Title: Buildings, WKB analysis, and spectral networks.

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Abstract: Buildings are higher dimensional analogues of trees. The goal of these lectures is to explain how the theory of harmonic maps to buildings affords a new perspective on certain aspects of the WKB analysis of differential equations that depend on a small parameter. We will also touch upon some motivation for developing this perspective, which derives from questions about compactifications of higher TeichmÃ¼ller spaces, stability in Fukaya categories, and the work of Gaiotto, Moore and Neitzke on spectral networks and wall-crossing phenomena. These talks are based on joint work with Ludmil Katzarkov, Alexander Noll and Carlos Simpson.

A central role in our discussion will be played by the notion of a versal pre-building associated with a given spectral cover of a Riemann surface. This notion generalizes to higher rank the leaf space of the foliation defined by a quadratic differential. We will see that spectral networks are closely related to the singular loci of versal buildings, and that distances in these buildings encode information about the asymptotic behavior at infinity of the Riemann-Hilbert correspondence.

Yesterday, we discussed compactifications of the character variety (after Thurston, Morgan-Shalen, Parreau...), and the moduli space of flat connections (after Hausel).

We intoduced the notion of WKB exponents, and discussed their relevance to the problem of compactifying the Riemann-Hilbert correspondence between these moduli spaces.

Today, we will discuss the problem of determining WKB exponents for connections on bundles of arbitrary rank.

Riemann-Hilbert Asymptotics

Recall our general set-up:

- X — Riemann surface
- $\mathcal{E}$ — rk. r hol. vector bundle on X; $\sigma : \wedge^r \mathcal{E} \simeq \mathcal{O}_X$
- ∇_0 — holomorphic connection
- $\varphi \in \mathrm{End}(\mathcal{E}) \otimes \Omega^1_X$ — End-valued 1-form (Higgs field); $\mathrm{tr}(\varphi) = 0$.
- $\hbar = 1/t \in \mathbb{R}$ — "small parameter"

$$\rightsquigarrow \nabla_\hbar := \nabla_0 + \varphi/\hbar$$

$$\rightsquigarrow \mathrm{Trans}(\hbar) : \pi_1(X) \to SL(r, \mathbb{C})$$

Problem

Describe the asymptotic behavior of the monodromy of $\mathrm{Trans}(\hbar)$ *as* $\hbar \to 0$. *Concretely, find* ν *so that* $\|Trans_\gamma(\hbar)\| \sim e^{\nu(\gamma)/\hbar}$.

Measuring the size of monodromy: WKB exponents

Fix a hermitian metric h on $\mathcal{E}$.
$\| - \| :=$ induced operator norm on $\mathrm{Hom}(\mathcal{E}_\mathrm{P}, \mathcal{E}_\mathrm{Q})$.

Definition

For γ a path in X, the WKB exponent $\nu(\gamma)$ is defined by

$$\nu(\gamma) = \limsup_{\hbar \to 0} \hbar \log \|\mathrm{Trans}_\gamma(\hbar)\|$$

More generally, we have the WKB dilation spectrum $\vec{\nu}$:

$$\vec{\nu}(\gamma) = (\alpha_1, ..., \alpha_r)$$

uniquely characterized by

$$\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_r$$

and

$$\alpha_1 + ... + \alpha_k = \limsup_{\hbar \to 0} \hbar \| \wedge^k \mathrm{Trans}_\gamma(\hbar)\|$$

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The space of metrics

Let $P \in X$. Using $\nabla_\hbar$ to transport the metric to $\mathcal{E}_P \simeq \mathbb{C}^r$:

$$\begin{pmatrix} \text{Hermitian metric} \\ h \text{ on } \mathcal{E} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \pi_1(X, P)\text{-equivariant map} \\ \tilde{h}_\hbar : \tilde{X} \to SL(\mathcal{E}_P, \sigma)/SU(\mathcal{E}_P, h_P) \end{pmatrix}$$

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Metric/Length structures on $SL(r,\mathbb{C})/SU(r)$

- A Riemannian metric.
- A Finsler distance: $d_{\mathrm{Fins}}(H,K) = |(\log \sup_{\|v\|_H=1} \|v\|_K)|$
- A vector valued distance: $\overrightarrow{d}(H,K) = (\log \lambda_1, ..., \log \lambda_r)$

Here λ_i = "dilation exponents": $\|e_i\|_H = \lambda_i \|e_i\|_K$, where $\{e_i\}$ is basis orthogonal for *both* H and K.

Observation

$\tilde{\gamma}$ a lift of γ to the universal cover; $\log \|\mathrm{Trans}_\gamma\| = d_{\mathrm{Fins}}(\tilde{h}(\tilde{\gamma}(0)), \tilde{h}(\tilde{\gamma}(1)).$

WKB exponent as a distance

Proposition

Fix γ a path in X. Then (there exists an ultrafilter ω s.t.):

$$\nu(\gamma) := \limsup_{\hbar \to 0} \hbar \|\mathrm{Trans}_\gamma(\hbar)\|$$

$$= \lim_{\omega} \hbar d_{Fins}(\tilde{h}_\hbar(P), \tilde{h}_\hbar(Q)) =: d_{\mathrm{Cone}_\omega}(\tilde{h}(P), \tilde{h}(Q))$$

where P, Q are endpoints of a lift of γ to $\tilde{X}$;
$\mathrm{Cone}_\omega := \mathrm{Cone}_\omega(SL(r, \mathbb{C})/SU(r))$

$\mathrm{Cone}_\omega(Y, d, *)$ (the asymptotic cone) = Gromov-Hausdorff limit of the sequence of rescaled metric spaces $(Y, \hbar_n d)$, $\hbar_n \to 0$.

Asymptotic Cones: looking from infinity

Definition

The asymptotic cone relative to ω is defined by:

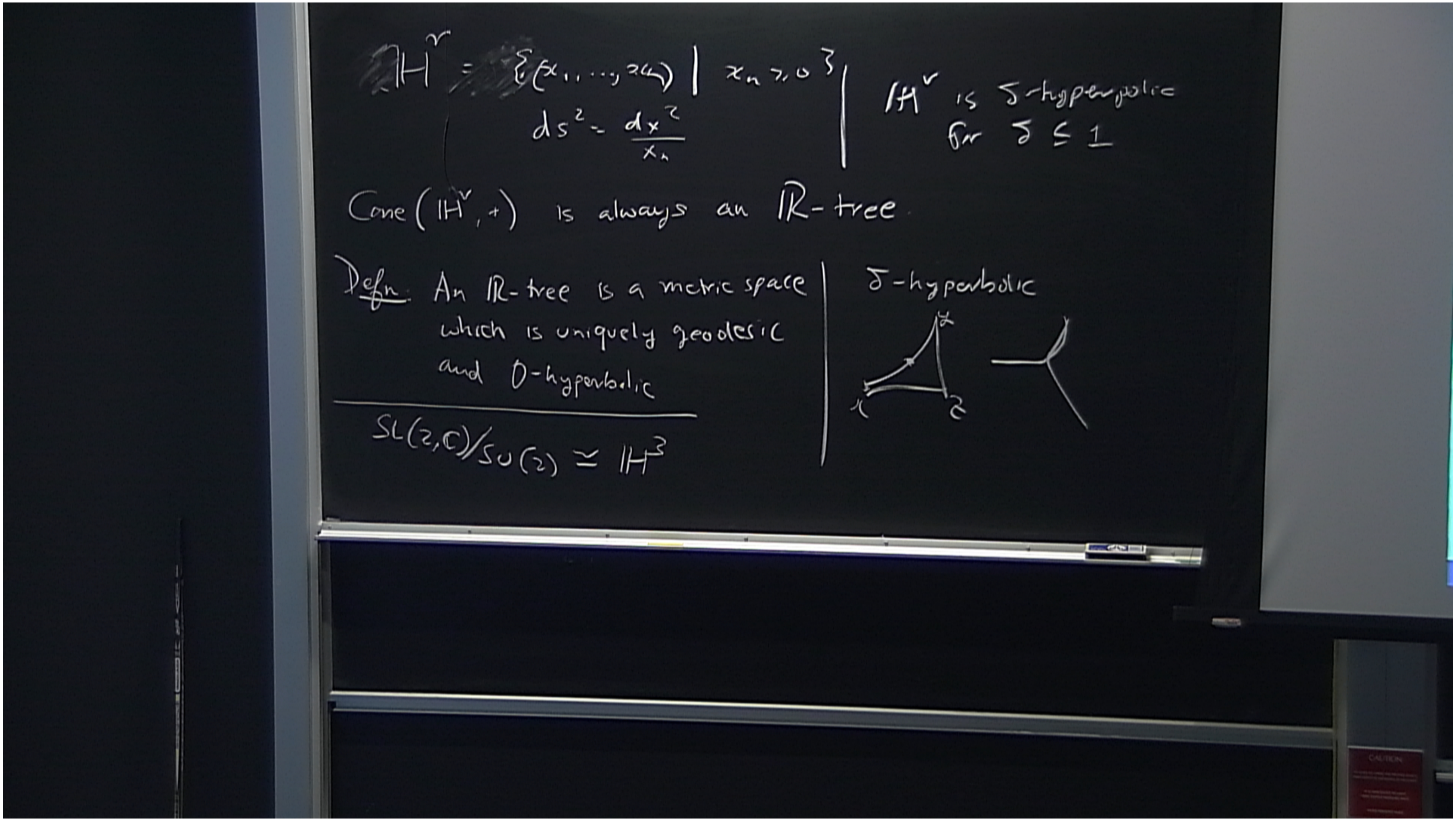
- $\mathrm{Cone}_\omega(Y, d, *) = \{\{y_n\} | \hbar_n d(y_n, *) \text{ is bounded } \}/\sim$
- $\{x_n\} \sim \{y_n\}$ iff $\lim_\omega \hbar_n d(x_n, y_n) = 0$
- $d_{\mathrm{Cone}_\omega}([\{x_n\}], [\{y_n\}]) = \lim_\omega \hbar_n d(x_n, y_n)$

Example

$\mathrm{Cone}(\mathbb{Z}^2) = \mathbb{R}^2$, where both sides have, say, the taxicab metric.

Example

$\mathrm{Cone}\{(x, y) \in \mathbb{R}^2 | \ y \geq x^2\}$ = non-negative part of y-axis.



$\mathbb{H}^n = \{(x_1, \dots, x_n) \mid x_n \geq 0\}$
$ds^2 = \frac{dx^2}{x_n}$
$\mathbb{H}^n$ is δ-hyperbolic for $\delta \leq 1$
Cone$(\mathbb{H}^n, +)$ is always an $\mathbb{R}$-tree
Defn. An $\mathbb{R}$-tree is a metric space which is uniquely geodesic and 0-hyperbolic
$SL(2,\mathbb{C})/SU(2) \simeq \mathbb{H}^3$
δ-hyperbolic
x
y
z

1. What extra geometric structure does $\mathrm{Cone}_\omega := \mathrm{Cone}_\omega(SL(r,\mathbb{C})/SU(r))$ have?

2. Does the entire dilation spectrum $\overrightarrow{\nu}$ have an interpretation in terms of the geometry of Cone_ω?

3. Does the limiting map $\tilde{h} : \tilde{X} \to \mathrm{Cone}_\omega$ enjoy any special properties?

It turns out that the answer lies in the geometry of flat submanifolds of the symmetric space, to which we turn next.

Apartments in symmetric spaces

Let $S \simeq G/K$ be a symmetric space of non-compact type.
E.g. $SL(r,\mathbb{C})/SU(r)$

Definition

An *apartment* A in S is a maximal totally geodesic submanifold of S that is isometric to a Euclidean space.

Apartments through $x \in S \longleftrightarrow \exp(\mathfrak{a})$ for a maximal abelian $\mathfrak{a} \subset \mathfrak{p}$.
Rank of symmetric space := dimension of any apartment

Example

In a rank 1 symmetric space like $SL(2,\mathbb{C})/SU(2)$, apartments = geodesics.

Theorem

1. *For any x, $y \in S$, there exists an apartment A containing x and y.*
2. *For any apartments A, A', their intersection is closed and convex in A (resp. A'), and there is an isometry $A \to A'$ fixing $A \cap A'$.*

Abstract Apartments = Coxeter Complexes

Apartments have extra structure preserved by the gluing maps: an action of the affine Weyl Group (specified by generating reflection hyperplanes)

$$W_{aff} \simeq W_{sph} \ltimes T$$

where $T \subseteq \mathbb{R}^{r-1}$ is a translation subgroup.

Figure : An $\mathbb{S}_2 \ltimes \mathbb{Z}$ apartment.

Red = chamber
Purple = sector

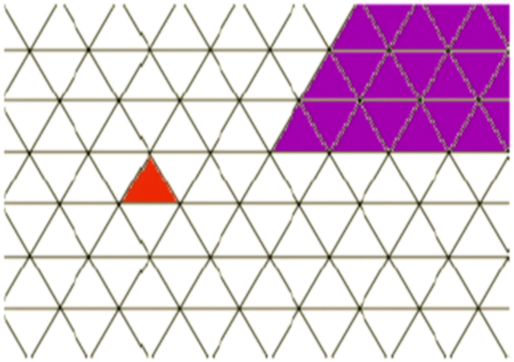


Figure : An $\mathbb{S}_3 \ltimes \mathbb{Z}^2$ apt.

Apartments in symmetric spaces

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$\mathbb{R}$-Buildings

Definition

Roughly, $\mathbb{R}$-*building* = metric space $\mathcal{B}$ + family of apartments $A \subset \mathcal{B}$ s.t.

- Every x, $y \in \mathcal{B} \implies \{x, y\}$ is contained in a single apartment A.
- For $\{x, y\} \subset A \cap A'$ there is a *Weyl group isometry* $A \to A'$ fixing $\{x, y\}$. $A \cap A'$ is convex, bounded by reflection hyperplanes.

Here x, y, are really "generalized chambers".

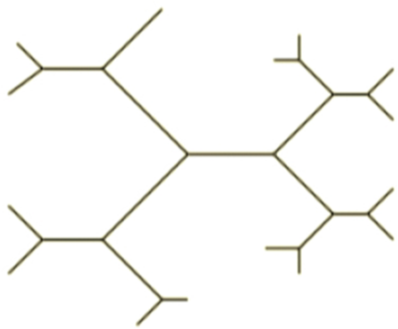


Figure : A rank 1 building = tree

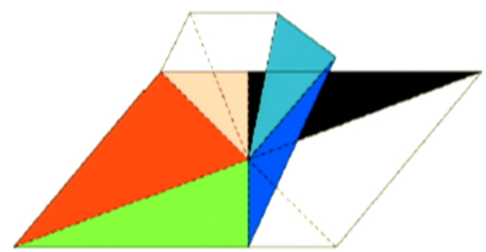


Figure : An $\tilde{A}_2$ building fragment

Cone_ω is an $\mathbb{R}$-building

Theorem (Kleiner-Leeb)

The asymptotic cone of a symmetric space is an $\mathbb{R}$-building.

Vector distance with values in $\mathbb{A}/W_{sph}$ makes sense.

$\rightsquigarrow$ WKB dilation spectrum $\overrightarrow{\nu}$ = vector distance in building Cone_ω.

$\rightsquigarrow$ Later we'll see that $\tilde{h} : \tilde{X} \to \mathrm{Cone}_\omega$ is harmonic.

Problem: Cone_ω is too big; does not depend only on spectral data (eigenvalues of φ). Need to understand better the relationship:

$$\begin{pmatrix} \text{Spectral covers} \\ \Sigma \to X \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{Equivariant harmonic} \\ \text{maps to buildings} \\ \tilde{X} \to \mathcal{B} \end{pmatrix}$$

Harmonic 1-Forms $\longleftrightarrow$ holomorphic 1-forms

$f : X \to \mathbb{R}$

harmonic

df is harmonic 1-form

$\longleftrightarrow$ holomorphic 1-form.

$f : X \to \mathbb{R}^r$

harmonic maps.

$(\lambda_1, \dots \lambda_n)$ is a sequence of holomorphic 1-forms

Spectral covers from harmonic maps

$\mathbb{A} := \{(x_1, ..., x_r) | \sum x_i = 0\} \simeq \mathbb{R}^{r-1}$

$h : X \to \mathbb{A}$ harmonic $\implies h^*(dx_i)$ harmonic 1-forms.
$\implies \lambda_i := (h^*_{\mathbb{C}}(dx_i))^{(1,0)} \in H^0(X, \omega_X)$ holomorphic; $\sum \lambda_i = 0$.

The symmetric polynomials $\sigma = (\sigma_1(dx_1, ..., dx_r), ..., \sigma_r(dx_1, ..., dx_r))$ are invariant under $W_{aff} \simeq \mathbb{S}_r \ltimes \mathbb{R}^r \curvearrowright \mathbb{A}$, and are therefore global objects on any building of type $(\mathbb{A}, W_{aff})$.

$h : X \to \mathcal{B}$ harmonic $\rightsquigarrow h^*(\sigma) =: \phi = (\phi_2, ..., \phi_r) \in \oplus_{k=2}^r H^0(X, \omega_X^k)$.
$\rightsquigarrow \Sigma_\phi := \{s | \; s^r + \phi_2 s^{r-2} + ... + \phi_r = 0\} \subset T_X^\vee$

Harmonic k-Forms $\longleftrightarrow$ holomorphic ...

$f : X \to \mathbb{R}$ harmonic

dF is harmonic 1-form $\longleftrightarrow$ holomorphic 1-form.

$f : X \longrightarrow \mathbb{R}^r \supset W_{sph}$

harmonic maps.

$\{x_1, \dots, x_n\}$ is a sequence of holomorphic 1-forms

$\mathbb{H}^n = \{(x_1, \dots, x_n) \mid x_n \geq 0\}$

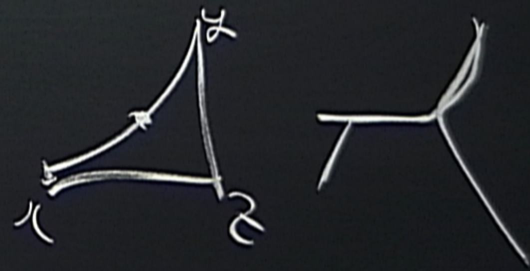
$$ds^2 = \frac{dx^2}{x_n}$$

$\mathbb{H}^n$ is δ-hyperbolic for $\delta \leq 1$

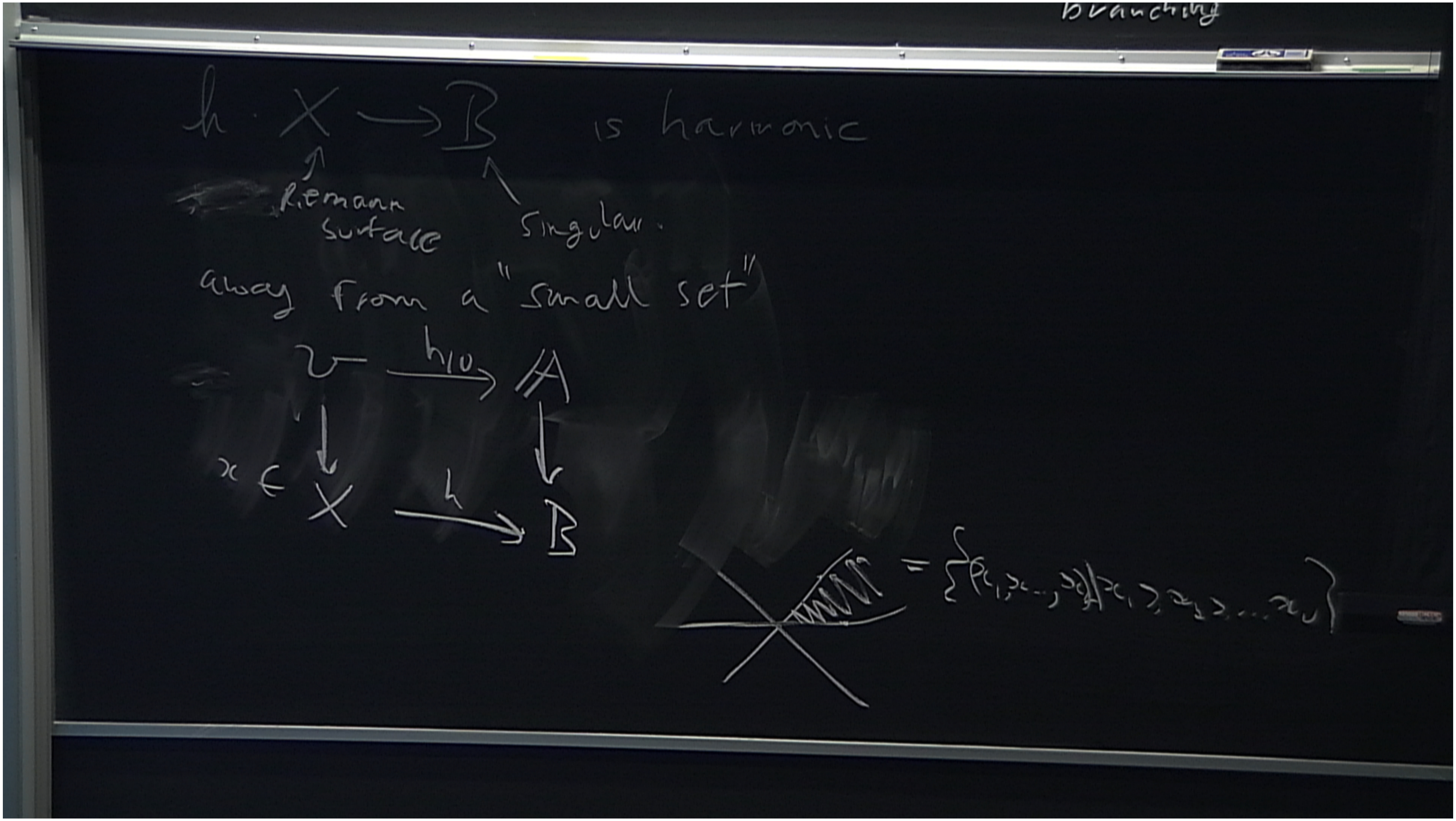
$\mathrm{Cone}(\mathbb{H}^n, +)$ is always an $\mathbb{R}$-tree.

Defn: An $\mathbb{R}$-tree is a metric space which is uniquely geodesic and 0-hyperbolic

δ-hyperbolic



$SL(2,\mathbb{C})/SU(2) \simeq \mathbb{H}^3 \Rightarrow \mathrm{Cone}(SL(2,\mathbb{C})/SU(2))$ an $\mathbb{R}$-tree with uncountable branching


branching
h: X → B is harmonic
Riemann surface
singular
away from a "small set"
h|U
x ∈ X
h
B

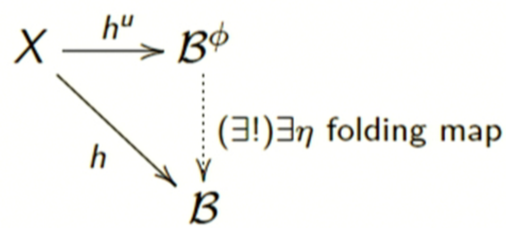
Versal buildings

Idea: Try to reverse this, and produce a harmonic map to a (pre)-building from a spectral cover.

Definition

Let $\phi \in \oplus_{k=2}^{r} H^0(X, \omega_X^k)$ be a point in the Hitchin base.

- A harmonic map $h : X \to \mathcal{B}$ is a *harmonic ϕ-map* if the h-induced spectral cover is Σ_ϕ.
- A harmonic ϕ-map $h^\phi : X \to \mathcal{B}^\phi$ is a *(uni)versal* harmonic ϕ-map if for any harmonic ϕ-map $h : X \to \mathcal{B}$



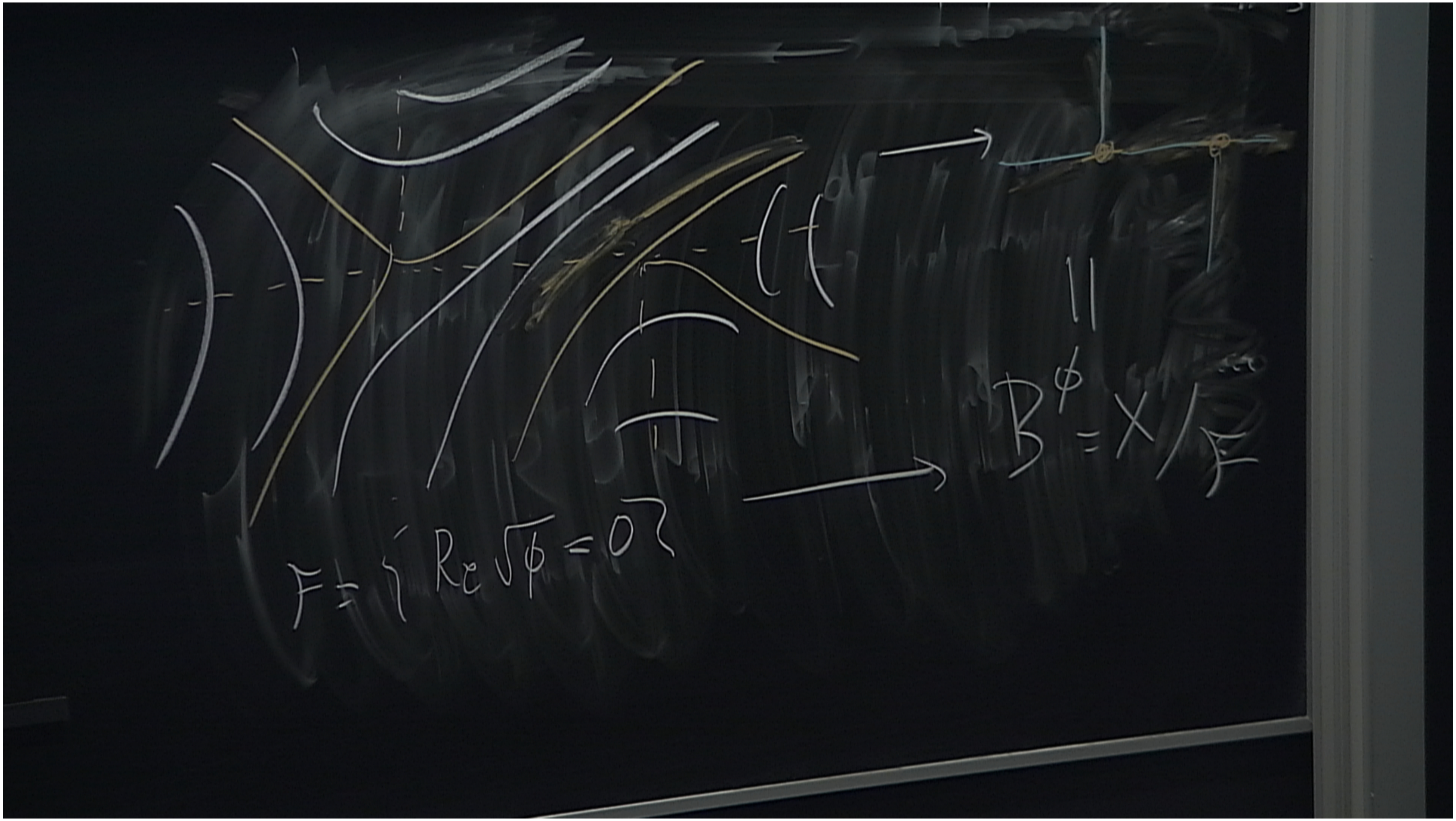
More precisely, should work with $\pi_1(X, x)$-equivariant maps $h : \tilde{X} \to \mathcal{B}$.

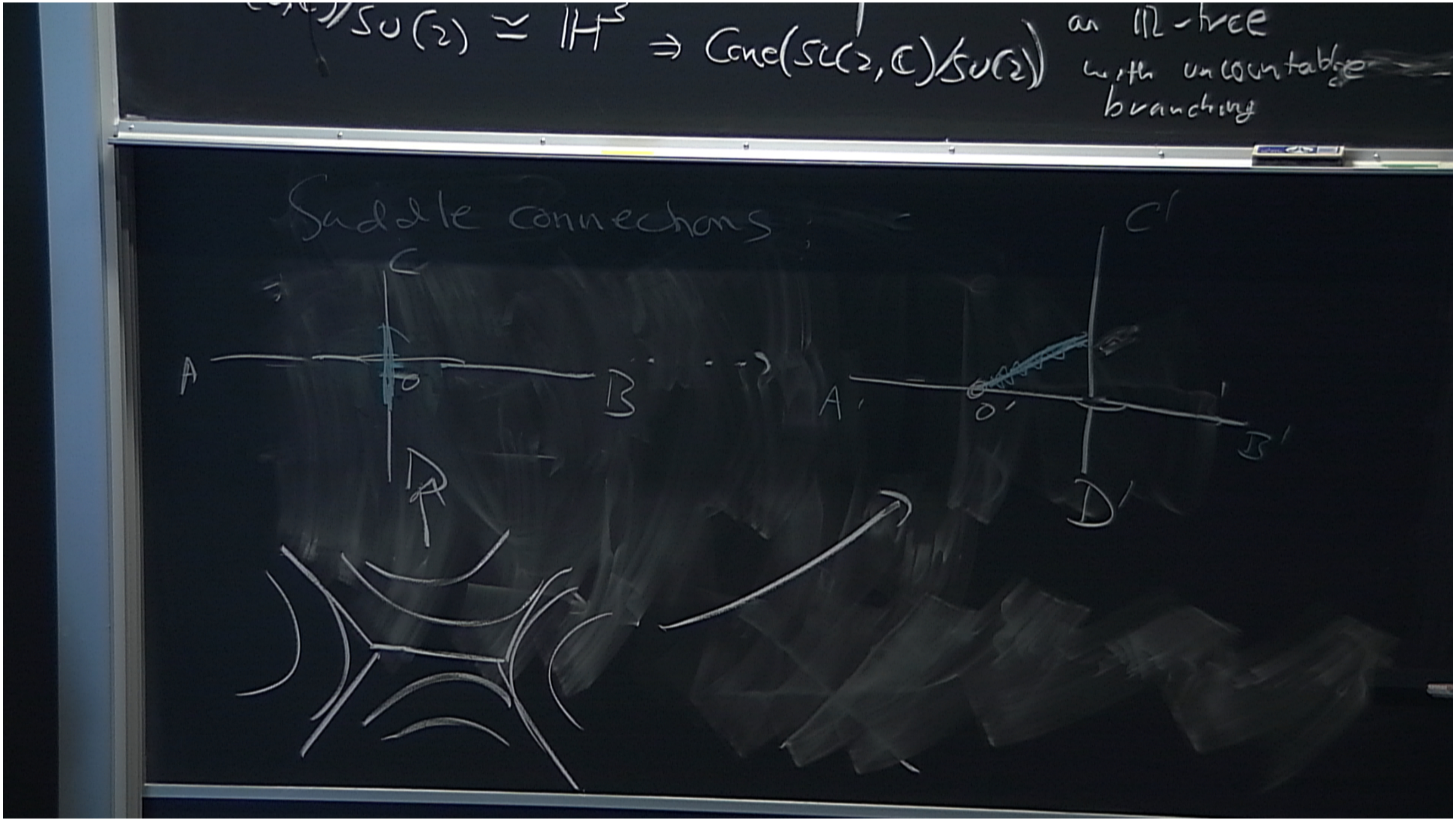
The SL_2 = rank 1 = tree case: foliations

$\phi \in$ Hitchin base $H^0(X, \omega_X^2) \leftrightarrow$ quadratic differentials.
$\rightsquigarrow$ foliation $\mathcal{F}$ defined by $Re(\sqrt{\phi}) = 0$.

Transverse measure on $\mathcal{F}$ given by $\int Re(\sqrt{(}\phi)$
$\rightsquigarrow$ metric on leaf space $X/\mathcal{F}$, an $\mathbb{R}$-tree = rank 1 $\mathbb{R}$-building.

- A versal $\mathcal{B}^\phi$ exists and is computed by the leaf space $\tilde{X}/\tilde{\mathcal{F}}$
- $\mathcal{B}^\phi$ is *universal* if there are no saddle connections.
- $\tilde{X}/\tilde{\mathcal{F}}$ computes the WKB exponents for any Riemann-Hilbert WKB problem $\nabla_0 + \varphi/\hbar$ with $\det(\varphi) = \phi$.


F = { Re √φ = 0 }
B^φ = X / F


SU(2) ≃ IH³ ⇒ Cone(SL(2,C)/SU(2))
an IR-tree
with uncountable
branching
Saddle connections:
C
A
O
B
C'
A'
O'
B'
D'

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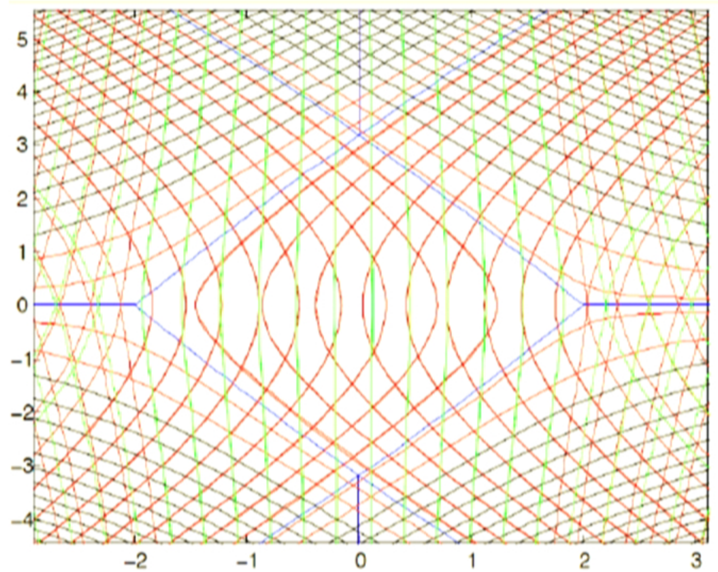
Higher rank: webs/multi-foliations



⇝ Versal Building $\mathcal{B}^u$ = proxy for the "multi-leaf-space"
Analogue of path tranverse to a foliation = non-critical path.

Back to WKB: the local picture

A path $\gamma : I \to X$ is non-critical if the eigenvalues λ_i of the Higgs field φ can be ordered along γ:

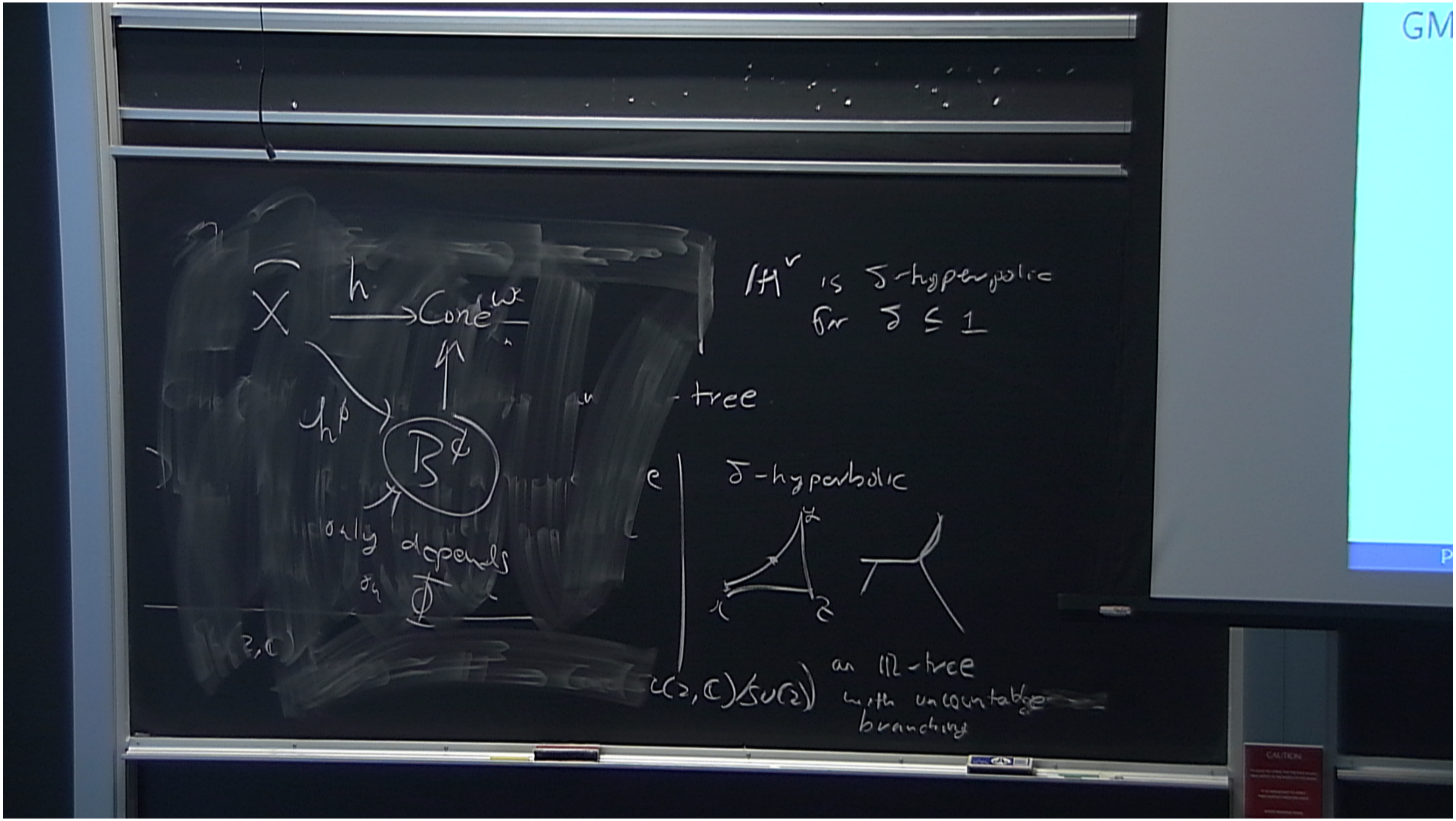
$$Re(\gamma^* \lambda_1, \partial_t) > Re(\gamma^* \lambda_2, \partial_t) > ... > Re(\gamma^* \lambda_r, \partial_t)$$

Theorem

Let γ be non-critical. Then $\vec{\nu}(\gamma) = (\int_\gamma Re\lambda_1, ..., \int_\gamma Re\lambda_r)$.

Together with "$\vec{\nu}(\gamma) =$ distance in Cone_ω", this can be used to argue that

- $h : \tilde{X} \to \mathrm{Cone}_\omega$ is a harmonic ϕ-map, where ϕ is the point in the Hitchin base (characteristic polynomial) determined by φ.
- There is an apartment in Cone_ω that contains the *entire* image of γ.


is δ-hyperbolic for δ ≤ 1
tree
δ-hyperbolic
an ℝ-tree with uncountable branching
only depends on
GM

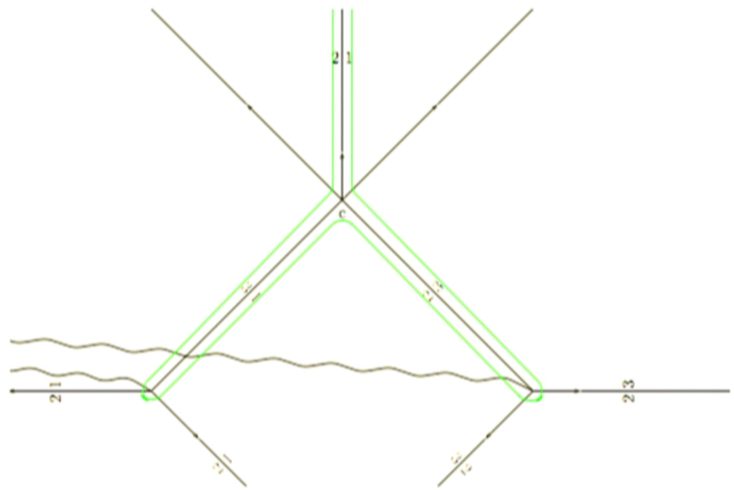
Spectral Networks

Gaiotto-Moore-Neitzke: $\Sigma \rightsquigarrow$ Spectral Networks:

- Rank 1: singular leaves of singular foliation $Re(\sqrt{\phi}) = 0$
- Higher rank: "singular leaves of singular web" $Re(\lambda_i - \lambda_j) = 0$.

i, j, k - sheets of Σ. Start at branch points , and draw ij leaves. If ij and jk collide, they generate an ik collision line.

GMN: to compute WKB exponent, modify local WKB picture by detours:



The BNR Example

Berk-Nevins-Roberts studied

$$\left(\hbar^3 \frac{d^3}{dx^3} - 3\hbar \frac{d}{dx} + x\right) \psi = 0$$

and pointed out the relevance of "collision Stokes lines" to WKB asymptotics. Spectral cover $\Sigma : p^3 - 3p + x = 0$.

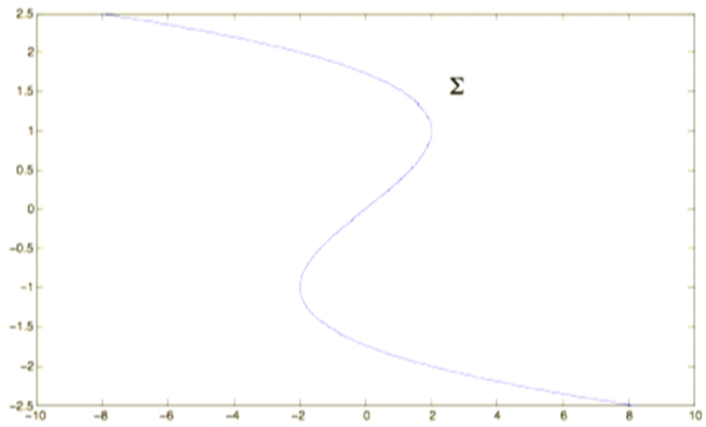


Figure : Real picture of spectral curve

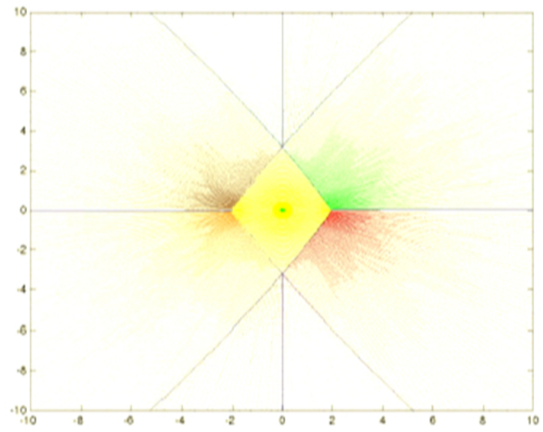


Figure : Spectral network $\mathcal{W}$ on $X = \mathbb{C}$.

A versal building exists

Let $X = \mathbb{C}$ and $\Sigma = \{(p, x) |\ p^3 - 3p + x = 0\} \subset T^\vee_X$.

Theorem

- *There exists a versal harmonic ϕ-map $h^\phi : X \to \mathcal{B}^\phi$.*
- *For any WKB problem $\nabla_0 + \varphi/\hbar$ with spectral cover of $\varphi = \Sigma$, h^ϕ computes the dilation spectrum $\overrightarrow{\nu}$.*
- *The inverse image under h of the singular locus of $\mathcal{B}^\phi$ coincides with the support of the extended spectral network.*

Extended means collision lines emanate in *both* directions from collision points.

Detours appear to be related to apartments "not seen by curve X".

Conjecture

The conclusions of the theorem are true for any smooth spectral cover Σ and any X that does not support BPS states.

Feautures that generalize to other $SL_3\mathbb{C}$ spectral covers:

- Near a collision point, the building looks like the BNR building.
- h^ϕ folds the Riemann surface along caustic curves.

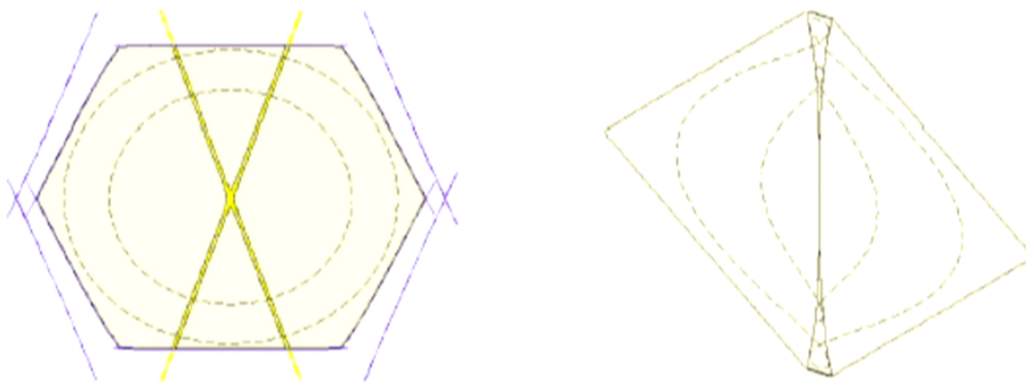


Figure : Folding along crossing caustics

The general theory for $SL_3\mathbb{C}$ (work in progress):

Enclosures = Weyl convex subets of $\mathbb{A}$

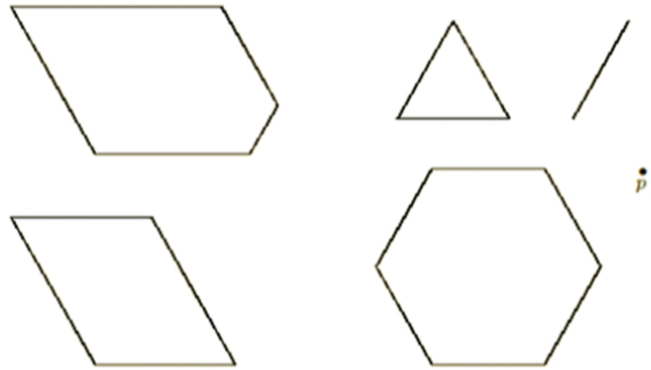


Figure : Some enclosures; $W_{aff} = \mathbb{S}_3 \ltimes \mathbb{R}^2$

These form a site with

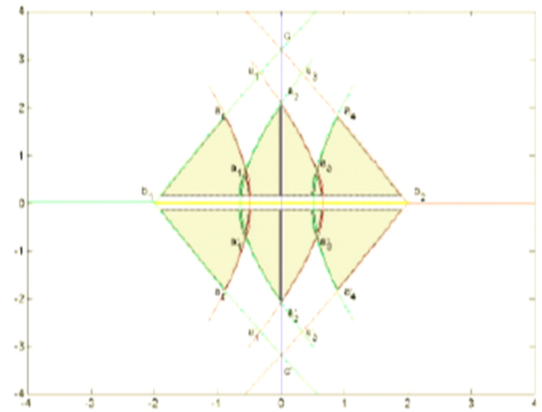
- maps = restrictions of elements of W_{aff}.
- open covers = finite collections $f_i : E_i \to E$ such that E is the union of the images of f_i

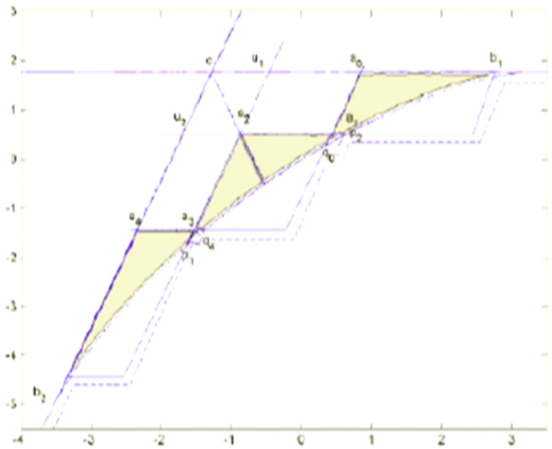
A construction is a finitely related sheaf on the site of enclosures.

Note that maps between enclosures, considered as representable sheaves, are *folding maps*; they are piece-wise affine on some finite subdivision.

Every building defines a construction in the obvious way. The category of constructions in closed under finite colimits and fiber products. It thus provides a flexible framework within which to situate any discussion involving constructions with buildings.

Given a construction F, and a point $p \in F$, the link F_x at F is a sheaf of the site of *spherical* enclosures. For $SL_3\mathbb{C}$, this is just a bipartite graph.



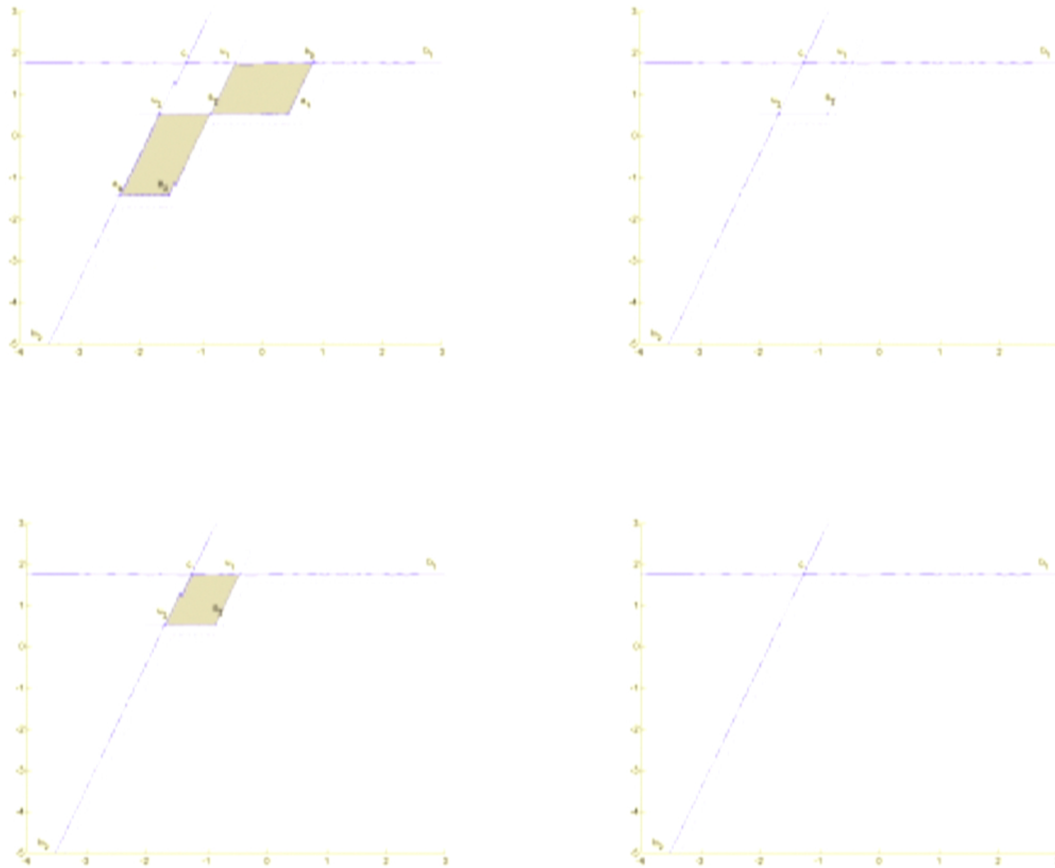


The locally defined maps

$$z \mapsto \left(\int_{z_0}^{z} Re(\lambda_1), \int_{z_0}^{z} Re(\lambda_2), \int_{z_0}^{z} Re(\lambda_3) \right)$$

give a singular flat structure on our Riemann surface (in fact, a $\mathbb{A}/W_{sph}$ valued distance function), and define the charts of an "initial construction" Z_0.

There is an initial relation R_0 on Z_0, which identifies pairs points (in a small neighborhood) in a small neighborhood on either side of each caustic.



Finally, we glue back in the part of the initial construction that we trimmed out to get the pre-building: