Title: Resurgent analysis and its applications to the Witten Laplacian

Date: May 28, 2015 11:00 AM

URL: http://pirsa.org/15050060

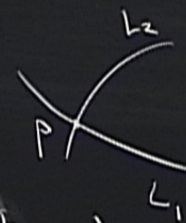
Abstract: The first lecture will be devoted to the review of the classical theory of the Witten Laplacian, the second -- to the concepts of resurgent analysis. The third -- to applications of the resurgent analysis to the Witten Laplacian. Time permitting, we will touch upon some foundational questions of resurgent analysis.

Witten Laplacian & Resurgence

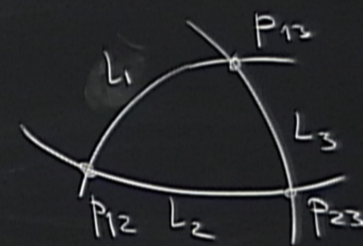
Fukaya categ.

(M, ω) $\quad \mathcal{O}b = $ Lagr. submflds L

$\mathrm{Mor}(L_1, L_2) =$



$$= \bigoplus_{p \in L_1 \cap L_2} (\text{coeff ring}) \cdot p$$



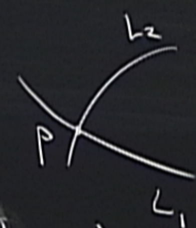
$$p_{12} \circ p_{23} = e^{-\frac{\mathrm{Area}\,\triangle}{\hbar}}\, p_{13}$$

Witten Laplacian & Resurgence

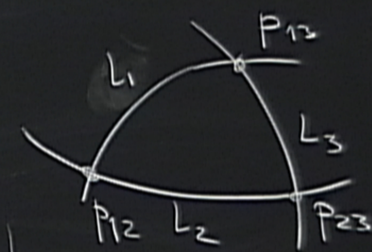
Fukaya categ.

(M, ω) Ob = Lagr. submflds L

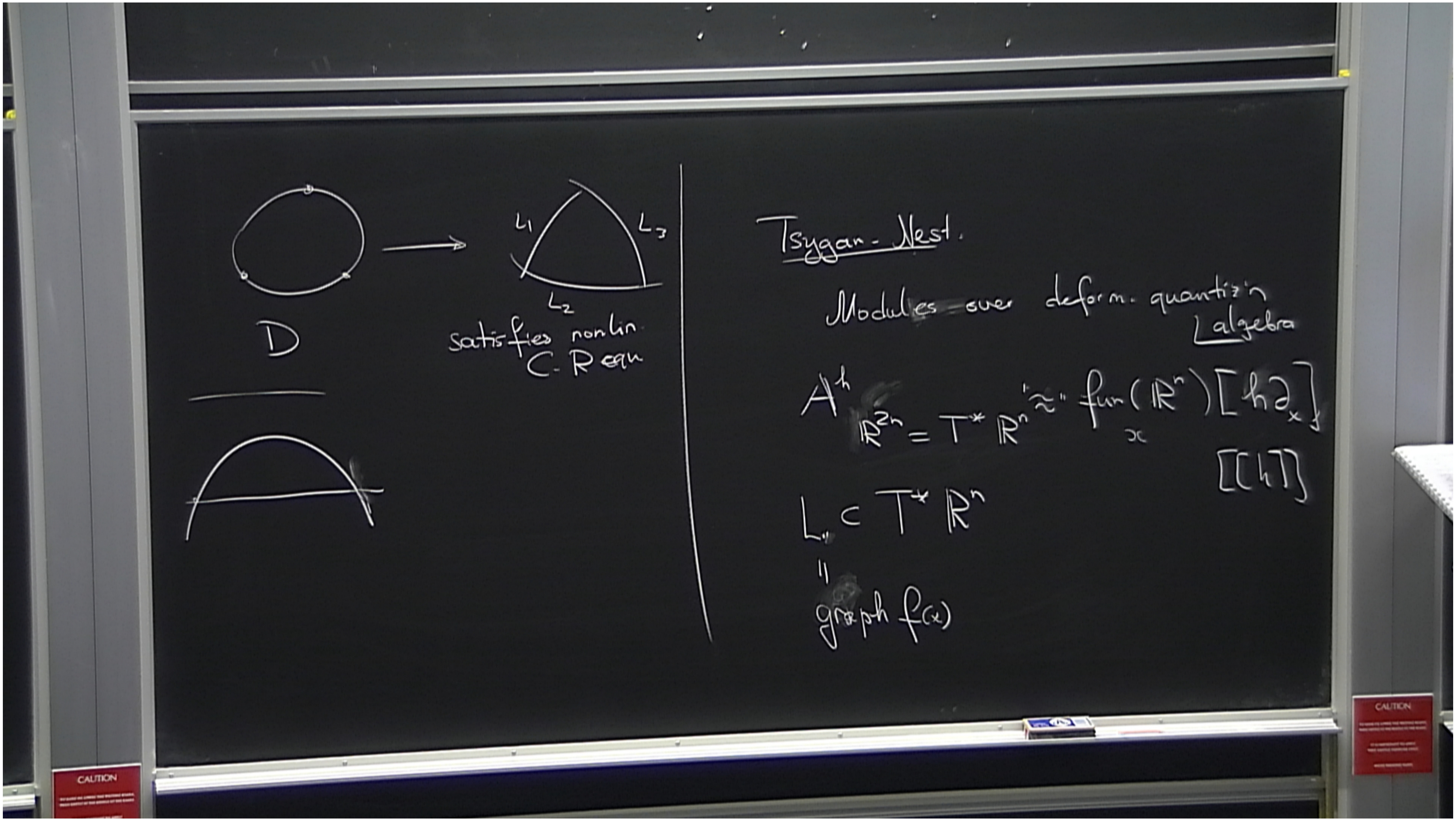
$\mathrm{Mor}(L_1, L_2)[\,] =$



$= \bigoplus_{p \in L_1 \cap L_2} (\text{coeff ring}) \cdot p\ [\quad]$

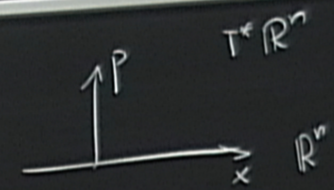


$p_{12} \circ p_{23} = \sum e^{-\frac{\mathrm{Area}(\triangle)}{\hbar}} \quad p_{13}$


D
L1
L3
L2
satisfies nonlin.
C-R eqn
Tsygan - Nest.
Modules over deform. quantiz'n
algebra
$A^h_{\mathbb{R}^{2n} = T^*\mathbb{R}^n}$ "≈" $fun_x(\mathbb{R}^n)[h\partial_x]$
$[[h]]$
$L_{\shortparallel} \subset T^*\mathbb{R}^n$
graph $f(x)$

$$M_L = \mathcal{A}^{\hbar}_{\mathbb{R}^{2n}}\, e^{if(x)/\hbar}$$

$$= \mathcal{A}^{\hbar}_{\mathbb{R}^{2n}} \Big/ \mathcal{A}^{\hbar}_{\mathbb{R}^{2n}}\,\big(\overset{p}{\widetilde{\hbar\partial_x}} - \nabla f\big)$$



$$R\mathrm{Hom}_{\mathcal{A}^{\hbar}_{\mathbb{R}^{2n}}}(M_L, M_{L'}) = \bigoplus_{p \in L \cap L'} \mathbb{C}[[\hbar]]\,[n]$$

Nadler-Zaslow

Tamarkin

Constructible sheaves on X $\longleftrightarrow$ $\mathrm{Fuk}(T^*X)$

$Y \overset{i}{\hookrightarrow} X$

L

$i_* L$

Tsygan-Nest.

Modules over deform. quantiz'n algebra

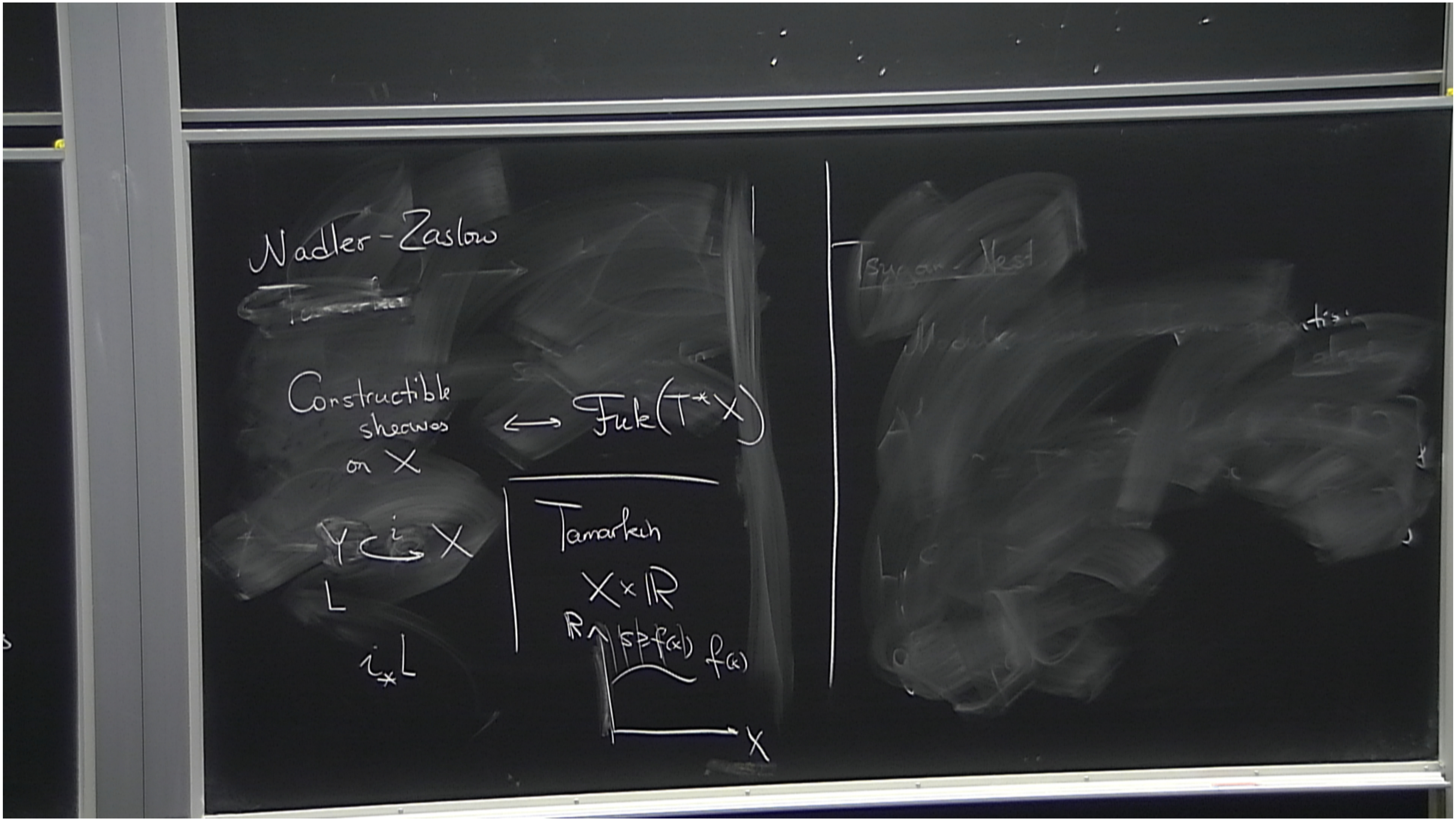
$A^{\hbar}_{\mathbb{R}^{2n} = T^*\mathbb{R}^n} \text{ "}\approx\text{" } \mathrm{fun}_x(\mathbb{R}^n)[\hbar \partial_x]$

$[[\hbar]]$

$L_f \subset T^*\mathbb{R}^n$

$\parallel$

graph $f(x)$


Nadler-Zaslow
Constructible sheaves on X ⟷ Fuk(T*X)
Y ⊂ X
i
L
i_* L
Tamarkin
X × ℝ
ℝ
s ≥ f(x)
f(x)
X

Nadler-Zaslow

Constructible sheaves on X $\longleftrightarrow$ $\mathrm{Fuk}(T^*X)$

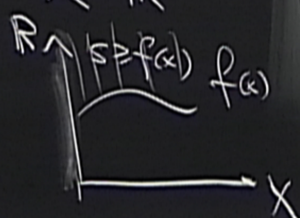
$$Y \overset{i}{\hookrightarrow} X$$

L

$i_* L$

Tamarkin

$X \times \mathbb{R}$



Witten Laplacian.

X, g compact, Riem. mfld

$f : X \to \mathbb{R}$ Morse fctn

$$d : \Omega^k(X) \to \Omega^{k+1}(X)$$

$$d_f = e^{-f/h} \circ d \circ e^{f/h}$$

$$\Delta_f^{(k)} = d_f^* d_f + d_f d_f^*$$

Nadler-Zaslow

Constructible sheaves on X $\longleftrightarrow$ $\mathrm{Fuk}(T^*X)$

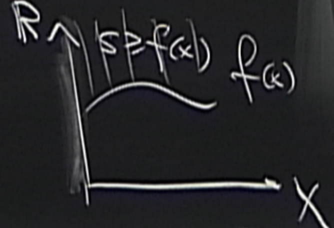
$Y \xhookrightarrow{i} X$

L

$i_* L$

Tamarkin

$X \times \mathbb{R}$



Witten Laplacian.

X, g compact, Riem. mfld

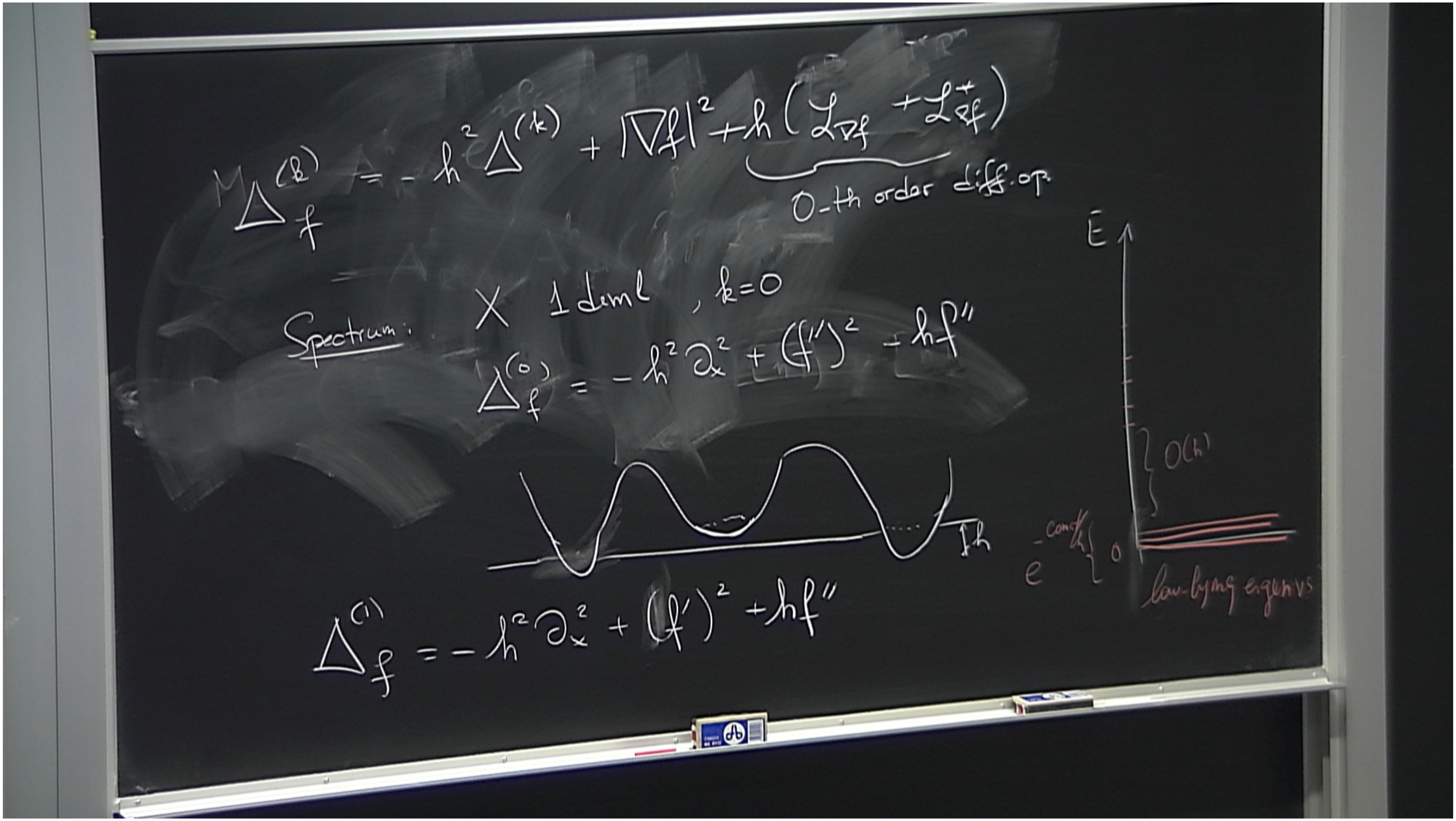
$f : X \to \mathbb{R}$ Morse fctn

$d : \Omega^k(X) \to \Omega^{k+1}(X)$

$h \in \mathbb{R}_{>0}$ small.

$$d_f = e^{-f/h} \circ d \circ e^{f/h}$$

$$\Delta_f^{(k)} = d_f^* d_f + d_f d_f^*$$


$\Delta_f^{(k)} = -h^2 \Delta^{(k)} + |\nabla f|^2 + h \underbrace{(\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^{+})}_{\text{0-th order diff. op.}}$
Spectrum: X 1 dim'l, $k=0$
$\Delta_f^{(0)} = -h^2 \partial_x^2 + (f')^2 - hf''$
$\Delta_f^{(1)} = -h^2 \partial_x^2 + (f')^2 + hf''$
h
E
$O(h)$
$e^{-\mathrm{const}/h}$
0
low-lying eigenvs

Witten Laplacian & Resurgence

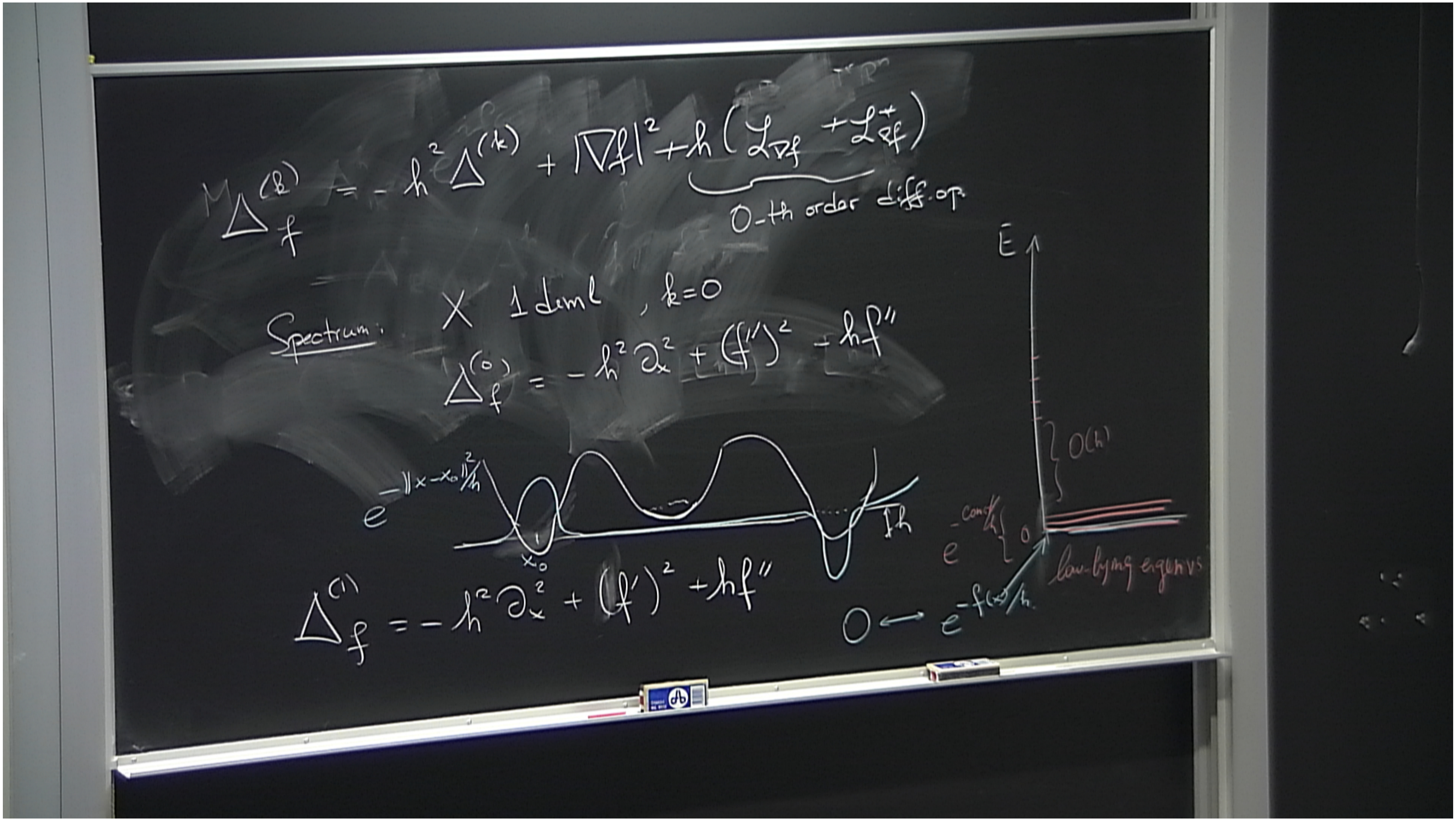
of low-lying eigenstates of $\Delta_f^{(k)}$ = # of crit. points of f with Morse index $\underline{\underline{k}}$

$\left(= \text{"\#" "+" in } \mathrm{Hess}_f \right)$

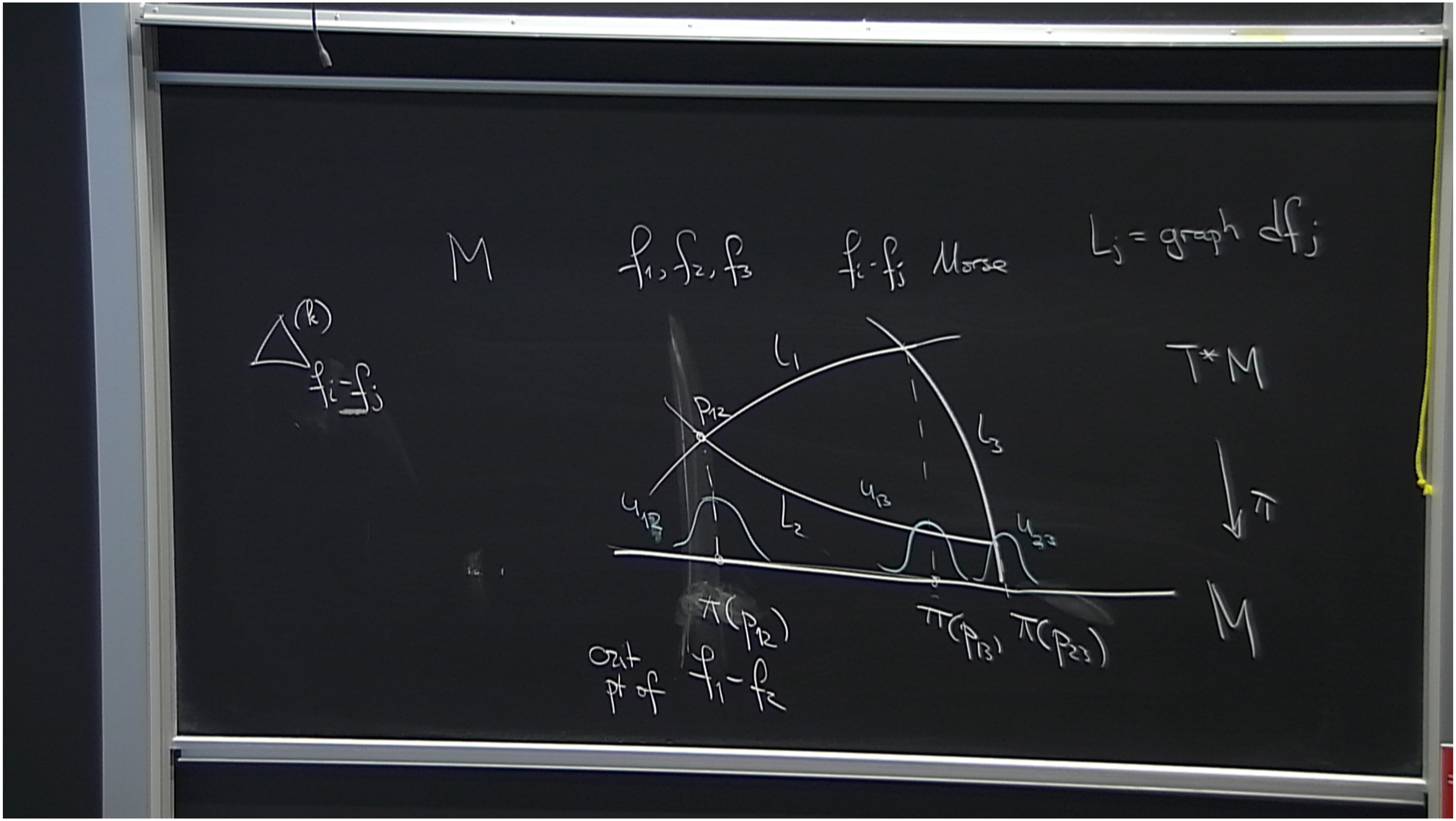
$\underline{\underline{\mathcal{E}x}}$ $f = x_1^2 + \dots + x_\ell^2 - x_{\ell+1}^2 - \dots - x_n^2$

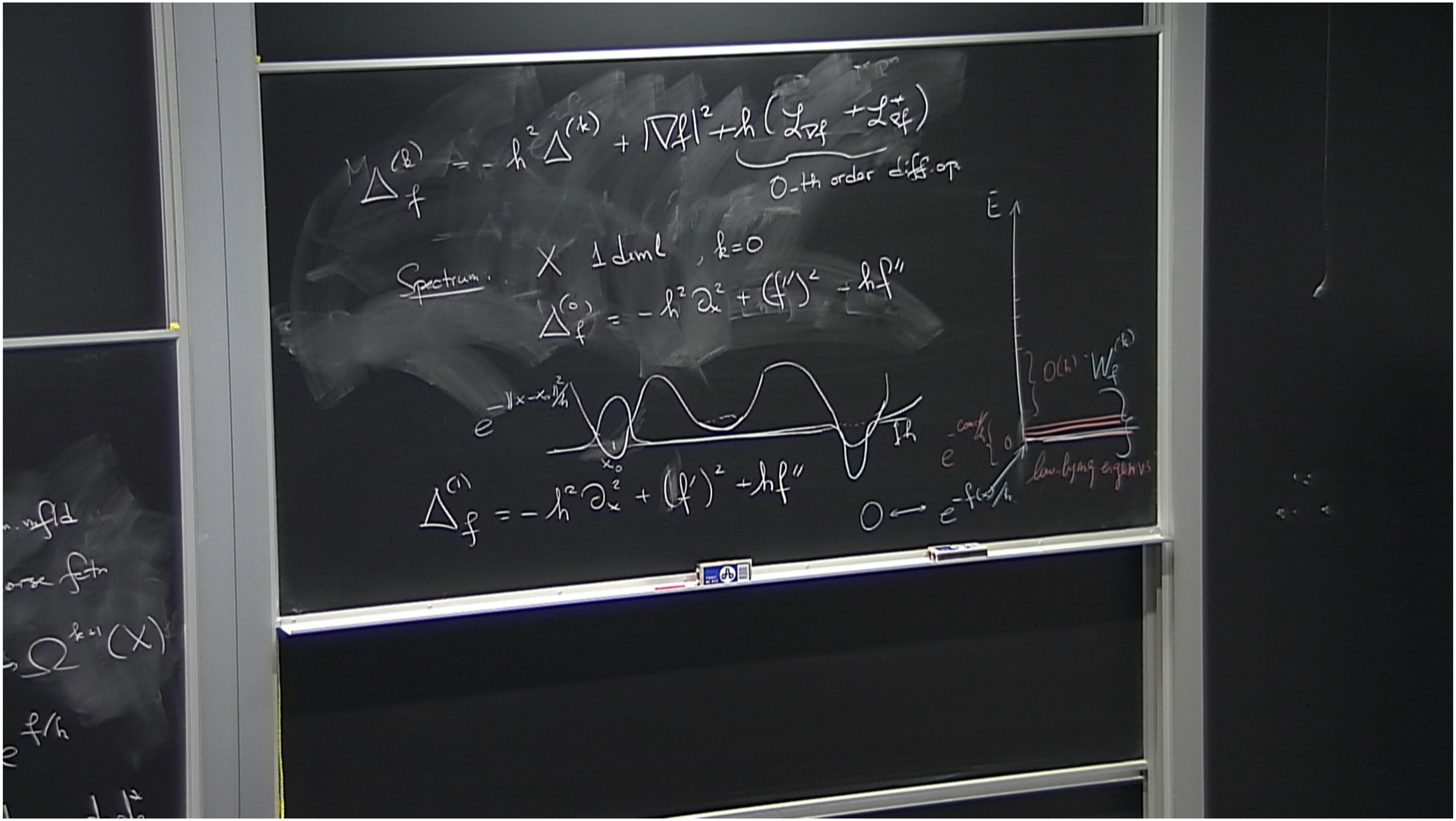
ℓ-forms with basis $dx_{i_1} \wedge \dots \wedge dx_{i_\ell}$ $\quad dx_1 \wedge \dots \wedge dx_\ell$

$$\Delta_f^{(\ell)} = -\hbar^2 \Delta^{(\ell)} + |\nabla f|^2 + \hbar \left(\dots -n \right)$$



$\Delta_f^{(k)} = -h^2 \Delta^{(k)} + |\nabla f|^2 + h(\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^{+})$
0-th order diff. op.
Spectrum:
X 1 diml, k=0
$\Delta_f^{(0)} = -h^2 \partial_x^2 + (f')^2 - hf''$
$e^{-\|x-x_0\|^2/h}$
x_0
$\sqrt{h}$
$\Delta_f^{(1)} = -h^2 \partial_x^2 + (f')^2 + hf''$
$0 \longleftrightarrow e^{-f(x)/h}$
E
O(h)
$e^{-\mathrm{const}/h}$
0
low-lying eigenvs


M
f1, f2, f3
fi - fj Morse
Lj = graph dfj
Δ(k)
fi - fj
L1
p12
L3
u13
u12
L2
u23
T*M
π
M
π(p12)
π(p13)
π(p23)
crit pt of f1 - f2



$\Delta_f^{(k)} = -h^2\Delta^{(k)} + |\nabla f|^2 + h(\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^{+})$
0-th order diff. op.
Spectrum. X 1 diml, k=0
$\Delta_f^{(0)} = -h^2\partial_x^2 + (f')^2 - hf''$
$e^{-\|x-x_0\|^2/h}$
x_0
$\Delta_f^{(1)} = -h^2\partial_x^2 + (f')^2 + hf''$
E
$O(h)$
$W_f^{(k)}$
$e^{-const/h}$
low-lying eigenvs
$0 \leftarrow e^{-f(x)/h}$
$\Omega^{k+1}(X)$
$e^{f/h}$

Conj: (Fukaya)

$$\frac{\langle u_{12} \wedge u_{23} , u_{13} \rangle_{L^2}}{\| u_{12} \|_{L_2} \| u_{23} \|_{L_2} \| u_{13} \|_{13}} \sim \sum_{\substack{\text{pseudohol} \\ \Delta\text{-les}}} \text{coeff}\; e^{-\frac{\text{Area } \Delta}{h}}$$

Witten Laplacian.

X, g compact, Riem

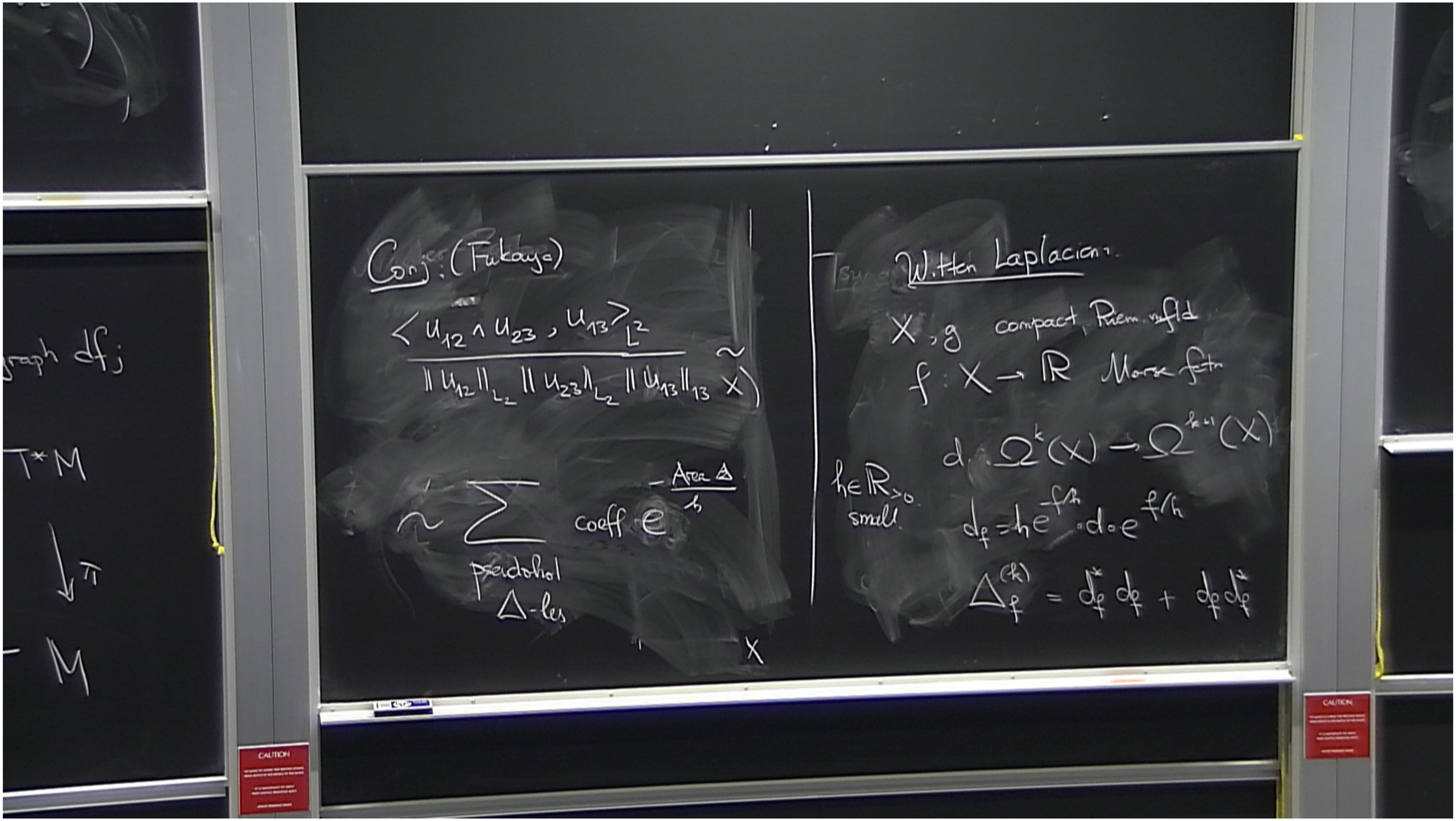
$f : X \to \mathbb{R}$ Mo

$h \in \mathbb{R}_{>0}$ small.

$d : \Omega^k(X) \to$

$d_f = h e^{-f/h} \circ d \circ e$

$\Delta_f^{(k)} = d_f^* d_f$


Conj: (Fukaya)
$\frac{\langle u_{12} \wedge u_{23}, u_{13} \rangle_{L^2}}{\|u_{12}\|_{L_2} \|u_{23}\|_{L_2} \|u_{13}\|_{13}} \sim$
$\sim \sum$ coeff $e^{-\frac{\text{Area}\,\Delta}{\hbar}}$
pseudohol
Δ-les
Witten Laplacian
X, g compact, Riem. mfld
$f: X \to \mathbb{R}$ Morse fctn
$d: \Omega^k(X) \to \Omega^{k+1}(X)$
$\hbar \in \mathbb{R}_{>0}$ small
$d_f = \hbar e^{-f/\hbar} \cdot d \cdot e^{f/\hbar}$
$\Delta_f^{(k)} = d_f^* d_f + d_f d_f^*$
graph df_j
T^*M
π
M
CAUTION

Conj: (Fukaya)

$$\frac{\langle u_{12}\wedge u_{23},\, u_{13}\rangle_{L^2}}{\|u_{12}\|_{L_2}\,\|u_{23}\|_{L_2}\,\|u_{13}\|_{13}} \sim \sum_{\substack{\text{pseudohol}\\ \Delta\text{-les}}} (\text{num} + O(h))\, e^{-\frac{\text{Area}\,\Delta}{h}}$$

Witten Laplacian.

X, g compact Riem. mfld

$f : X \to \mathbb{R}$ Morse fctn

$d : \Omega^k(X) \to \Omega^{k+1}(X)$

$h \in \mathbb{R}_{>0}$ small.

$d_f = h e^{-f/h} \circ d \circ e^{f/h}$

$\Delta_f^{(k)} = d_f^* d_f + d_f d_f^*$

$$\langle du_1, u_2 \rangle = e^{-\text{Area}/\hbar}(\text{num} + O(\hbar)).$$