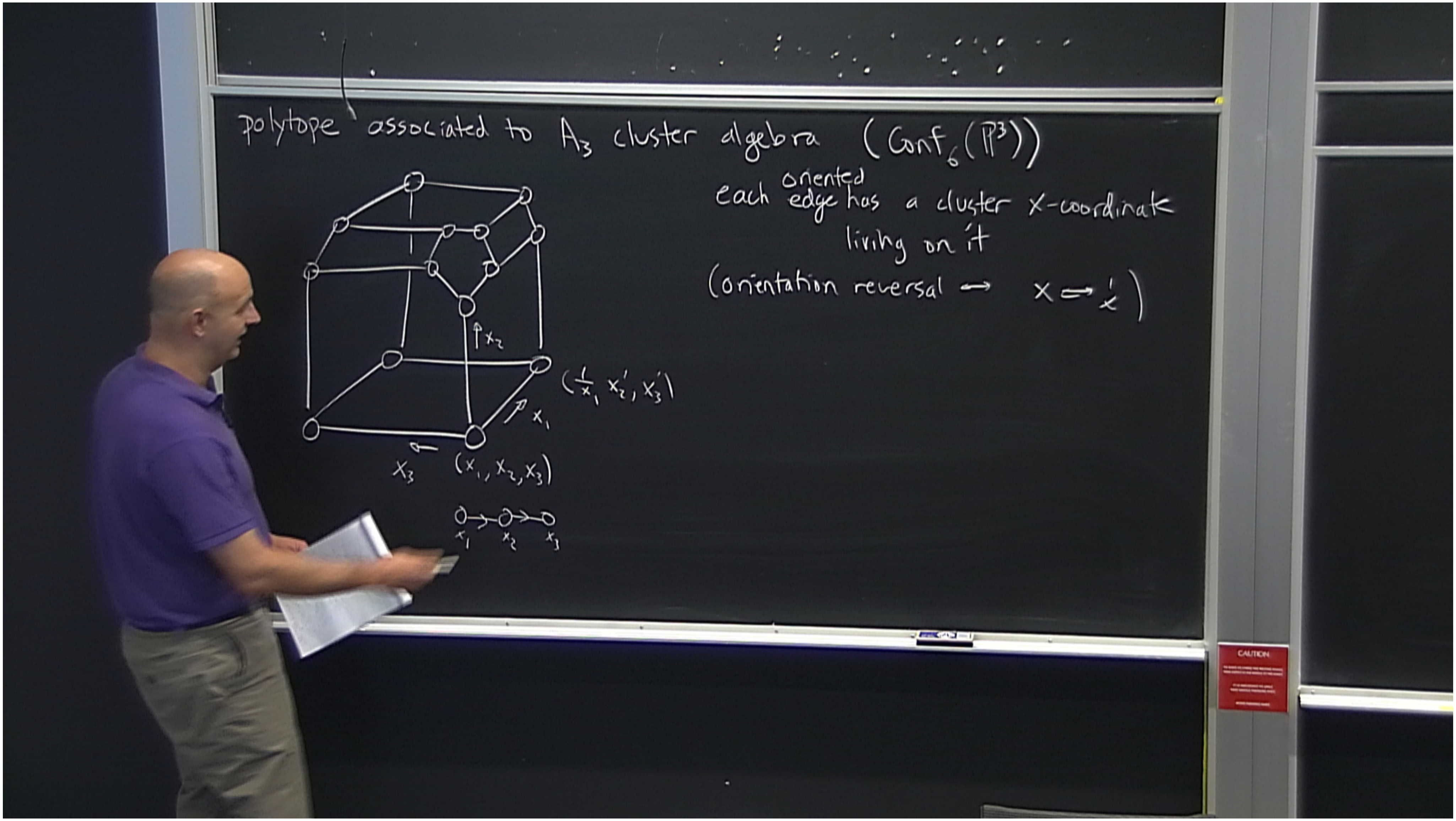
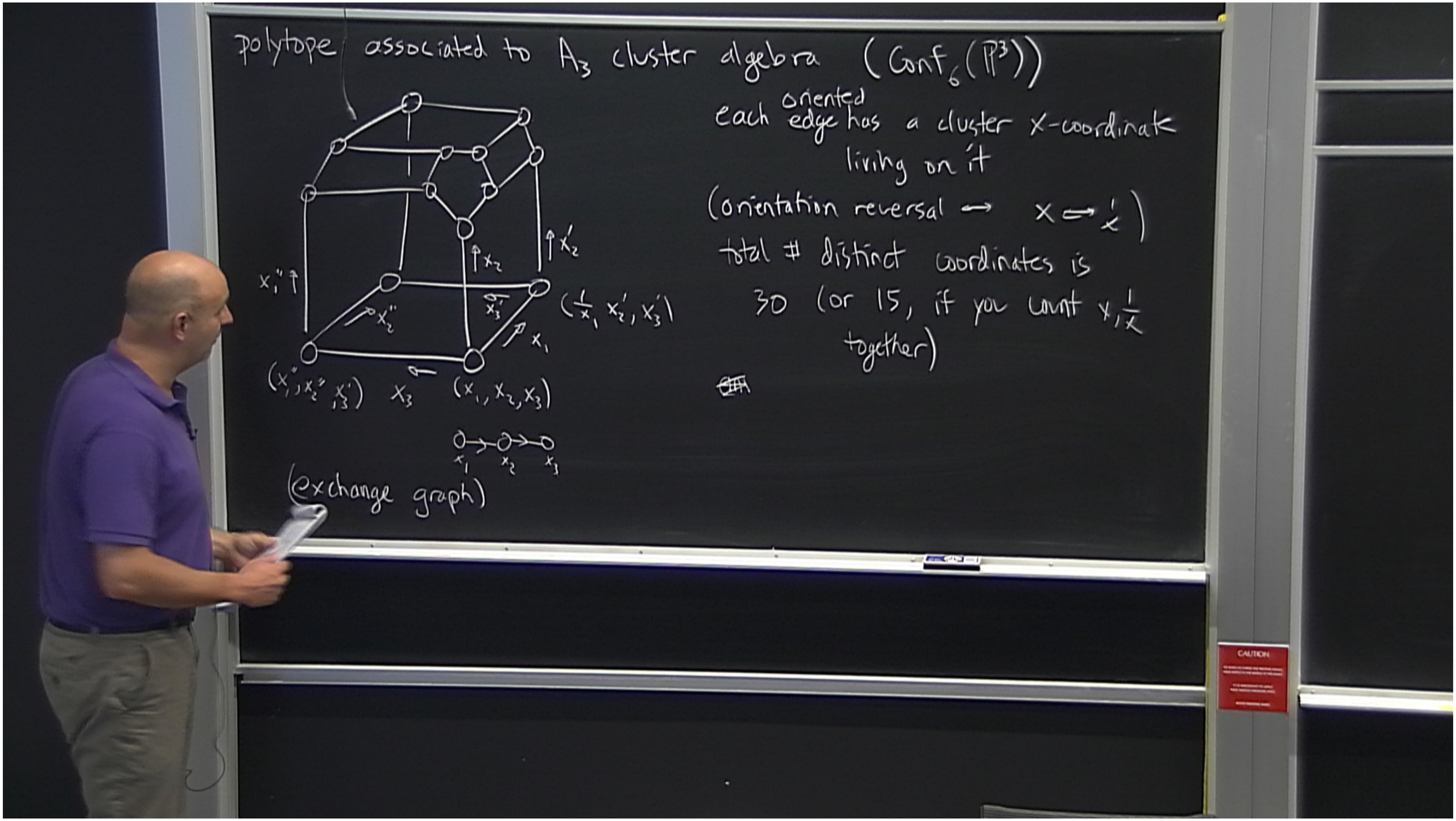
Title: Cluster Algebras and Scattering Amplitudes

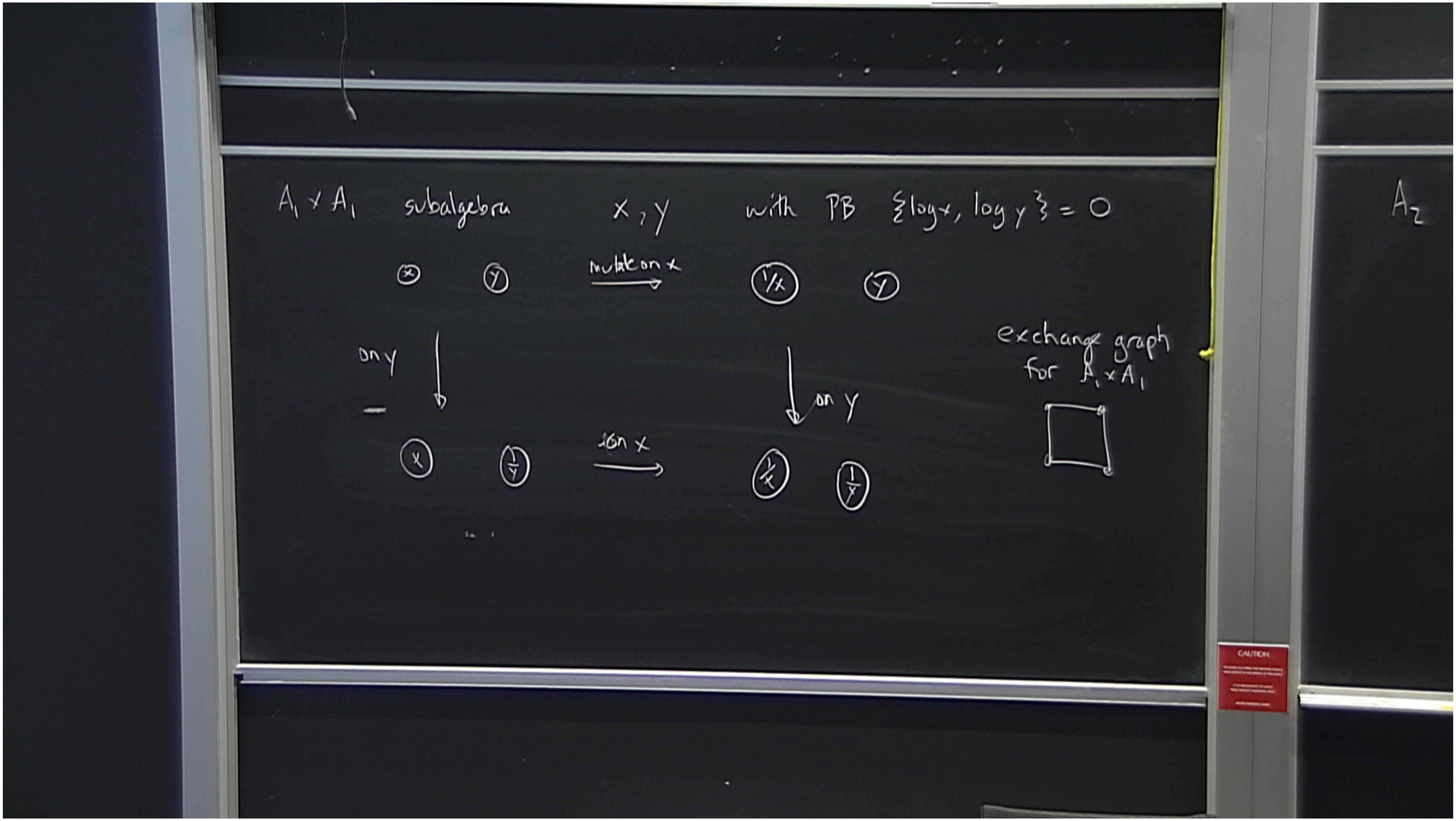
Date: May 27, 2015 03:30 PM

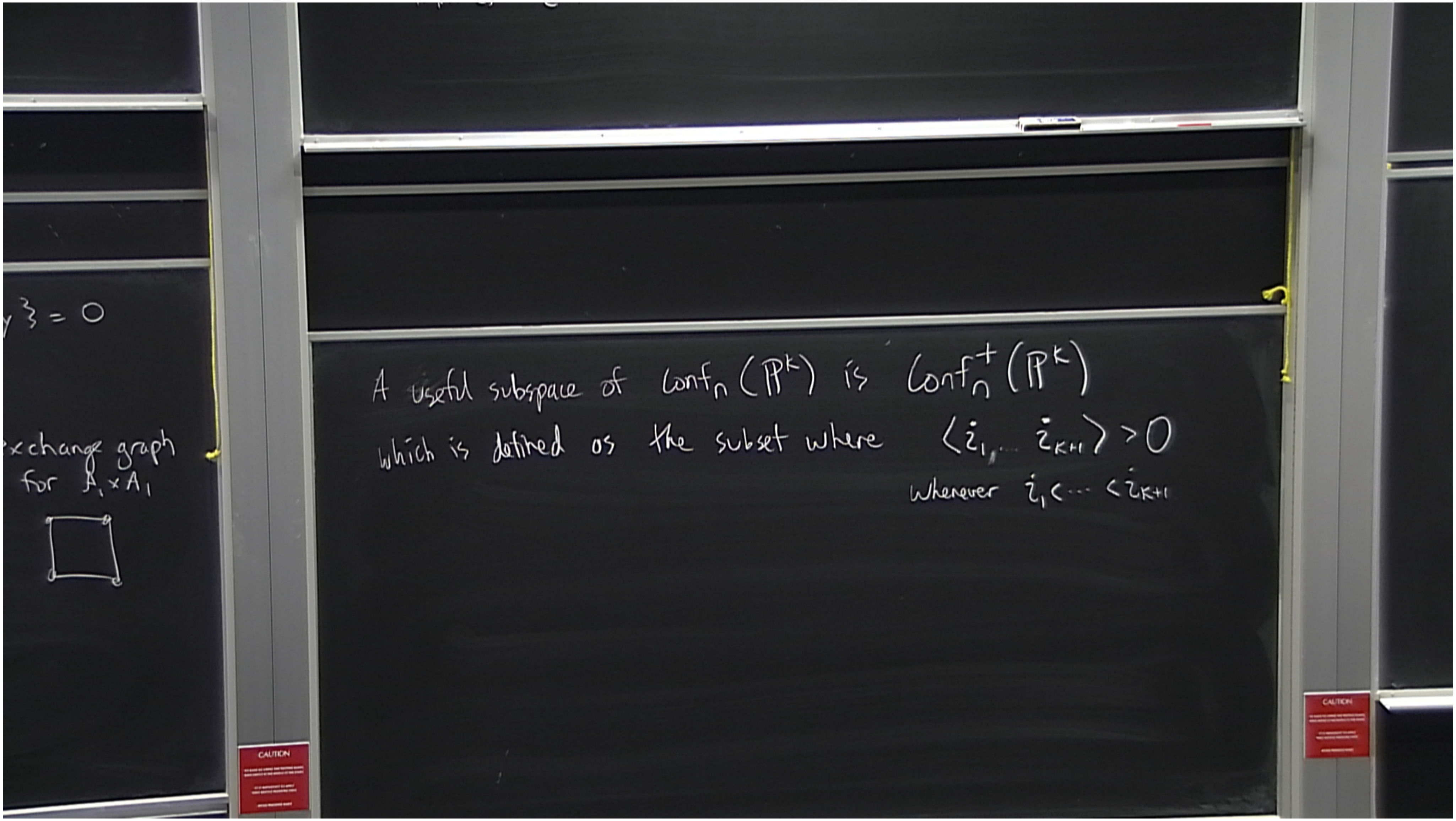
URL: http://pirsa.org/15050058

Abstract: Supersymmetric gauge theory computes a very special class of (generalized) polylogarithm functions known as scattering amplitudes that have remarkable mathematical properties. In particular, there is a rich connection between these amplitudes and the G(4,n) Grassmannian cluster algebra. To explain this connection I will review some basic facts about the Hopf algebra of polylogarithms and cluster Poisson varieties. I will then define cluster polylogarithm functions which roughly speaking are polylogarithm functions whose arguments are cluster X-coordinates of some cluster algebra A. I will describe an additional property of certain scattering amplitudes, that they are "local" in the algebra A, and describe the classification of cluster polylogarithm functions with this property. The computation of new amplitudes can be greatly aided by knowledge of the class of functions in terms of which they may be expressed, as I will illustrate via an example.

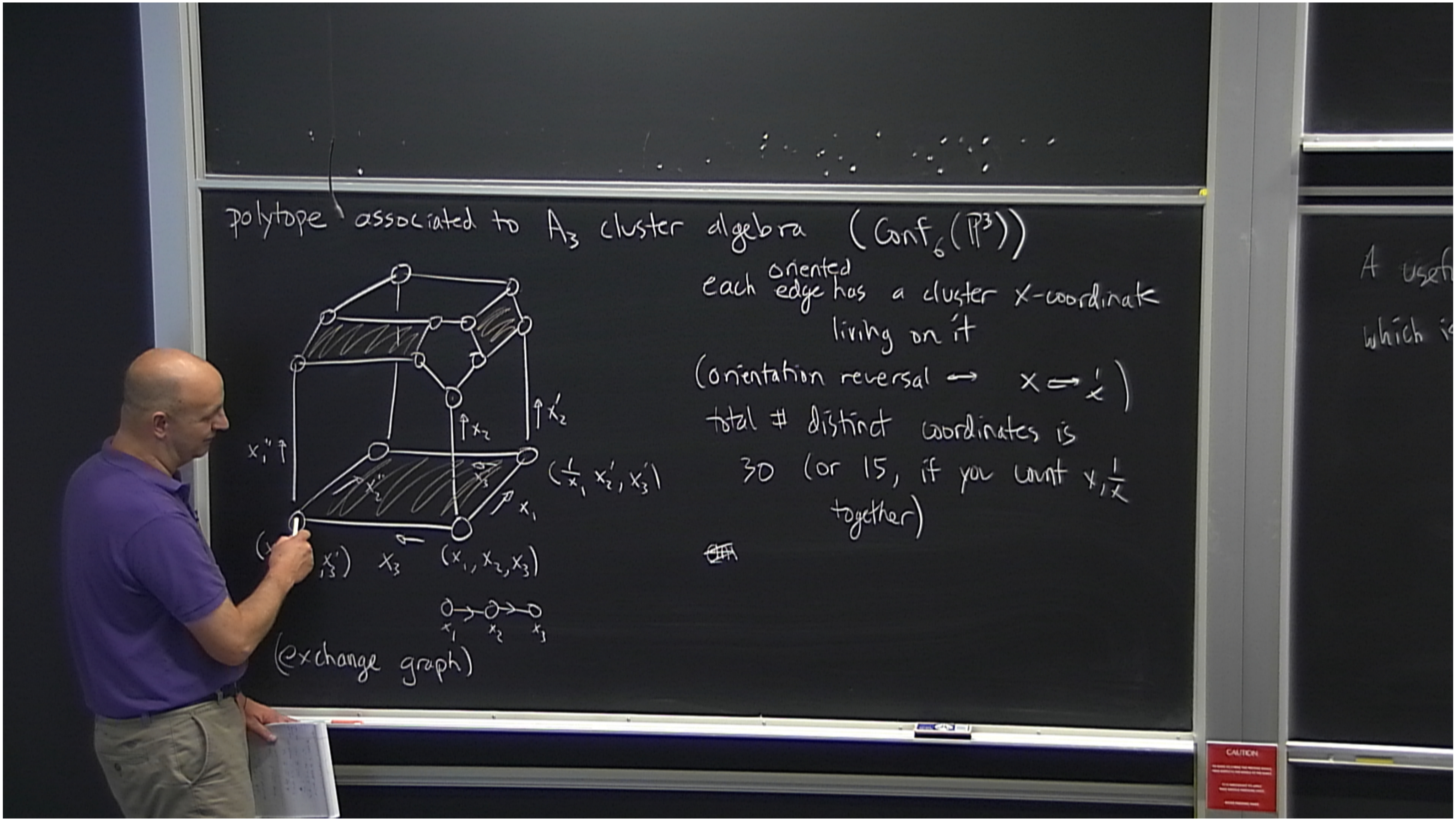

polytope associated to A3 cluster algebra (Conf6(P3))
each oriented edge has a cluster X-coordinate living on it
(orientation reversal ⟷ X ⟷ 1/X)
x2
(1/x1, x2', x3')
x1
x3
(x1, x2, x3)
x1 x2 x3


polytope associated to A_3 cluster algebra $(\mathrm{Conf}_6(\mathbb{P}^3))$
each oriented edge has a cluster X-coordinate living on it
(orientation reversal ⟷ $x \leftrightarrow \frac{1}{x}$)
total # distinct coordinates is 30 (or 15, if you count $x, \frac{1}{x}$ together)
(x_1'', x_2'', x_3')
x_3
(x_1, x_2, x_3)
$(\frac{1}{x_1}, x_2', x_3')$
x_1
x_2
x_2'
x_1''
x_2''
x_3'
$x_1 \quad x_2 \quad x_3$
(exchange graph)


A1 × A1 subalgebra x, y with PB {log x, log y} = 0
mutate on x
on y
on y
on x
exchange graph for A1 × A1
A2



y } = 0
xchange graph
for $A_1 \times A_1$
A useful subspace of $Conf_n(\mathbb{P}^k)$ is $Conf_n^+(\mathbb{P}^k)$
which is defined as the subset where $\langle i_1, \dots i_{k+1} \rangle > 0$
whenever $i_1 < \dots < i_{k+1}$
CAUTION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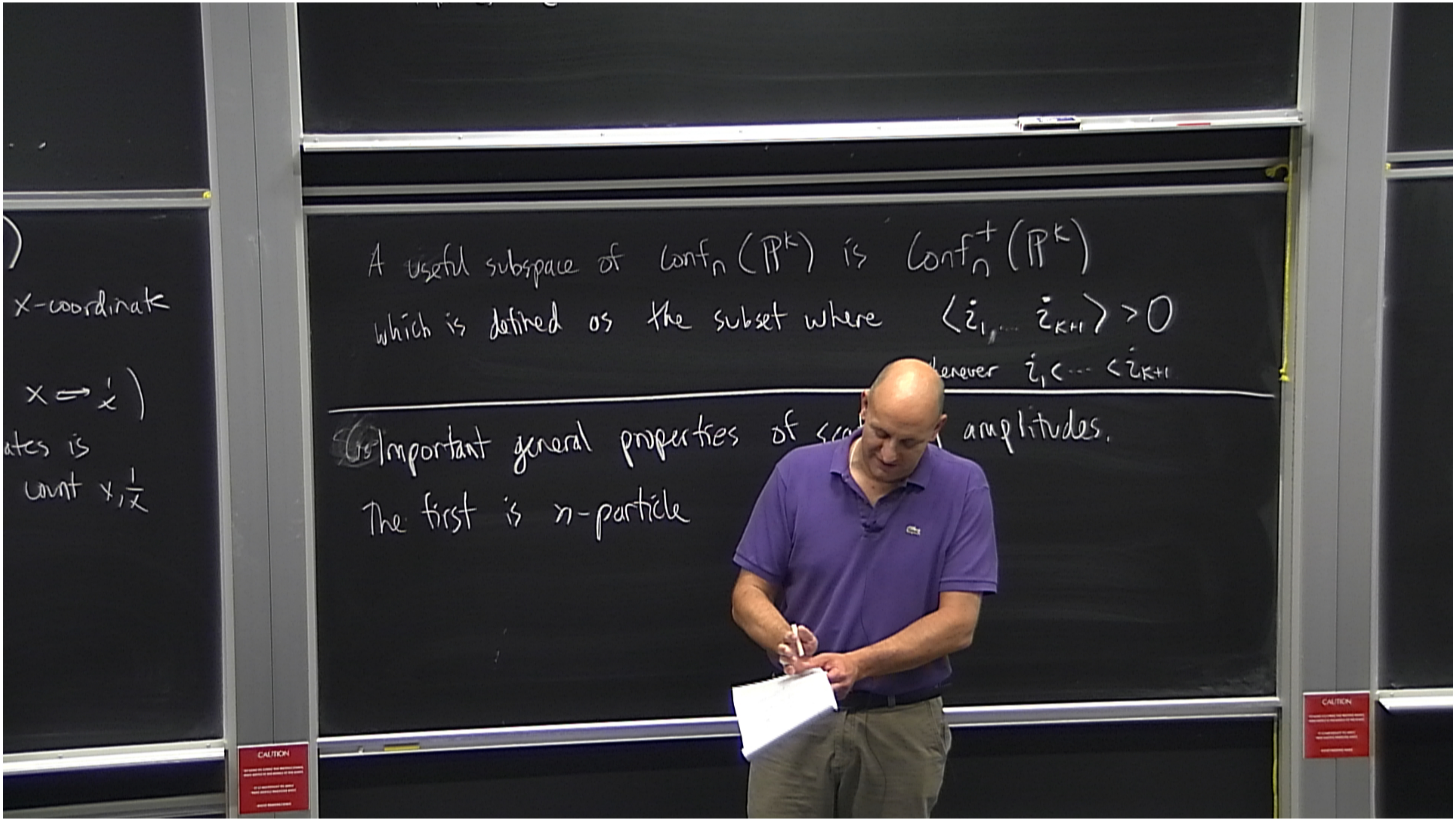
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Important general properties of amplitudes.
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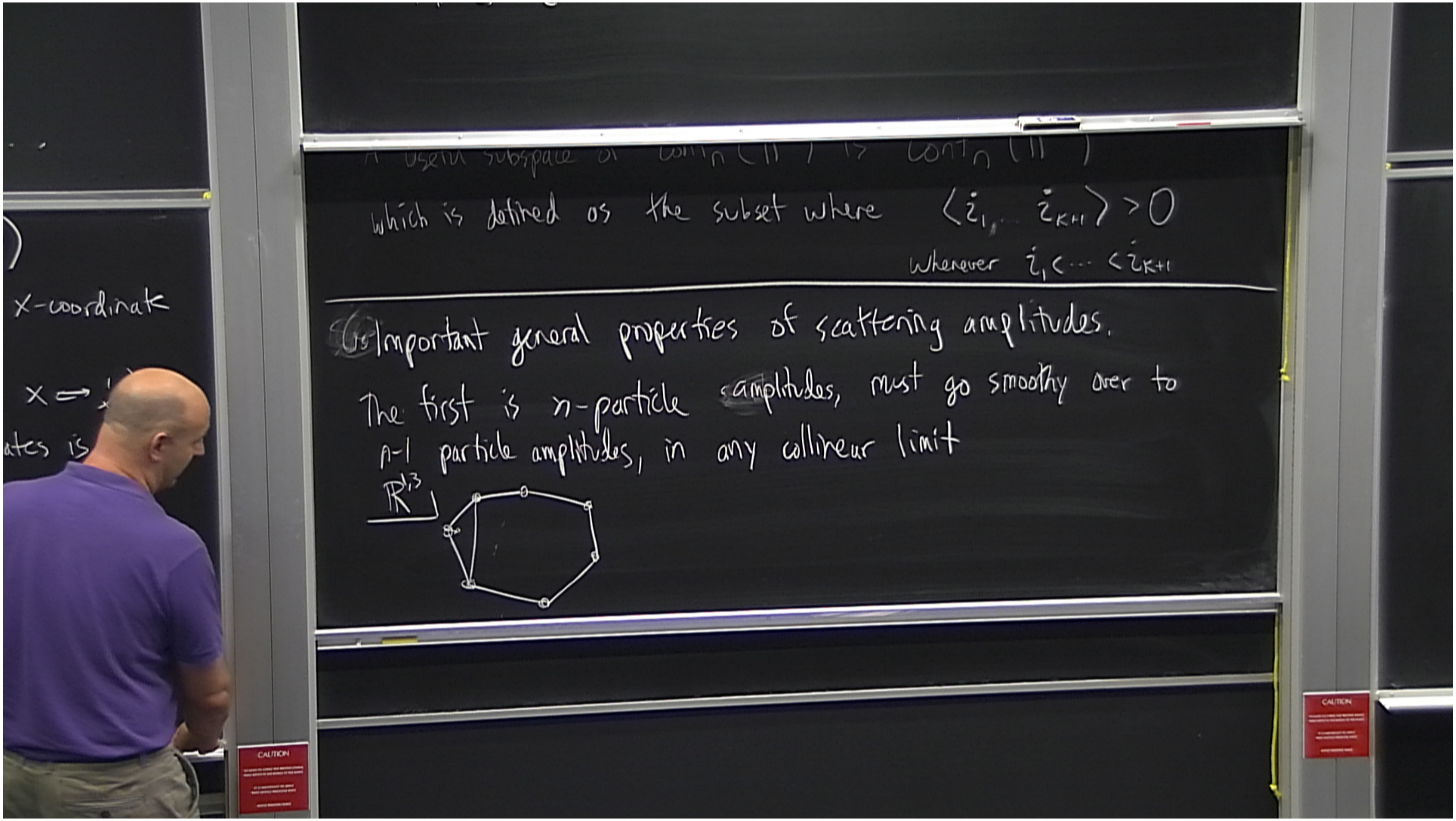
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Important general properties of scattering amplitudes.
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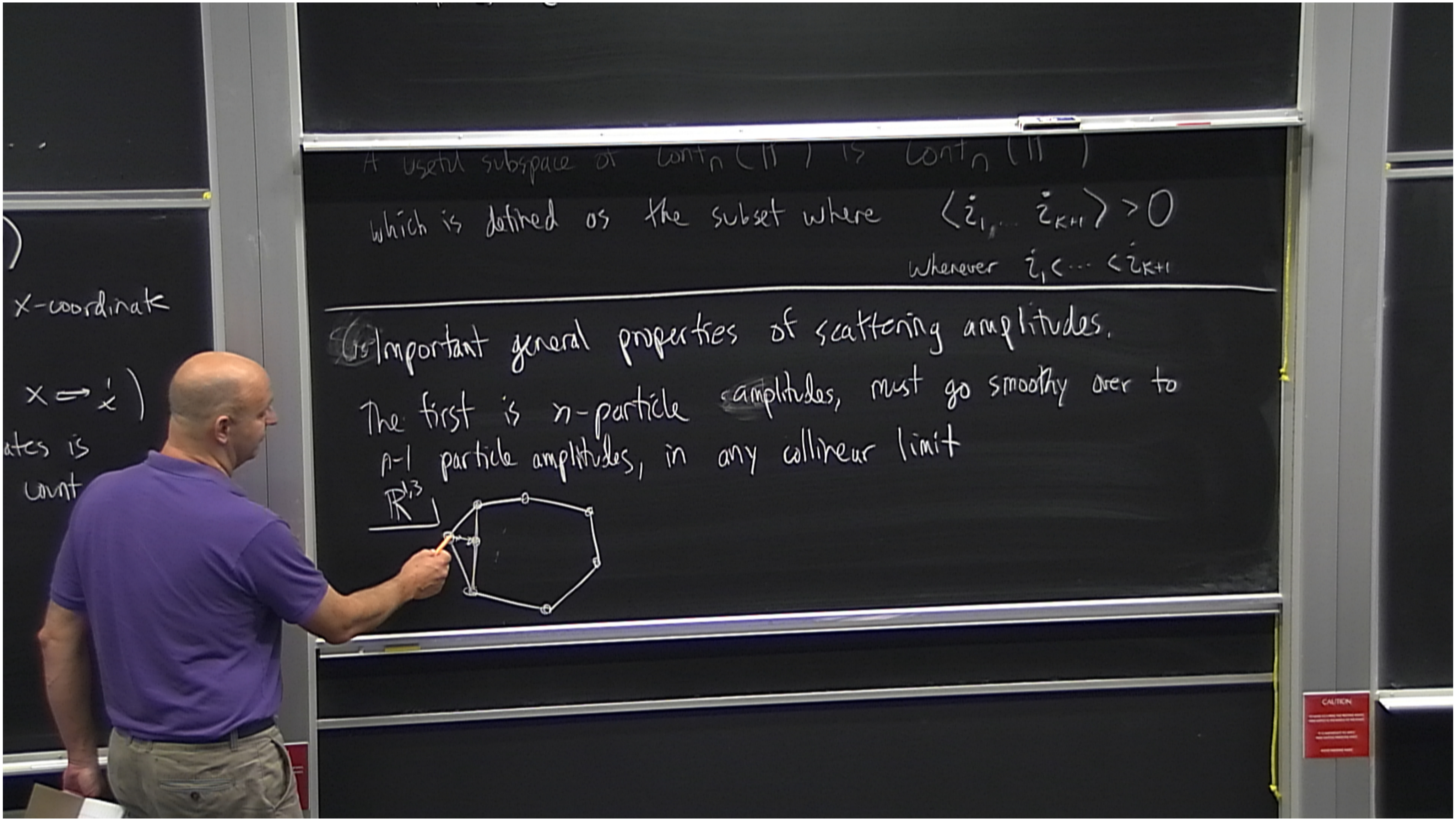
R1,3
x-coordinate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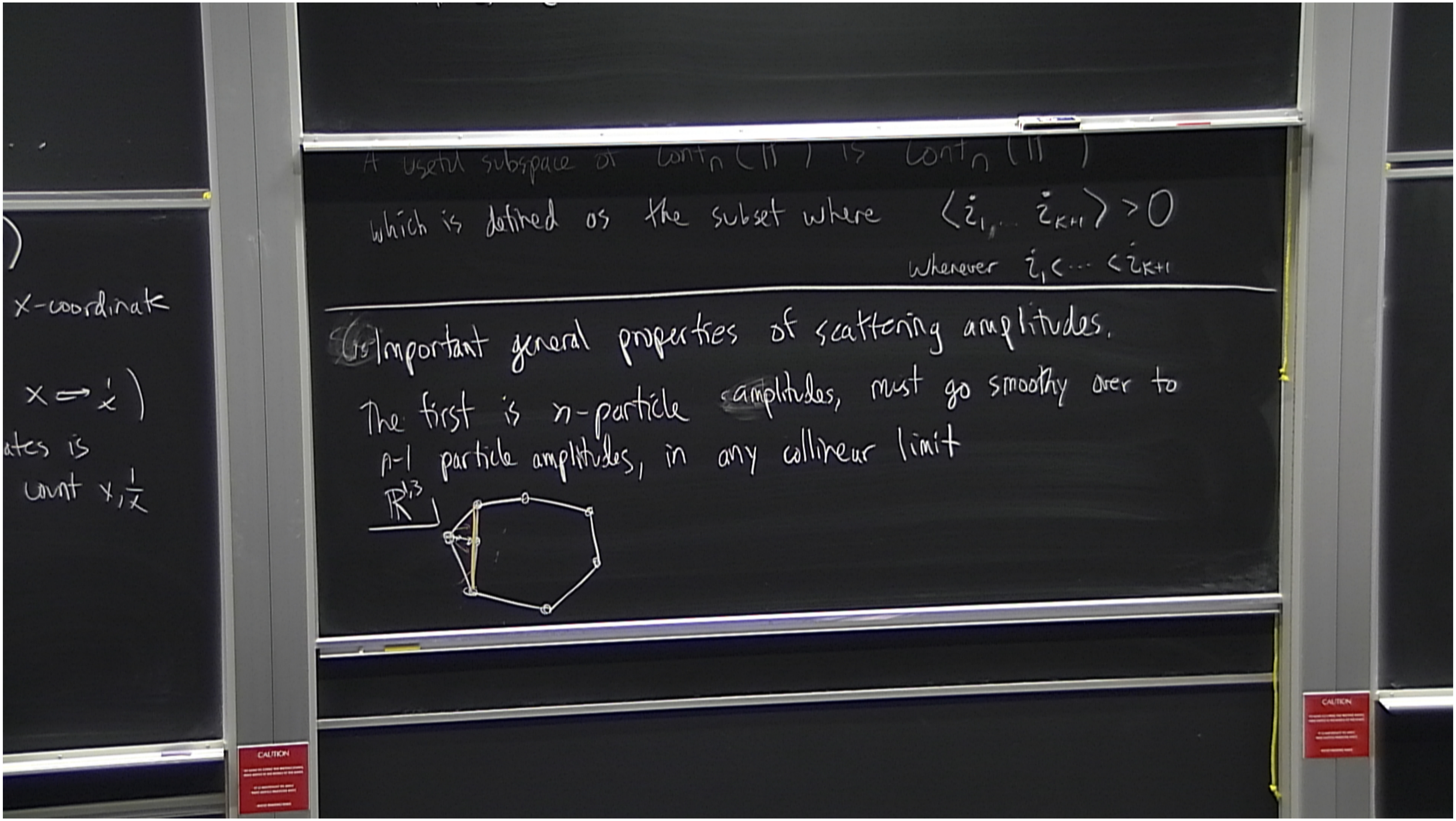
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x ⇒ 1/x)
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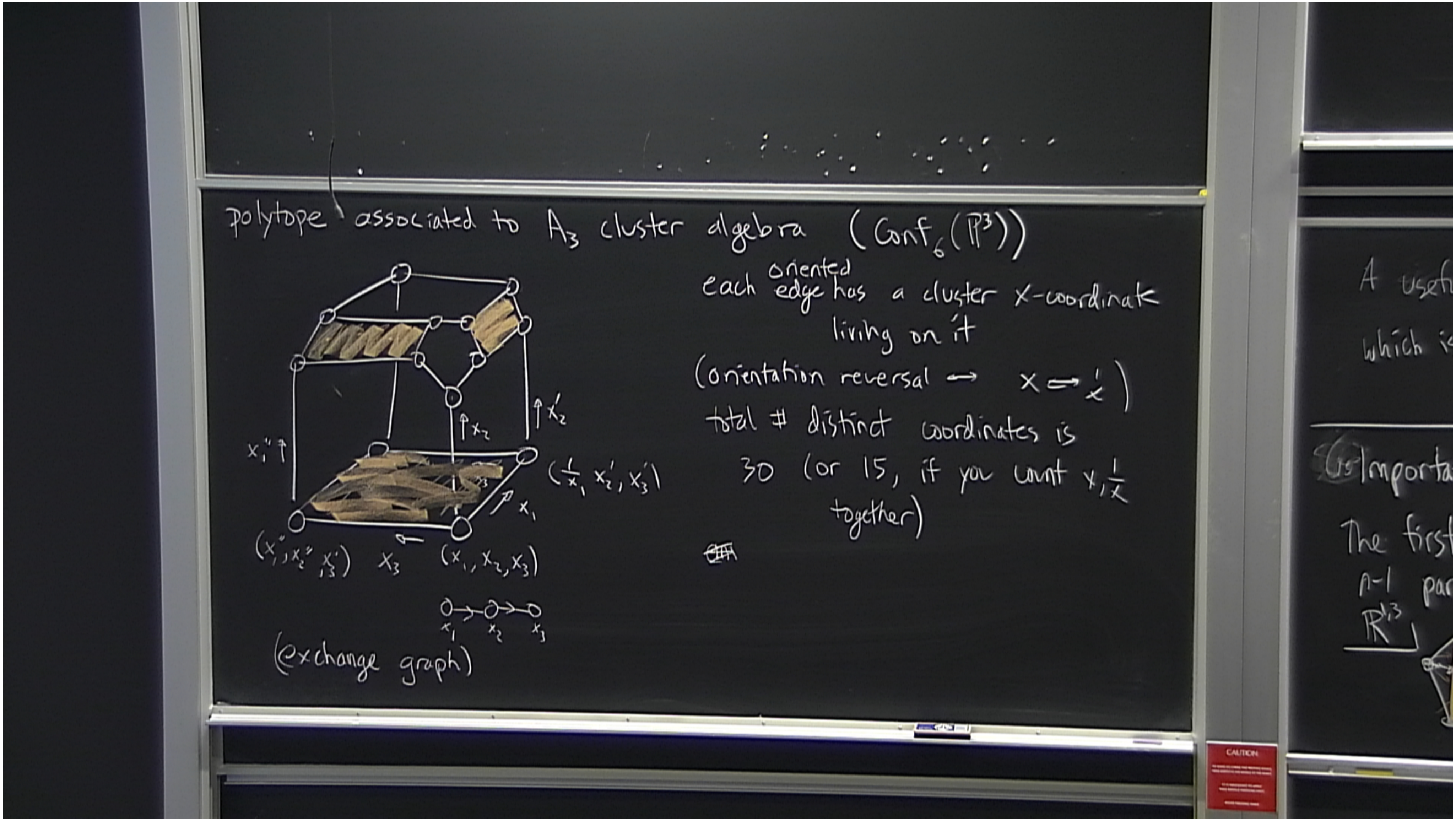
count x, 1/x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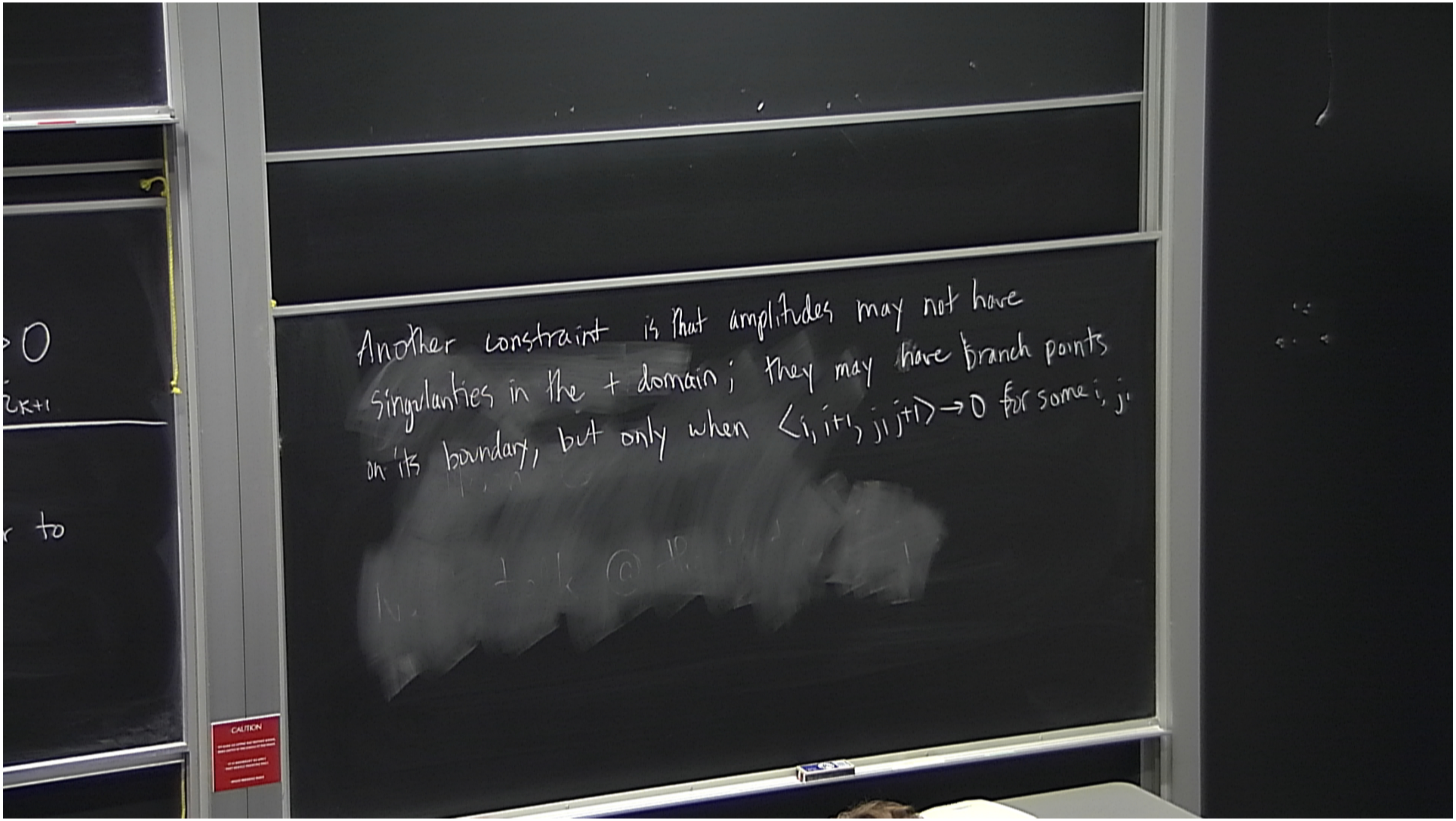
x1
x2
x2'
x1''
x1 x2 x3
(exchange graph)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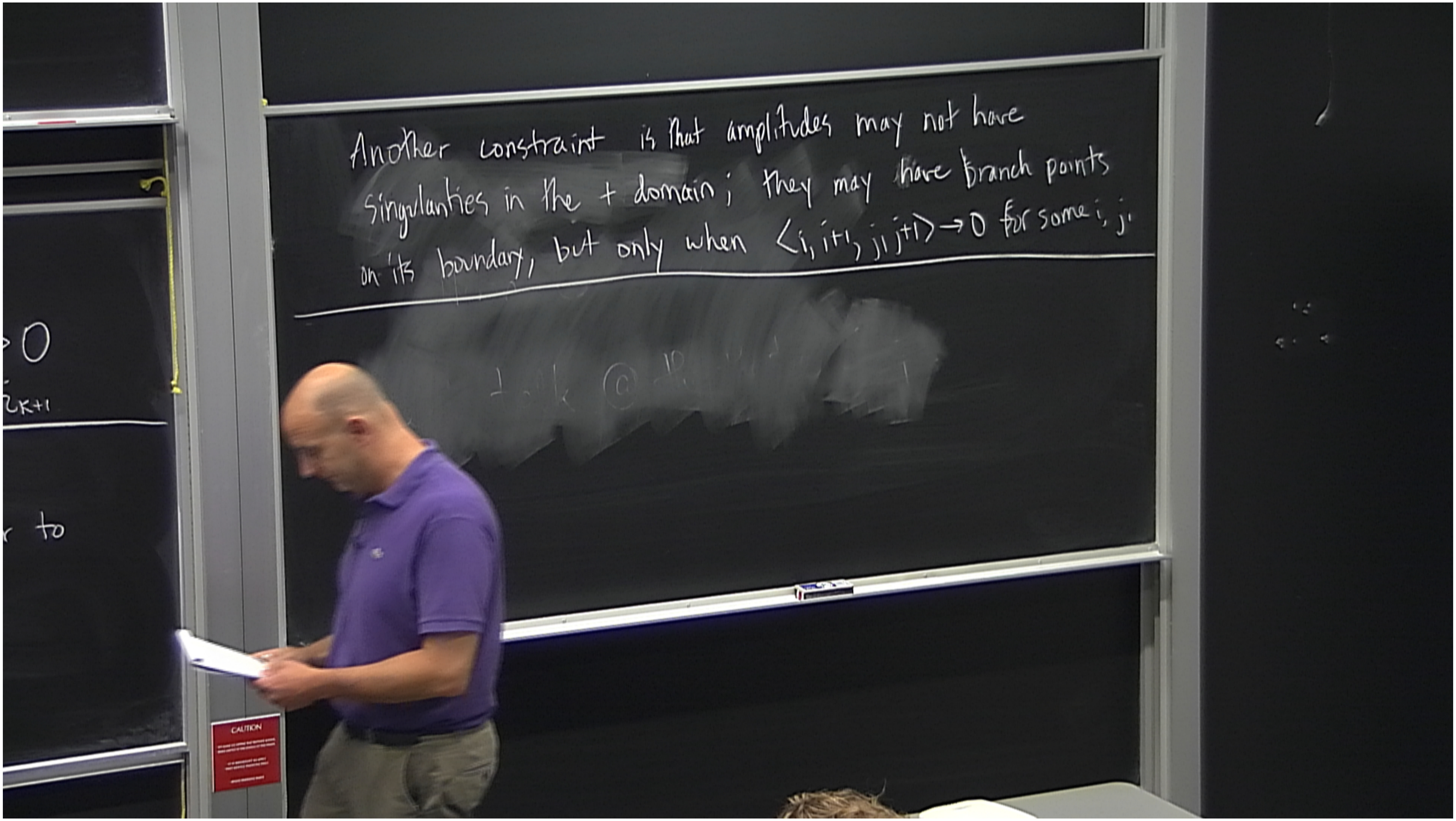
on its boundary, but only when ⟨i, i+1, j, j+1⟩ → 0 for some i, j.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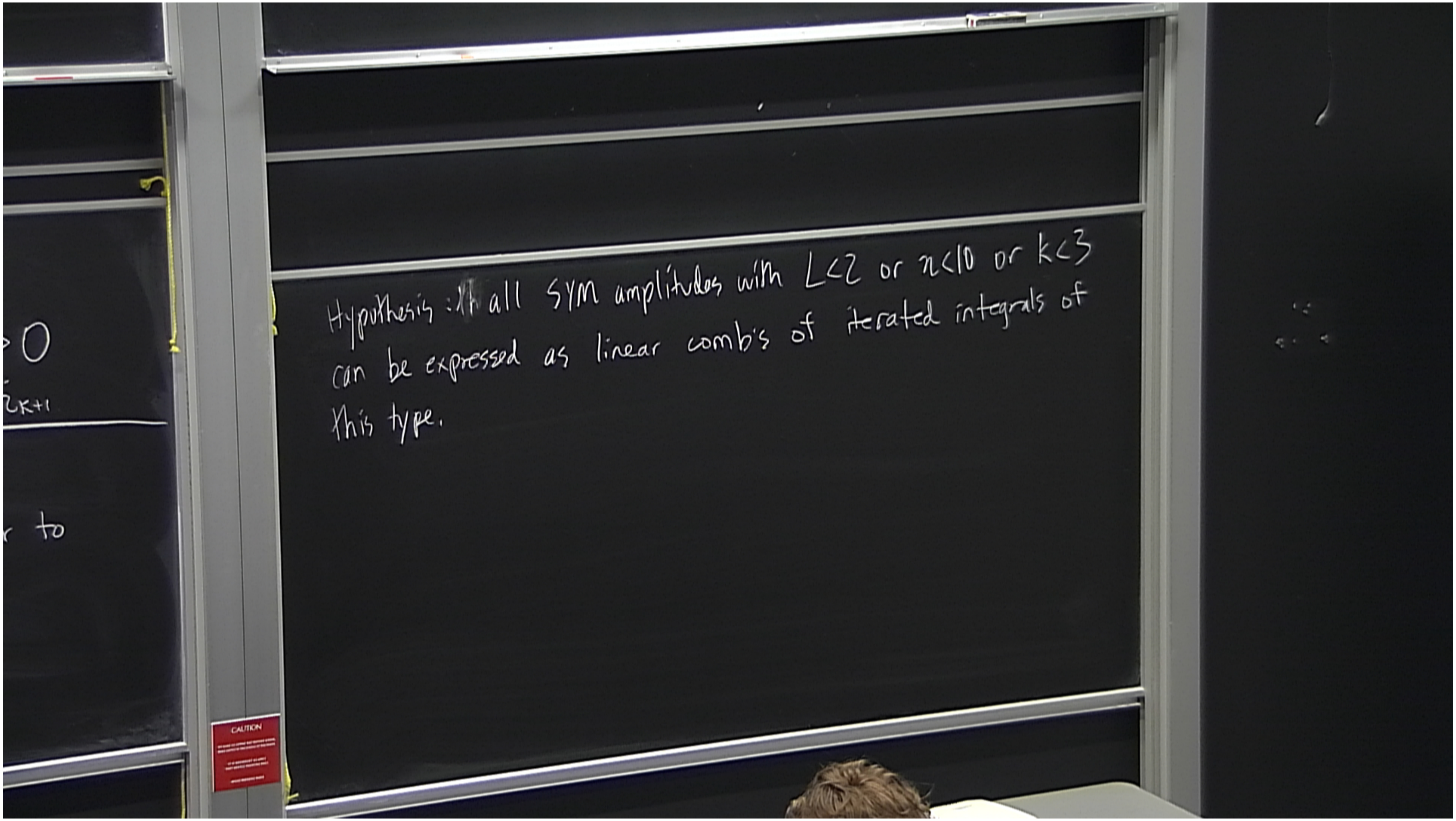
CAUTION


Hypothesis: all SYM amplitudes with L<2 or n<10 or k<3
can be expressed as linear comb's of iterated integrals of
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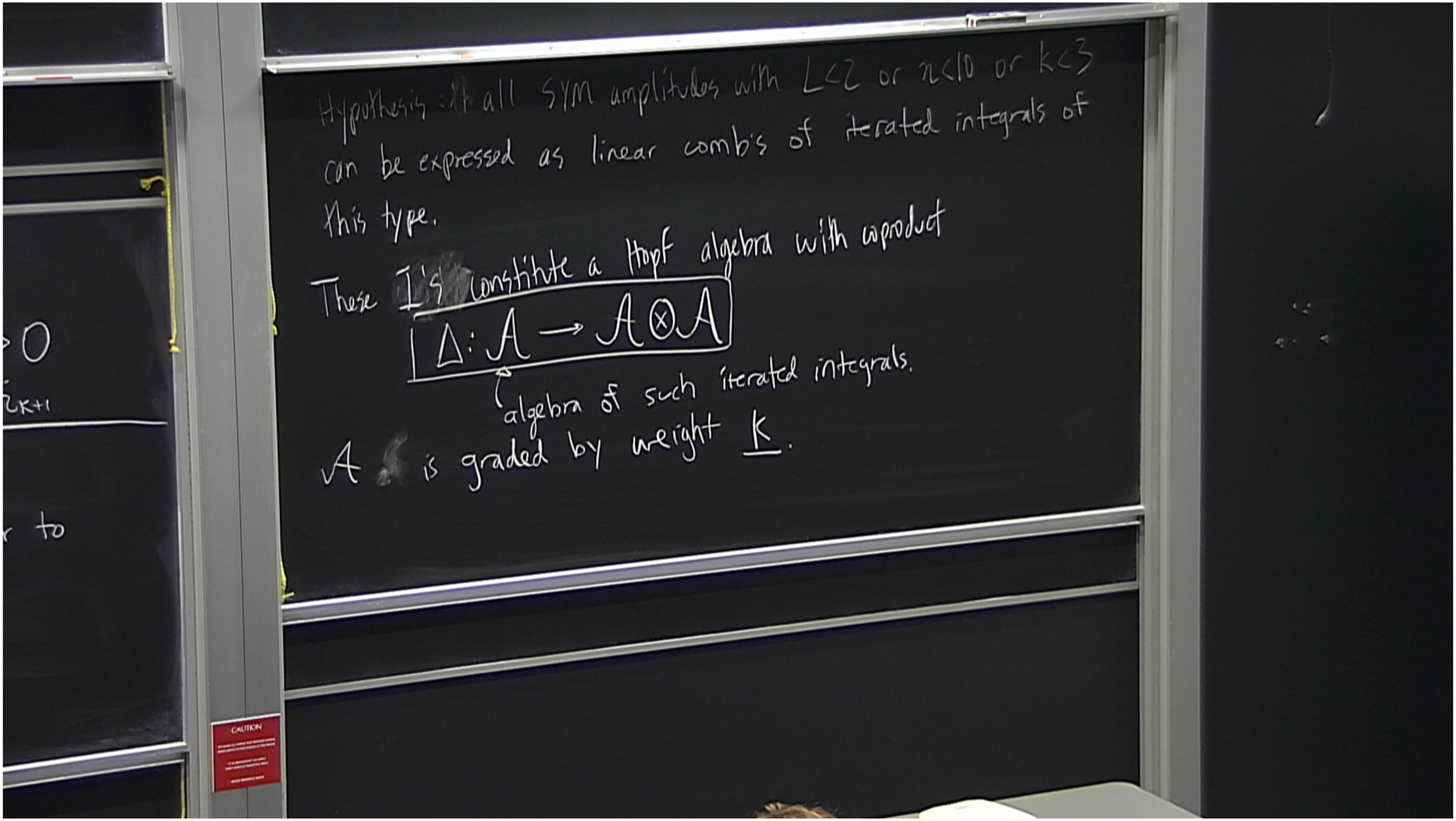
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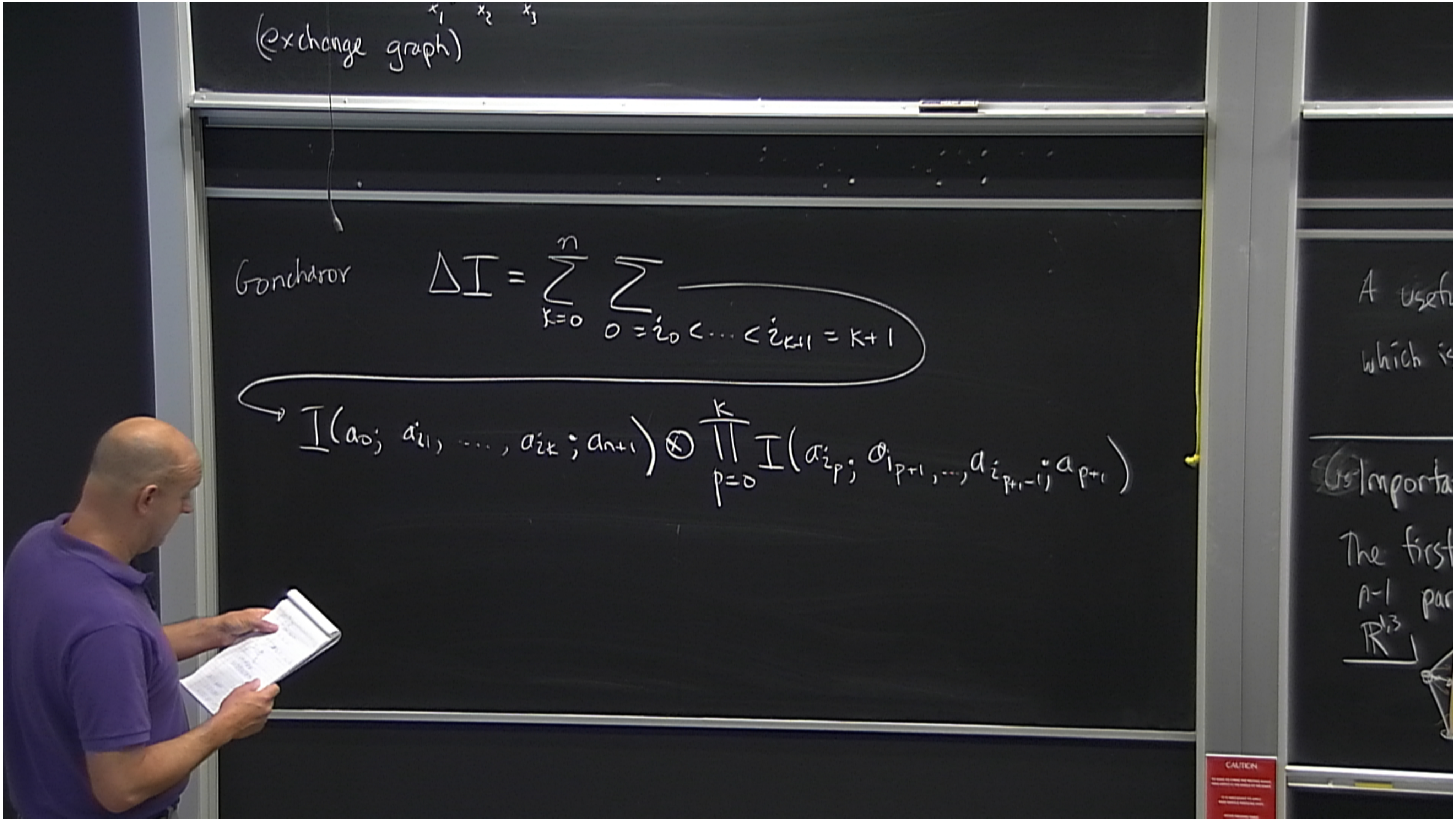
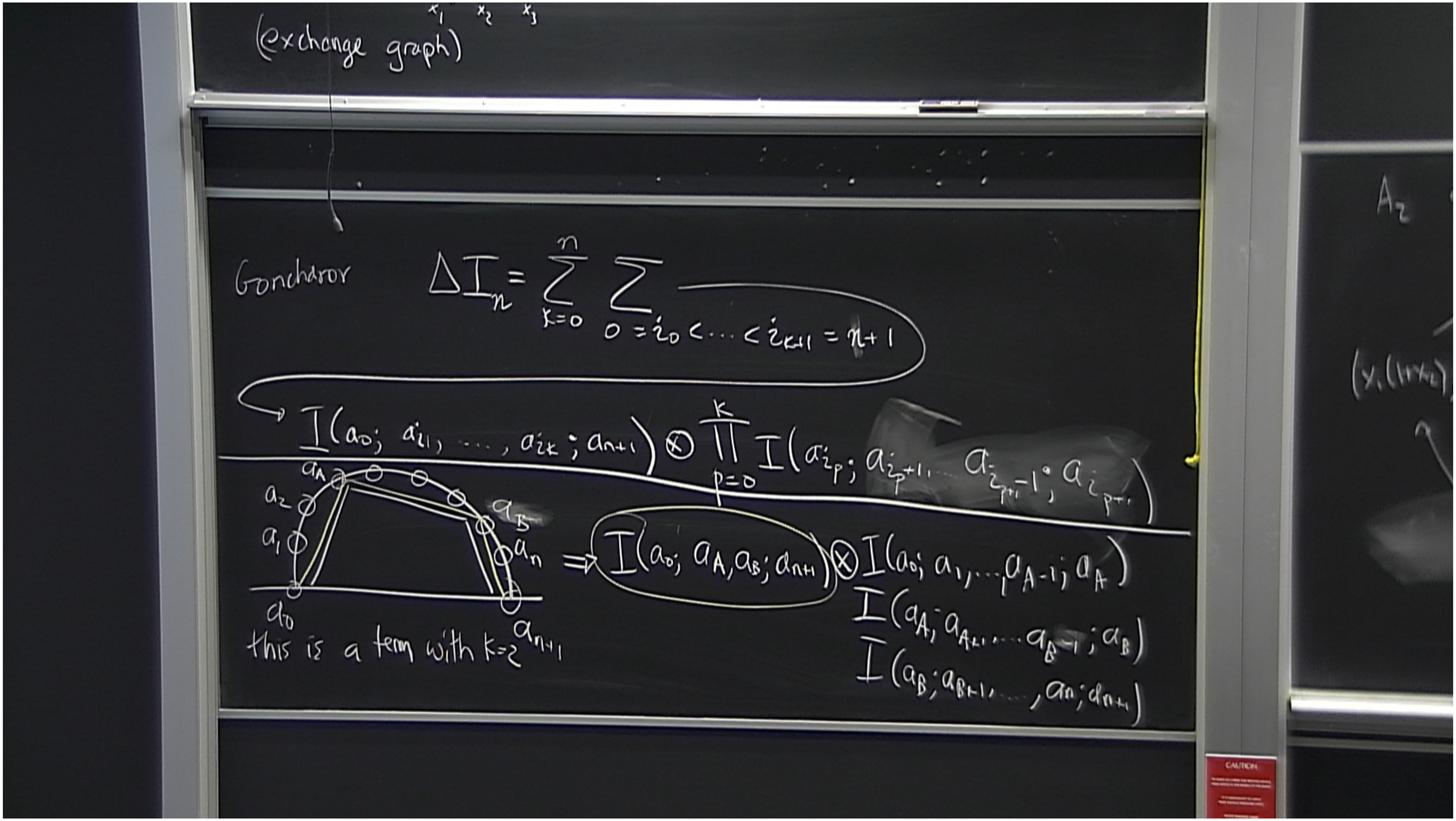
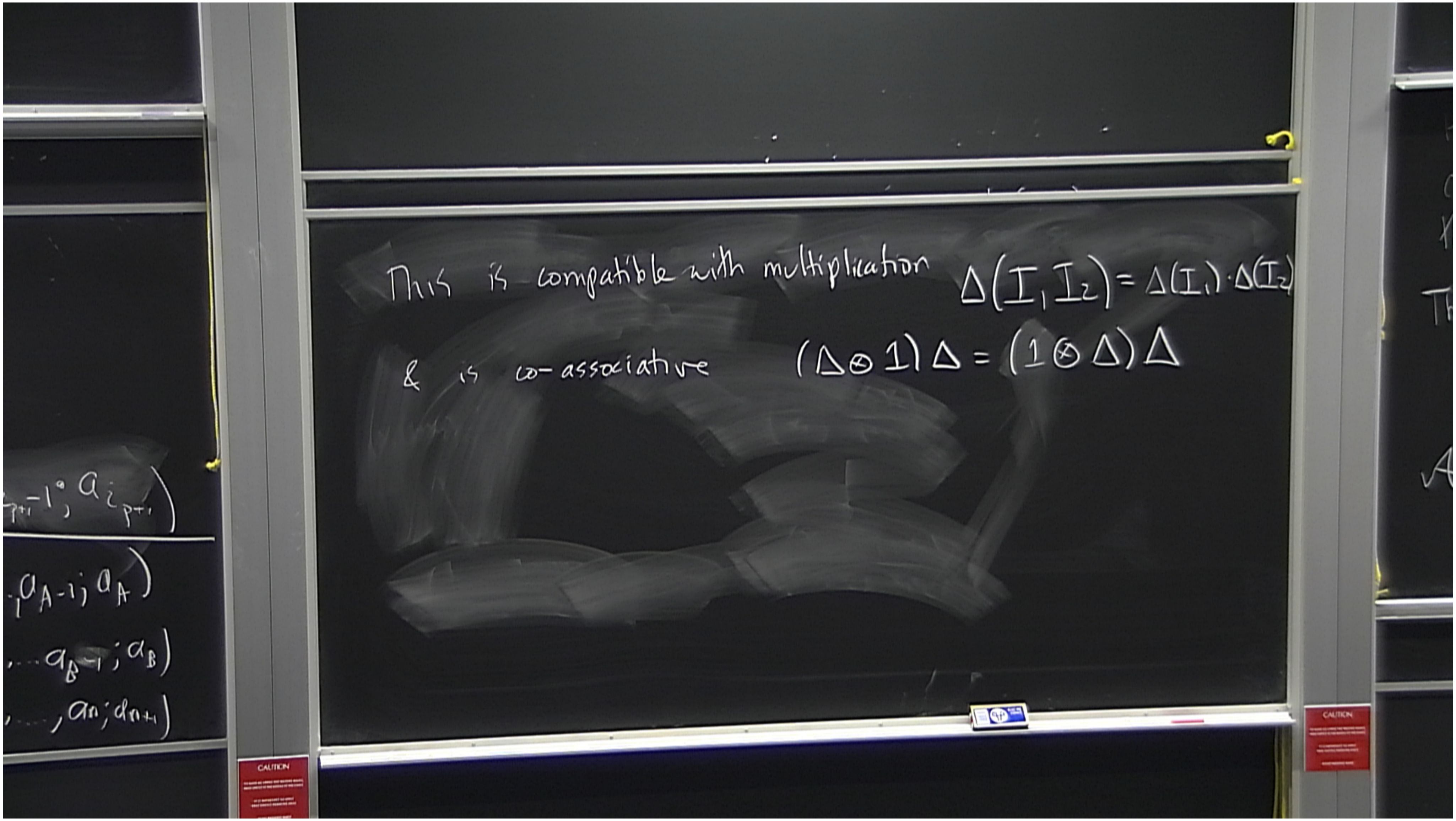
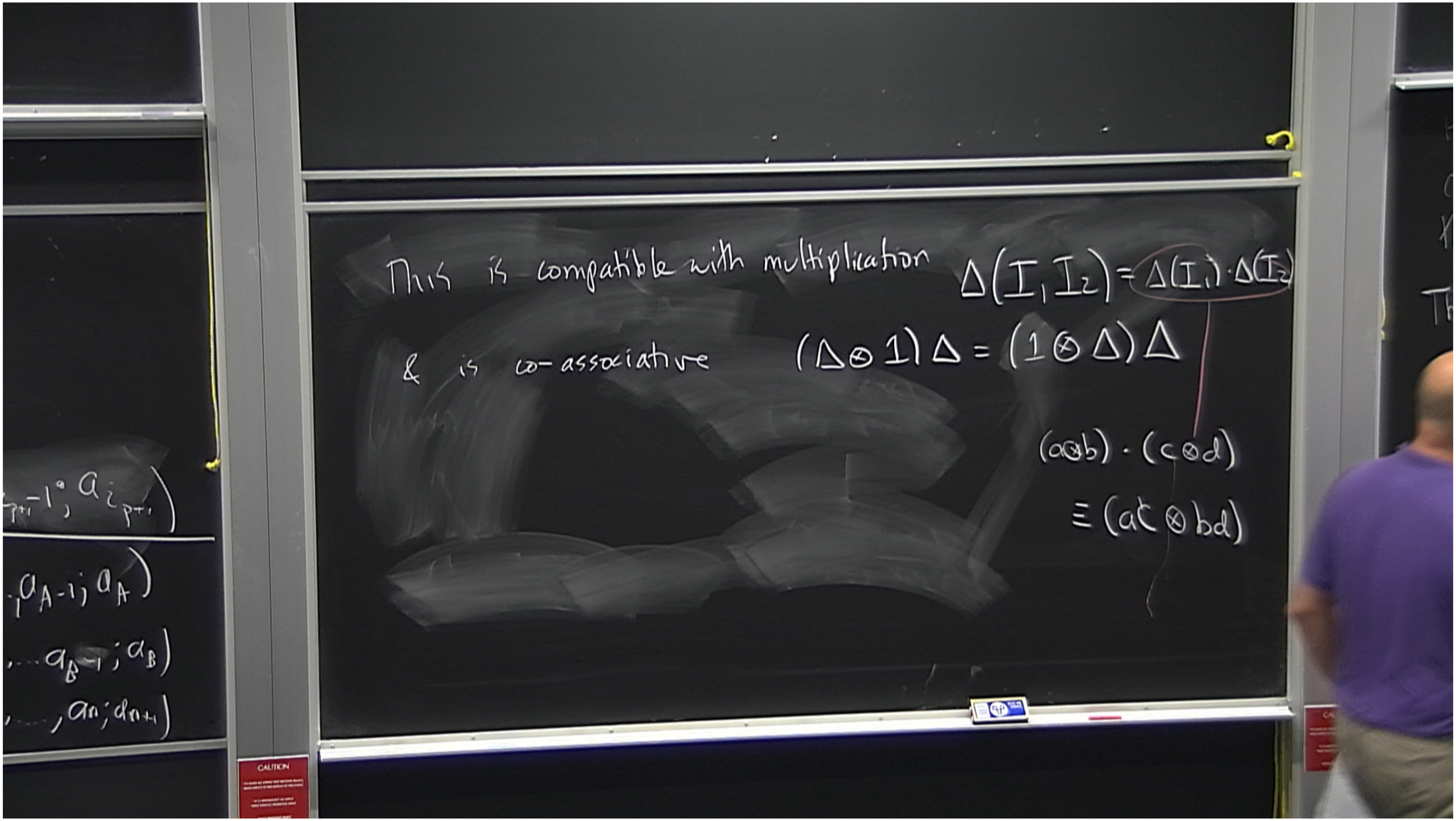
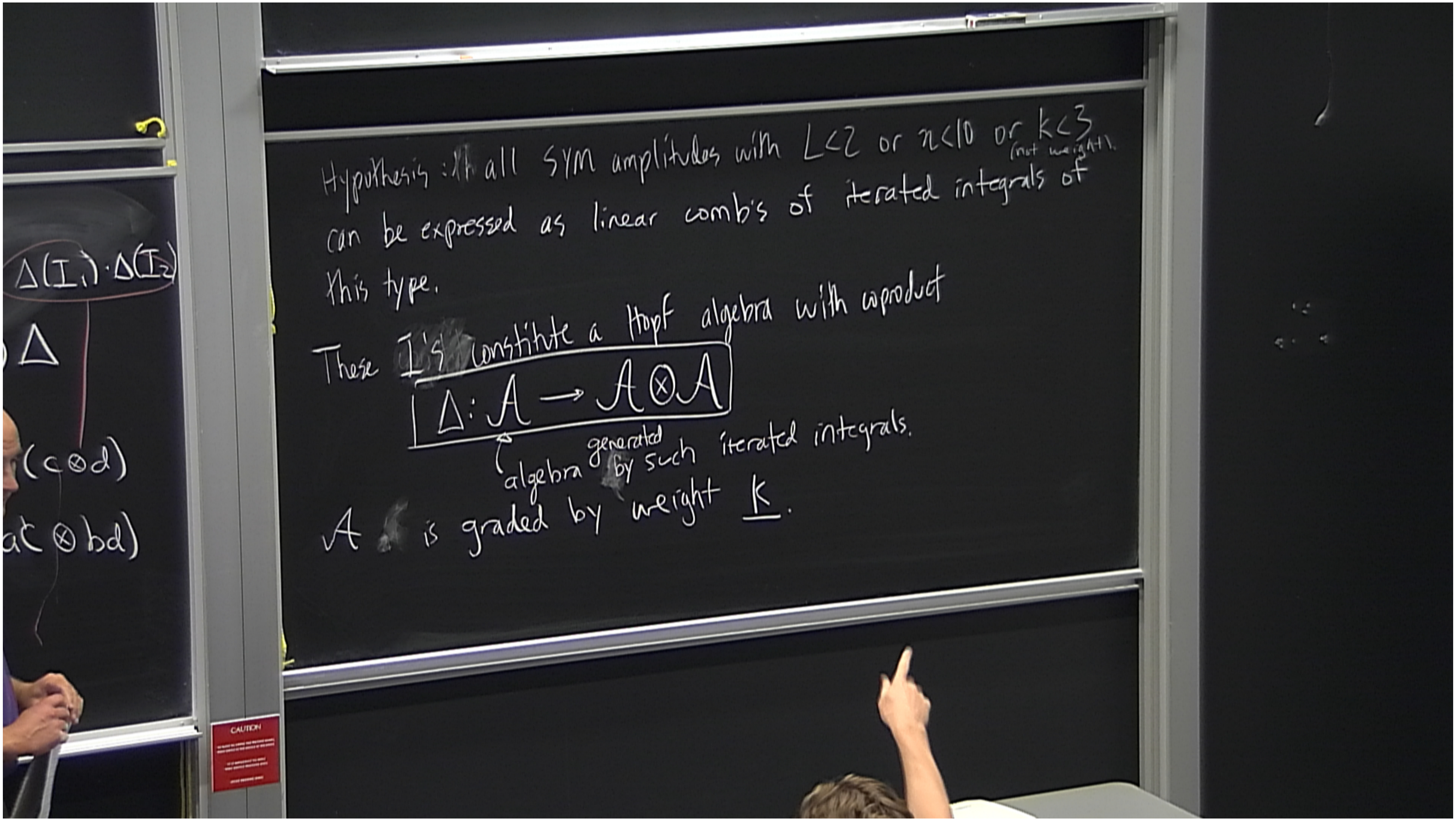
These I's constitute a Hopf algebra with coproduct
Δ: A → A⊗A
algebra of such iterated integrals.
A is graded by weight K.


(exchange graph)
Goncharov
$\Delta I = \sum_{k=0}^{n} \sum_{0 = i_0 < \cdots < i_{k+1} = k+1}$
$I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}})$
CAUTION


(exchange graph)
Goncharov
ΔI_n = Σ_{k=0}^{n} Σ_{0=i_0<…<i_{k+1}=n+1}
I(a_0; a_{i_1}, …, a_{i_k}; a_{n+1}) ⊗ ∏_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, … a_{i_{p+1}-1}; a_{i_{p+1}})
⇒ I(a_0; a_A, a_B; a_{n+1}) ⊗ I(a_0; a_1, … a_{A-1}; a_A)
I(a_A; a_{A+1}, … a_{B-1}; a_B)
I(a_B; a_{B+1}, …, a_n; a_{n+1})
this is a term with k=2


This is compatible with multiplication Δ(I1 I2) = Δ(I1)·Δ(I2)
& is co-associative (Δ⊗1)Δ = (1⊗Δ)Δ


This is compatible with multiplication $\Delta(I_1 I_2) = \Delta(I_1)\cdot\Delta(I_2)$
& is co-associative $(\Delta\otimes 1)\Delta = (1\otimes\Delta)\Delta$
$(a\otimes b)\cdot(c\otimes d)$
$\equiv (ac\otimes bd)$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can be expressed as linear comb's of iterated integrals of
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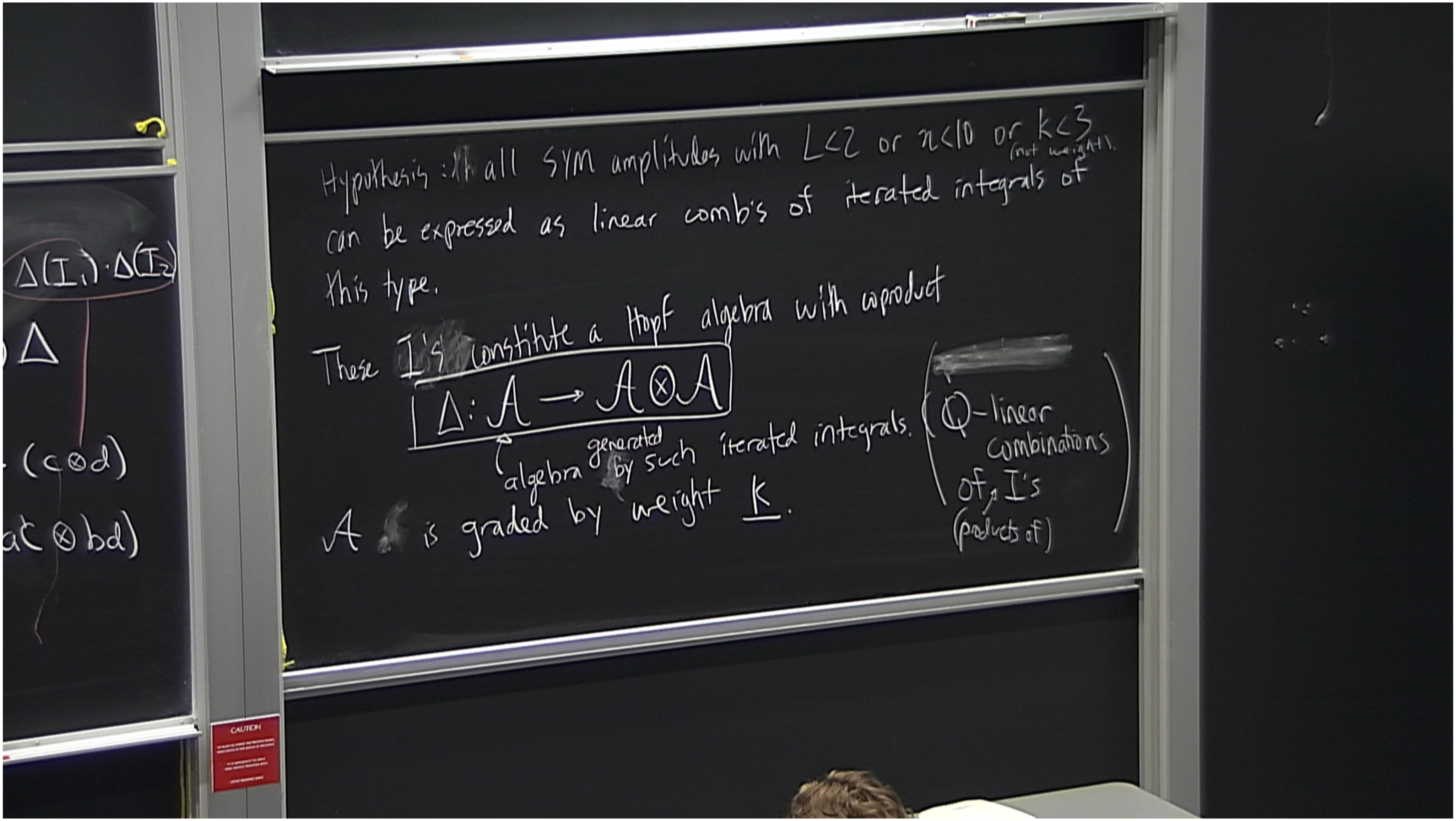
These I's constitute a Hopf algebra with coproduct
Δ: A → A⊗A
algebra generated by such iterated integrals.
A is graded by weight K.
Δ(I1)·Δ(I2)
(c⊗d)
ac⊗bd)
CAUTION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(Q-linear combinations of (products of) I's)
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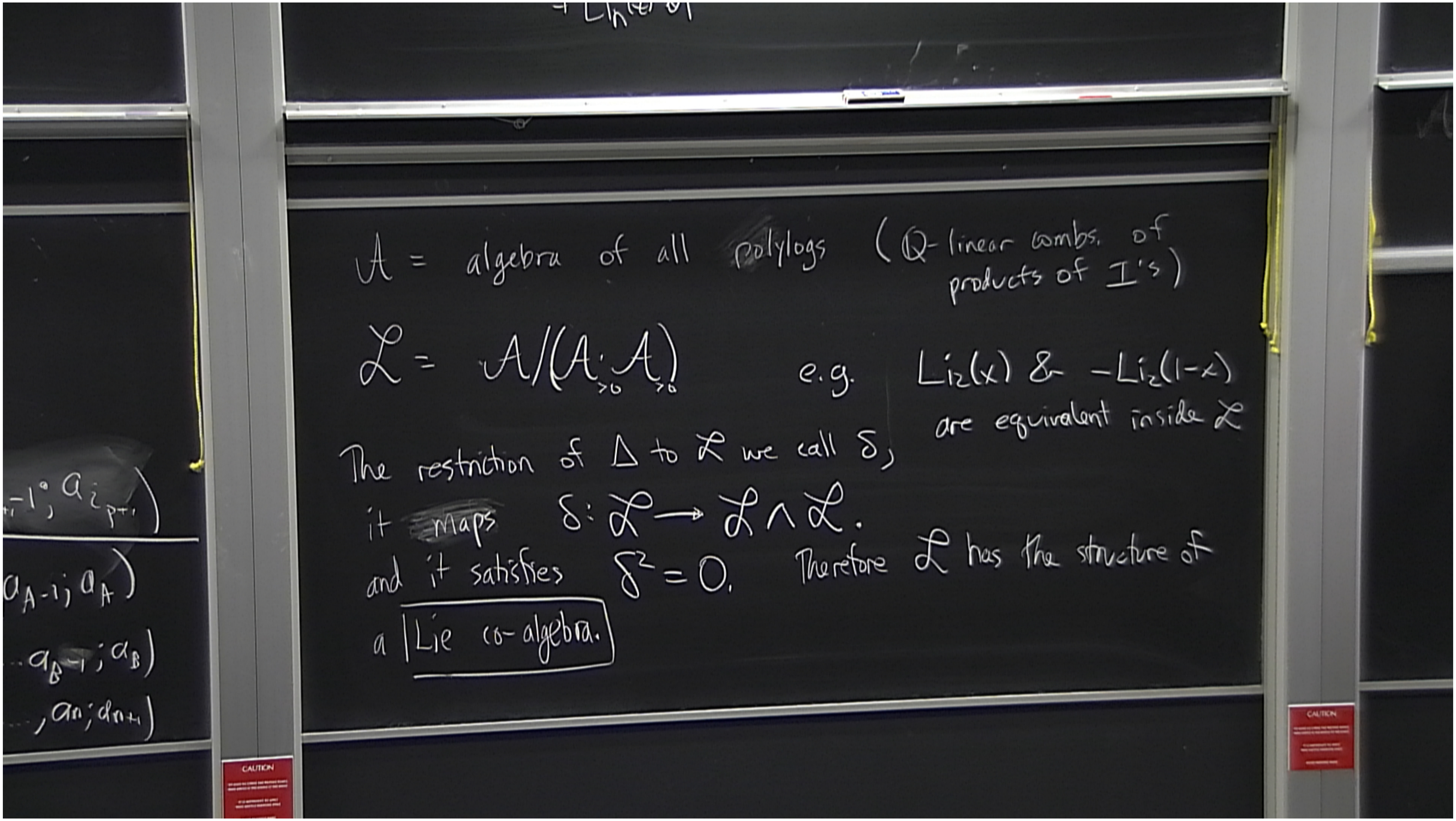
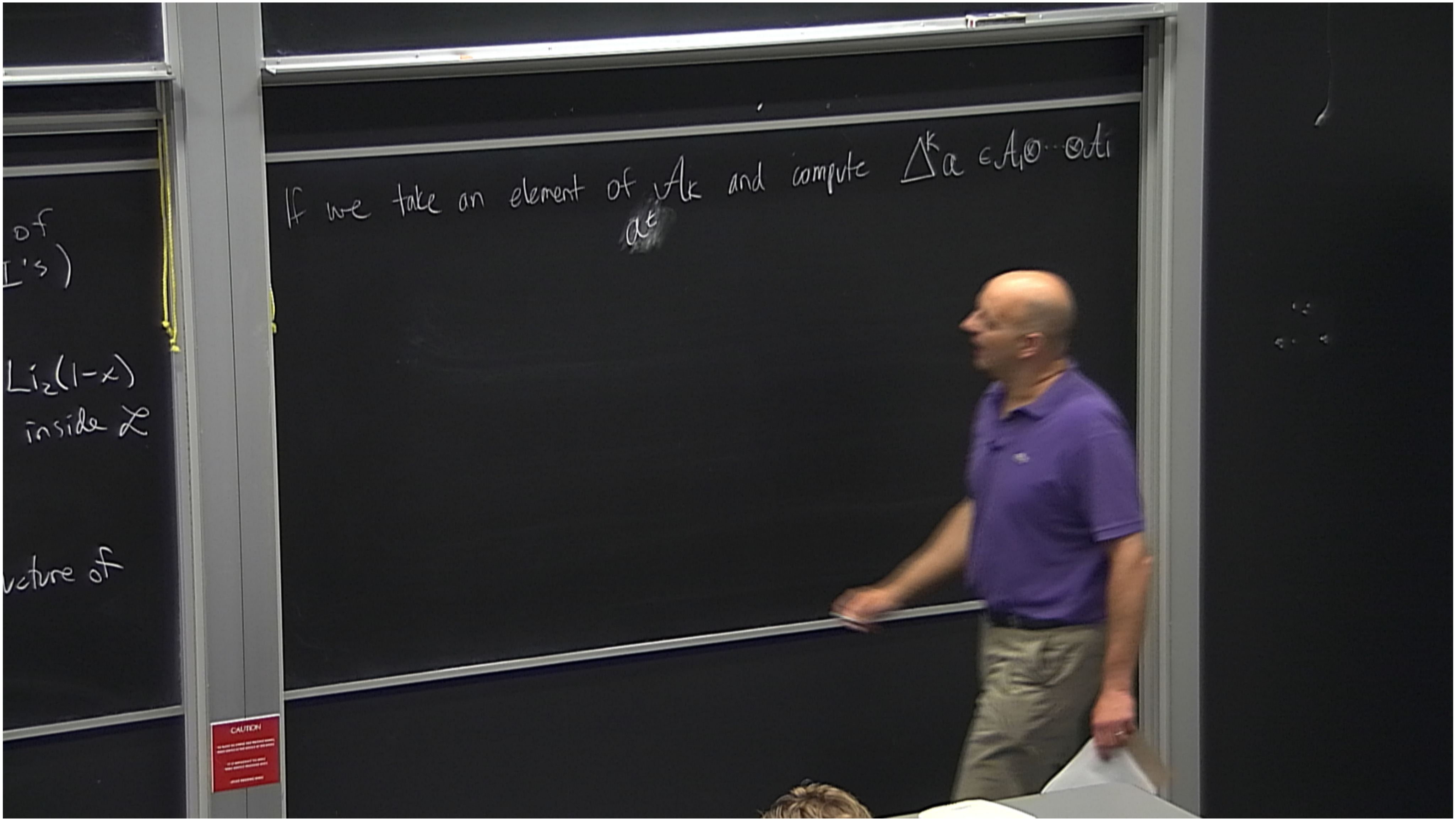

$\mathcal{A}$ = algebra of all polylogs (ℚ-linear combs. of products of I's)
$\mathcal{L} = \mathcal{A}/(\mathcal{A}_{>0} \cdot \mathcal{A}_{>0})$
e.g. $Li_2(x)$ & $-Li_2(1-x)$ are equivalent inside $\mathcal{L}$
The restriction of Δ to $\mathcal{L}$ we call δ, it maps $\delta: \mathcal{L} \to \mathcal{L} \wedge \mathcal{L}$.
and it satisfies $\delta^2 = 0$. Therefore $\mathcal{L}$ has the structure of a Lie co-algebra.


If we take an element of A_k and compute $\Delta^k a \in A_1 \otimes \cdots \otimes A_1$
of
I's)
$Li_2(1-x)$
inside $\mathcal{L}$
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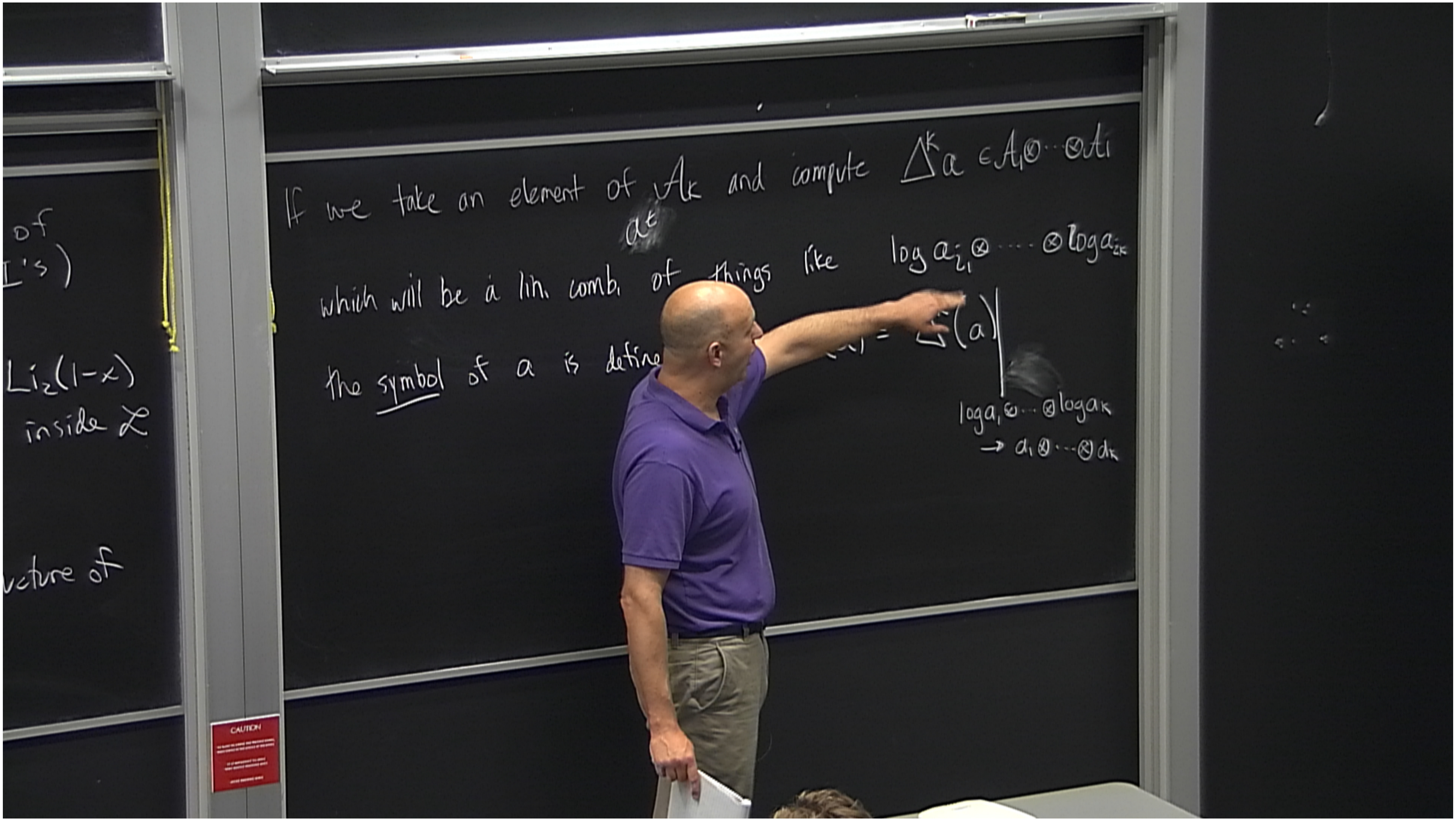
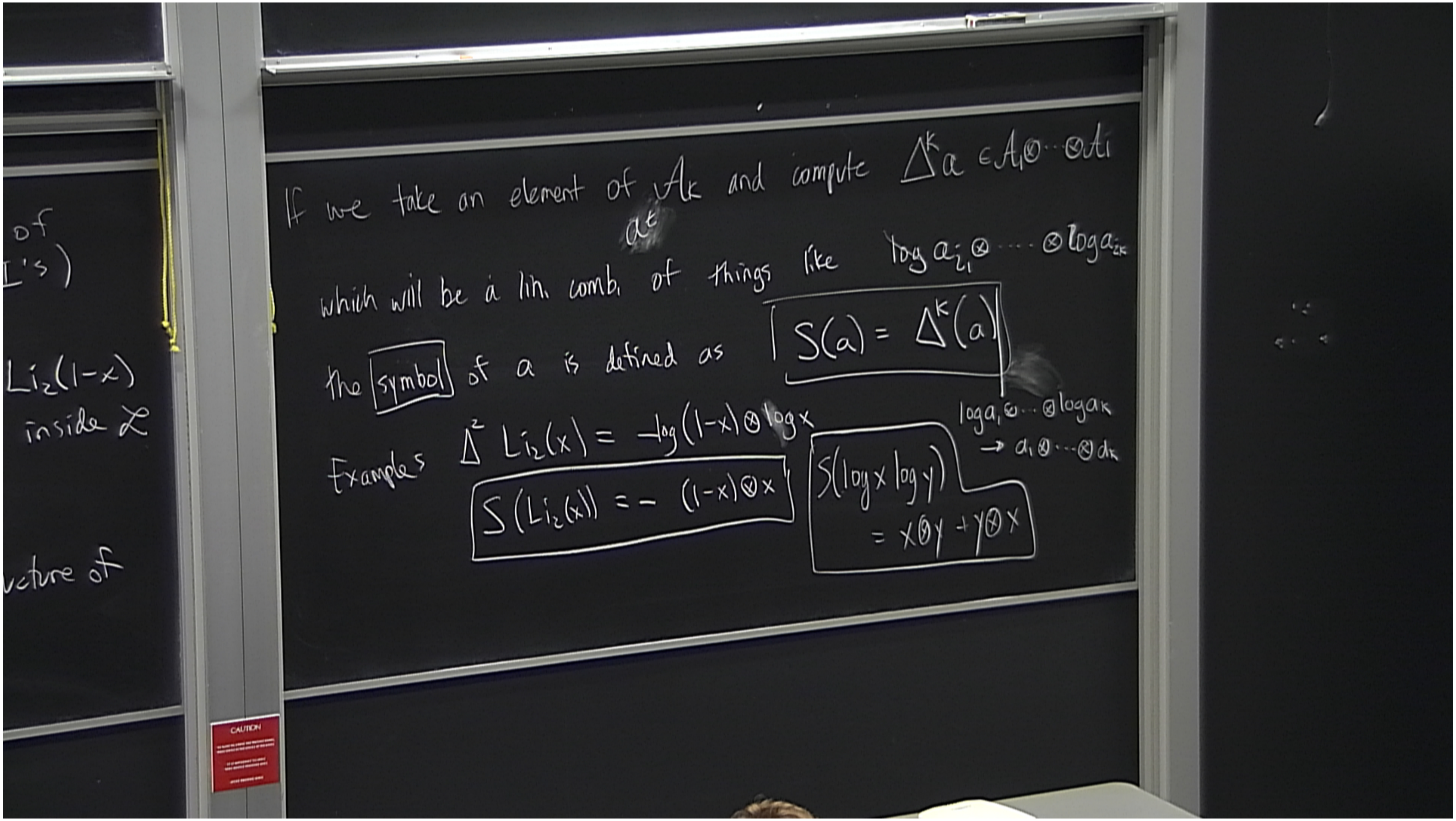
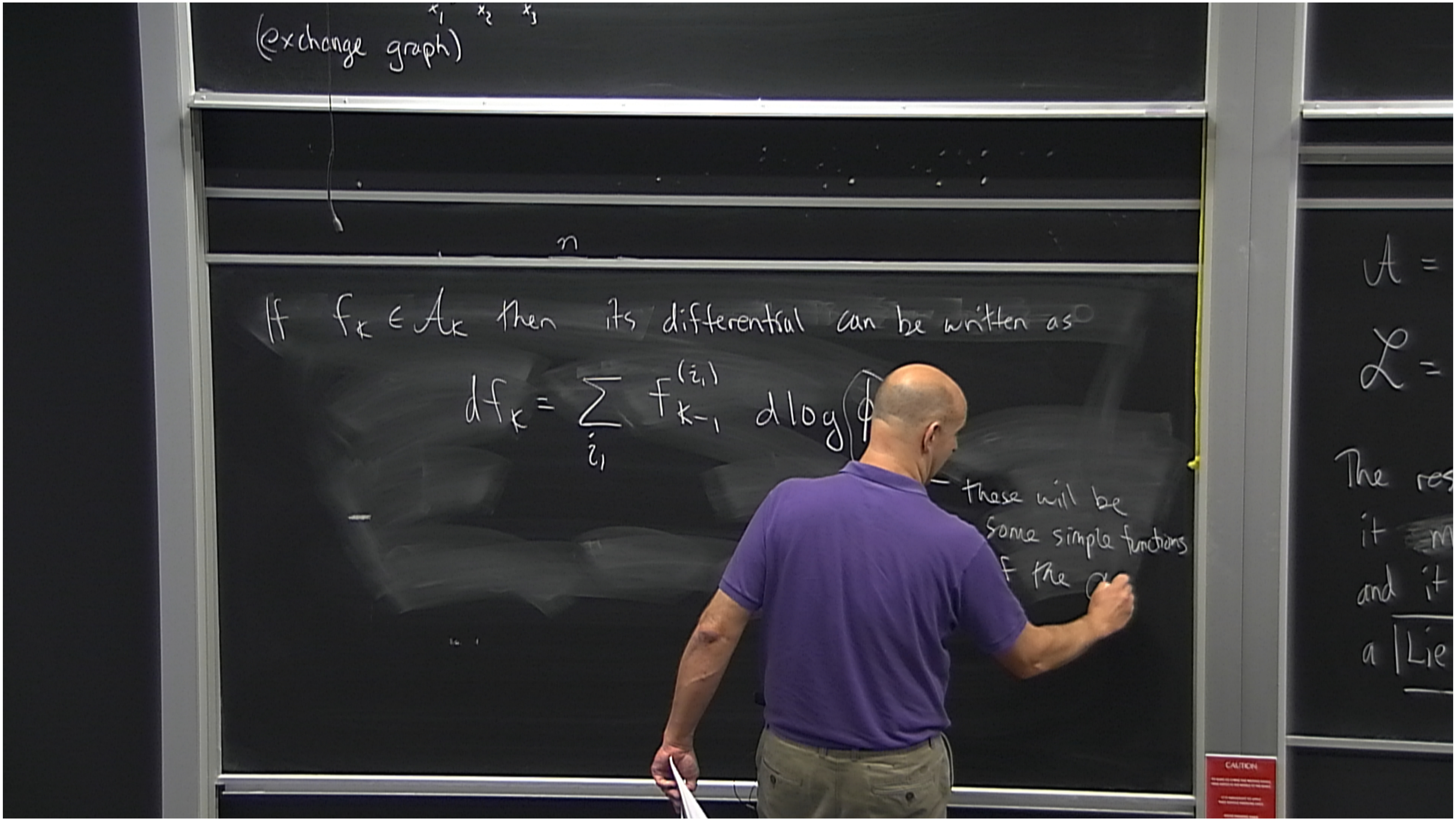
CAUTION



If we take an element of A_k and compute Δ^k a ∈ A_1 ⊗ ··· ⊗ A_1
which will be a lin. comb. of things like log a_{i_1} ⊗ ···· ⊗ log a_{i_k}
the symbol of a is define
log a_1 ⊗ ··· ⊗ log a_k
→ a_1 ⊗ ··· ⊗ a_k
Li_2(1-x)
inside
CAUTION


If we take an element of $\mathcal{A}_k$ and compute $\Delta^k a \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_1$
which will be a lin. comb. of things like $\log a_{i_1} \otimes \cdots \otimes \log a_{i_k}$
the symbol of a is defined as $S(a) = \Delta^k(a)$
$\log a_1 \otimes \cdots \otimes \log a_k \to a_1 \otimes \cdots \otimes a_k$
Examples $\Delta^2 Li_2(x) = -\log(1-x) \otimes \log x$
$S(Li_2(x)) = -(1-x) \otimes x$
$S(\log x \log y) = x \otimes y + y \otimes x$
inside


(exchange graph)
If $f_k \in \mathcal{A}_k$ then its differential can be written as
$df_k = \sum_{i_1} f_{k-1}^{(i_1)} \, d\log \phi$
these will be
some simple functions
of the
$\mathcal{A} =$
$\mathcal{L} =$
The res
it
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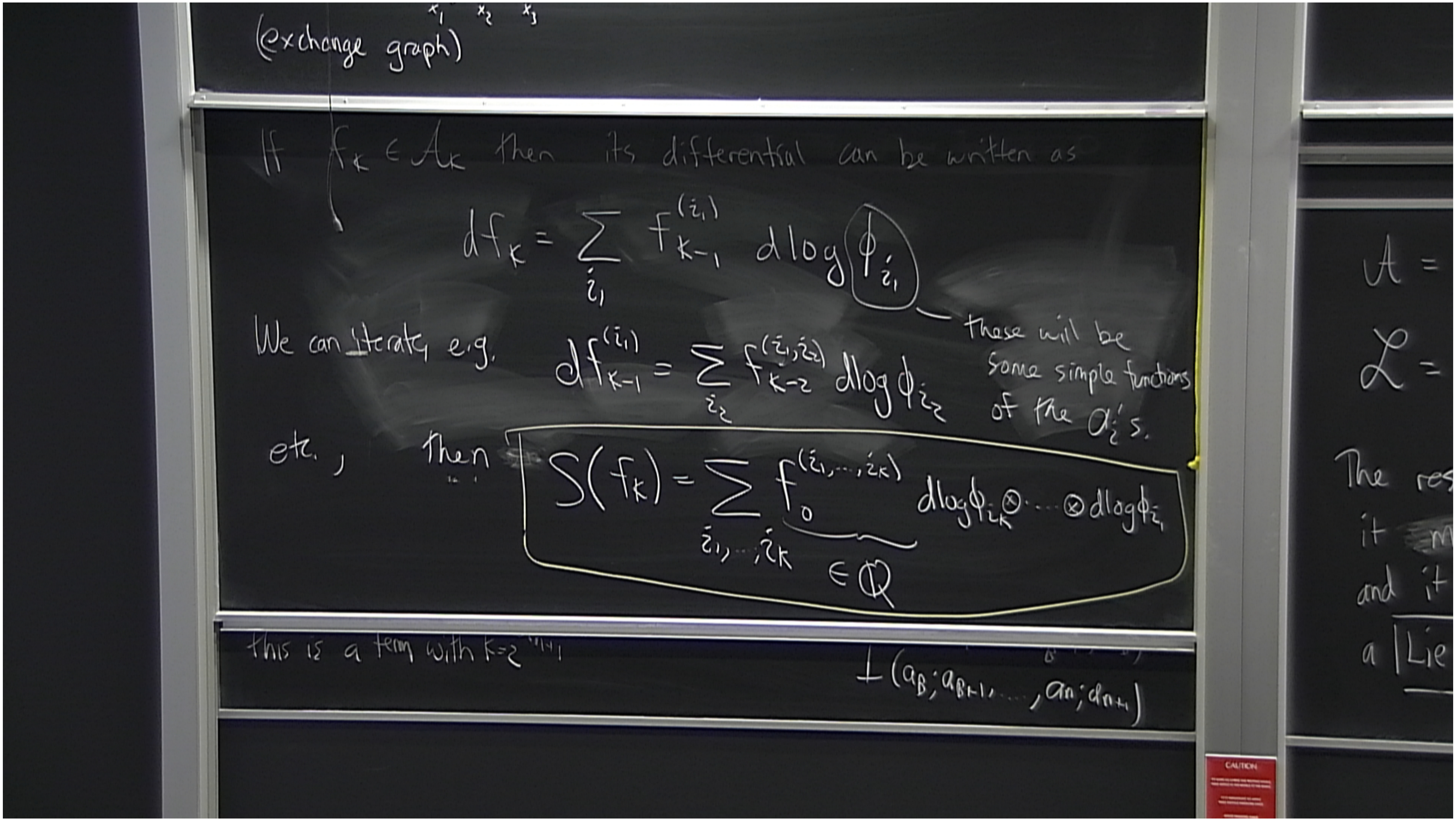
CAUTION



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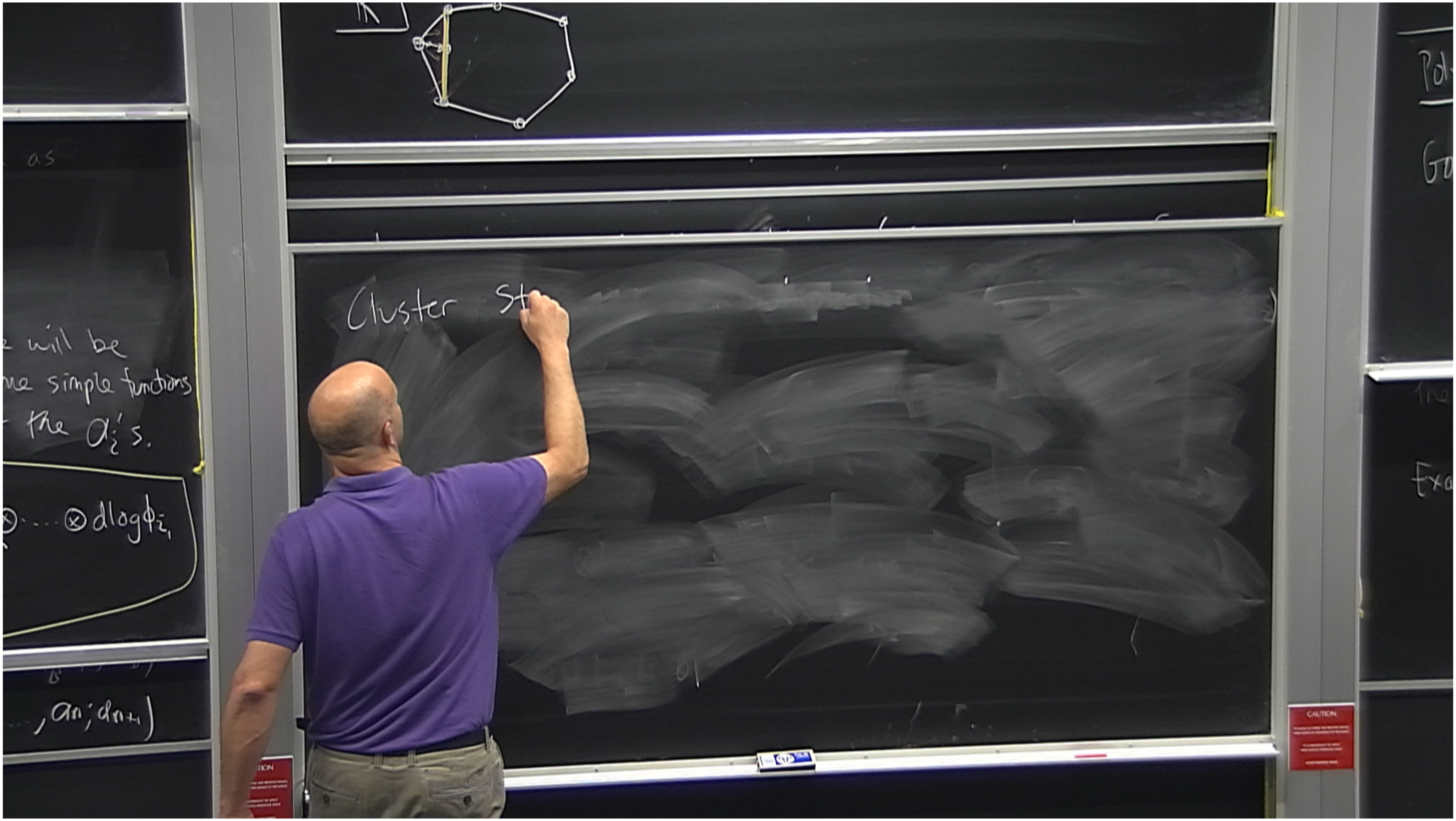
If $f_k \in \mathcal{A}_k$ then its differential can be written as
$df_k = \sum_{i_1} f_{k-1}^{(i_1)} \, d\log \phi_{i_1}$
these will be some simple functions of the a_i's.
We can iterate, e.g. $df_{k-1}^{(i_1)} = \sum_{i_2} f_{k-2}^{(i_1,i_2)} \, d\log \phi_{i_2}$
etc., then $S(f_k) = \sum_{i_1,\dots,i_k} f_0^{(i_1,\dots,i_k)} \, d\log\phi_{i_k} \otimes \cdots \otimes d\log\phi_{i_1}$
$\in \mathbb{Q}$
this is a term with $k=2$
$I(a_B; a_{B+1}, \dots, a_n; a_{n+1})$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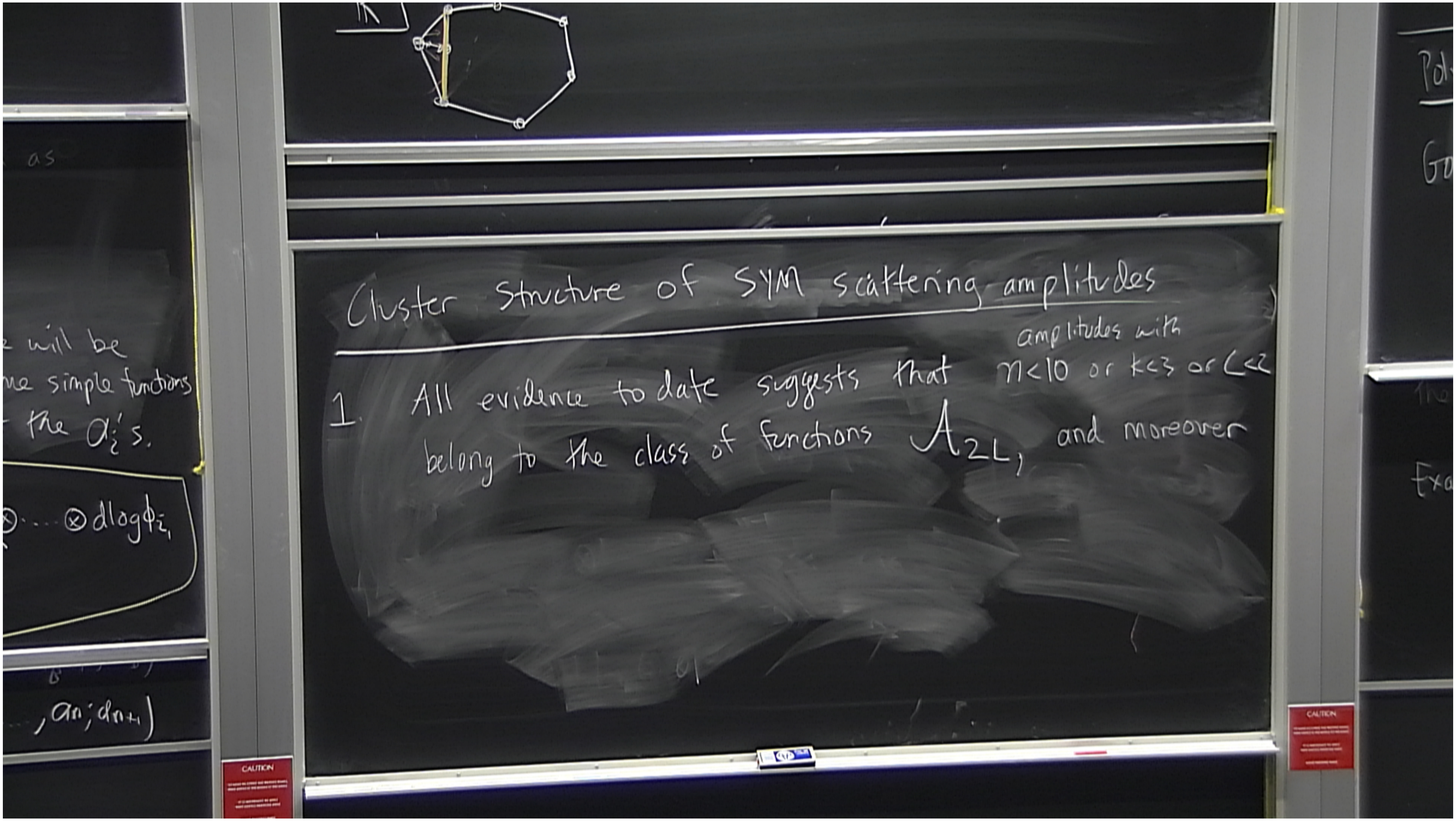
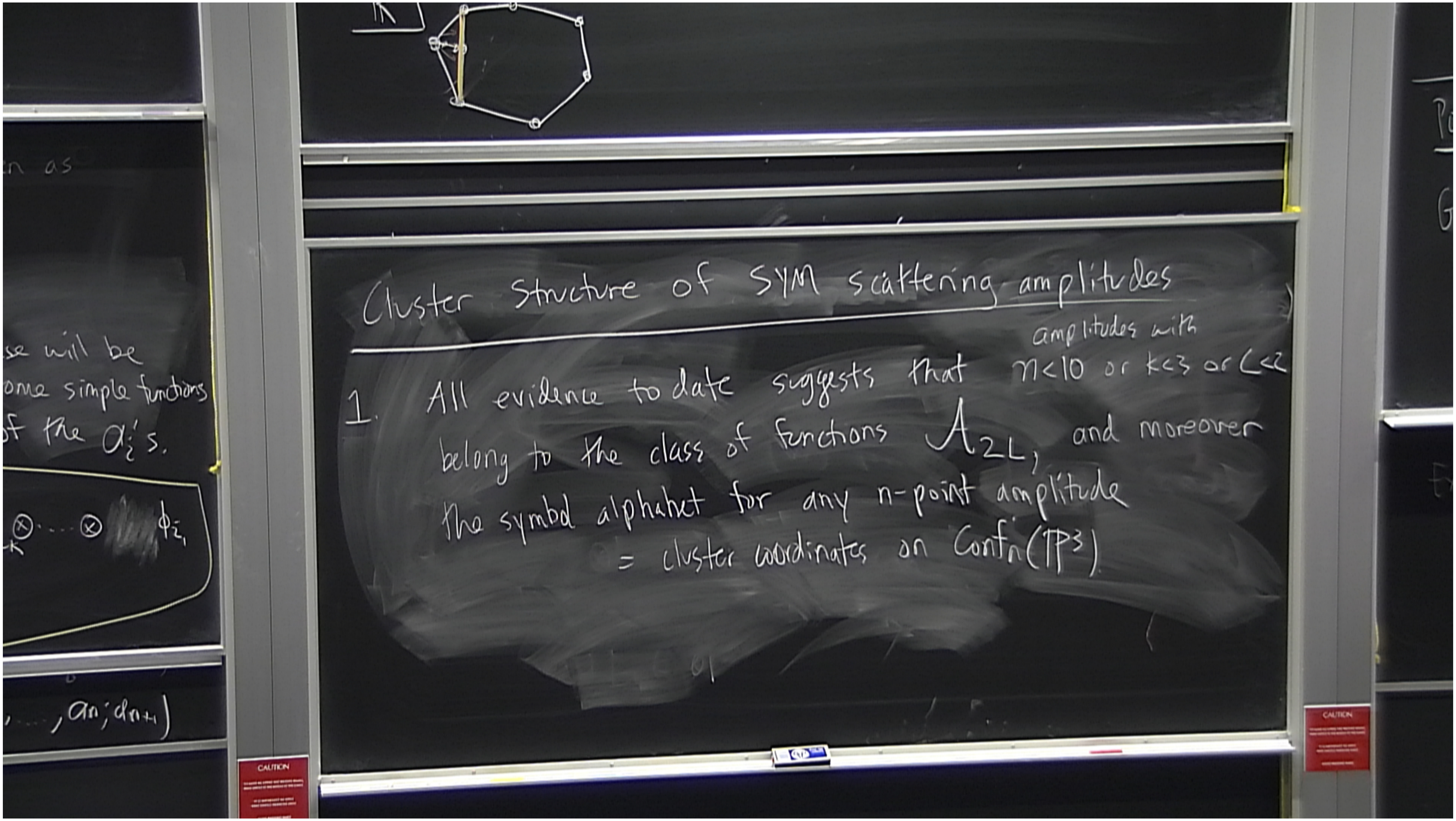

Cluster Structure of SYM scattering amplitudes
1. All evidence to date suggests that amplitudes with n<10 or k<3 or L<2
belong to the class of functions A2L, and moreover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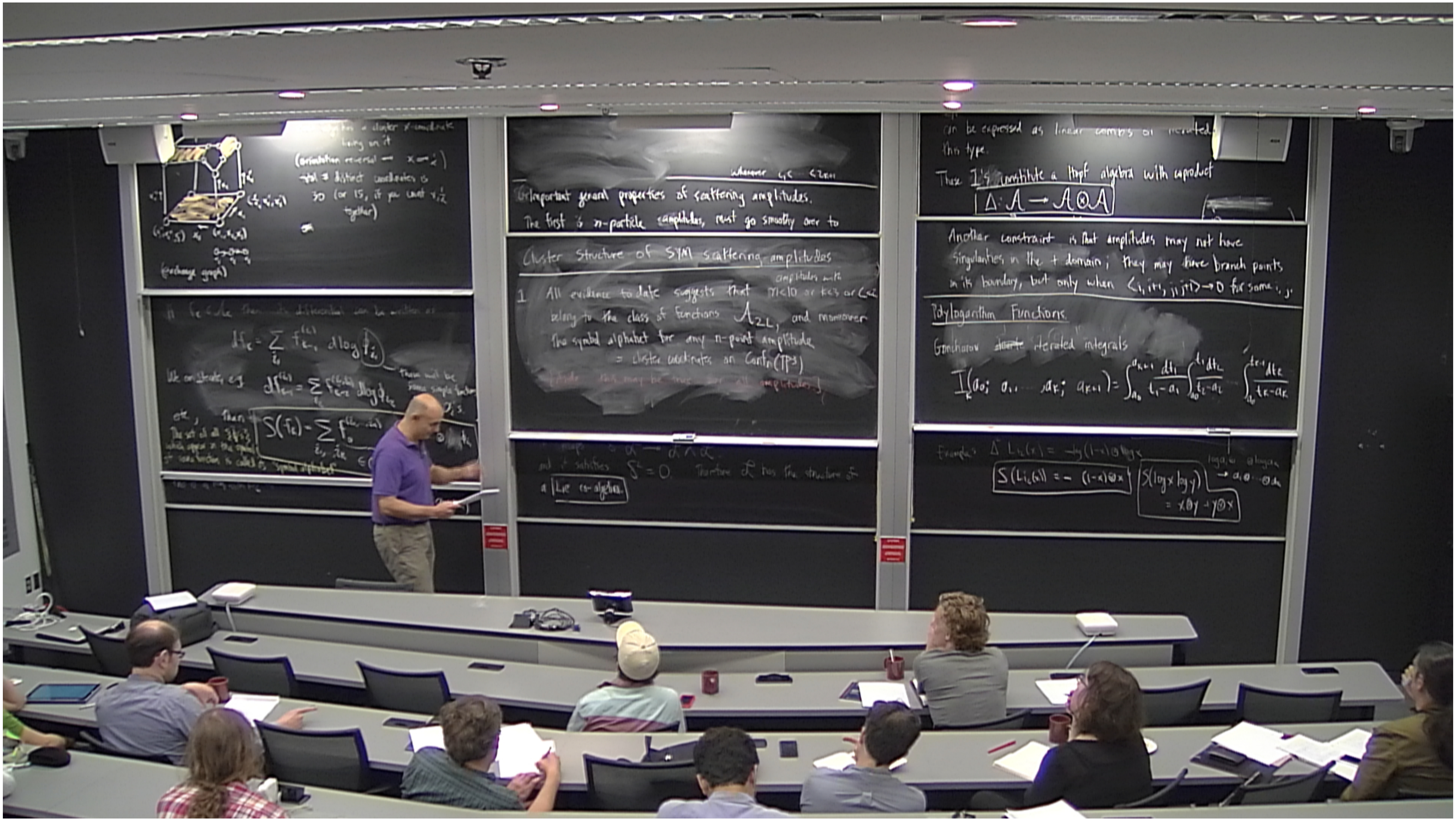
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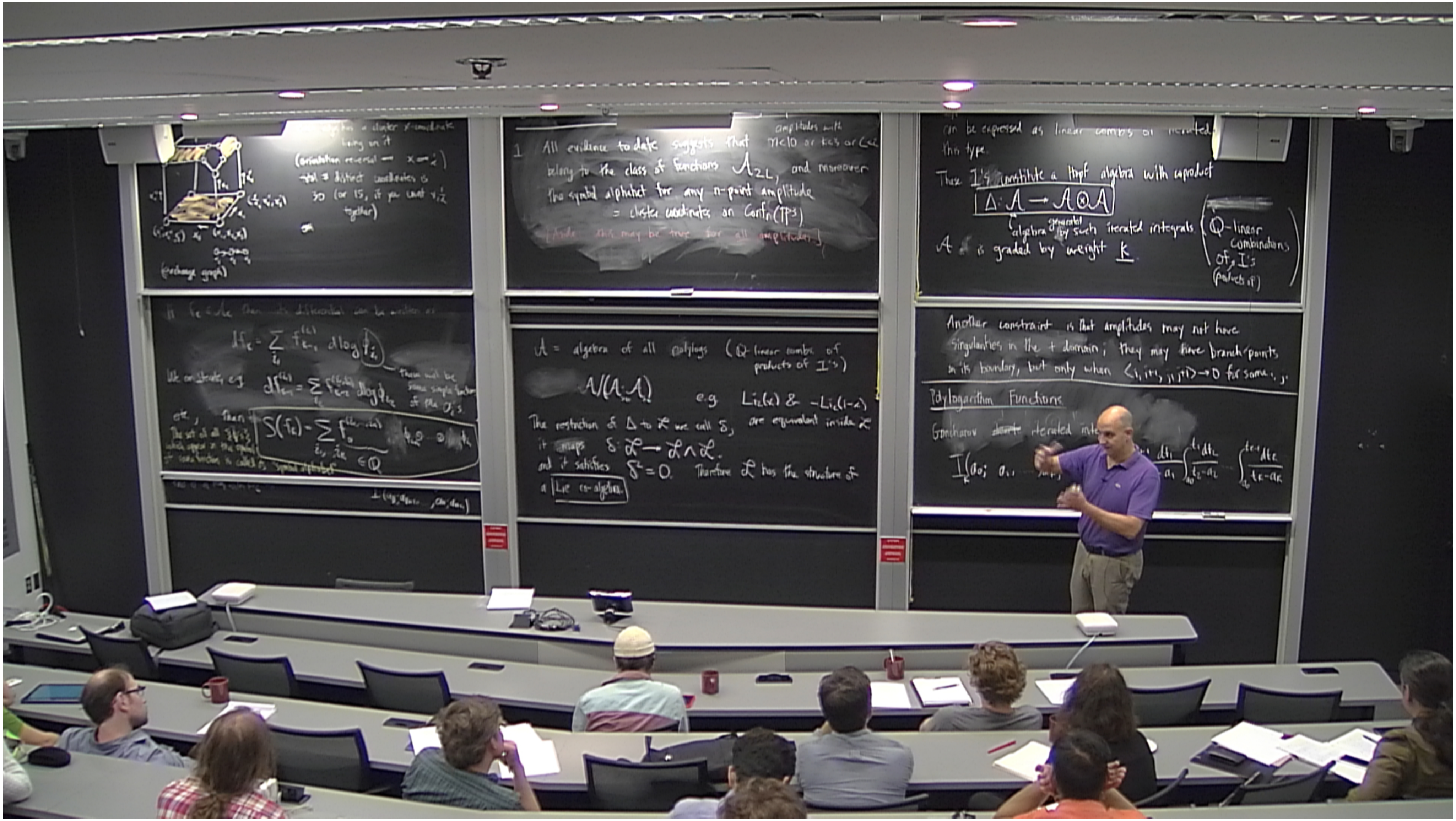
$\Delta: A \to A \otimes A$
Another constraint is that amplitudes may not have singularities in the + domain; they may have branch points on its boundary, but only when ⟨i, i+1, j, j+1⟩ → 0 for some i, j.
Polylogarithm Functions
Goncharov iterated integrals
$I_k(a_0; a_1 \ldots a_k; a_{k+1}) = \int_{a_0}^{a_{k+1}} \frac{dt_1}{t_1-a_1} \int_{a_0}^{t_1} \frac{dt_2}{t_2-a_2} \cdots \int_{a_0}^{t_{k-1}} \frac{dt_k}{t_k-a_k}$
$S(Li_2(x)) = -(1-x)\otimes x$
$S(\log x \log y) = x\otimes y + y \otimes x$
a Lie co-algebra
$S(f_k) = \sum f$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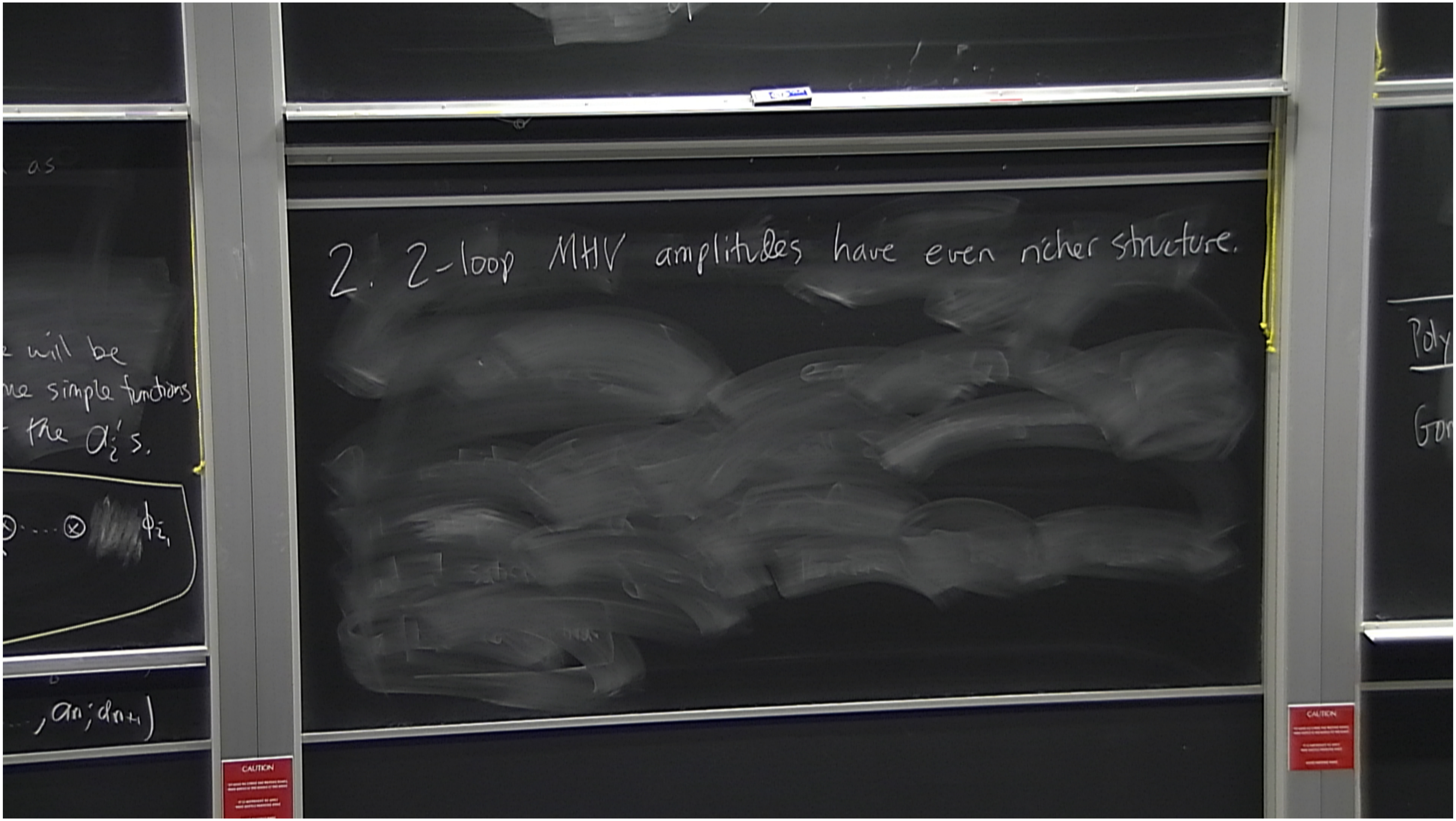
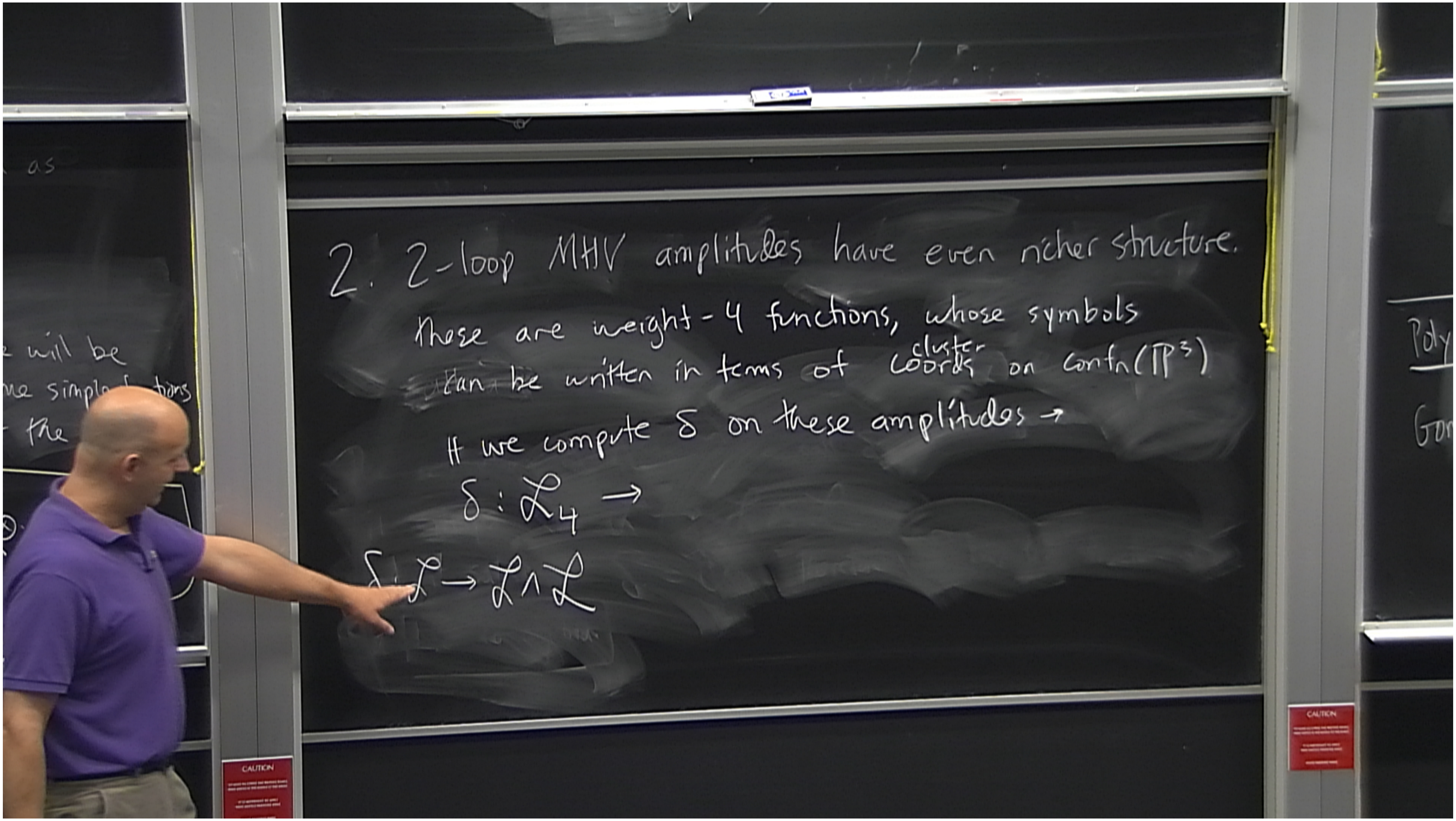

2. 2-loop MHV amplitudes have even richer structure.
will be
simple functions
the a_i's.
, a_n; a_{n+1})
CAUTION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If we compute δ on these amplitudes →
δ : L4 →
δ : L → L ∧ L

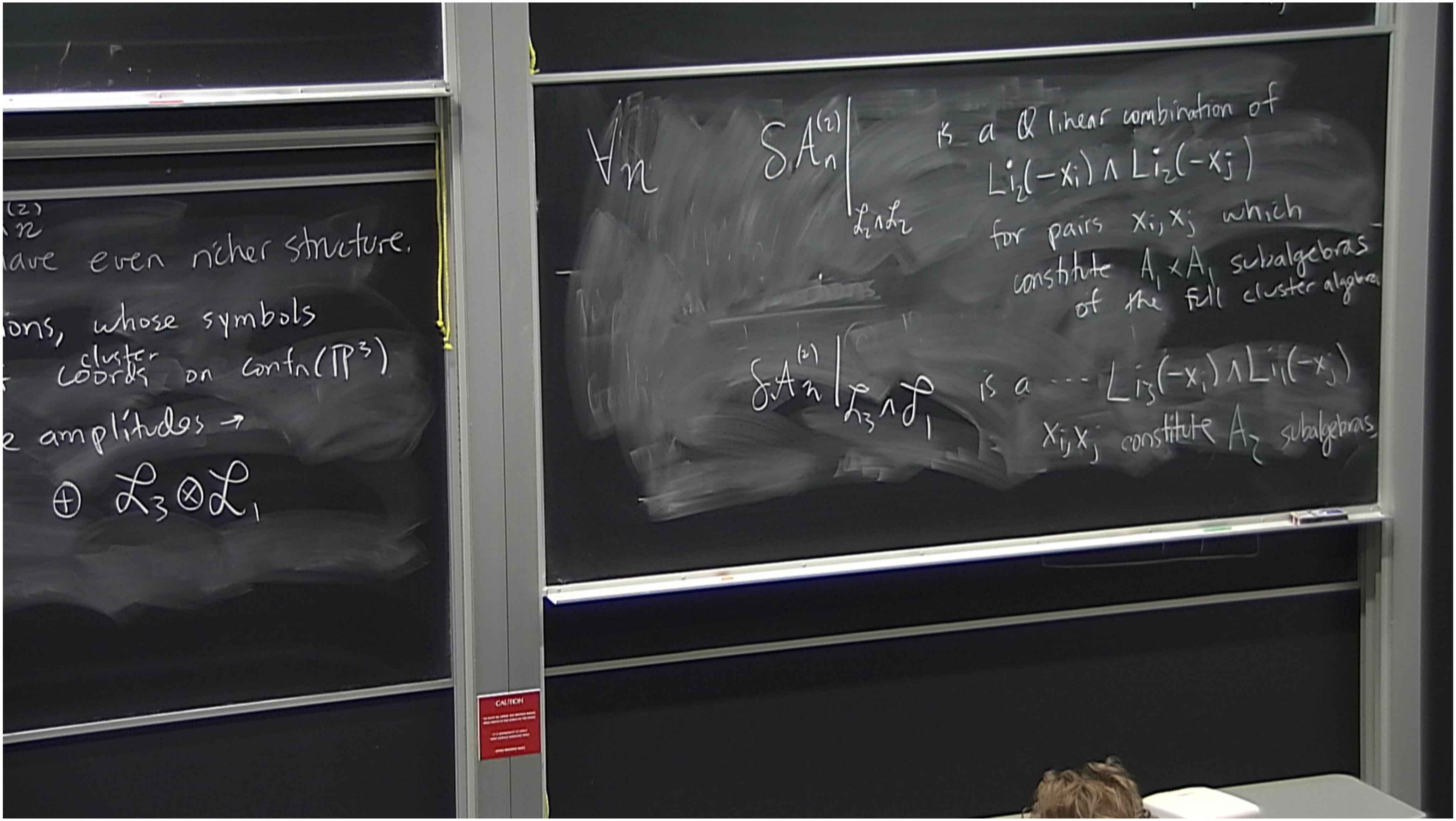
2. 2-loop MHV amplitudes $A_n^{(2)}$ have even richer structure.

These are weight-4 functions, whose symbols can be written in terms of cluster coords on $\mathrm{Conf}_n(\mathbb{P}^3)$

If we compute δ on these amplitudes →

$$\delta : \mathcal{L}_4 \to \mathcal{L}_2 \wedge \mathcal{L}_2 \oplus \mathcal{L}_3 \otimes \mathcal{L}_1$$

$$\delta : \mathcal{L} \to \mathcal{L} \wedge \mathcal{L}$$


(2)
n
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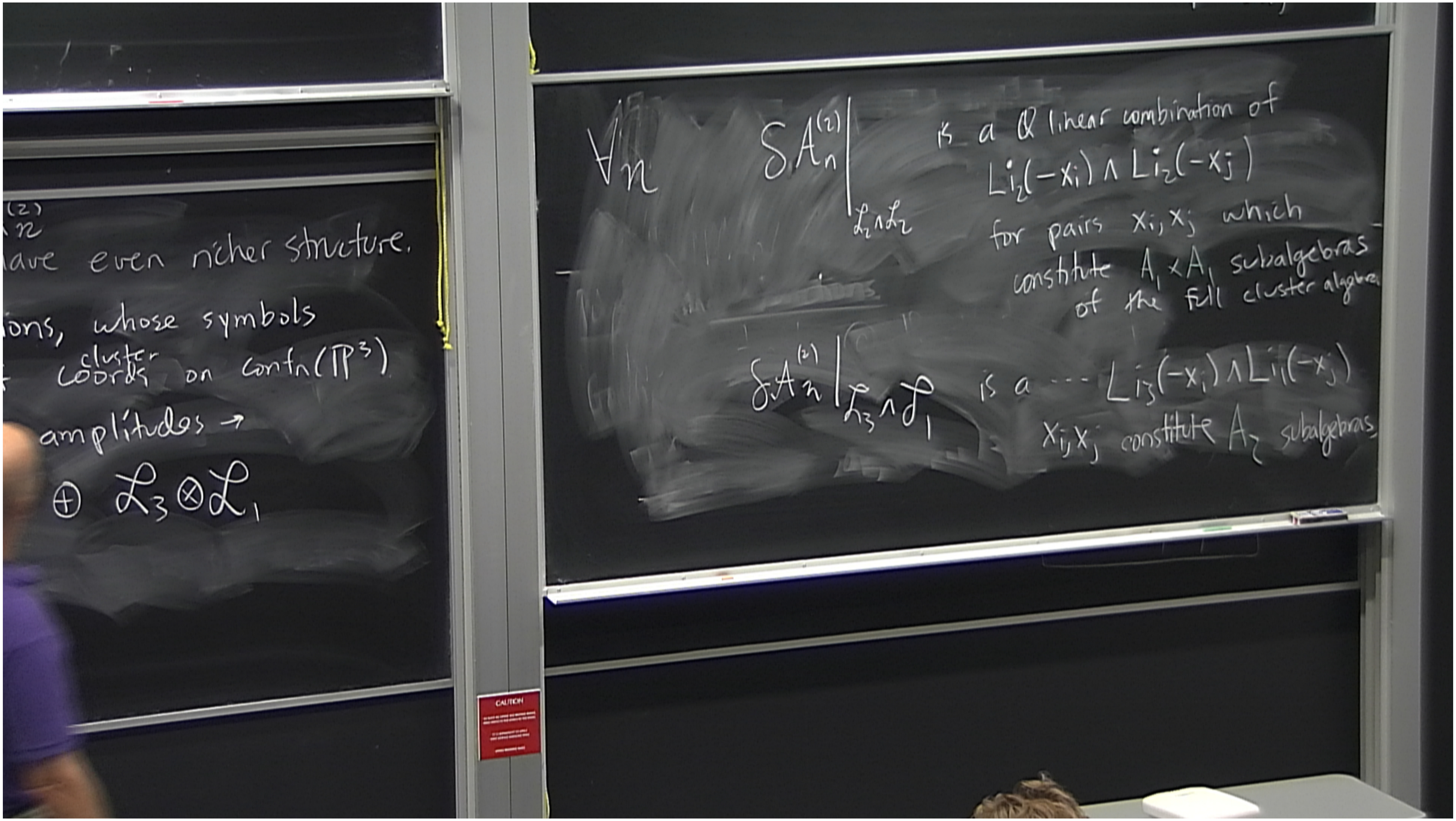
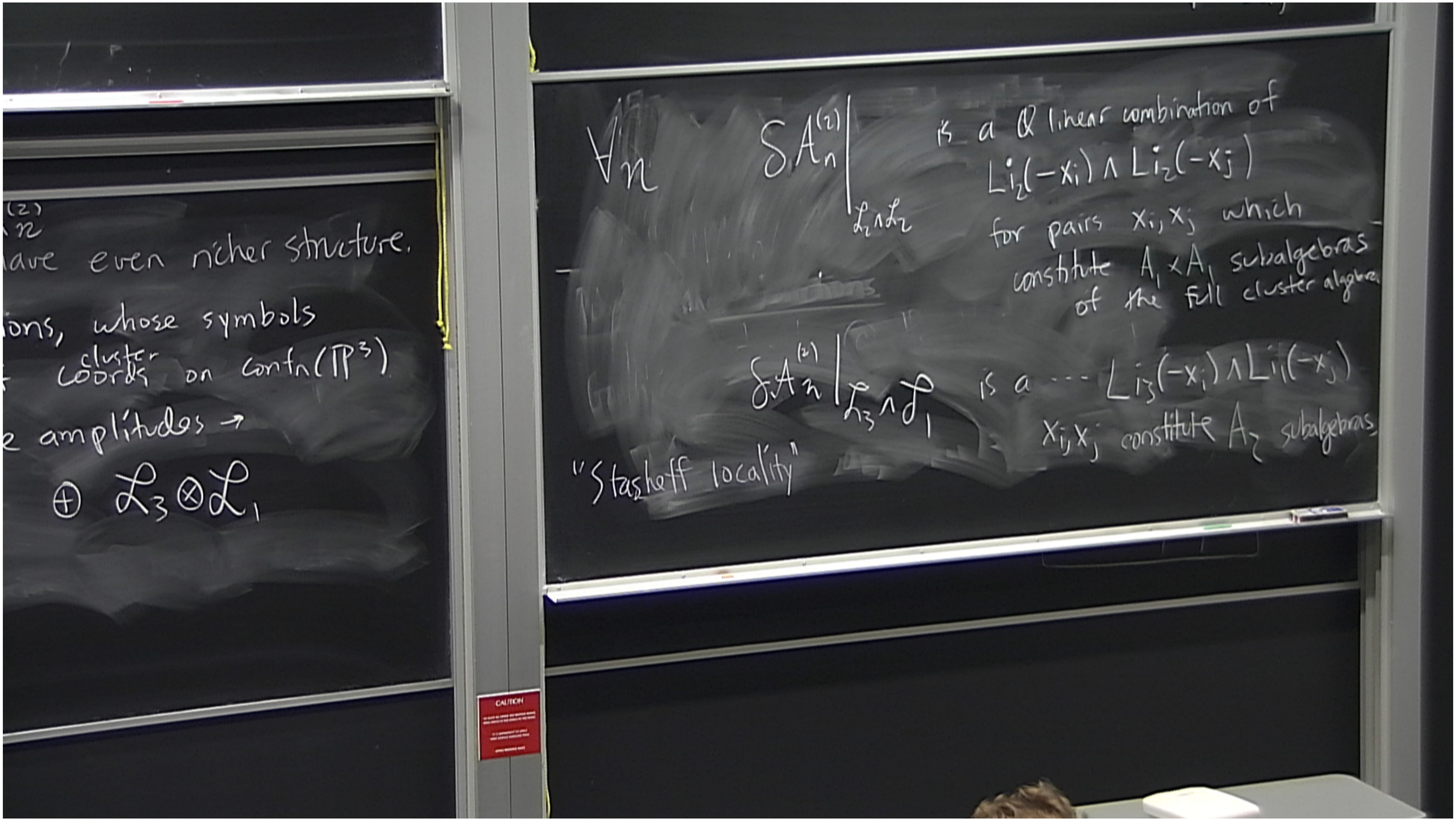
cluster coords on Conf$_n$($\mathbb{P}^3$)
e amplitudes →
⊕ $\mathcal{L}_3 \otimes \mathcal{L}_1$
∀n $\delta A_n^{(2)}\big|_{\mathcal{L}_2 \wedge \mathcal{L}_2}$ is a ℚ linear combination of
$Li_2(-x_i) \wedge Li_2(-x_j)$
for pairs x_i, x_j which
constitute $A_1 \times A_1$ subalgebras
of the full cluster algebra
$\delta A_n^{(2)}\big|_{\mathcal{L}_3 \wedge \mathcal{L}_1}$ is a ··· $Li_3(-x_i) \wedge Li_1(-x_j)$
x_i, x_j constitute A_2 subalgebras
CAUTION


(2)
n
have even richer structure.
ions, whose symbols
cluster coords on Conf_n(P^3)
amplitudes →
⊕ L_3 ⊗ L_1
∀n δA_n^(2)| L_2∧L_2 is a Q linear combination of
Li_2(−x_i) ∧ Li_2(−x_j)
for pairs x_i, x_j which
constitute A_1 × A_1 subalgebras
of the full cluster algebra
δA_n^(2)| L_3∧L_1 is a ··· Li_3(−x_i) ∧ Li_1(−x_j)
x_i, x_j constitute A_2 subalgebras

$^{(2)}$
n
...ave even richer structure.
...ions, whose symbols
... cluster coords on $\mathrm{Conf}_n(\mathbb{P}^3)$
...e amplitudes →
$\oplus\ \mathcal{L}_3 \otimes \mathcal{L}_1$

$\forall n \quad \delta A_n^{(2)}\Big|_{\mathcal{L}_2 \wedge \mathcal{L}_2}$ is a $\mathbb{Q}$ linear combination of $\mathrm{Li}_2(-x_i) \wedge \mathrm{Li}_2(-x_j)$ for pairs x_i, x_j which constitute $A_1 \times A_1$ subalgebras of the full cluster algebra

$\delta A_n^{(2)}\Big|_{\mathcal{L}_3 \wedge \mathcal{L}_1}$ is a $\cdots$ $\mathrm{Li}_3(-x_i) \wedge \mathrm{Li}_1(-x_j)$ x_i, x_j constitute A_2 subalgebras



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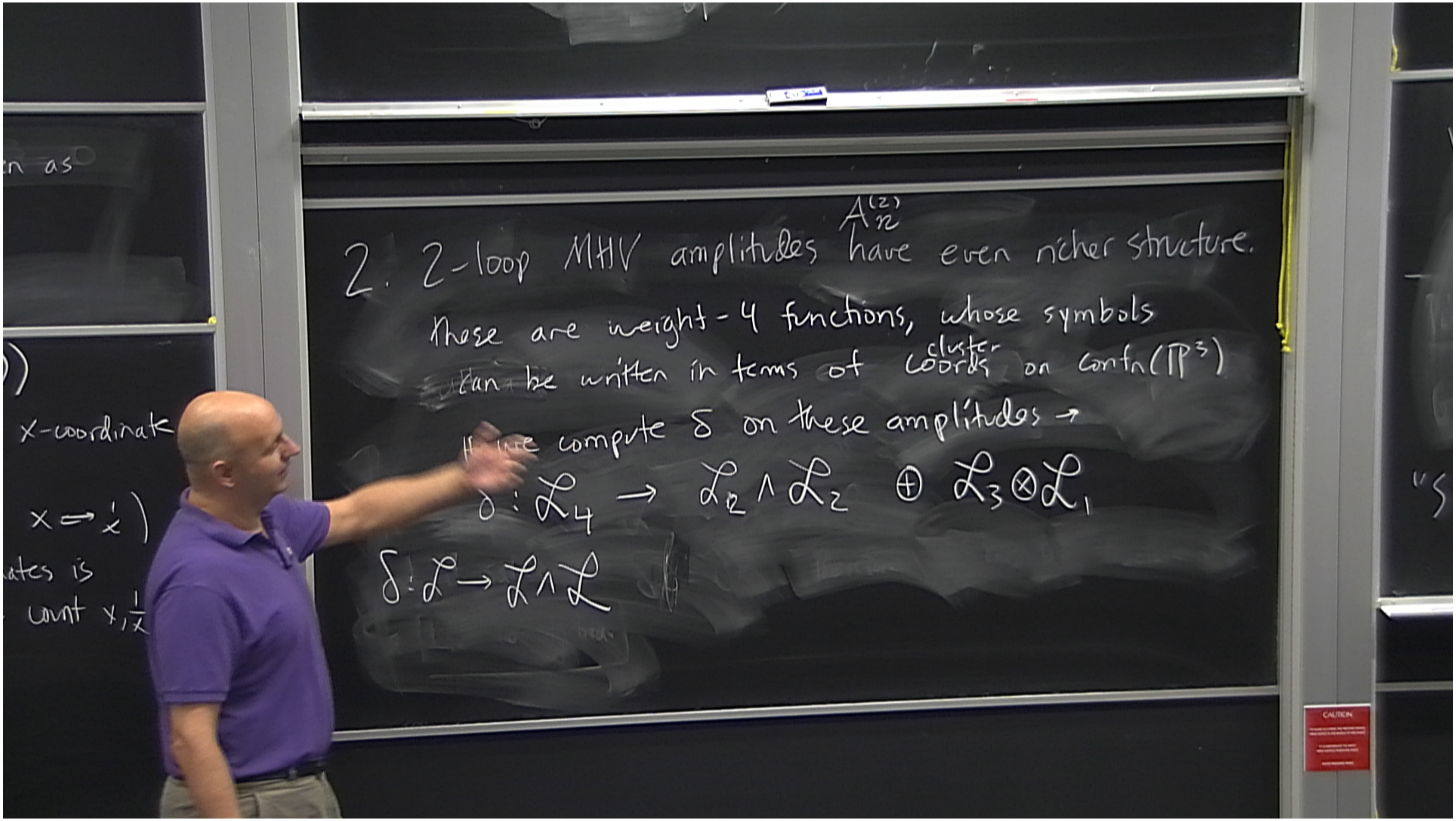
x_i, x_j constitute A_2 subalgebras
"Stasheff locality"


2. 2-loop MHV amplitudes $A_n^{(2)}$ have even richer structure.
These are weight-4 functions, whose symbols
can be written in terms of cluster coords on Conf$_n(\mathbb{P}^3)$
If we compute δ on these amplitudes →
$\delta : \mathcal{L}_4 \to \mathcal{L}_2 \wedge \mathcal{L}_2 \oplus \mathcal{L}_3 \otimes \mathcal{L}_1$
$\delta : \mathcal{L} \to \mathcal{L} \wedge \mathcal{L}$
x-coordinate
ates is
count x, 1/x