Title: Cluster Algebras and Scattering Amplitudes

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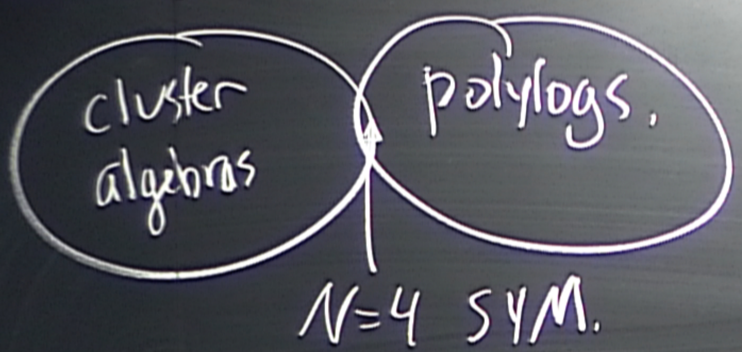
Abstract: Supersymmetric gauge theory computes a very special class of (generalized) polylogarithm functions known as scattering amplitudes that have remarkable mathematical properties. In particular, there is a rich connection between these amplitudes and the G(4,n) Grassmannian cluster algebra. To explain this connection I will review some basic facts about the Hopf algebra of polylogarithms and cluster Poisson varieties. I will then define cluster polylogarithm functions which roughly speaking are polylogarithm functions whose arguments are cluster X-coordinates of some cluster algebra A. I will describe an additional property of certain scattering amplitudes, that they are "local" in the algebra A, and describe the classification of cluster polylogarithm functions with this property. The computation of new amplitudes can be greatly aided by knowledge of the class of functions in terms of which they may be expressed, as I will illustrate via an example.

rov, Golden, Paulos, Vergu, Volovich



Theme: (some) scattering amps. in $\mathcal{N}=4$ have "cluster structure"

compute $\longrightarrow$ analyze $\longrightarrow$ identify "principles" $\longrightarrow$ compute (and aud

Teaser

A_2 cluster algebra

generated by x_i satisfying $\boxed{x_{i+1} = \frac{1+x_i}{x_{i-1}}}$

$$x_1,\ x_2,\ x_3 = \frac{1+x_2}{x_1},\ x_4 = \frac{1+\frac{1+x_2}{x_1}}{x_2} = \frac{1+x_1+x_2}{x_1 x_2}$$

$$x_5 = \frac{1+x_1}{x_2},\quad x_6 = \frac{1+\frac{1+x_1}{x_2}}{\frac{1+x_1+x_2}{x_1 x_2}} = x_1,\ x_7 = x_2, \text{ etc.}$$

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Abel 1881 $$\sum_{i=1}^{5} \mathrm{Li}_2(-x_i) \approx 0$$

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\,dt$$

$\approx$ means modulo π^2, $\log^2$ terms.

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2013 Goncharov et al discovered $\sum_{i=1}^{40} Li_3(-x_i) = 0$

$x \in$ cluster variables of D_4 algebra.

SYM theory gives many non-zero "cluster functions"

Scattering Amplitudes

discrete labels $\Rightarrow$

n = # of particles

L = loop order $\quad L = 0, 1, 2, \ldots$

$k = 0, \ldots, n-4$ specifies the types of interacting particles.

interacting particles.

A scattering amplitude is a function of the n energy & momentum vectors $p_i = (p^0, p^1, p^2, p^3) \in \mathbb{R}^{1,3}$, $i = 1, \ldots, n$

satisfying $p_i^2 = 0 \;\; \forall i$ with respect to $(-1, +1, +1, +1)$ metric.

& $\sum_{i=1}^{n} p_i = 0$ total energy & momentum conservation.

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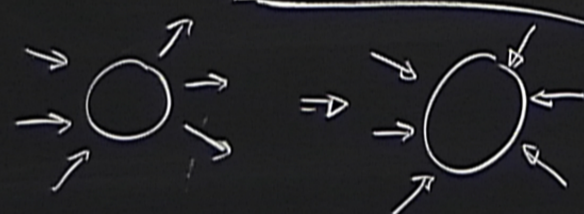
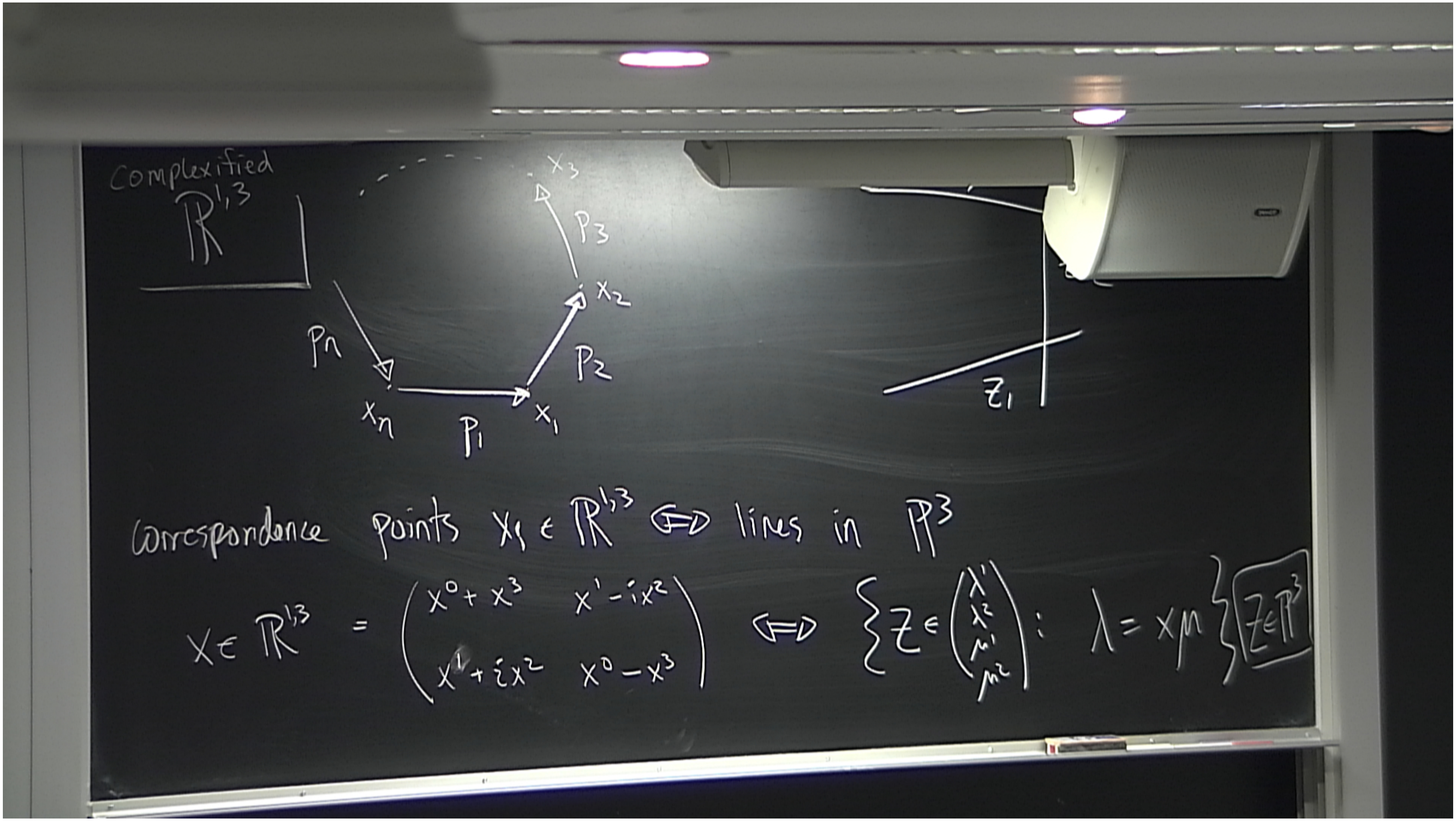
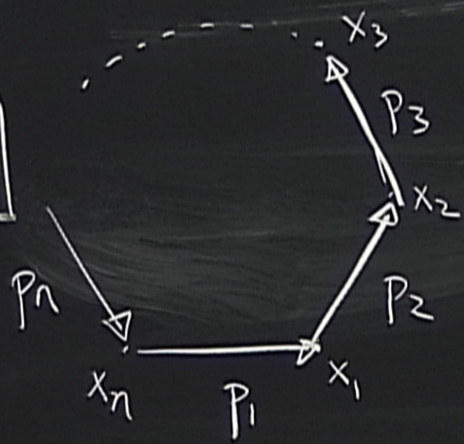


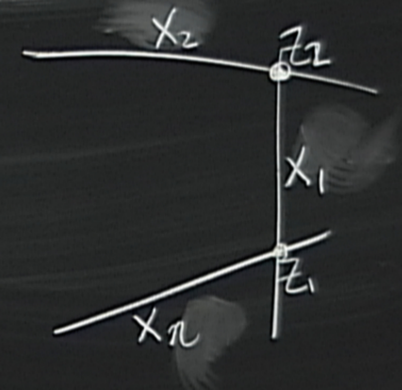
its always convenient to allow complexified momenta.


Complexified
$\mathbb{R}^{1,3}$
x_3
p_3
x_2
p_2
p_n
x_n
p_1
x_1
Z_1
correspondence points $x_i \in \mathbb{R}^{1,3} \Leftrightarrow$ lines in $\mathbb{P}^3$
$X \in \mathbb{R}^{1,3} = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix} \Leftrightarrow \left\{ Z \in \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \mu^1 \\ \mu^2 \end{pmatrix} : \lambda = x\mu \right\}$ $\boxed{Z \in \mathbb{P}^3}$

We have a nontrivial submanifold of $\mathbb{C}^{4n}$:

complexified $\mathbb{R}^{1,3}$





Correspondence points $x_i \in \mathbb{R}^{1,3} \Leftrightarrow$ lines in $\mathbb{P}^3$

$$x \in \mathbb{R}^{1,3} = \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{pmatrix} \qquad \left\{ Z \in \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \mu^1 \\ \mu^2 \end{pmatrix} : \lambda = x\mu \right\} \boxed{Z \in \mathbb{P}^3}$$

The lines in $\mathbb{P}^3$ pairwise intersect, ie Z_i intersects Z_{i+1} & Z_{i-1} $\forall i$

suppose that some Z lies on the line corresponding to X_i & the line X_{i+1}

that means $Z = \begin{pmatrix}\lambda \\ \mu\end{pmatrix}$

$$\lambda = X_i \mu$$
$$\lambda = X_{i+1} \mu$$

$$0 = \underbrace{(X_i - X_{i+1})}\mu$$

requires $(X_i - X_{i+1})$ to have det 0.

$\Longrightarrow$ $X_i - X_{i+1}$ is null

One more ingredient: whenever we talk about an amplitude in (planar) SYM, we mean the coefficient of some particular term prop. to $\mathrm{Tr}[T^{a_1} \cdots T^{a_n}]$

$T^a \in$ Lie algebra of the gauge group.

group.

So we can assemble the n points $Z_i \in \mathbb{P}^3$ into

$$\underbrace{\left(\begin{array}{ccc} | & & | \\ Z_1 & \cdots & Z_n \\ | & & | \end{array}\right)}_{n} \Big\} 4$$

"magical" symmetry of SYM (dual conformal symmetry) = left-multiplication by $SL(4,\mathbb{C})$.

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$\boxed{\text{Kinematic domain for SYM theory}}$

$$Gr(4,n)/(\mathbb{C}^*)^{n-1} \cong \boxed{\mathrm{Confn}(\mathbb{P}^3).}$$

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$\boxed{\text{Kinematic domain for SYM theory}}$

$$Gr(4,n)/(\mathbb{C}^*)^{n-1} \cong \boxed{\mathrm{Conf}_n(\mathbb{P}^3)}.$$

manifold of dimension $3(n-5)$

Coordinates on this space.

$$\langle ijkl \rangle = \det \begin{pmatrix} | & | & | & | \\ z_i & z_j & z_k & z_l \\ | & | & | & | \end{pmatrix}$$

cross-ratios, e.g. $$x_1 = \frac{\langle 1234 \rangle \langle 3456 \rangle}{\langle 1346 \rangle \langle 2345 \rangle}$$

$$x_2 = \frac{\langle 1456 \rangle \langle 2346 \rangle}{\langle 1246 \rangle \langle 3456 \rangle}, \quad x_3 = \frac{\langle 1346 \rangle \langle 1256 \rangle}{\langle 1236 \rangle \langle 1456 \rangle}$$

x_i are coordinates on $\mathrm{Conf}_6(\mathbb{P}^3)$

Aside

Claim (conjecture) you may see something like

$$\mathrm{Li}_4\left(\frac{\langle 1256\rangle\langle 2578\rangle\left(\langle 1237\rangle\langle 4568\rangle - \langle 1238\rangle\langle 4567\rangle\right)}{\langle 1237\rangle\langle 1258\rangle\langle 2456\rangle\langle 5678\rangle}\right)$$

you'll never see this with exchanged.

$$\mathrm{Li}_k(z) = \int_0^z \frac{dt}{t}\,\mathrm{Li}_{k-1}(t)$$

$$\mathrm{Li}_1(z) = -\log(1-z).$$

Cluster Algebras

Cluster Poisson Varieties Fock & Goncharov.

algebras generated by a preferred set of "cluster coordinates" grouped into "clusters"

clusters can be represented by quivers.

$$x_1 \longrightarrow x_2 \longrightarrow x_3$$

$\exists$ a "mutation rule" to generate new clusters/quivers

$\boxed{\text{example}}$

conservation.

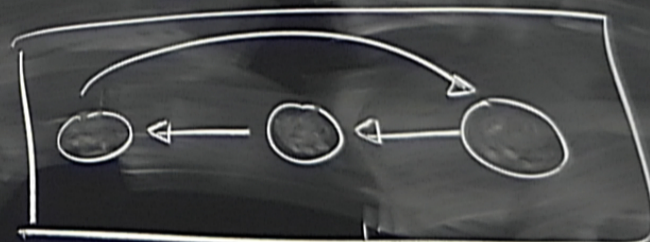
First, pick some node j to "mutate on":

2nd, for each path $i \to j \to k$, add an arrow $i \to k$

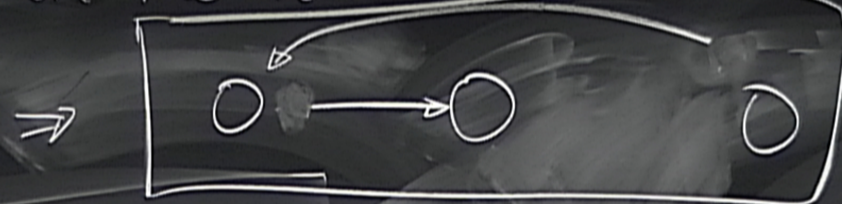
3rd, reverse all arrows into or out of j

4th, delete all 2-cycles.

Example Let's mutate $(x_1) \to (x_2) \to (x_3)$ on x_2



Now mutate on the leftmost node



I need to explain how the quiver labels transform under mutation.

Let B_{ij} = # of arrows (with sign) from $i \to j$

ex. $x_1 \longrightarrow x_2 \longrightarrow x_3$ $\qquad B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

$\langle ijkl \rangle = \det \begin{pmatrix} \cdots \\ 1 & 1 & 1 & 1 \end{pmatrix}$

if we mutate the collection $(x_1, x_2 \dots)$ on x_j then

$$x_i' = \begin{cases} 1/x_i & \text{if } i = j \\ x_i\left(1 + x_j^{\operatorname{sign} B_{ij}}\right)^{B_{ij}} & \text{if } i \neq j \end{cases}$$

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B' = the "B-matrix" associated to the new, mutated quiver.

The point on a cluster Poisson variety with

$$\{\log X_i, \log X_j\} = B_{ij}$$

(That means, you choose a collection of log-canonical coordinates on your cluster Poisson variety).

You can check that $\{\log X'_i, \log X'_j\} = B'_{ij}$ under any mutation

Cluster Algebras   Cluster Poisson Varieties   Fock &

∃ Classification of finite cluster algebras "of Grassmannian type"
$\mathrm{Conf}_n(\mathbb{P}^k)$.

finite = # of distinct quivers is finite.

& the number of distinct coords is finite.

finite mutation type = # of distinct quivers is finite is finite

of " coordinates is infinite

Now

I need

let

ex.

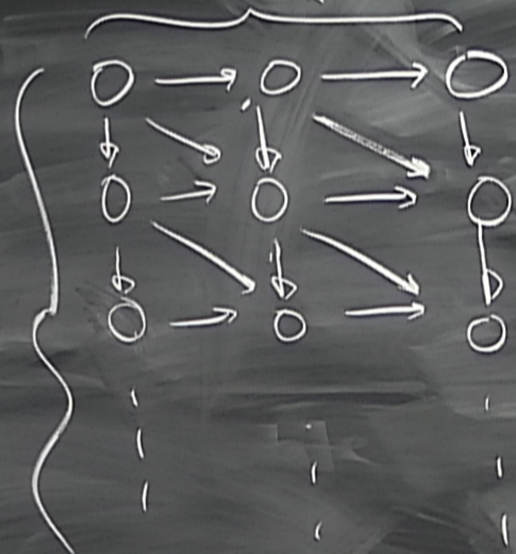
an algebra is finite iff it has quivers which, via mutation, can be brought to the form of a Dynkin diagram of one of the Lie algebras.

<u>example</u> $\mathrm{Conf}_6(\mathbb{P}^3)$ $\quad x_1 \longrightarrow x_2 \longrightarrow x_3 \quad A_3$

14 distinct quivers,

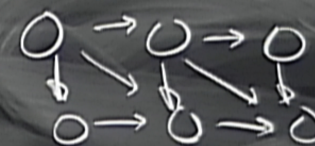
30 " cluster coordinates.

The $\mathrm{Conf}_n(\mathbb{P}^k)$ can be generated from the seed quiver



n-k-2 rows

$\mathrm{Conf}_7(\mathbb{P}^3)$ relevant to 7-particle amplitudes



by a sequence of mutations you can reach the E_6 Dynkin diagram 833 quivers 770 variables

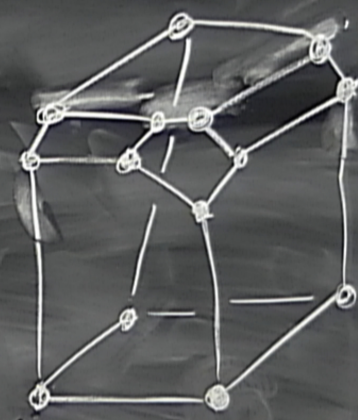
ex. $x_1 \longrightarrow x_2 \longrightarrow x_3$

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

506 distinct quivers

∞ variables.

It's useful to visualize
the combinatorics of
mutation as
a polytope



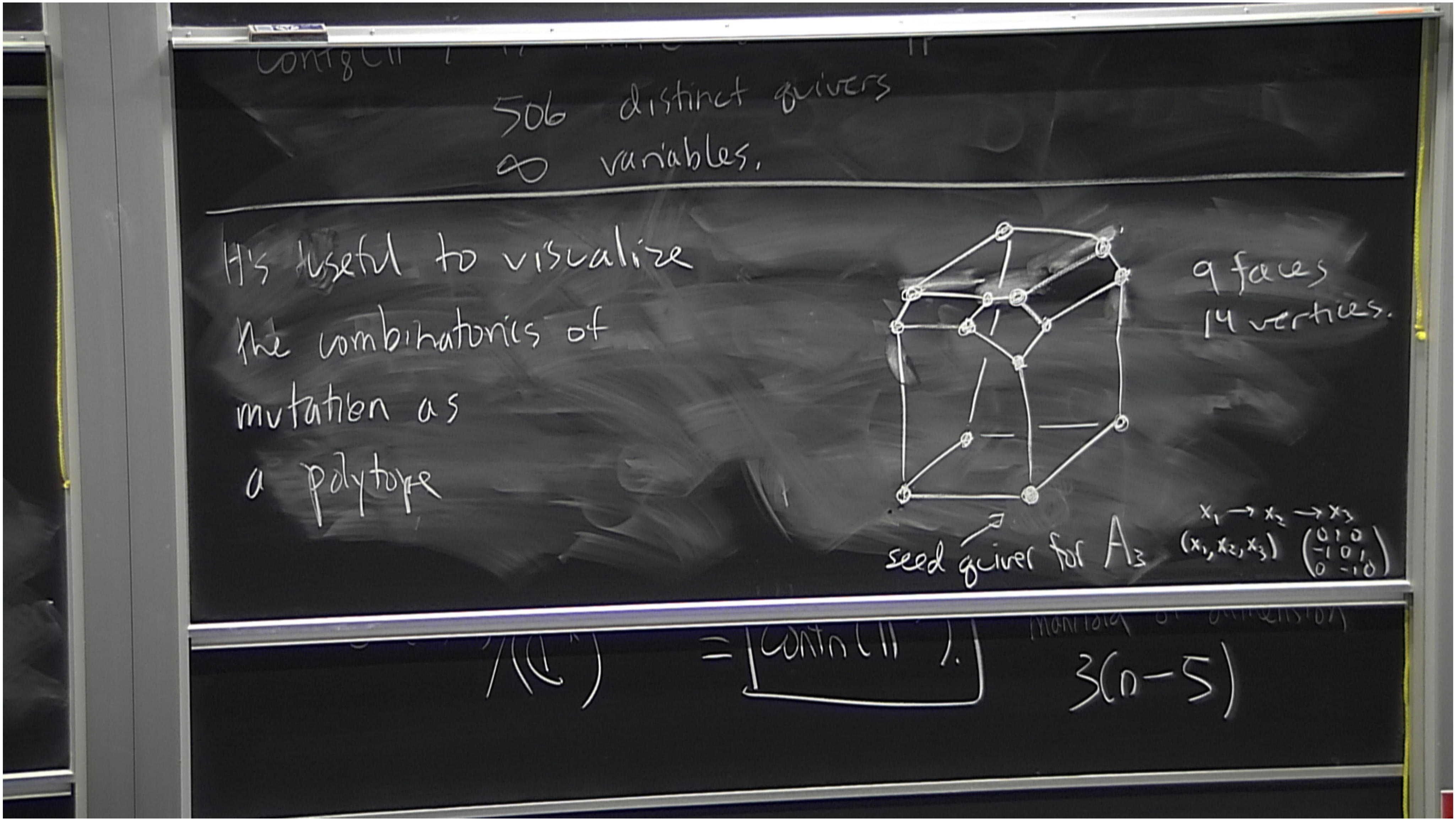
seed quiver for A_3

$x_1 \to x_2 \to x_3$

(x_1, x_2, x_3) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

$= \boxed{\mathrm{Conf}_n(\ldots)}.$

$3(n-5)$


506 distinct quivers
∞ variables.
It's useful to visualize
the combinatorics of
mutation as
a polytope
9 faces
14 vertices.
x1 → x2 → x3
(x1, x2, x3)
seed quiver for A3
3(n−5)