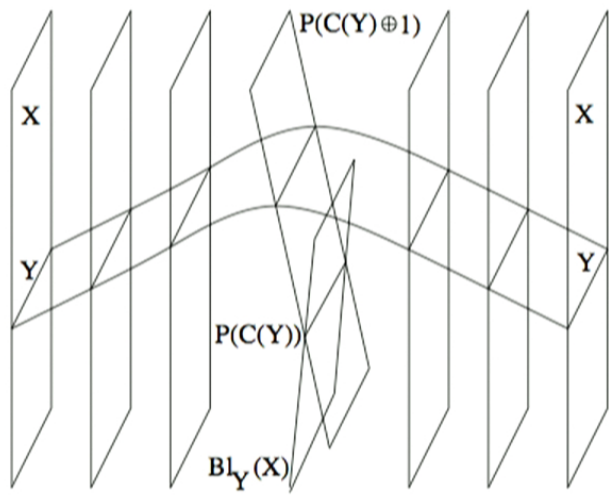
- start with $Z^{V_\Gamma} \times \mathbb{P}^1$, deformation coordinate $\zeta \in \mathbb{P}^1$, and

$$\tilde{\omega}_\Gamma^{(Z)} = \prod_{e \in E_\Gamma} \frac{1}{(\|x_{s(e)} - x_{t(e)}\|^2 + |\zeta|^2)^{D-1}} \bigwedge_{v \in V_\Gamma} dx_v \wedge d\bar{x}_v \wedge d\zeta \wedge d\bar{\zeta}$$

- divergence locus in the central fiber $\zeta = 0$

$$\cup_{e \in E_\Gamma} \Delta_e^{(Z)} \subset Z^{V_\Gamma} \times \{0\}$$

- starting with $Z^{V_\Gamma} \times \mathbb{P}^1$ blowups along $\Delta_\gamma^{(Z)} \times \{0\}$, induced biconnected subgraphs
- obtain smooth projective variety $\mathscr{D}(Z[\Gamma])$ fibered over $\mathbb{P}^1$: fiber over $\zeta \neq 0 \in \mathbb{P}^1$ equal to Z^{V_Γ}; fiber over $\zeta = 0$ has a component $\overline{F(X,\Gamma)}$ plus other components projectivizations $\mathbb{P}(C \oplus 1)$ of normal cones of blowups



- in $\mathscr{D}(Z[\Gamma])$ the chain of integration $\tilde{\sigma}_\Gamma^{(Z,y)}$ becomes separated from the locus of divergence

Connes–Kreimer Hopf algebra $\mathscr{H} = \mathscr{H}(\mathscr{T})$ (depends on theory)

- Free commutative algebra in generators Γ 1PI Feynman graphs
- Grading: loop number (or internal lines)

$$\deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0$$

- Coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathscr{V}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

- Antipode: inductively

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

Algebraic renormalization (Connes-Kreimer; Ebrahimi-Fard, Guo, Kreimer)

- Rota–Baxter algebra of weight $\lambda = -1$: $\mathscr{R}$ commutative unital algebra; $T : \mathscr{R} \to \mathscr{R}$ linear operator with

$$T(x)T(y) = T(xT(y)) + T(T(x)y) + \lambda T(xy)$$

- Example: $T =$ projection onto polar part of Laurent series
- T determines splitting $\mathscr{R}_+ = (1 - T)\mathscr{R}$, $\mathscr{R}_- =$ unitization of $T\mathscr{R}$; both $\mathscr{R}_\pm$ are algebras
- Feynman rule $\phi : \mathscr{H} \to \mathscr{R}$ commutative algebra homomorphism from CK Hopf algebra $\mathscr{H}$ to Rota–Baxter algebra $\mathscr{R}$ weight -1

$$\phi \in \mathrm{Hom}_{\mathrm{Alg}}(\mathscr{H}, \mathscr{R})$$

- Note: ϕ does *not know* that $\mathscr{H}$ Hopf and $\mathscr{R}$ Rota-Baxter, only commutative algebras

Rota-Baxter algebra for $(\mathscr{Y}, \mathscr{D})$

- even forms with log poles $\Omega^{\text{even}}_{\mathscr{Y}}(\log \mathscr{D})$: commutative algebra
- polar part operator

$$T(\eta) = \sum_{j=1}^{n} \frac{df_j}{f_j} \wedge \mathrm{Res}_{D_j}(\eta)$$

f_j = local equation for D_j

- $(\Omega^{\text{even}}_{\mathscr{Y}}(\log \mathscr{D}), T)$ = Rota–Baxter algebra of weight -1

$$T(\eta \wedge T(\xi)) + T(T(\eta) \wedge \xi) - T(\eta) \wedge T(\xi) = T(\eta \wedge \xi)$$

- obtain Rota-Baxter algebra of configuration spaces
- Regularization: given a Feynman graph Γ and the (non-holomorphic closed) form $\omega_\Gamma^{(Z)}$: pull back to wornderful compactification and replace by cohomologous algebraic form η_Γ with log poles
- algebraic Feynman rules: the assignment

$$\phi : \Gamma \mapsto \omega_\Gamma^{(Z)} \mapsto \eta_\Gamma$$

defines a morphism of commutative algebras from the Hopf algebra of Feynman graphs to the Rota–Baxter algebra of configuration spaces

- Renormalization: apply BPHZ to this algebraic Feynman rule

Birkhoff factorization

$$\phi_-(\Gamma) = -T(\eta_\Gamma + \sum_{\gamma\subset\Gamma} \phi_-(\gamma) \wedge \eta_{\Gamma/\gamma})$$

$$\phi_+(\Gamma) = (1-T)(\eta_\Gamma + \sum_{\gamma\subset\Gamma} \phi_-(\gamma) \wedge \eta_{\Gamma/\gamma}) = \eta_{\Gamma,\mathscr{D}} + \sum_{\gamma\subset\Gamma} (\phi_-(\gamma) \wedge \eta_{\Gamma/\gamma})_{\mathscr{D}},$$

with $\eta_{\mathscr{D}} = \eta - T(\eta)$

Renormalized integral

$$\int_{\tilde{\sigma}_{\Gamma,\mathbb{C}}} \eta_{\Gamma,\mathscr{D}} + \sum_{\gamma\subset\Gamma} (\phi_-(\gamma) \wedge \eta_{\Gamma/\gamma})_{\mathscr{D}}$$

free of divergences and integral of algebraic differential form: genuine period