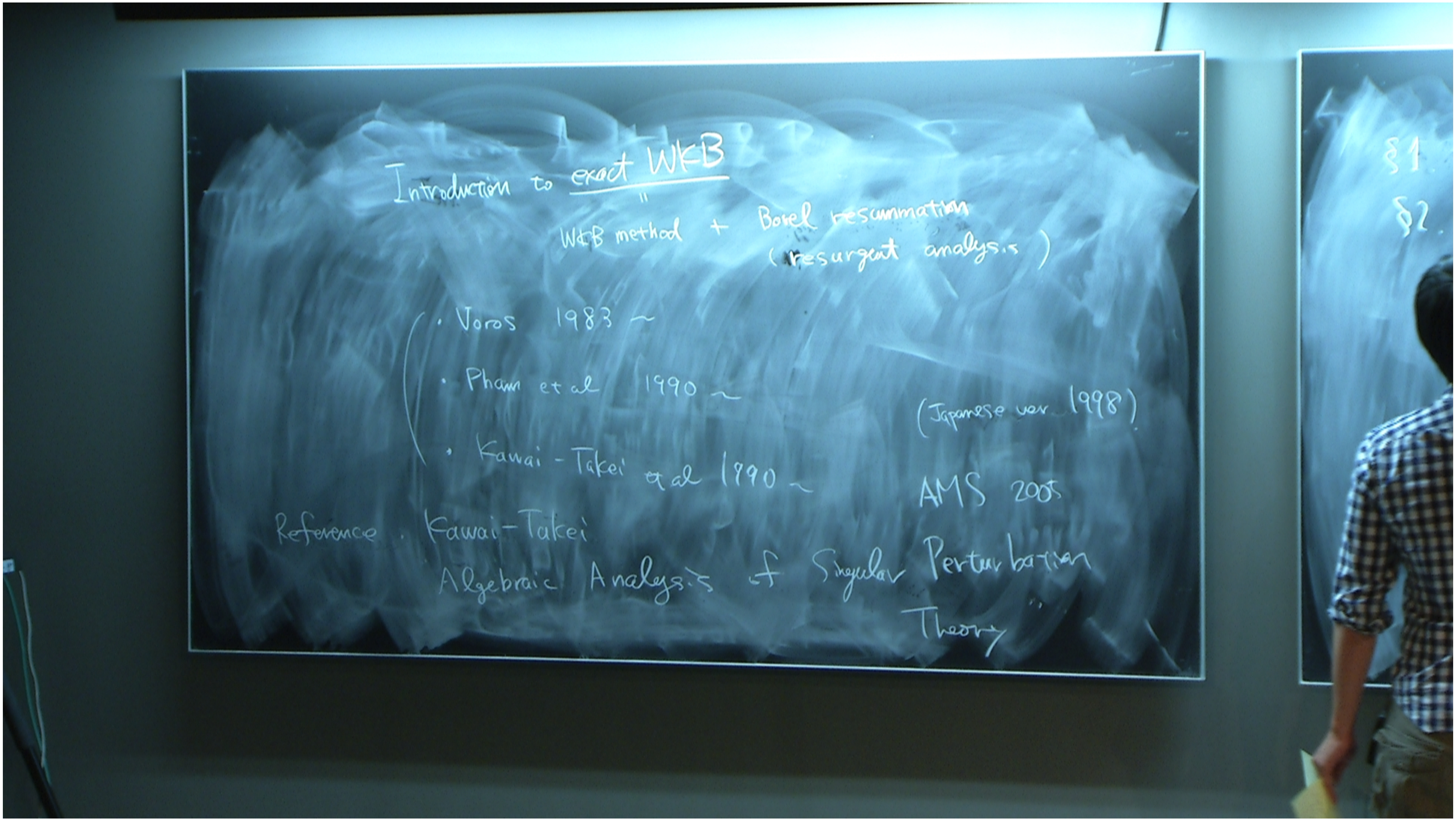
Title: Introduction to exact WKB analysis I

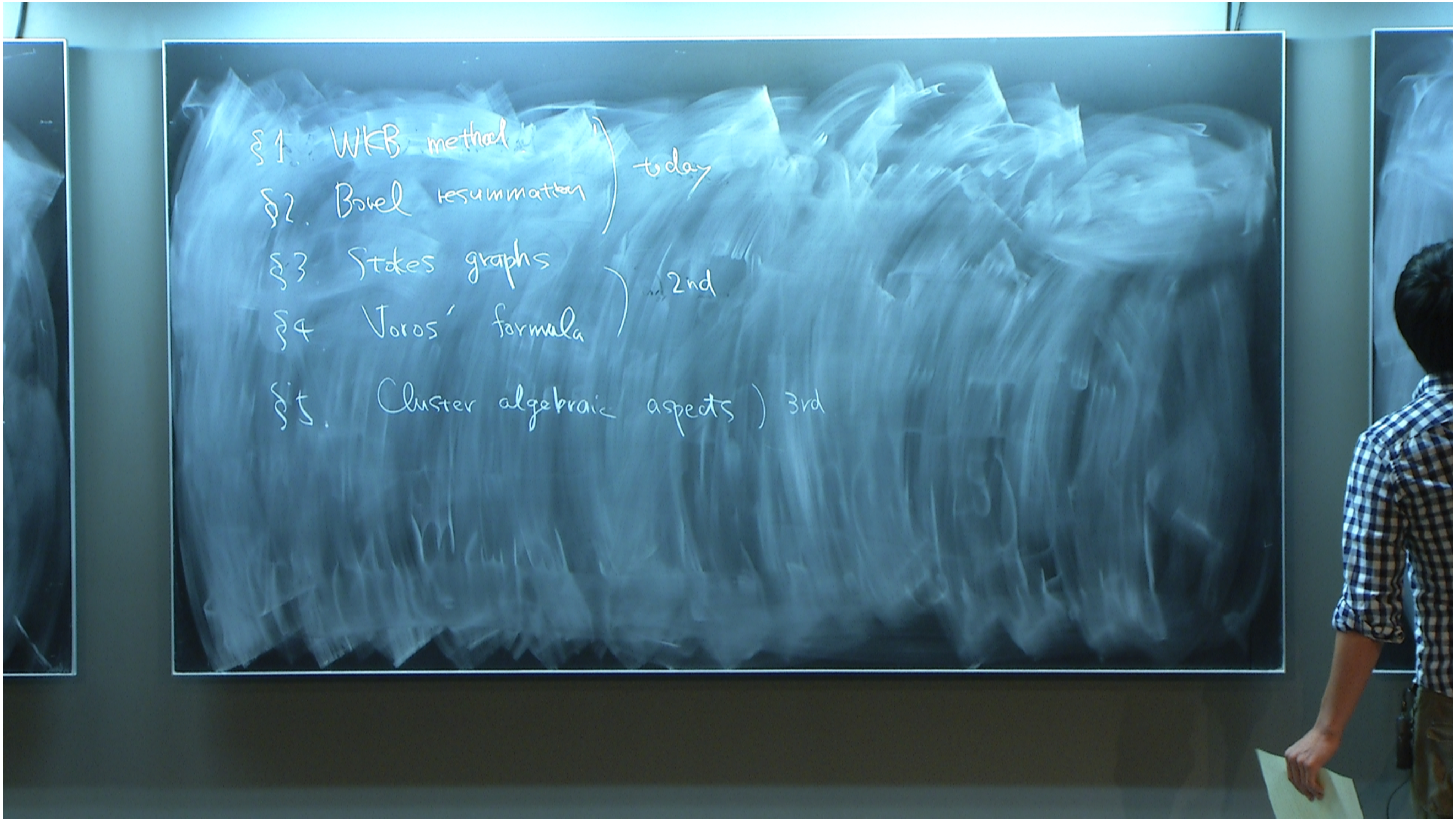
Date: May 25, 2015 11:00 AM

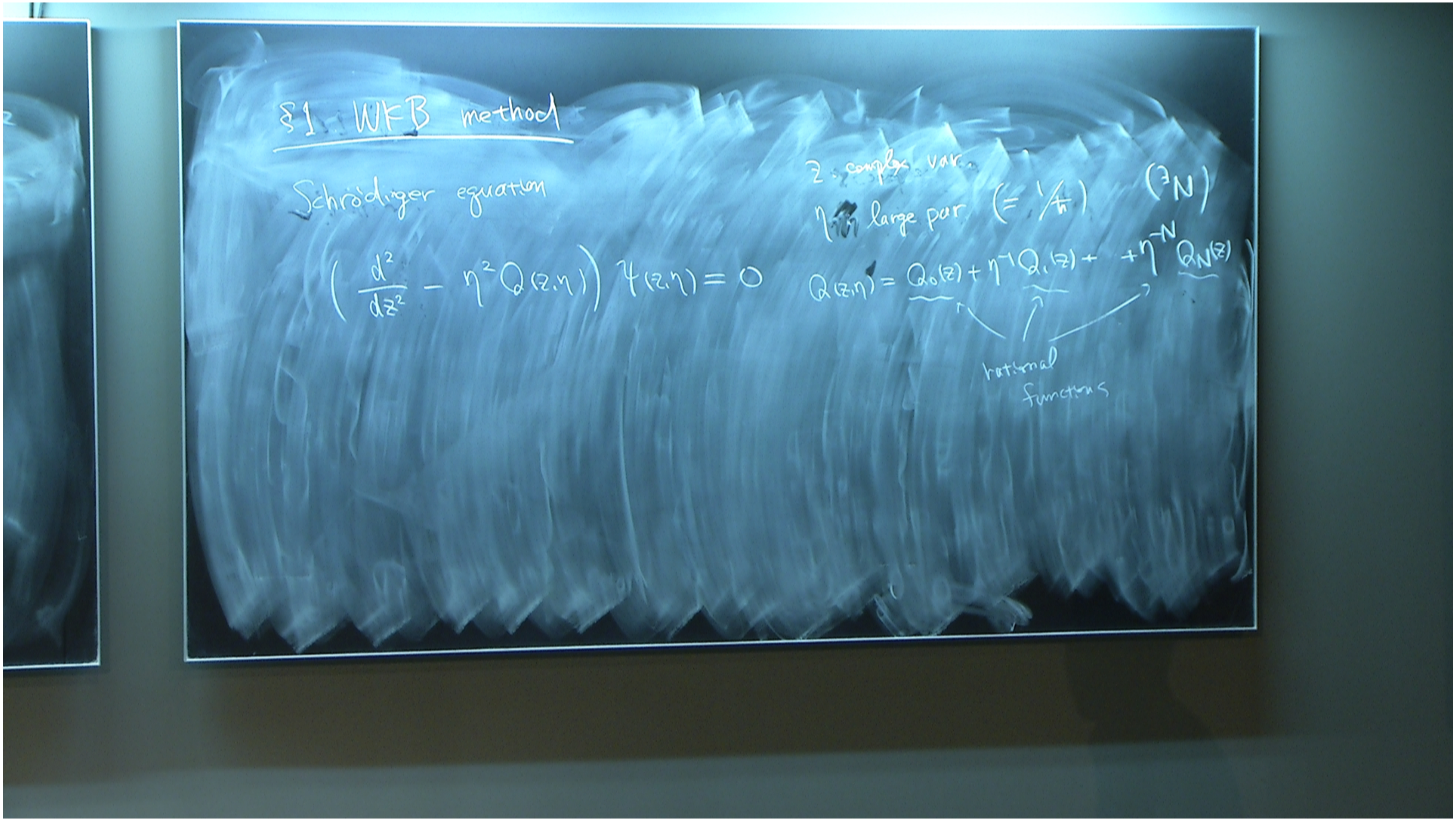
URL: http://pirsa.org/15050048

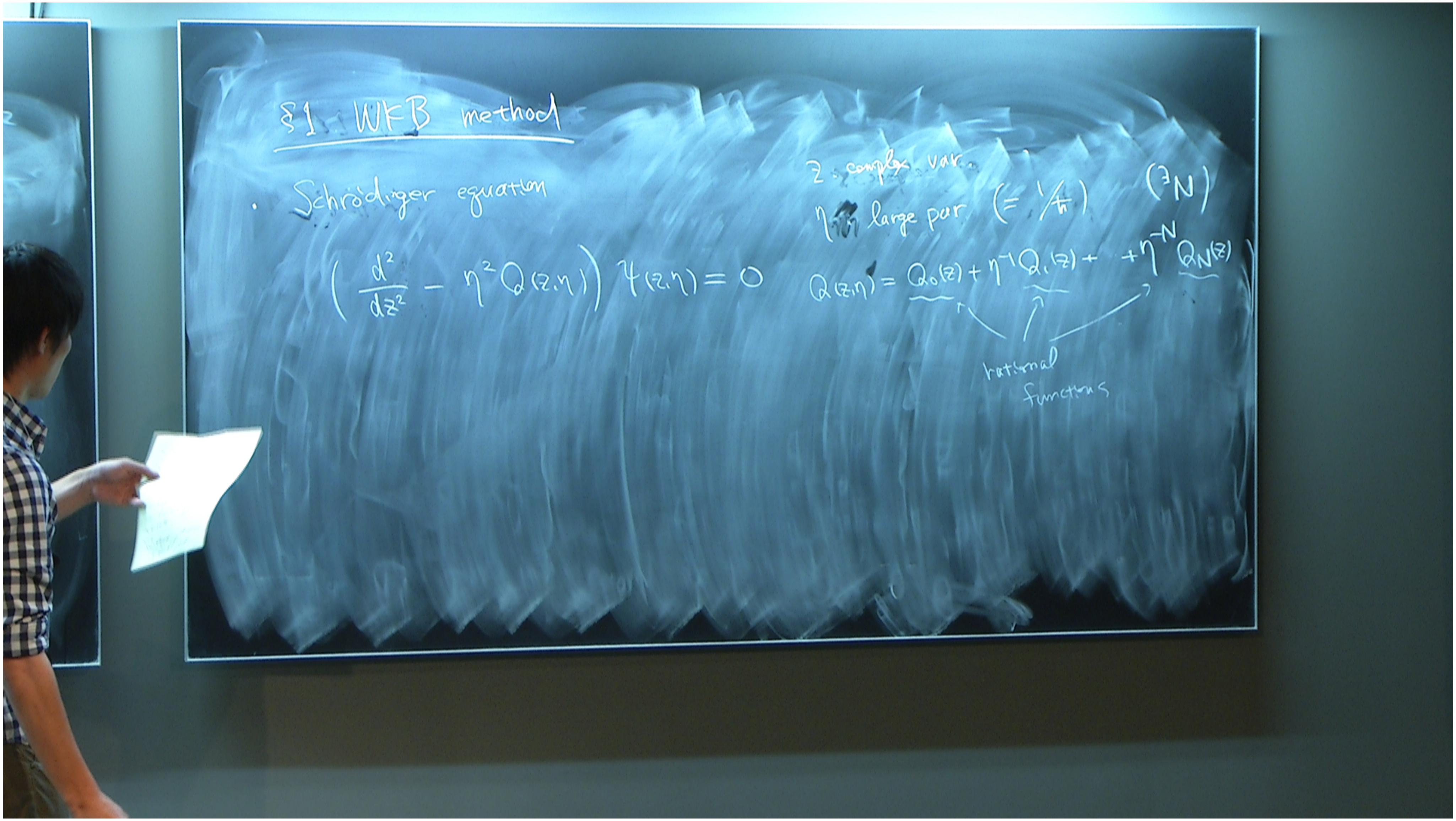
Abstract: Exact WKB analysis, developed by Voros et.al., is an effective method for the global study of differential equations (containing a large parameter) defined on a complex domain. In the first and second lecture I'll give an introduction to exact WKB analysis, and recall some basic facts about WKB solutions, Borel resummation, Stokes graphs etc.

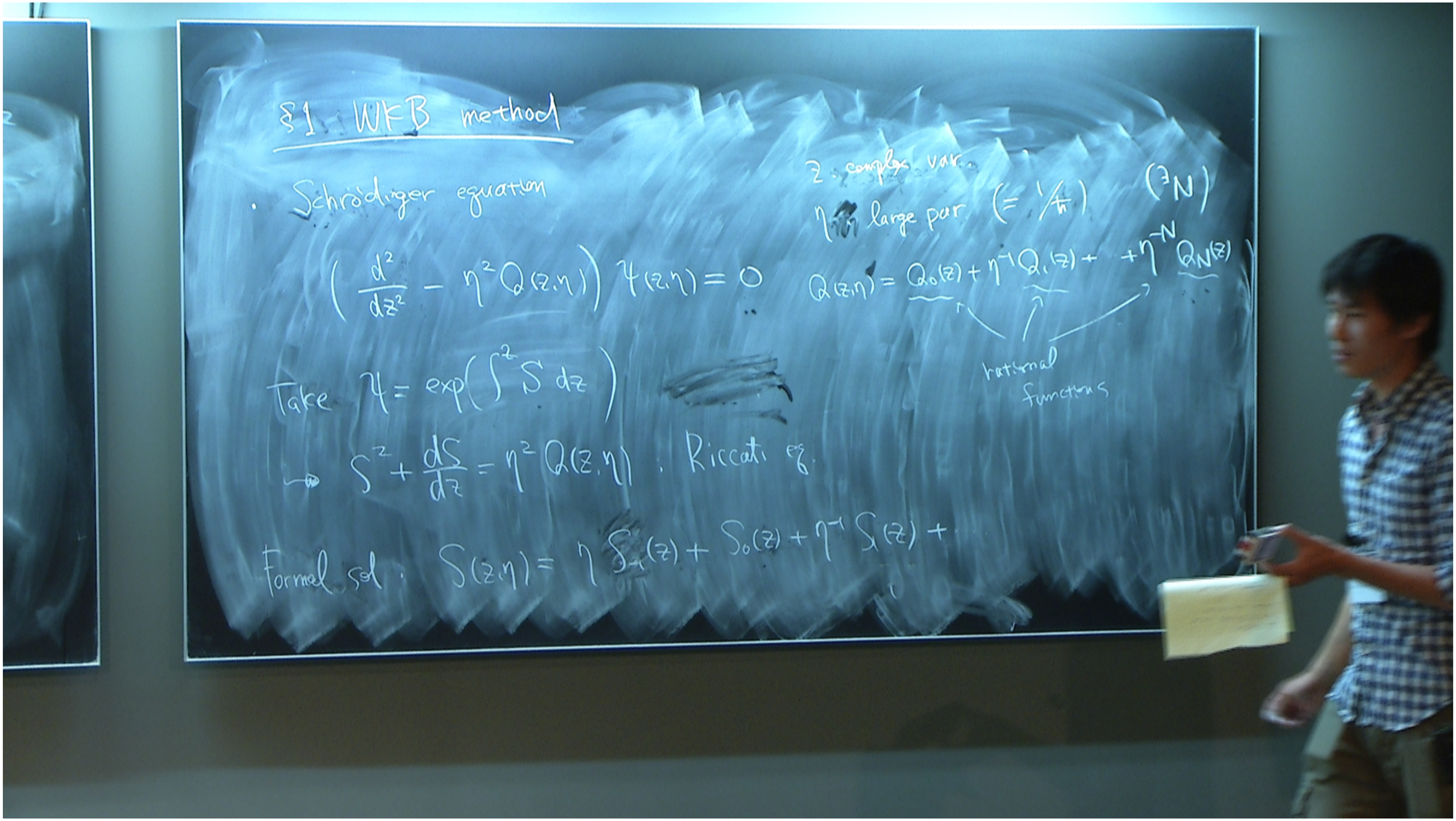
On the other hand, cluster algebras are a particular class of commutative subalgebras of the field of rational functions with distinguished generators. I'll explain about a hidden cluster algebraic structure in exact WKB analysis in the third lecture. This is a joint work with Tomoki Nakanishi (Nagoya).

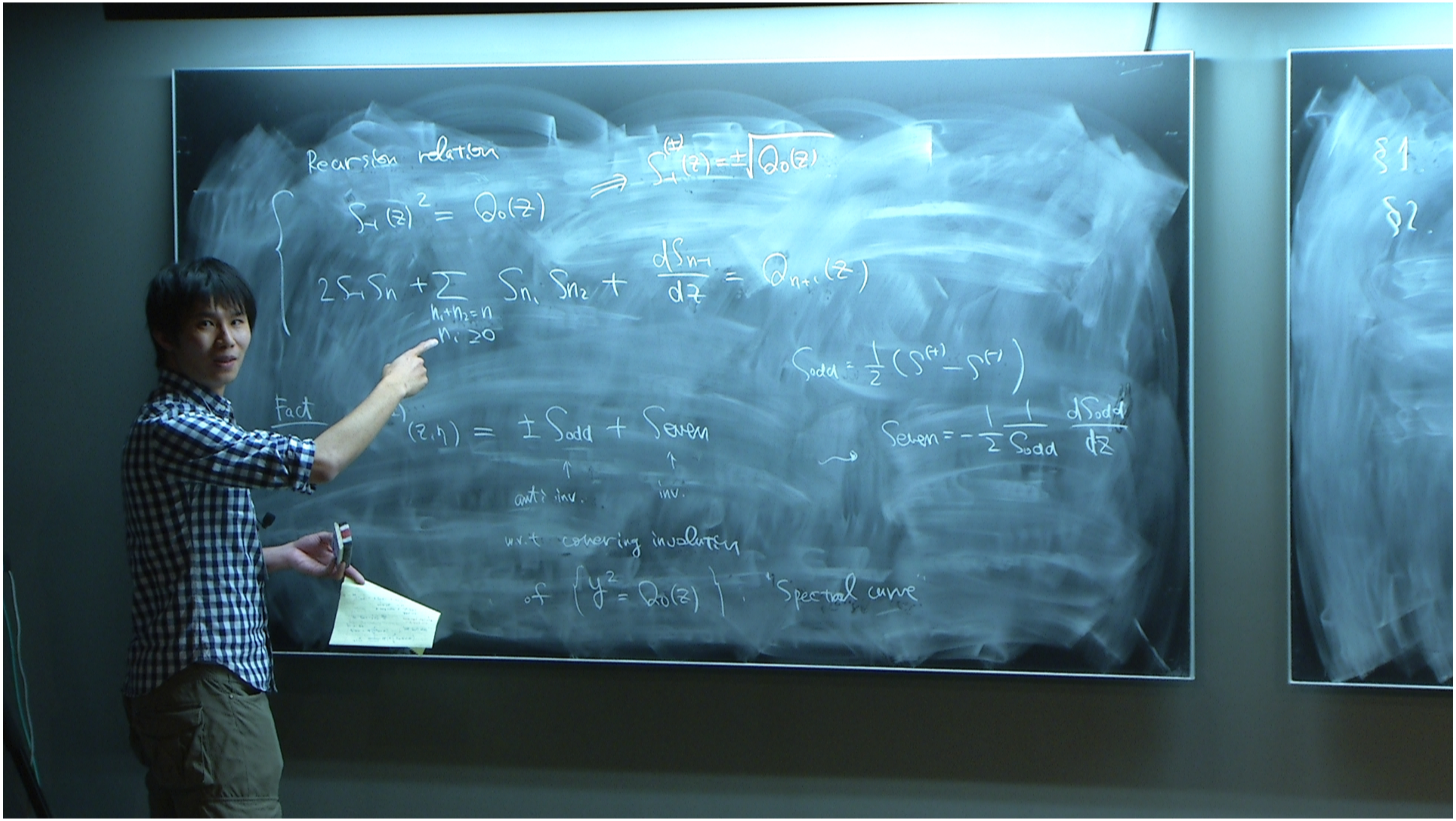

Introduction to exact WKB
WKB method + Borel resummation
(resurgent analysis)
· Voros 1983 ~
· Pham et al 1990 ~
· Kawai - Takei et al 1990 ~
(Japanese ver. 1998)
AMS 2005
Reference : Kawai - Takei
"Algebraic Analysis of Singular Perturbation Theory"
§1
§2

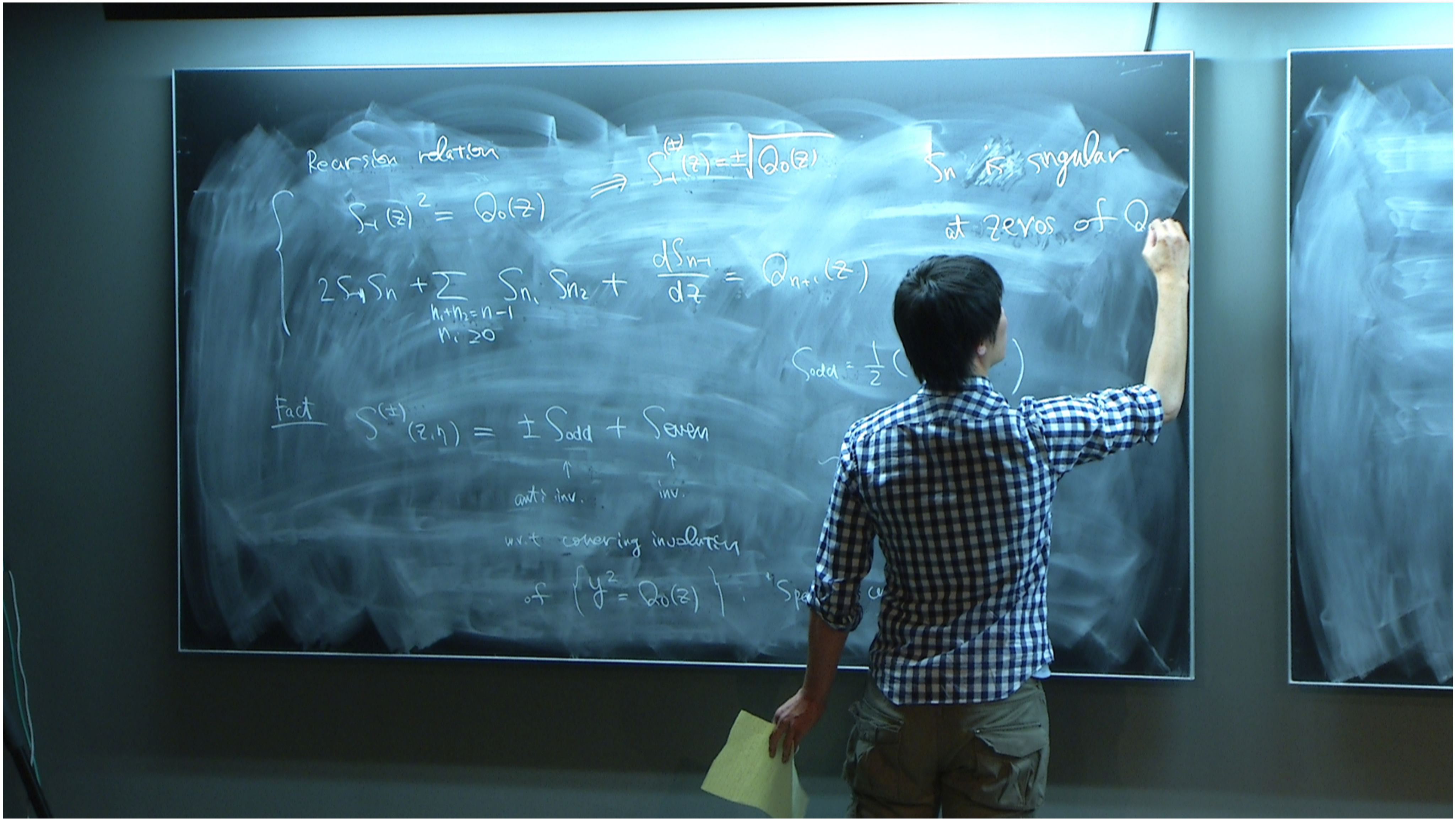

§1 WKB method
§2 Borel resummation
today
§3 Stokes graphs
§4 Voros' formula
2nd
§5 Cluster algebraic aspects
3rd

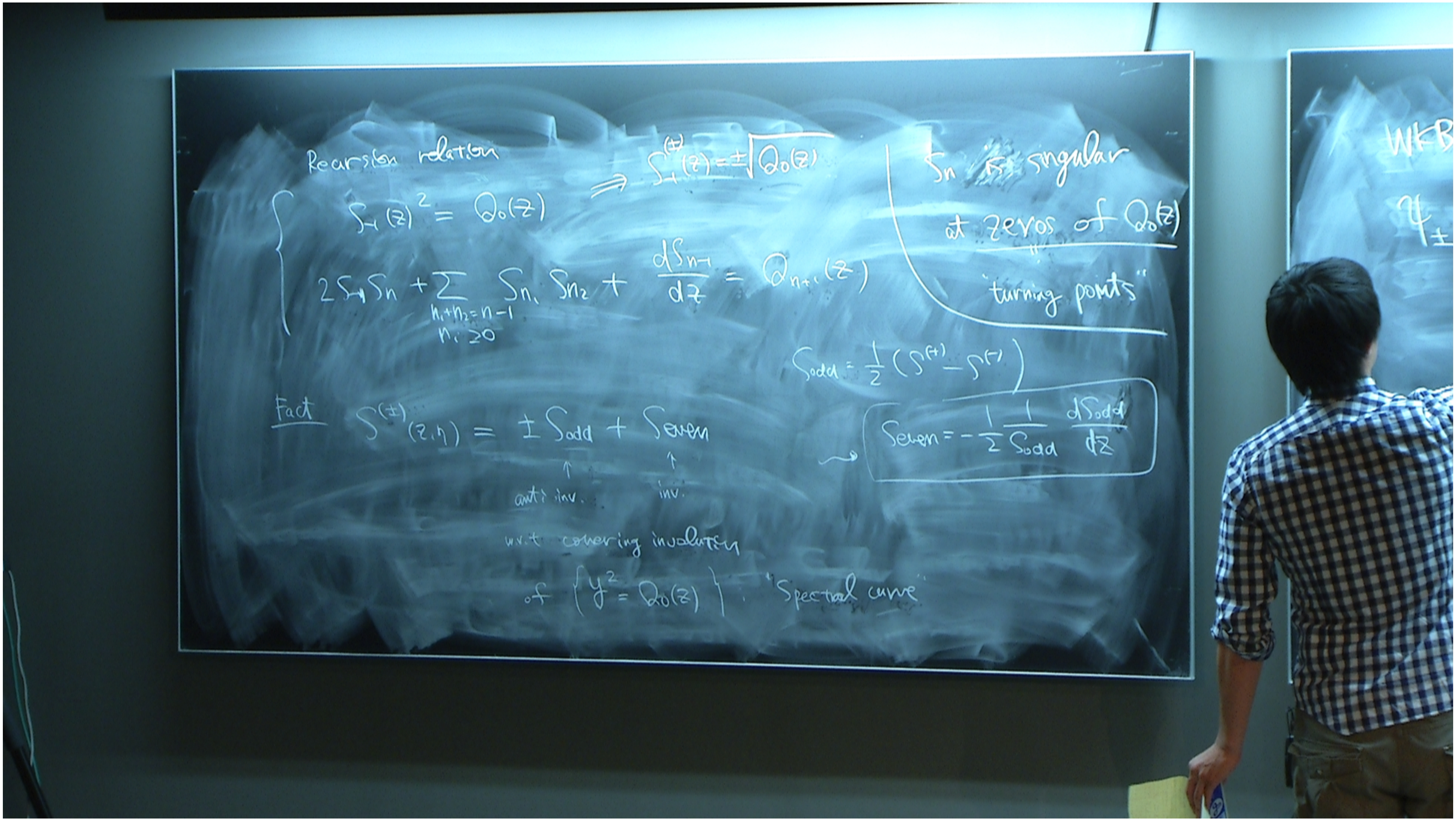

§1 WKB method
Schrödinger equation
$\left(\frac{d^2}{dz^2} - \eta^2 Q(z,\eta)\right)\psi(z,\eta) = 0$
z : complex var.
η : large par $(= 1/\hbar)$
$(\exists N)$
$Q(z,\eta) = Q_0(z) + \eta^{-1}Q_1(z) + \cdots + \eta^{-N}Q_N(z)$
rational functions


§1. WKB method
Schrödinger equation
(d²/dz² − η² Q(z,η)) ψ(z,η) = 0
z : complex var.
η : large par (= 1/ℏ)
(∃N)
Q(z,η) = Q₀(z) + η⁻¹Q₁(z) + ⋯ + η⁻ᴺQ_N(z)
rational functions


§1 WKB method
· Schrödinger equation
$\left(\frac{d^2}{dz^2} - \eta^2 Q(z,\eta)\right)\psi(z,\eta) = 0$
z : complex var.
η : large par. $(= 1/\hbar)$
$(\exists N)$
$Q(z,\eta) = Q_0(z) + \eta^{-1}Q_1(z) + \cdots + \eta^{-N}Q_N(z)$
rational functions
Take $\psi = \exp\left(\int^z S\,dz\right)$
$\to S^2 + \frac{dS}{dz} = \eta^2 Q(z,\eta)$: Riccati eq.
Formal sol : $S(z,\eta) = \eta S_{-1}(z) + S_0(z) + \eta^{-1}S_1(z) + \cdots$


Recursion relation
$\Rightarrow S_{-1}^{(\pm)}(z) = \pm\sqrt{Q_0(z)}$
$S_{-1}(z)^2 = Q_0(z)$
$2S_{-1}S_n + \sum_{\substack{n_1+n_2=n \\ n_i \geq 0}} S_{n_1}S_{n_2} + \frac{dS_{n-1}}{dz} = Q_{n+1}(z)$
$S_{odd} = \frac{1}{2}(S^{(+)} - S^{(-)})$
Fact
$(z,\eta) = \pm S_{odd} + S_{even}$
anti inv.
inv.
$\rightsquigarrow S_{even} = -\frac{1}{2}\frac{1}{S_{odd}}\frac{dS_{odd}}{dz}$
w.r.t covering involution
of $\{y^2 = Q_0(z)\}$ "Spectral curve"
§1
§2


Recursion relation
$S_{-1}(z)^2 = Q_0(z)$
$2S_{-1}S_n + \sum_{\substack{n_1+n_2=n-1 \\ n_i \geq 0}} S_{n_1}S_{n_2} + \frac{dS_{n-1}}{dz} = Q_{n+1}(z)$
$\Rightarrow S_{-1}^{(\pm)}(z) = \pm\sqrt{Q_0(z)}$
S_n singular
at zeros of Q_0
$S_{odd} = \frac{1}{2}($
Fact
$S^{(\pm)}(z,\eta) = \pm S_{odd} + S_{even}$
anti inv.
inv.
w.r.t covering involution
of $\{y^2 = Q_0(z)\}$


Recursion relation
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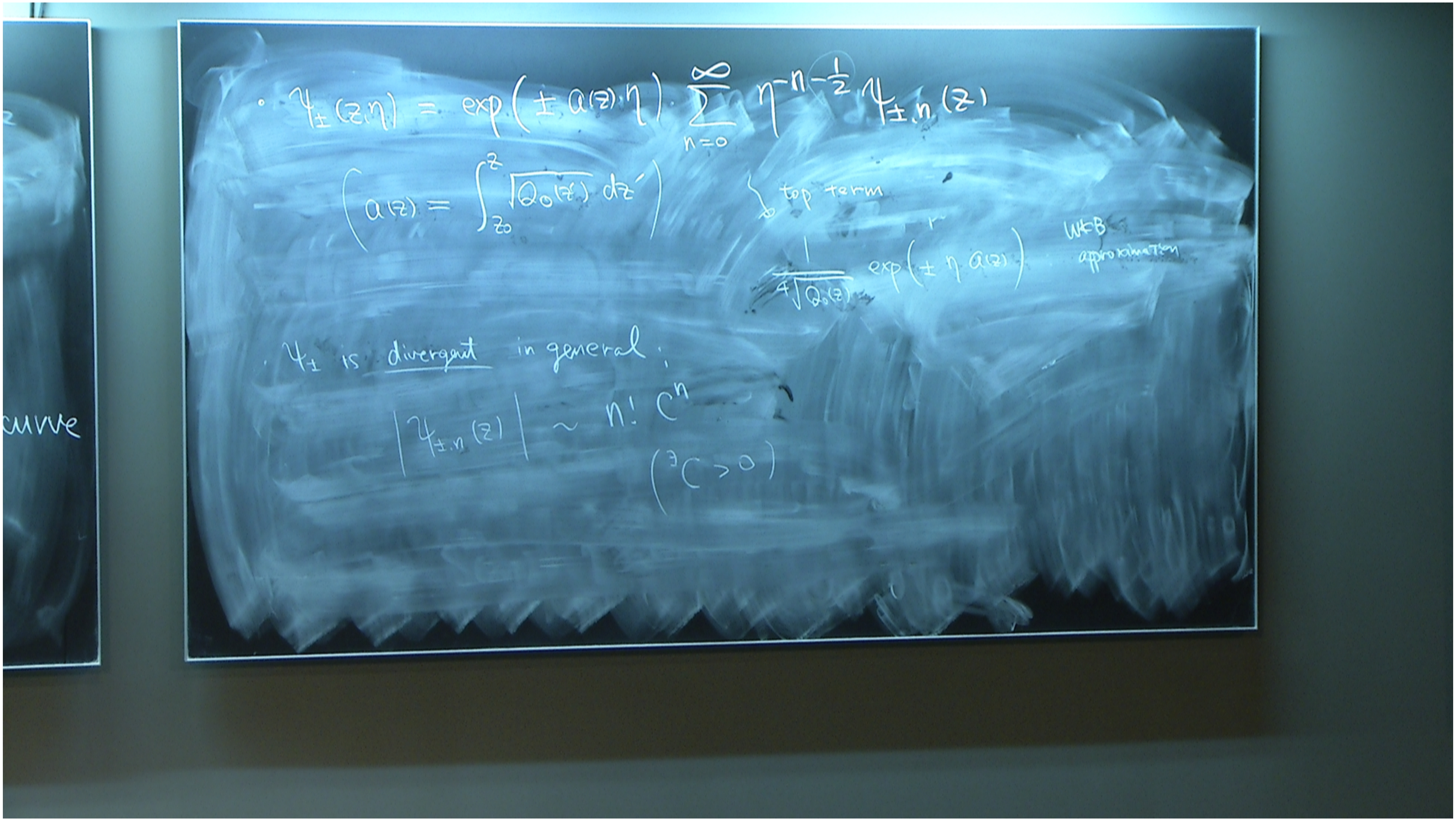
$\Rightarrow S_{-1}^{(\pm)}(z) = \pm\sqrt{Q_0(z)}$
S_n is singular
at zeros of $Q_0(z)$
"turning points"
$S_{odd} = \frac{1}{2}(S^{(+)} - S^{(-)})$
Fact $S^{(\pm)}(z,\eta) = \pm S_{odd} + S_{even}$
anti inv.
inv.
w.r.t covering involution
of $\{y^2 = Q_0(z)\}$ "Spectral curve"
$S_{even} = -\frac{1}{2}\frac{1}{S_{odd}}\frac{dS_{odd}}{dz}$
WKB

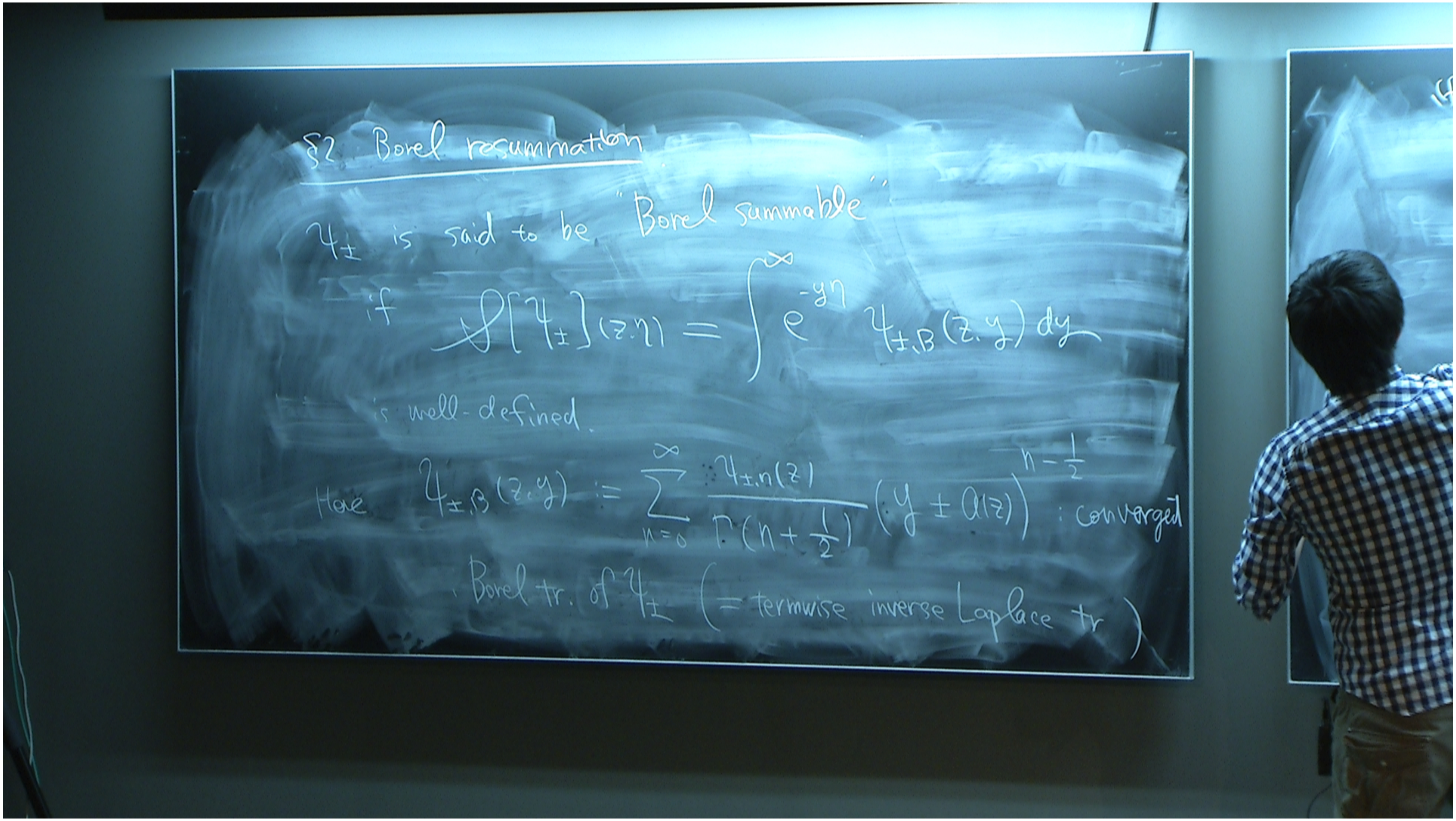
WKB solutions:

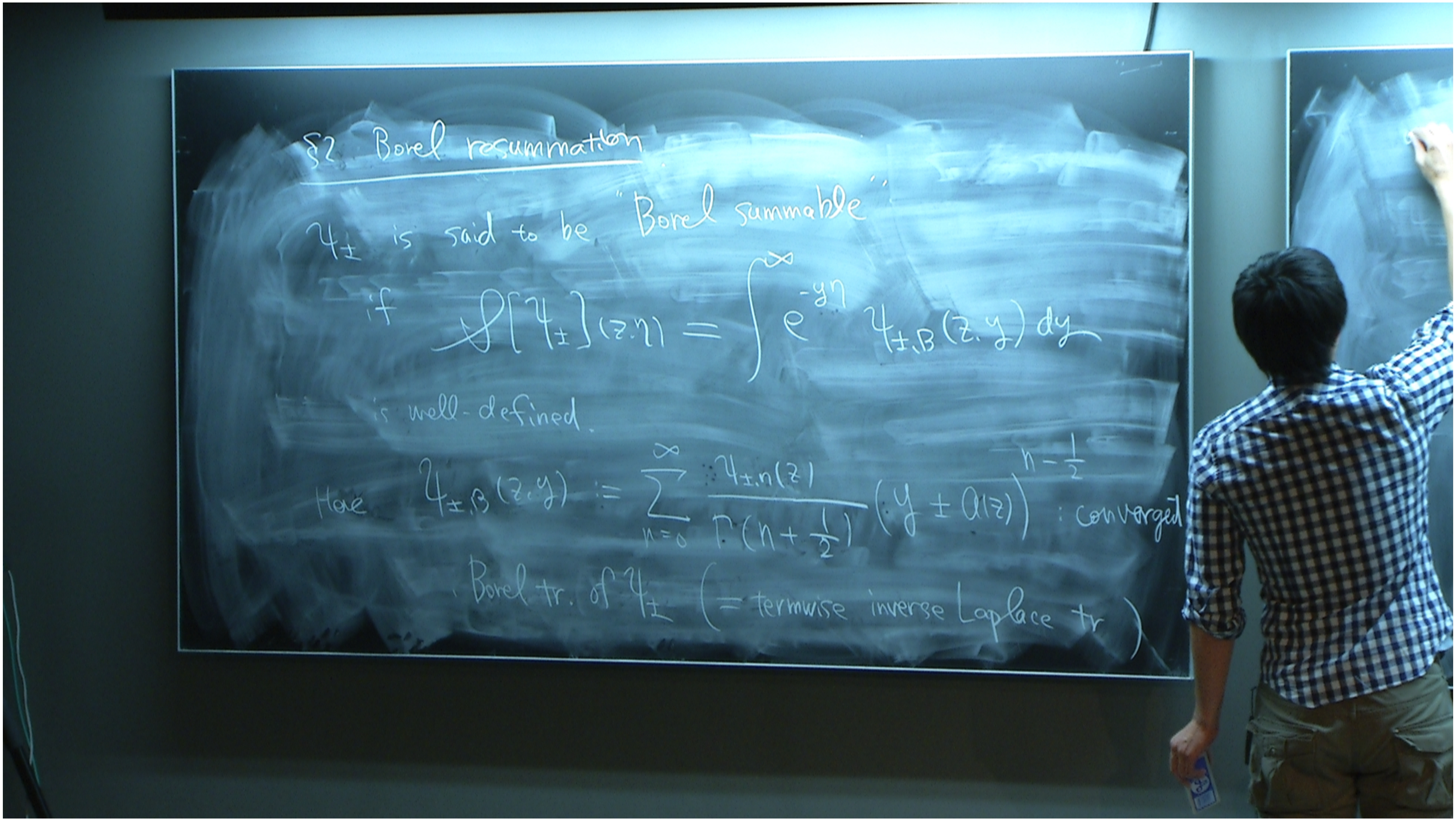
$$\psi_{\pm}(z,\eta) = \exp\left(\int_{z_0}^{z} S^{(\pm)}(z',\eta)\, dz' \right) \qquad z_0 \text{: generic pt}$$

$$\underset{\text{up to const}}{=} \frac{1}{\sqrt{S_{odd}(z,\eta)}} \exp\left(\pm \int_{z_0}^{z} S_{odd}(z',\eta)\, dz' \right)$$

along a path on spectral curve avoiding turning points


curve
$\psi_\pm(z,\eta) = \exp(\pm a(z)\eta) \sum_{n=0}^{\infty} \eta^{-n-\frac{1}{2}} \psi_{\pm,n}(z)$
$\left(a(z) = \int_{z_0}^{z} \sqrt{Q_0(z')}\, dz' \right)$
top term
$\frac{1}{\sqrt[4]{Q_0(z)}} \exp(\pm \eta\, a(z))$
WKB approximation
$\psi_\pm$ is divergent in general;
$|\psi_{\pm,n}(z)| \sim n!\, C^n$
$({}^\exists C > 0)$


§2. Borel resummation
$\psi_\pm$ is said to be "Borel summable"
if $\mathcal{S}[\psi_\pm](z,\eta) = \int^{\infty} e^{-y\eta}\, \psi_{\pm,B}(z,y)\, dy$
is well-defined.
Here $\psi_{\pm,B}(z,y) := \sum_{n=0}^{\infty} \frac{\psi_{\pm,n}(z)}{\Gamma(n+\frac{1}{2})} (y \pm a(z))^{n-\frac{1}{2}}$: converged
Borel tr. of $\psi_\pm$ (= termwise inverse Laplace tr.)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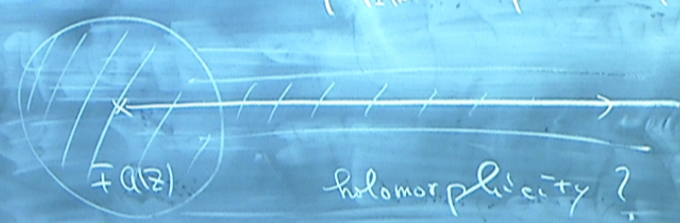
Borel tr. of $\psi_\pm$ (= termwise inverse Laplace tr.)

$$|\psi_{\pm,B}(z,y)| \le C_1 e^{C_2|y|}$$



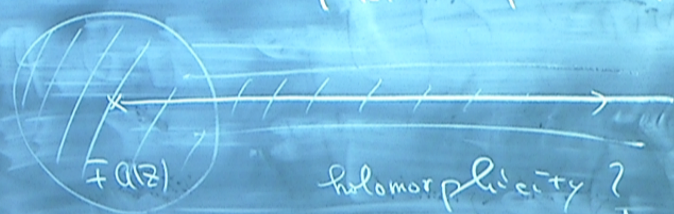
holomorphicity ?

If $\psi_\pm$ is Borel summable

then $\mathcal{S}[\psi_\pm](z,\eta)$ gives a analytic solution of Sch eq.
(not formal)

and $\mathcal{S}[\psi_\pm](z,\eta) \underset{\text{asymp. exp}}{\sim} \psi_\pm(z,\eta)$ for $\eta \to +\infty$.

y | $\left|\psi_{\pm,B}(z,y)\right| \le C_1 e^{C_2|y|}$



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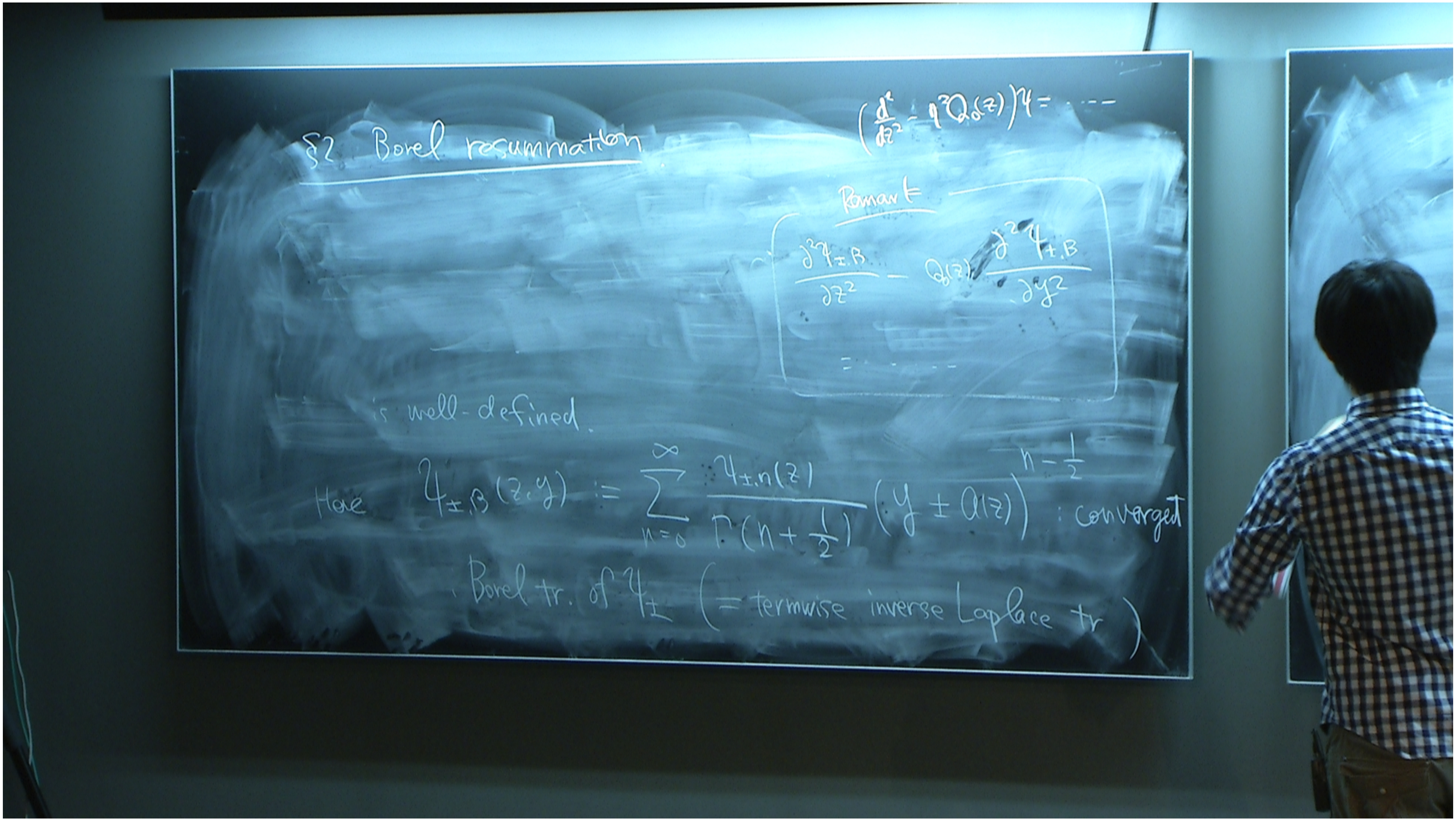
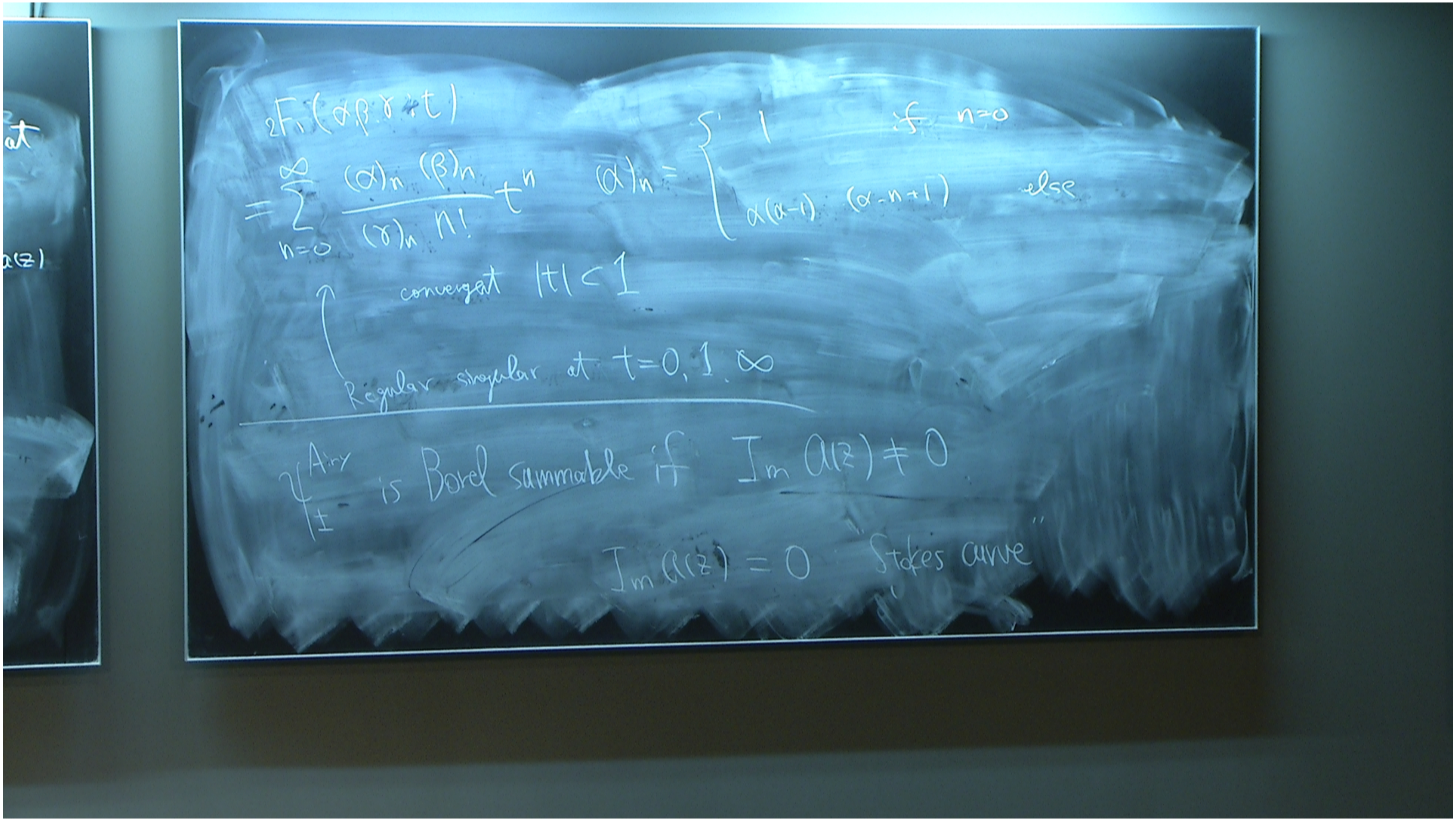

§2 Borel resummation
$\left(\frac{d^2}{dz^2} - \eta^2 Q_0(z)\right)\psi = \cdots$
Remark
$\frac{\partial^2 \psi_{\pm,B}}{\partial z^2} - Q_0(z) \frac{\partial^2 \psi_{\pm,B}}{\partial y^2}$
$= \cdots$
is well-defined.
Here $\psi_{\pm,B}(z,y) := \sum_{n=0}^{\infty} \frac{\psi_{\pm,n}(z)}{\Gamma(n+\frac{1}{2})} (y \pm a(z))^{n-\frac{1}{2}}$: converget
Borel tr. of $\psi_{\pm}$ (= termwise inverse Laplace tr)

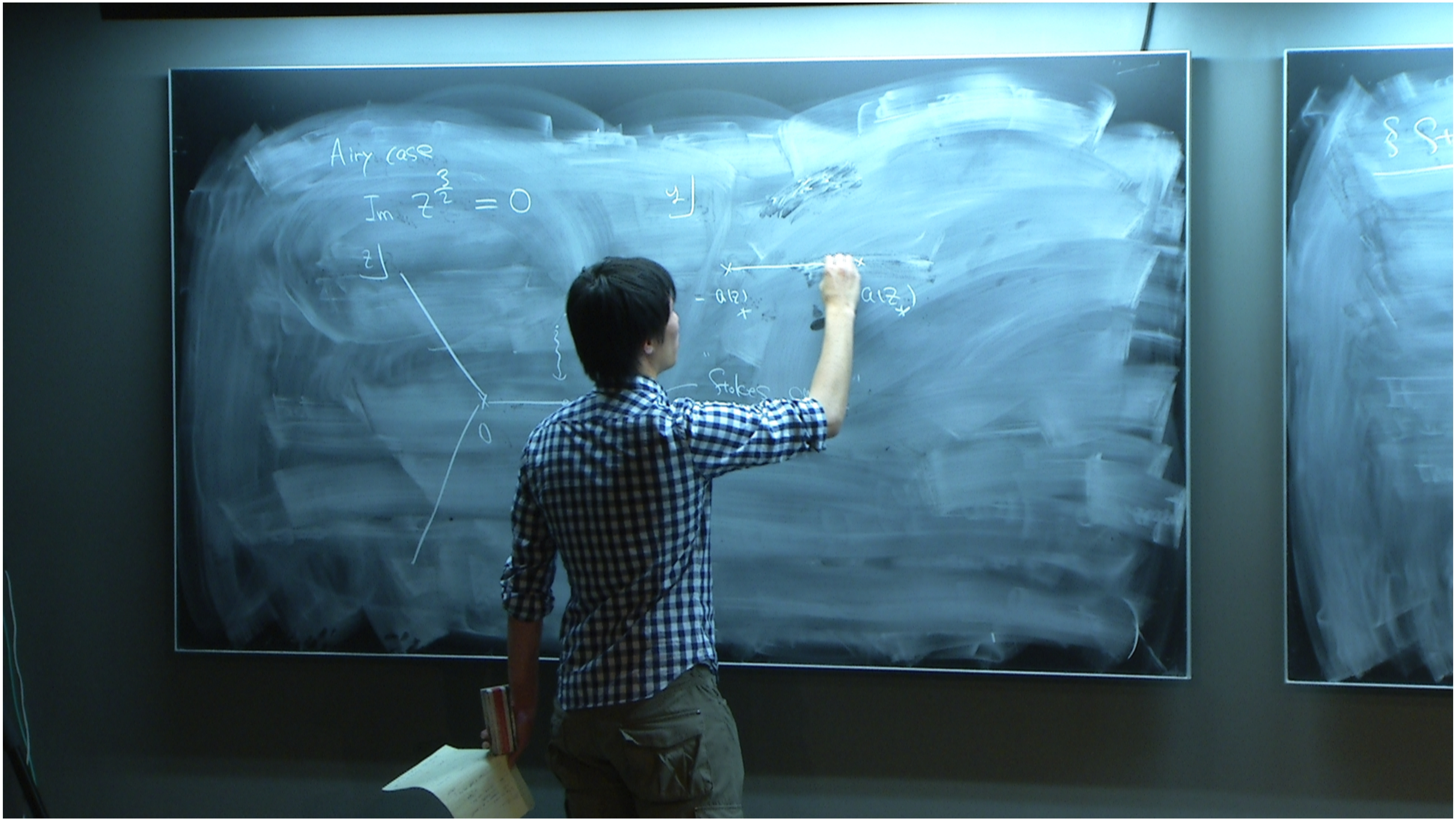
Fact: The Borel tr. of Airy WKB sol is given by

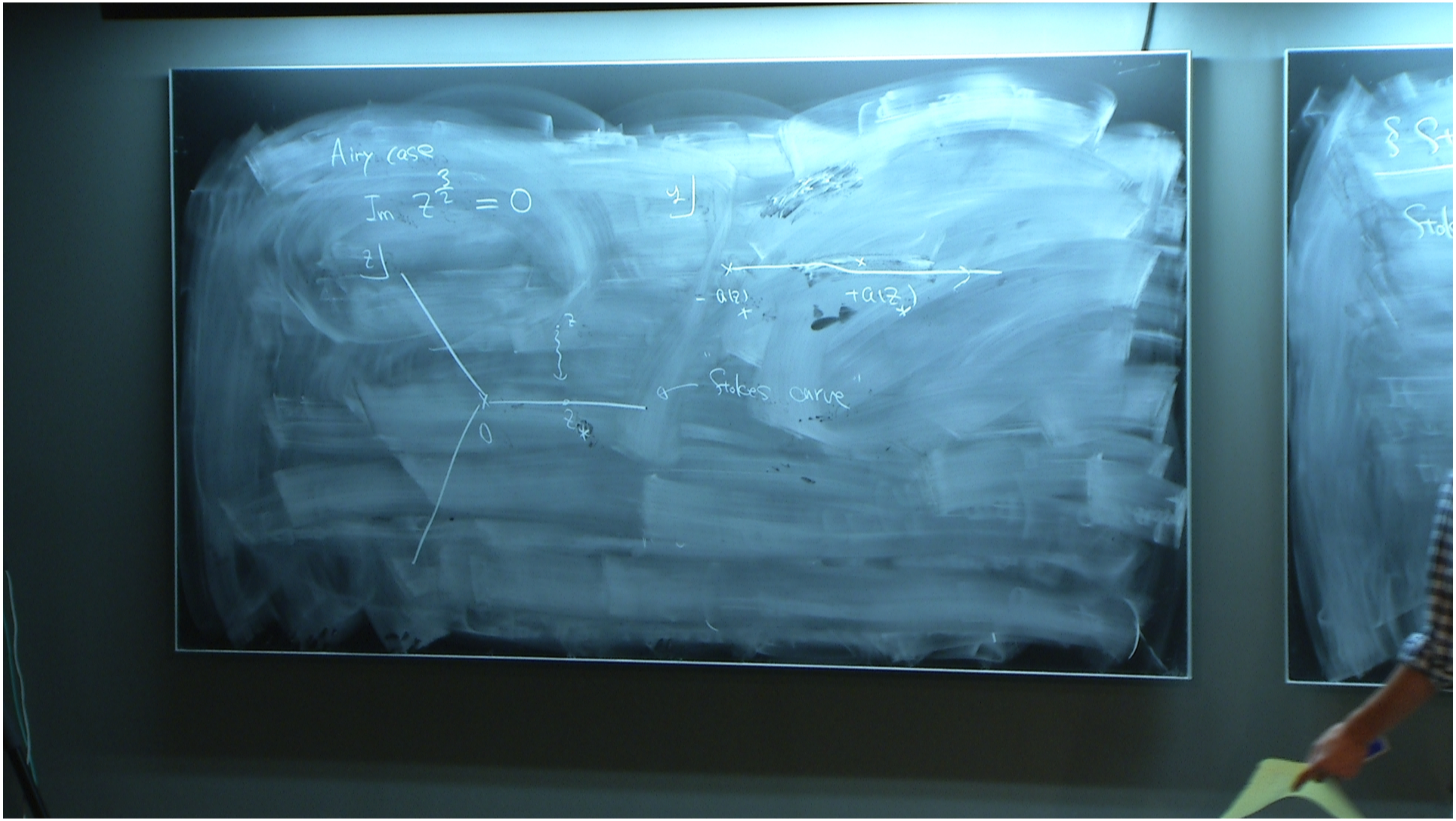
$$\psi_{+,B}(z,y) = (\text{some factor})\ {}_2F_1(\alpha,\beta,\gamma;t)$$

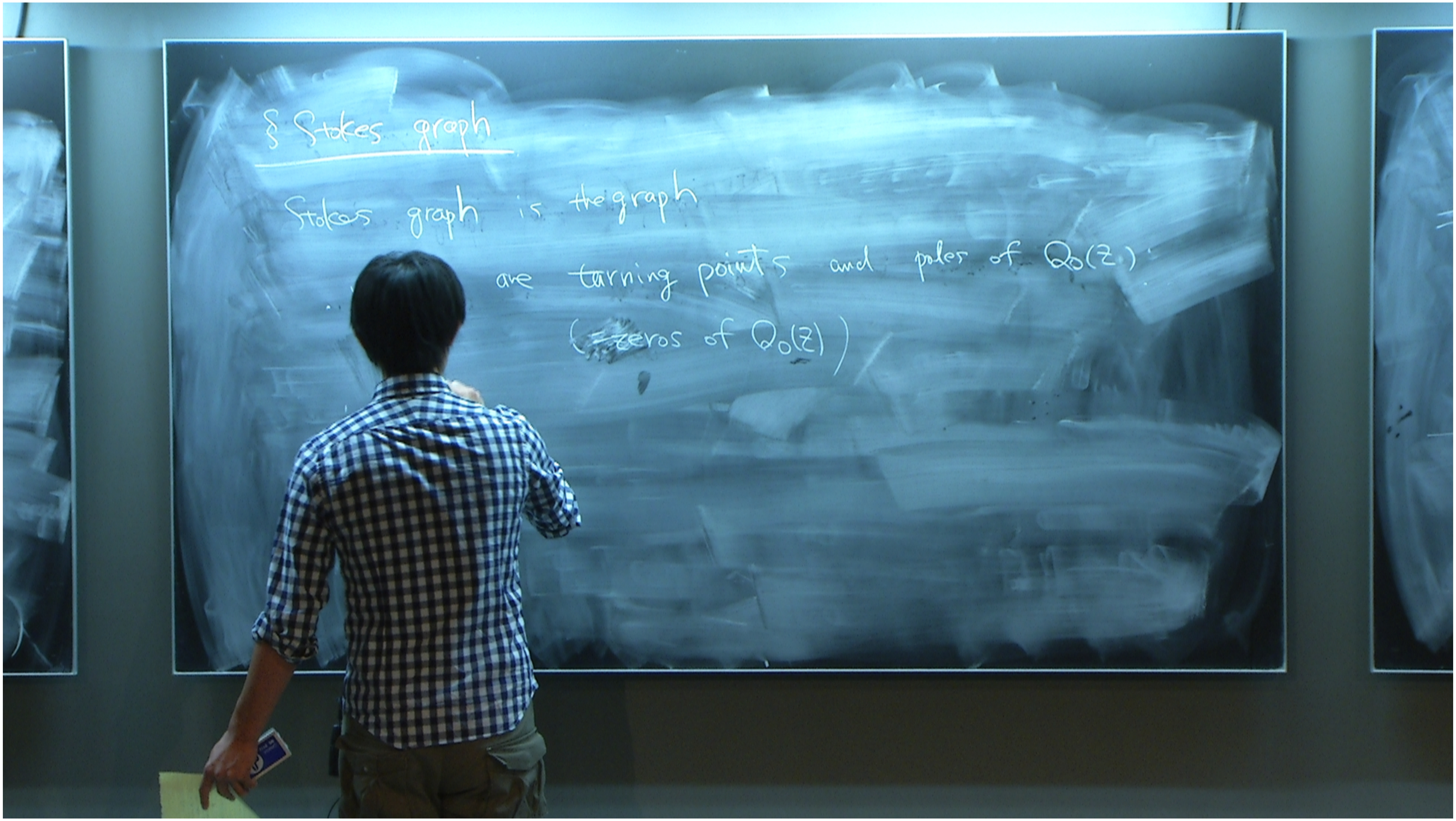
$$\exists\, \alpha,\beta,\gamma \in \mathbb{C}$$

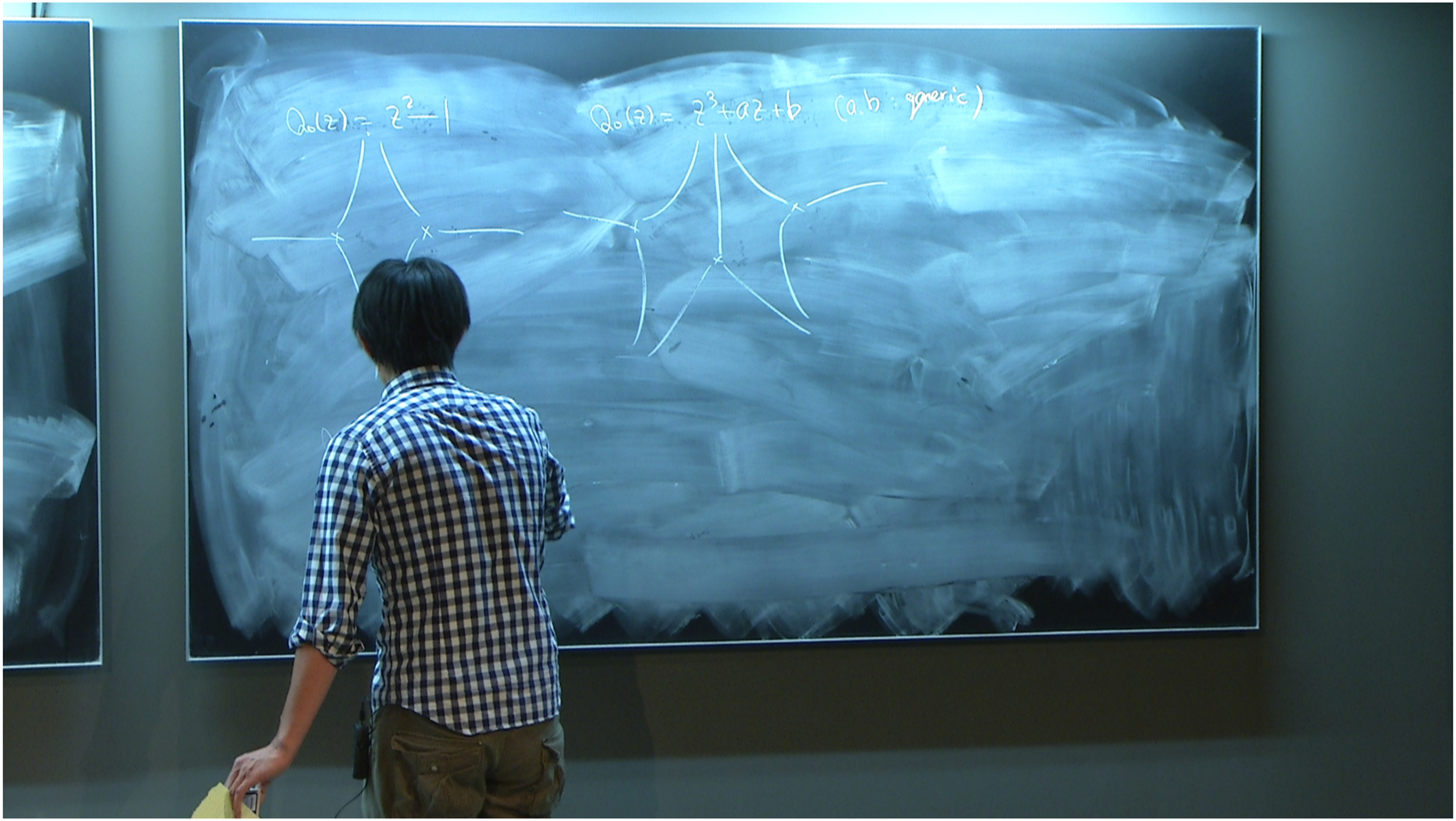
$$t = \frac{3}{4}\frac{y}{z^{3/2}} + \frac{1}{2}$$

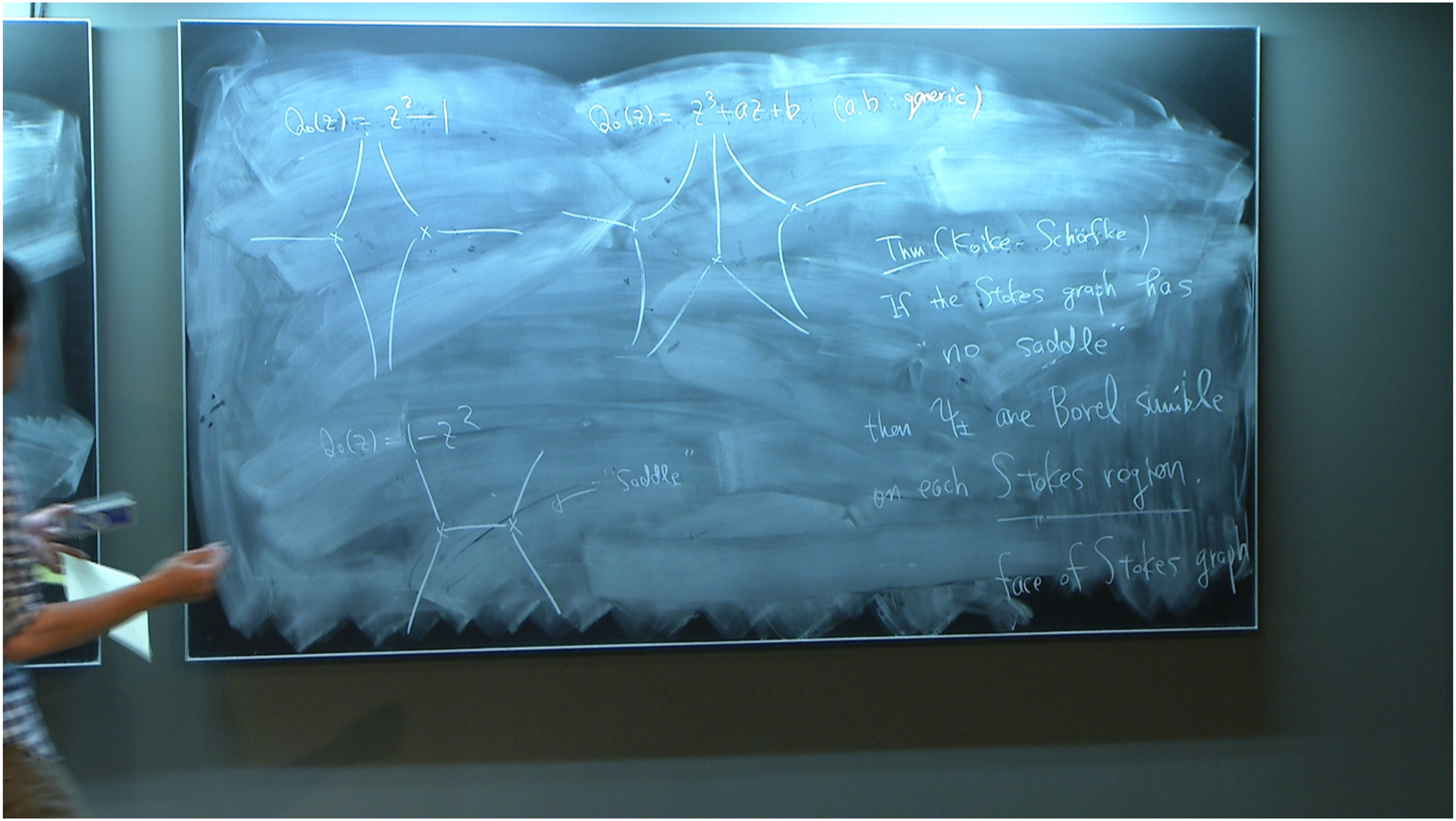

${}_2F_1(\alpha,\beta,\gamma;t)$
$= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n\, n!} t^n$
$(\alpha)_n = \begin{cases} 1 & \text{if } n=0 \\ \alpha(\alpha-1)\cdots(\alpha-n+1) & \text{else} \end{cases}$
convergent $|t|<1$
Regular singular at $t=0,1,\infty$
$\psi_{\pm}^{\mathrm{Airy}}$ is Borel summable if $\mathrm{Im}\, a(z) \neq 0$
$\mathrm{Im}\, a(z) = 0$ "Stokes curve"


Airy case
Im $z^{\frac{3}{2}} = 0$
Stokes


Airy case
Im $z^{\frac{3}{2}} = 0$
z
y
0
z_*
$-a(z_*)$
$+a(z_*)$
"Stokes curve"


§ Stokes graph
Stokes graph is the graph
are turning points and poles of $Q_0(z)$
(= zeros of $Q_0(z)$)


$Q_0(z) = z^2 - 1$
$Q_0(z) = z^3 + az + b$ (a, b : generic)


$Q_0(z) = z^2 - 1$
$Q_0(z) = z^3 + az + b$ (a, b : generic)
Thm (Koike-Schäfke)
If the Stokes graph has
"no saddle"
then $\psi_\pm$ are Borel summable
on each Stokes region.
face of Stokes graph
$Q_0(z) = 1 - z^2$
"saddle"