Title: Introduction to exact WKB analysis I

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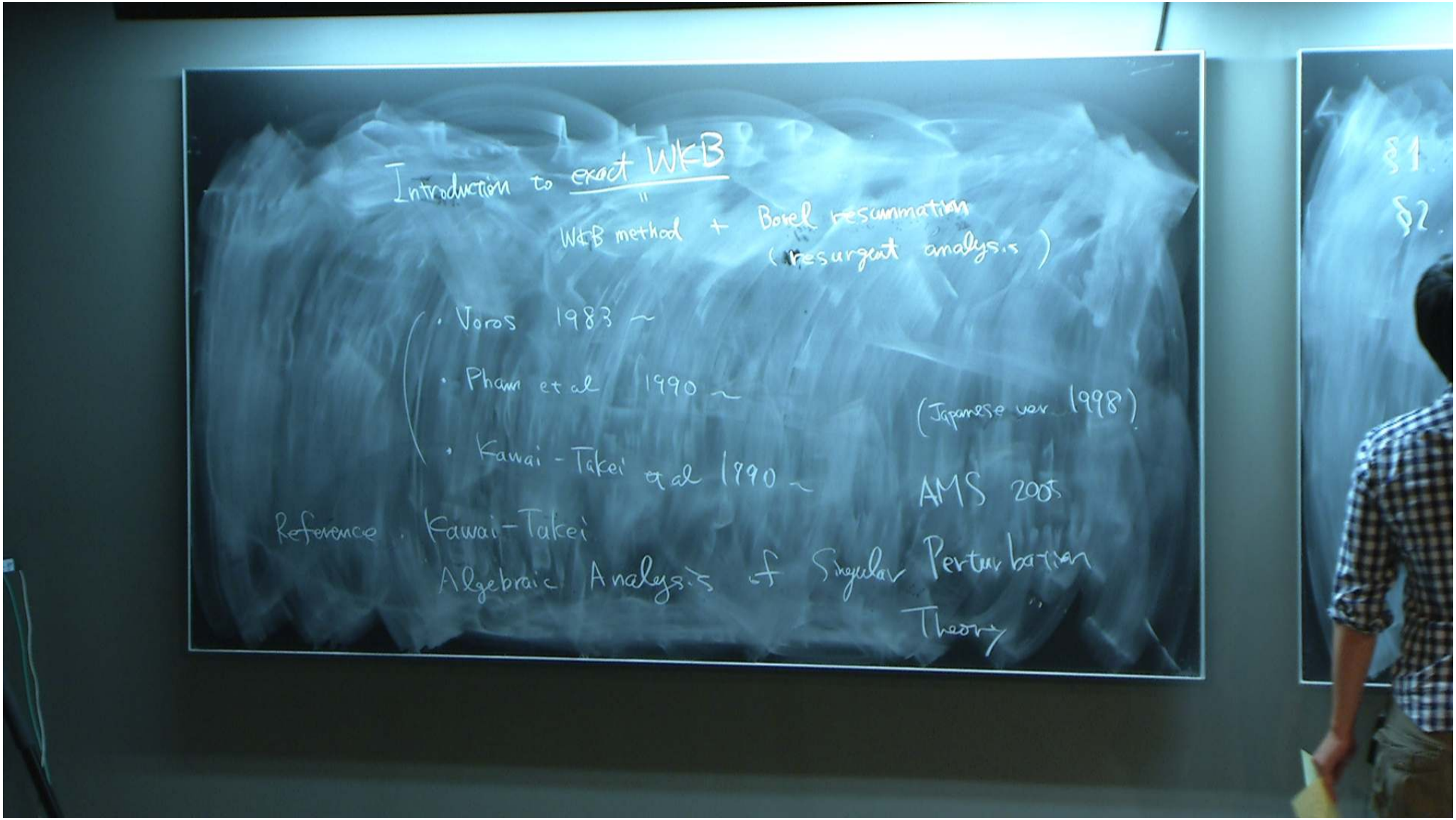
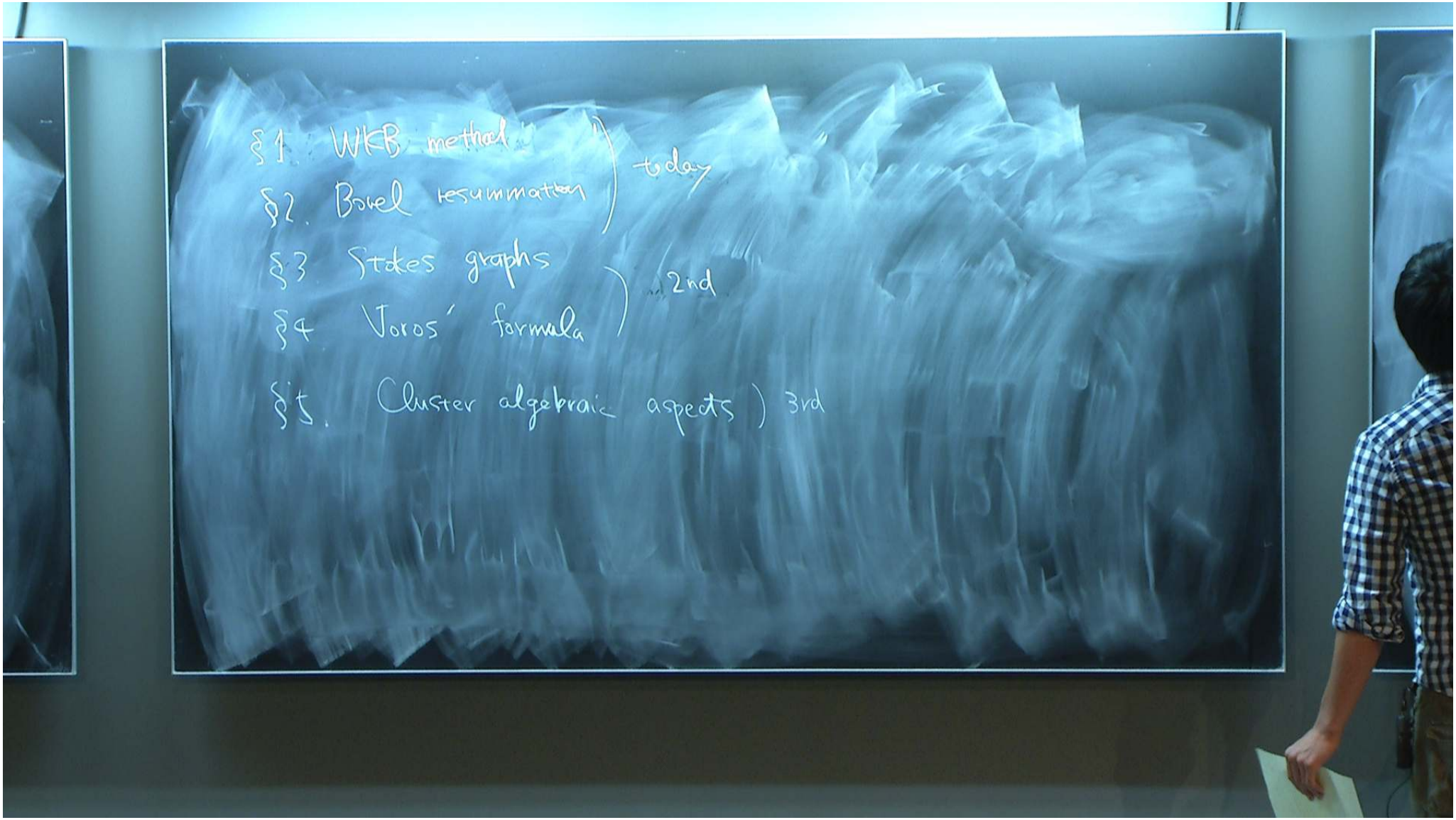
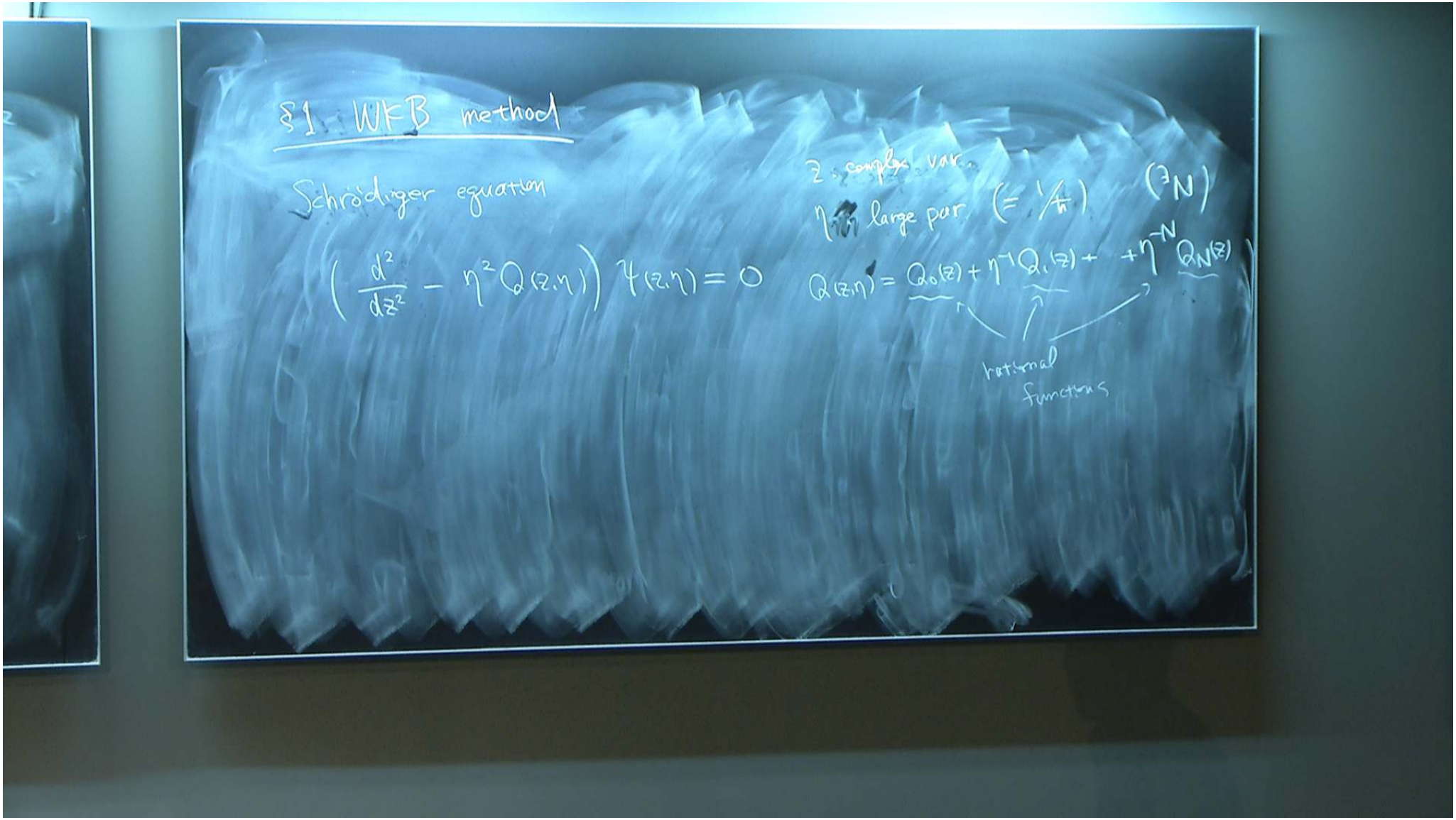
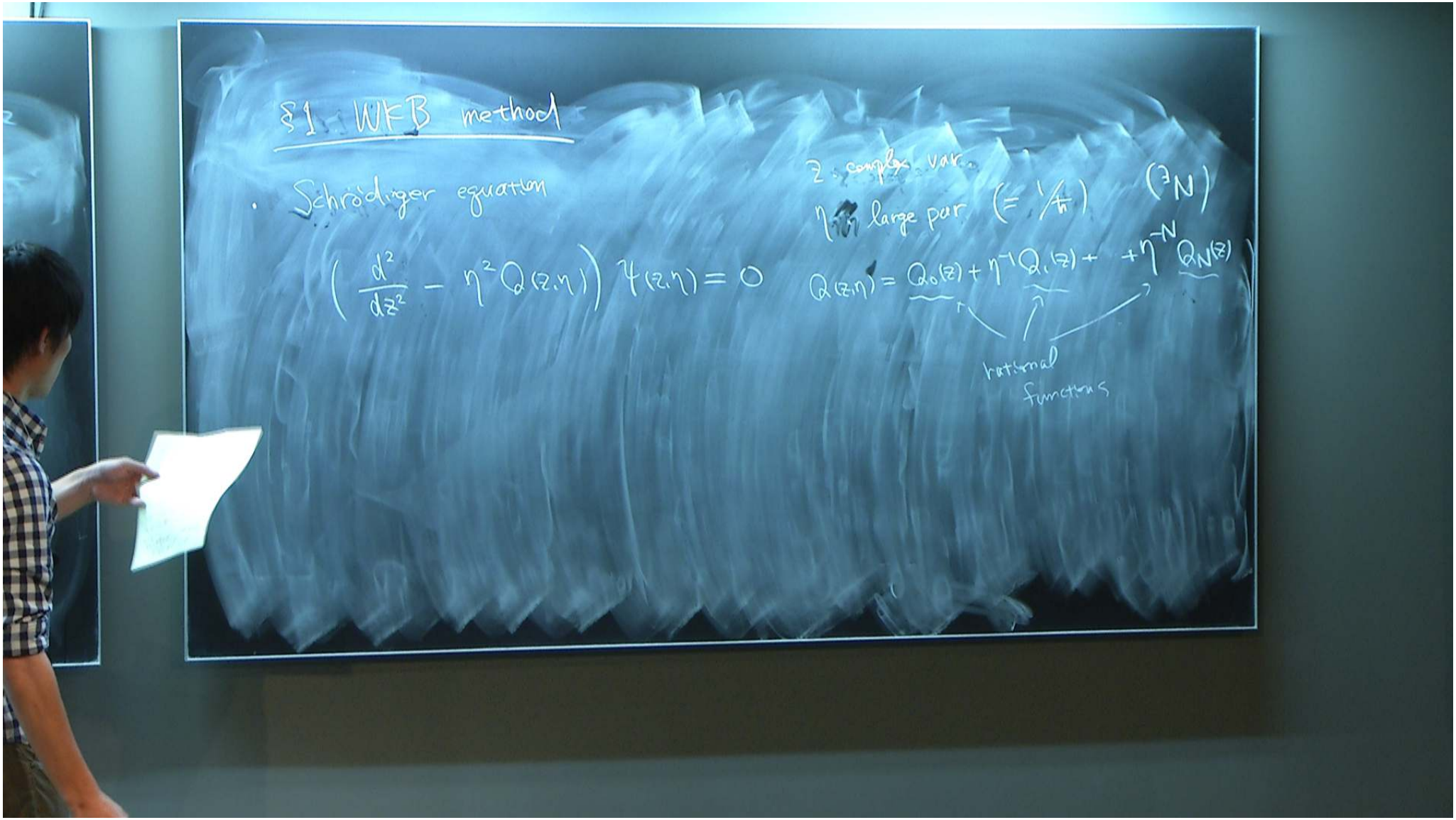
Abstract: Exact WKB analysis, developed by Voros et.al., is an effective method for the global study of differential equations (containing a large parameter) defined on a complex domain. In the first and second lecture I'll give an introduction to exact WKB analysis, and recall some basic facts about WKB solutions, Borel resummation, Stokes graphs etc.

On the other hand, cluster algebras are a particular class of commutative subalgebras of the field of rational functions with distinguished generators. I'll explain about a hidden cluster algebraic structure in exact WKB analysis in the third lecture. This is a joint work with Tomoki Nakanishi (Nagoya).


Introduction to exact WKB
WKB method + Borel resummation
(resurgent analysis)
· Voros 1983 ~
· Pham et al 1990 ~
· Kawai-Takei et al 1990 ~
(Japanese ver 1998)
AMS 2005
Reference : Kawai-Takei
Algebraic Analysis of Singular Perturbation
Theory
§1
§2


§1 WKB method
§2 Borel resummation
today
§3 Stokes graphs
§4 Voros' formula
2nd
§5 Cluster algebraic aspects
3rd


§1 WKB method
Schrödinger equation
$\left(\frac{d^2}{dz^2} - \eta^2 Q(z,\eta)\right)\psi(z,\eta) = 0$
z : complex var.
η large par. $(= 1/\hbar)$ $(^{?}N)$
$Q(z,\eta) = Q_0(z) + \eta^{-1}Q_1(z) + \cdots + \eta^{-N}Q_N(z)$
rational functions


§1 WKB method
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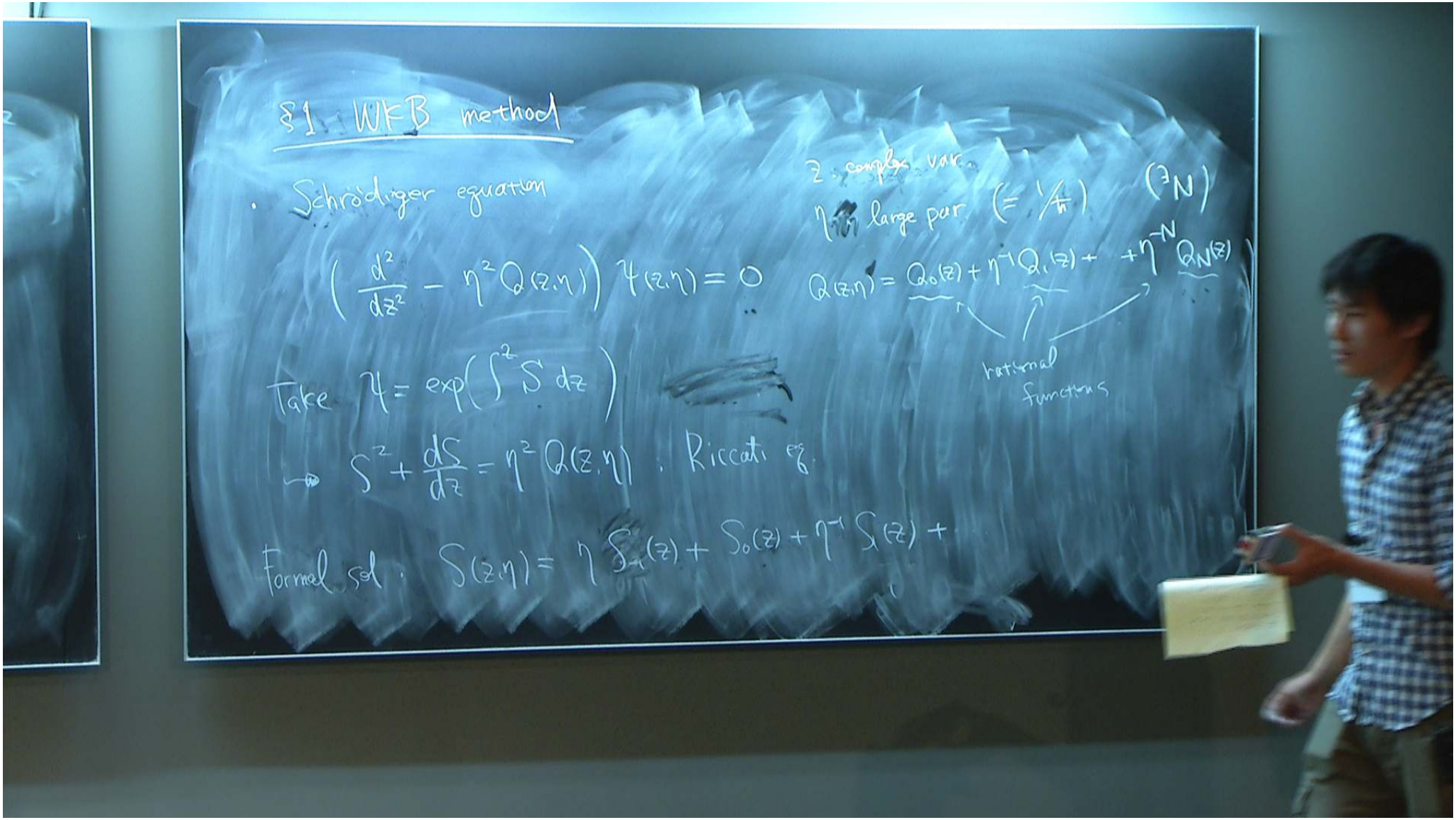
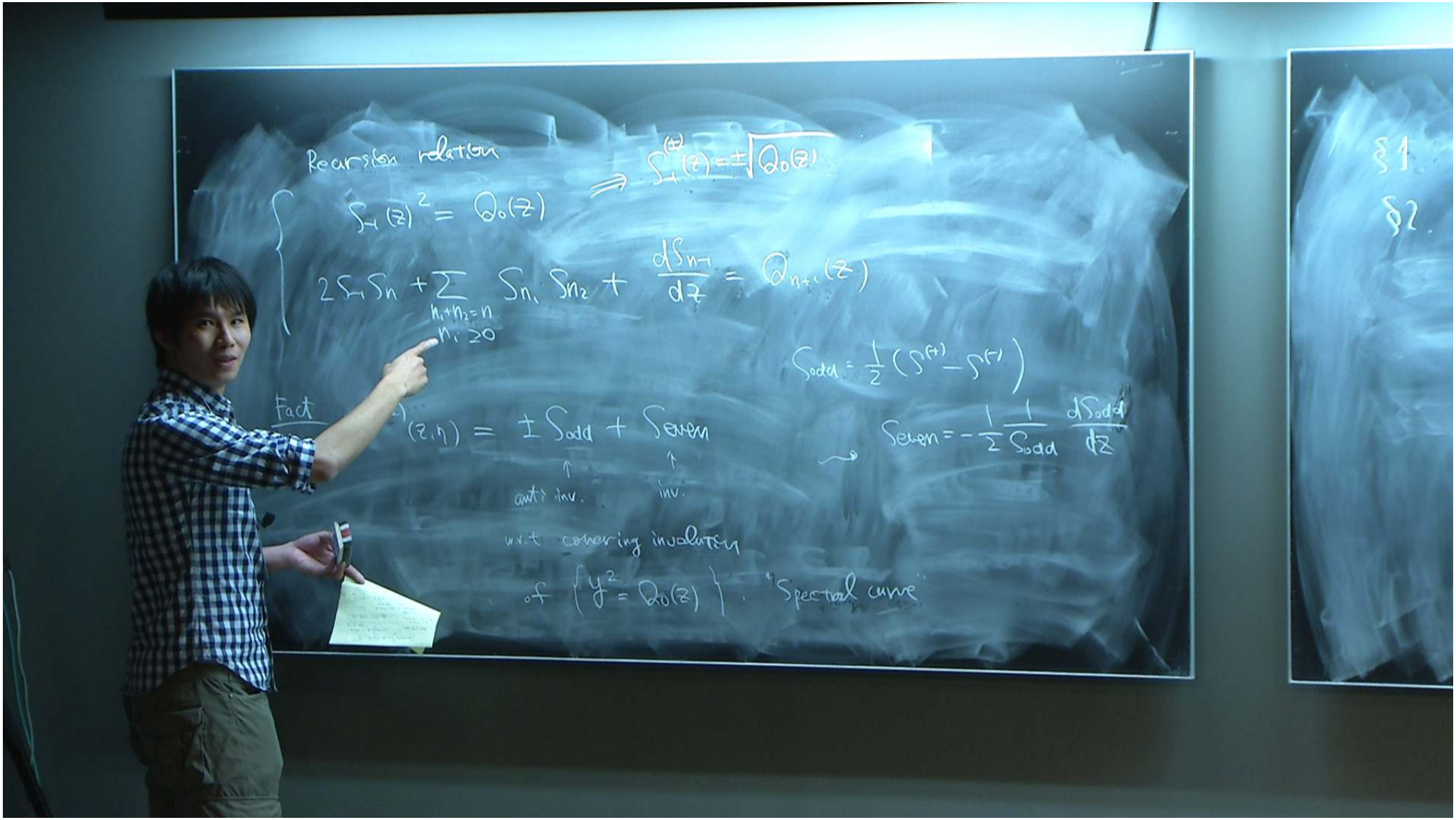
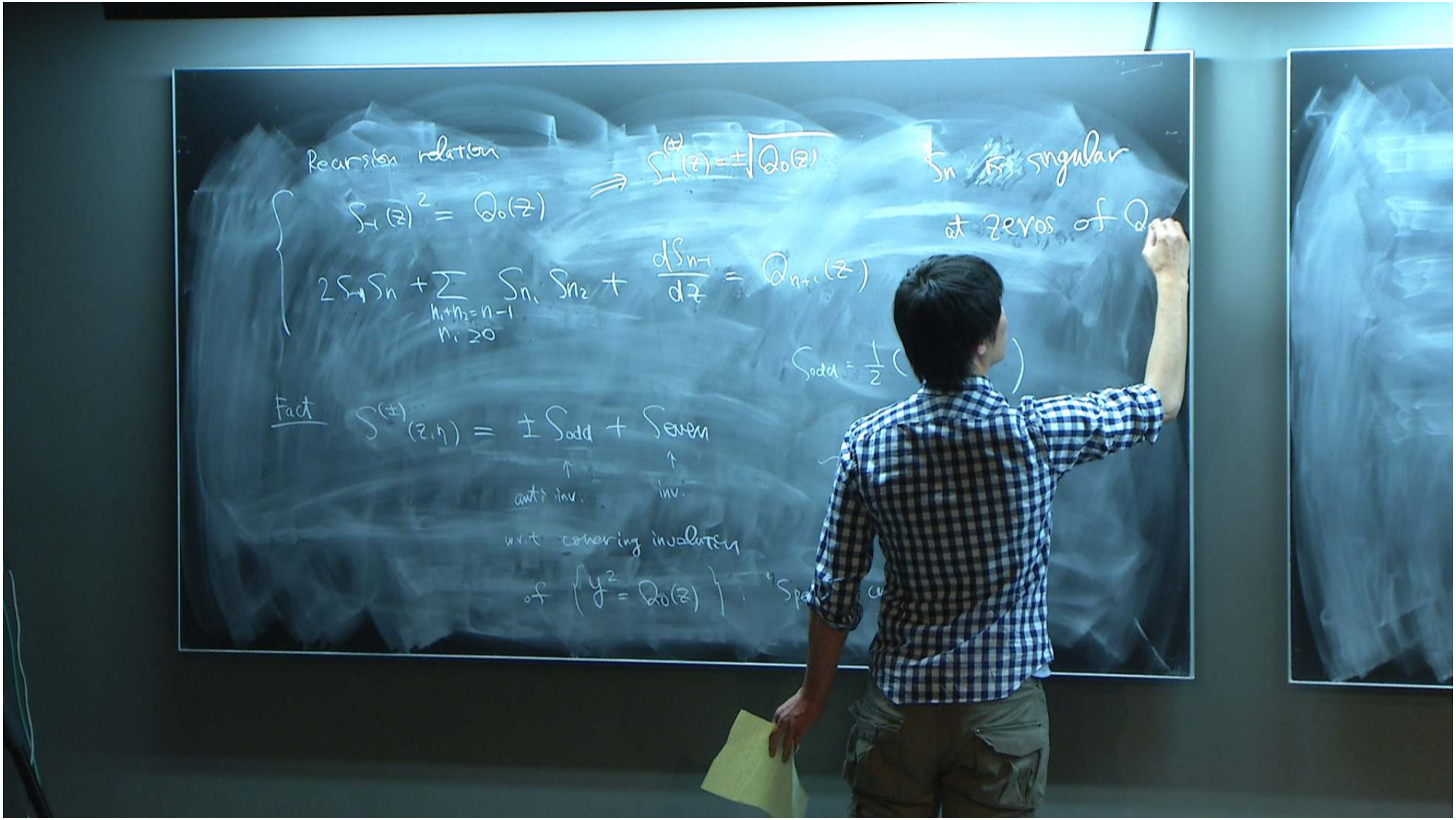
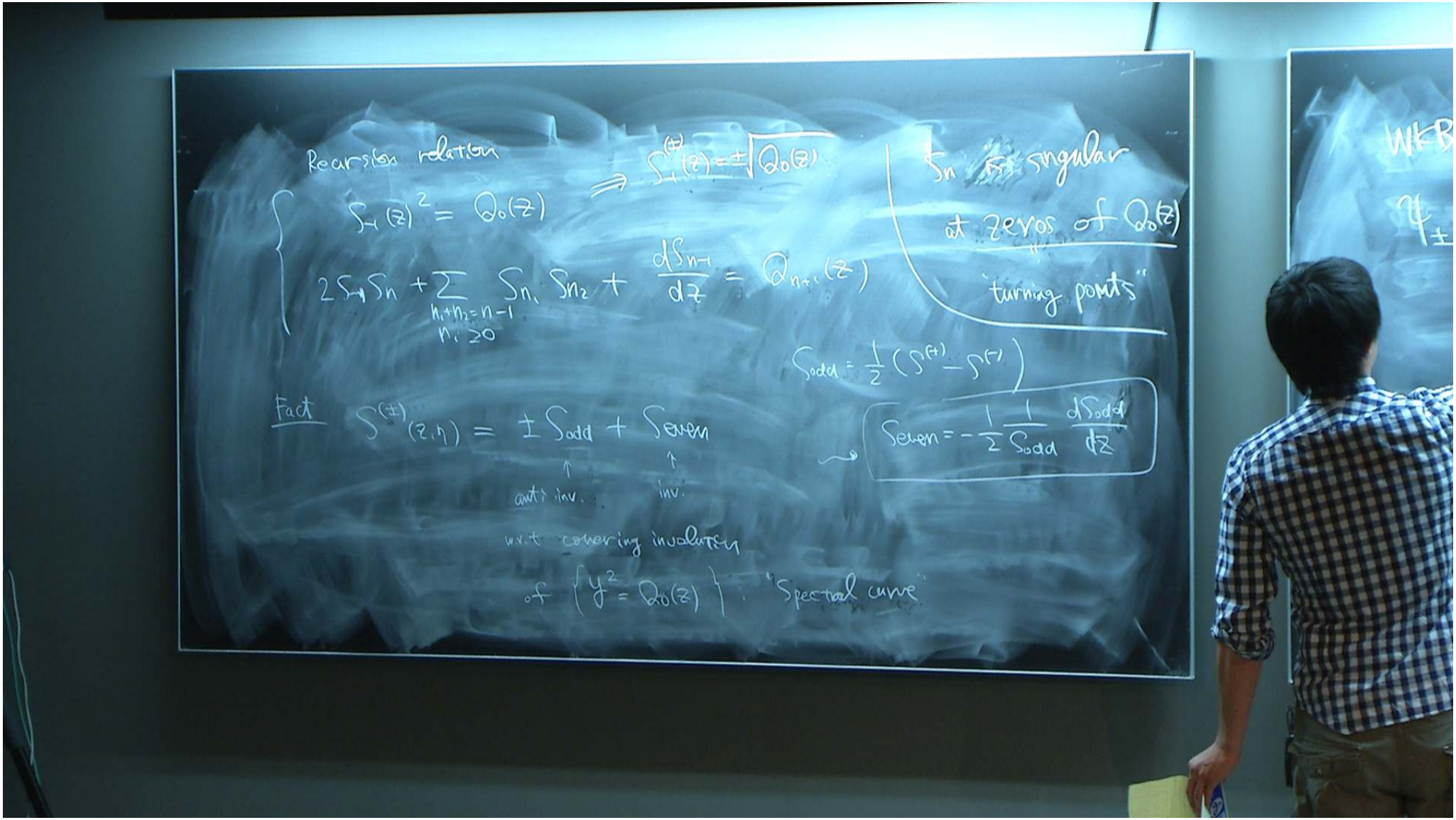
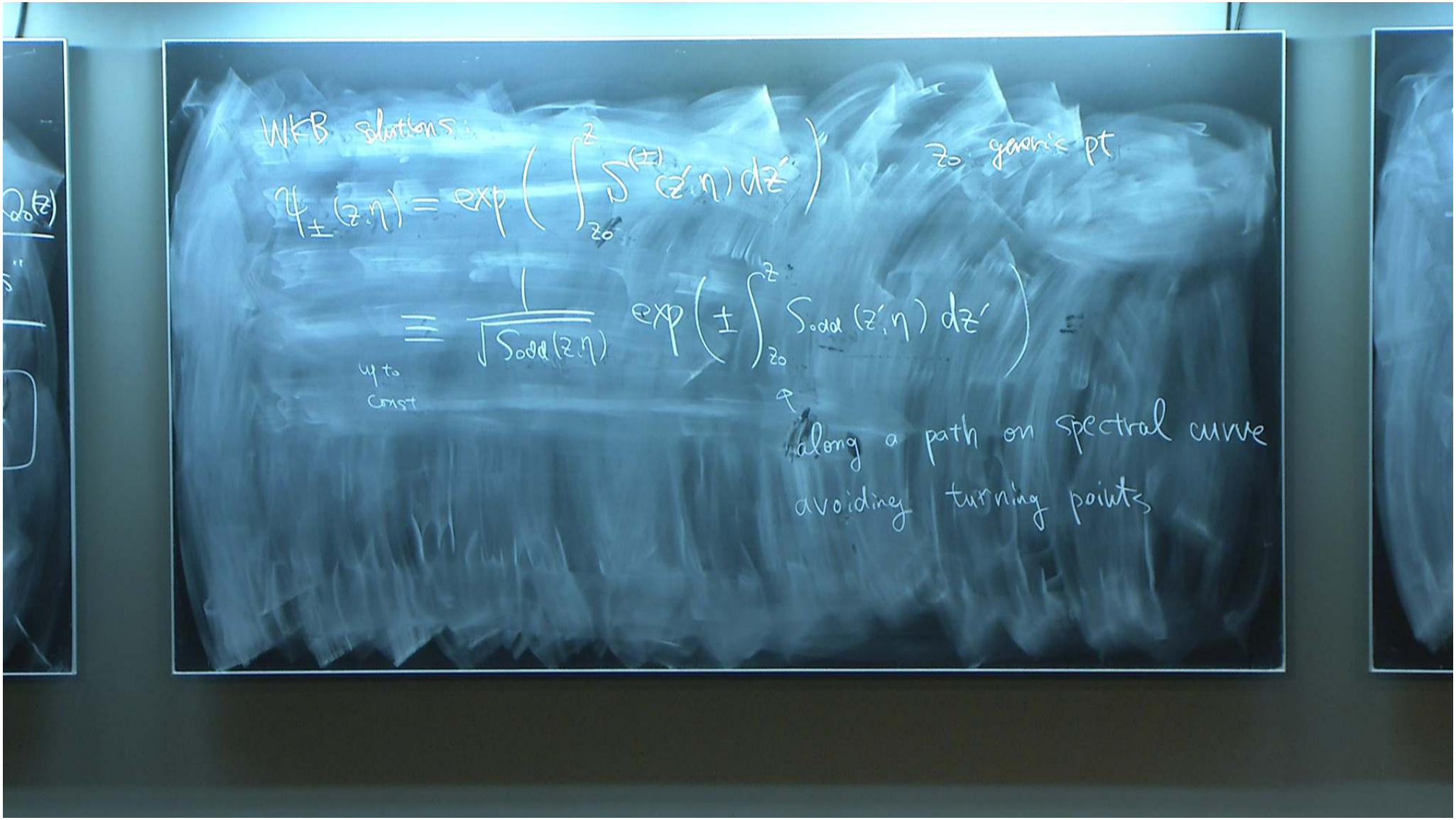
rational functions

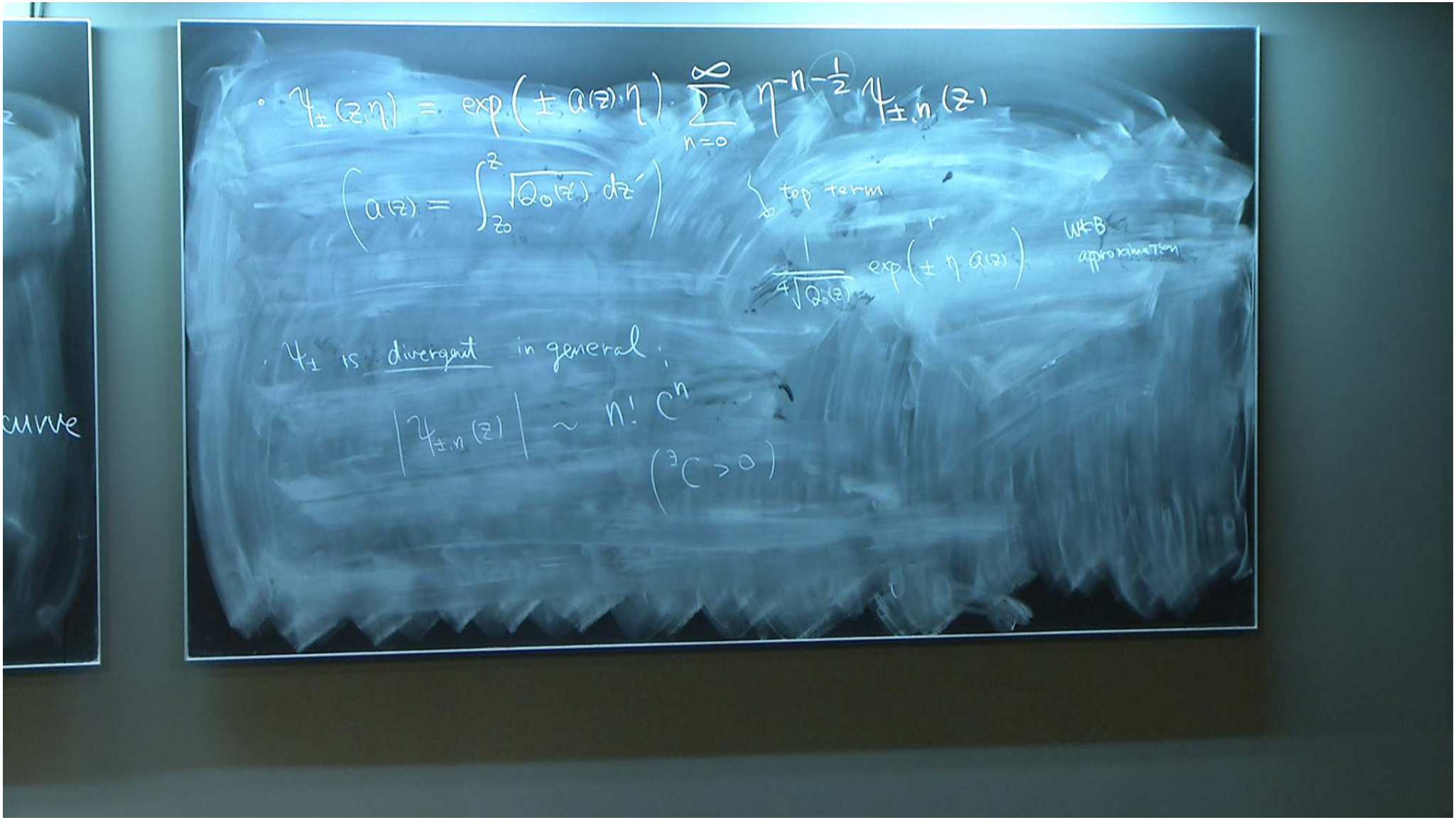

§1 WKB method
· Schrödinger equation
$\left(\frac{d^2}{dz^2} - \eta^2 Q(z,\eta)\right)\psi(z,\eta) = 0$
z : complex var.
η : large par. $(= 1/\hbar)$
$Q(z,\eta) = Q_0(z) + \eta^{-1}Q_1(z) + \cdots + \eta^{-N}Q_N(z)$
rational functions
Take $\psi = \exp\left(\int^z S\,dz\right)$
$\rightsquigarrow S^2 + \frac{dS}{dz} = \eta^2 Q(z,\eta)$: Riccati eq.
Formal sol : $S(z,\eta) = \eta S_1(z) + S_0(z) + \eta^{-1}S_1(z) + \cdots$

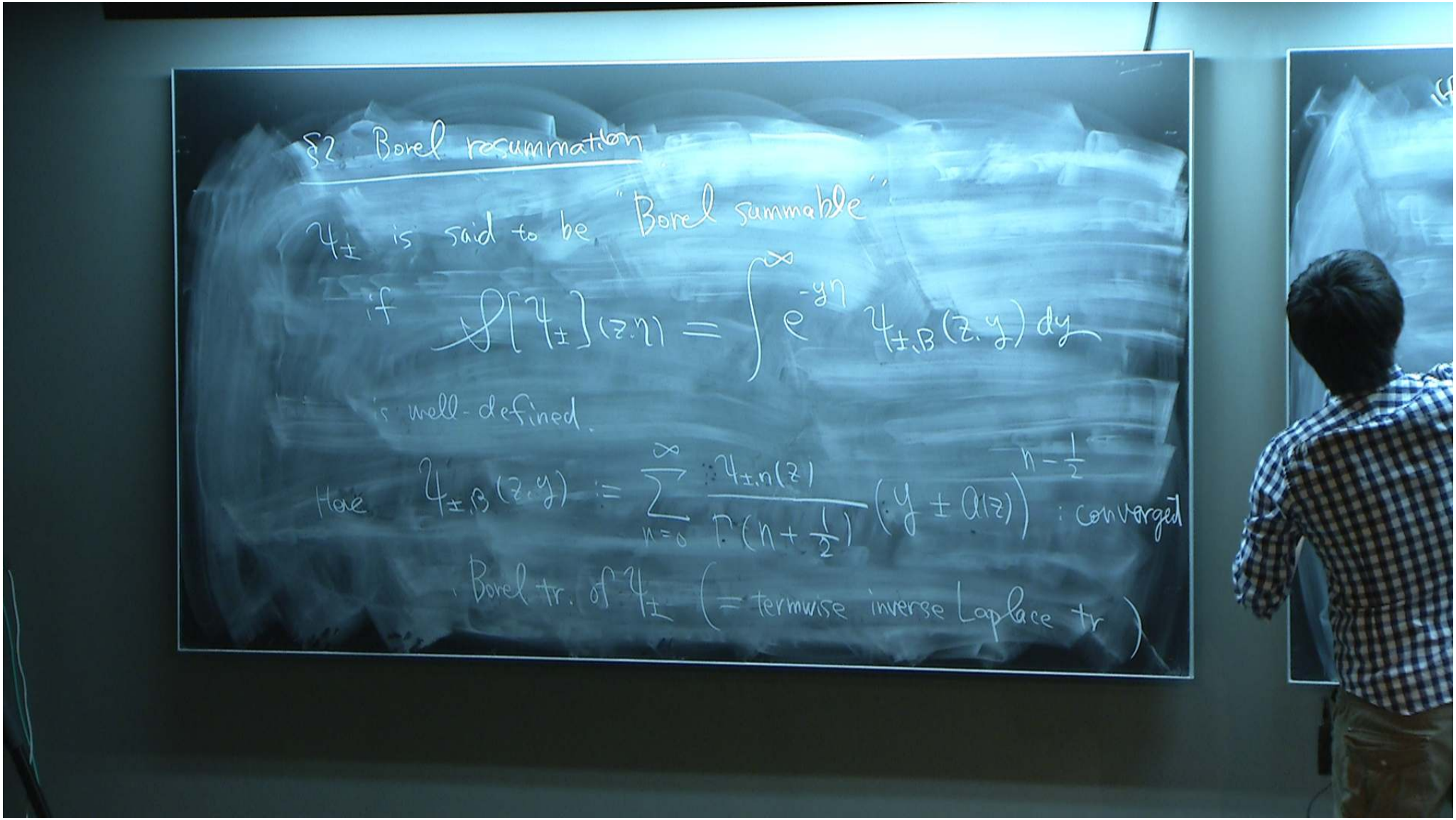

Recursion relation
$S_{-1}(z)^2 = Q_0(z)$
$\Rightarrow S_{-1}^{(\pm)}(z) = \pm\sqrt{Q_0(z)}$
$2S_{-1}S_n + \sum_{\substack{n_1+n_2=n \\ n_i \geq 0}} S_{n_1} S_{n_2} + \frac{dS_{n-1}}{dz} = Q_{n+1}(z)$
$S_{odd} = \frac{1}{2}(S^{(+)} - S^{(-)})$
Fact
$S_{even} = -\frac{1}{2}\frac{1}{S_{odd}}\frac{dS_{odd}}{dz}$
$(z,\eta) = \pm S_{odd} + S_{even}$
anti-inv.
inv.
w.r.t covering involution
of $\{y^2 = Q_0(z)\}$ "Spectral curve"
§1.
§2.

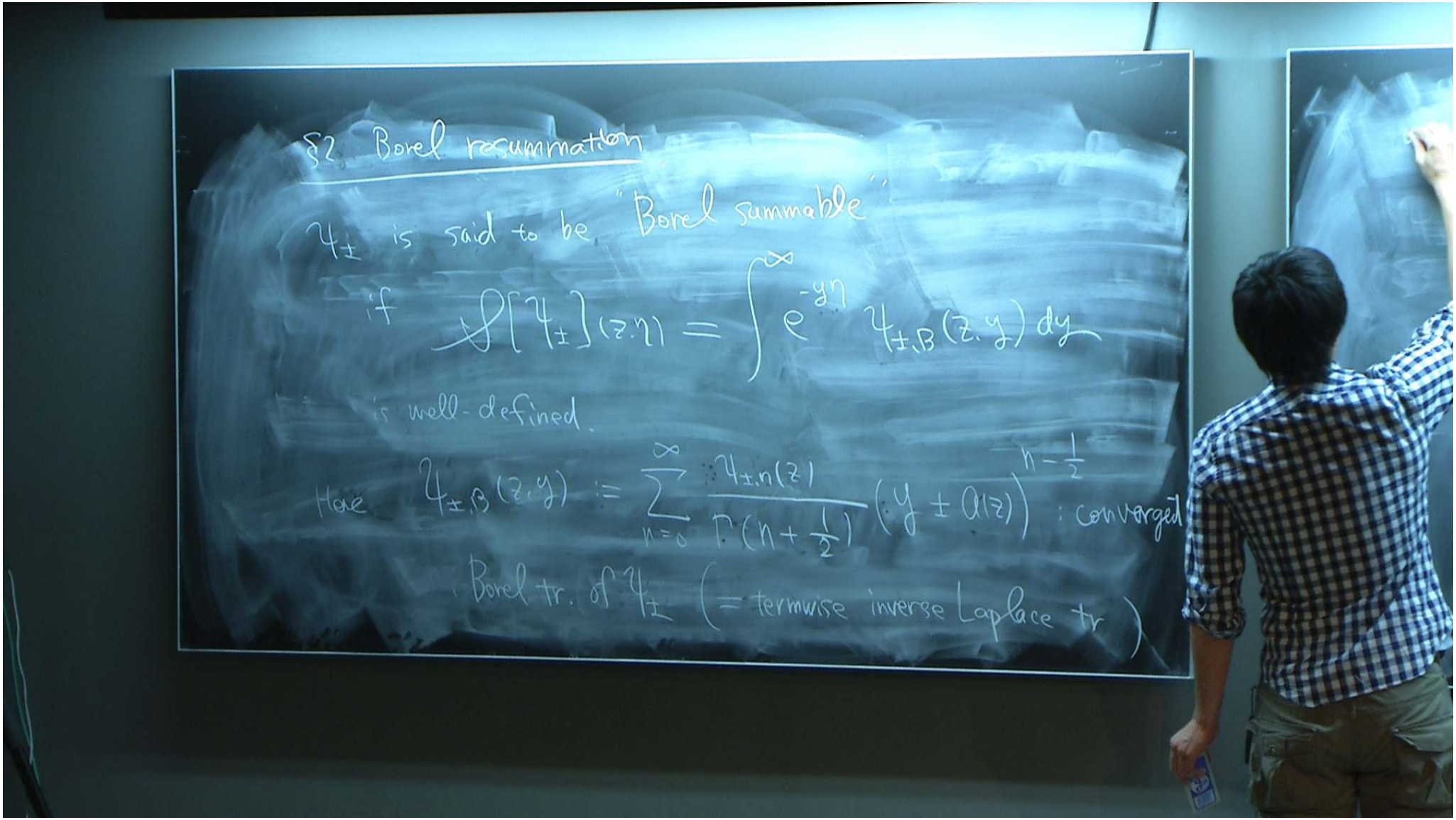

Recursion relation
$S_{-1}(z)^2 = Q_0(z)$
$\Rightarrow S_{-1}^{(\pm)}(z) = \pm\sqrt{Q_0(z)}$
S_n ... singular
at zeros of Q_0
$2S_{-1}S_n + \sum_{\substack{n_1+n_2=n-1 \\ n_i \geq 0}} S_{n_1} S_{n_2} + \frac{dS_{n-1}}{dz} = Q_{n+1}(z)$
$S_{odd} = \frac{1}{2}($ $)$
Fact $S^{(\pm)}(z,\eta) = \pm S_{odd} + S_{even}$
↑ anti-inv.
↑ inv.
w.r.t. covering involution
of $\{y^2 = Q_0(z)\}$


Recursion relation
$S_{-1}(z)^2 = Q_0(z)$
$\Rightarrow S_{-1}^{(\pm)}(z) = \pm\sqrt{Q_0(z)}$
$2S_{-1}S_n + \sum_{\substack{n_1+n_2=n-1 \\ n_i \geq 0}} S_{n_1} S_{n_2} + \frac{dS_{n-1}}{dz} = Q_{n+1}(z)$
S_n is singular at zeros of $Q_0(z)$
"turning points"
$S_{odd} = \frac{1}{2}(S^{(+)} - S^{(-)})$
Fact $S^{(\pm)}(z,\eta) = \pm S_{odd} + S_{even}$
anti-inv.
inv.
$S_{even} = -\frac{1}{2}\frac{1}{S_{odd}}\frac{dS_{odd}}{dz}$
w.r.t covering involution
of $\{y^2 = Q_0(z)\}$ "Spectral curve"


WKB solutions:
$\psi_\pm(z,\eta) = \exp\left(\int_{z_0}^{z} S^{(\pm)}(z',\eta)\,dz'\right)$
z_0 generic pt
$\equiv \frac{1}{\sqrt{S_{odd}(z,\eta)}} \exp\left(\pm\int_{z_0}^{z} S_{odd}(z',\eta)\,dz'\right)$
up to Const
along a path on spectral curve avoiding turning points


$\psi_\pm(z,\eta) = \exp\left(\pm a(z)\cdot\eta\right)\cdot\sum_{n=0}^{\infty}\eta^{-n-\frac{1}{2}}\psi_{\pm,n}(z)$
$\left(a(z) = \int_{z_0}^{z}\sqrt{Q_0(z')}\,dz'\right)$
top term
$\frac{1}{\sqrt[4]{Q_0(z)}}\exp\left(\pm\eta\,a(z)\right)$
WKB approximation
$\psi_\pm$ is divergent in general:
$\left|\psi_{\pm,n}(z)\right| \sim n!\,C^n$
$(\exists C>0)$
curve


§2 Borel resummation
$\varphi_\pm$ is said to be "Borel summable"
if $\mathcal{B}[\varphi_\pm](z,\eta) = \int_0^\infty e^{-y\eta}\, \varphi_{\pm,B}(z,y)\, dy$
is well-defined.
Here $\varphi_{\pm,B}(z,y) := \sum_{n=0}^{\infty} \frac{\varphi_{\pm,n}(z)}{\Gamma(n+\frac{1}{2})} (y \pm a(z))^{n-\frac{1}{2}}$: converged
Borel tr. of $\varphi_\pm$ (= termwise inverse Laplace tr)


§2 Borel resummation
$\varphi_\pm$ is said to be "Borel summable"
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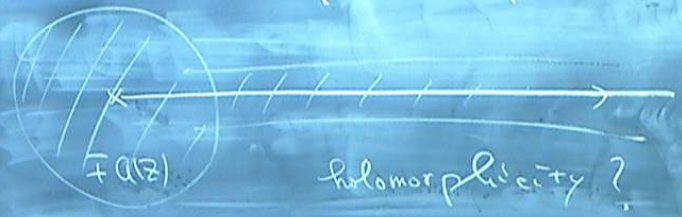
Borel tr. of $\varphi_\pm$ (= termwise inverse Laplace tr.)

$|\psi_{\pm}(z,y)| \leq C_1 e^{C_2|y|}$



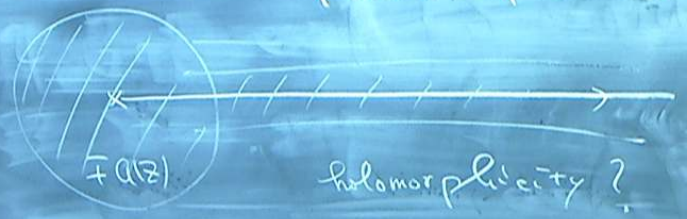
If $\psi_{\pm}$ is Borel summable

then $\mathcal{S}[\psi_{\pm}](z,\eta)$ gives a analytic solution of Sch eq.

(not formal)

and $\mathcal{S}[\psi_{\pm}](z,\eta) \underset{\text{asymp. exp}}{\sim} \psi_{\pm}(z,\eta)$ for $\eta \to +\infty$

$|\psi_{\pm}(z,y)| \leq C_1 e^{C_2|y|}$



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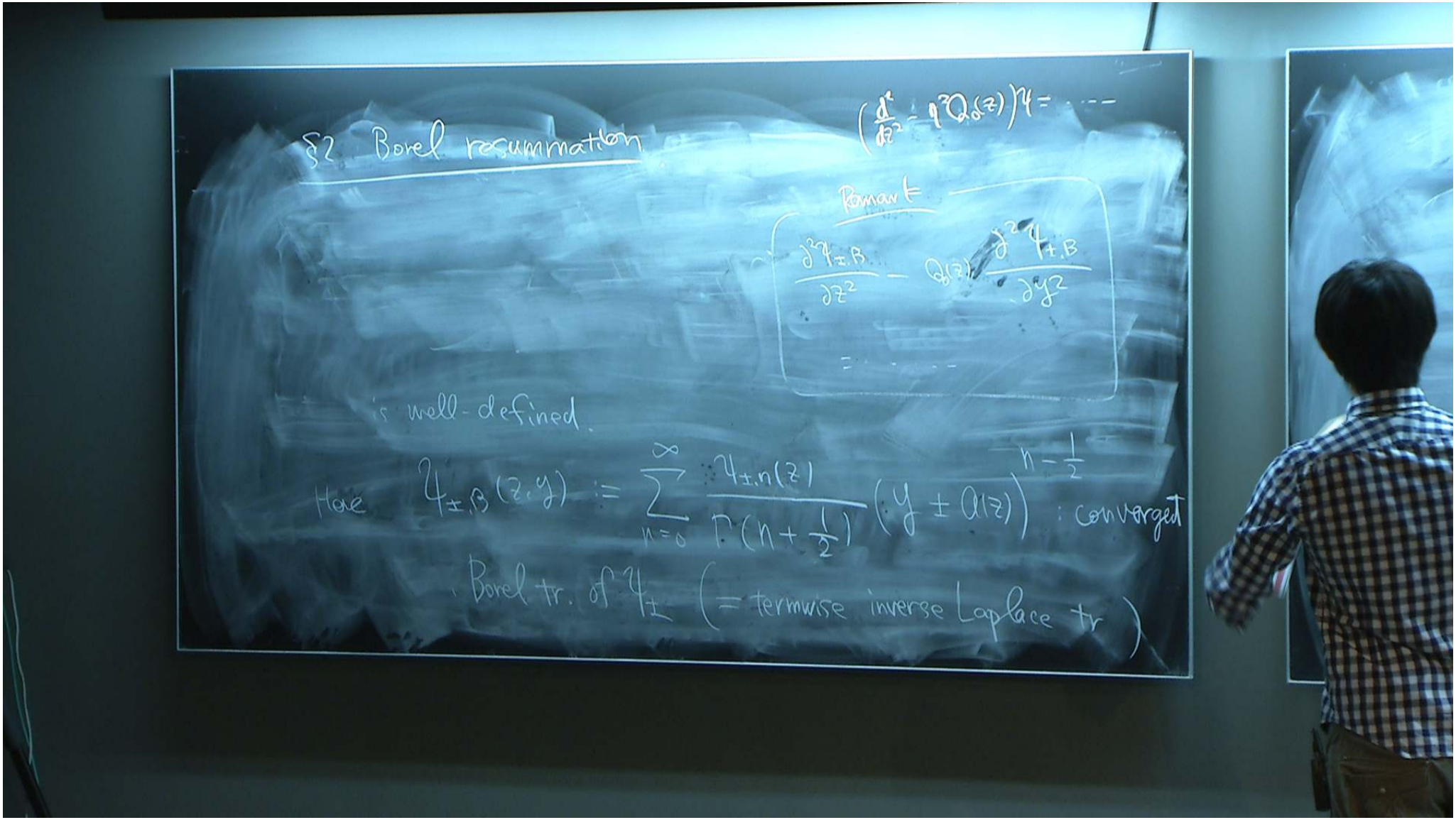
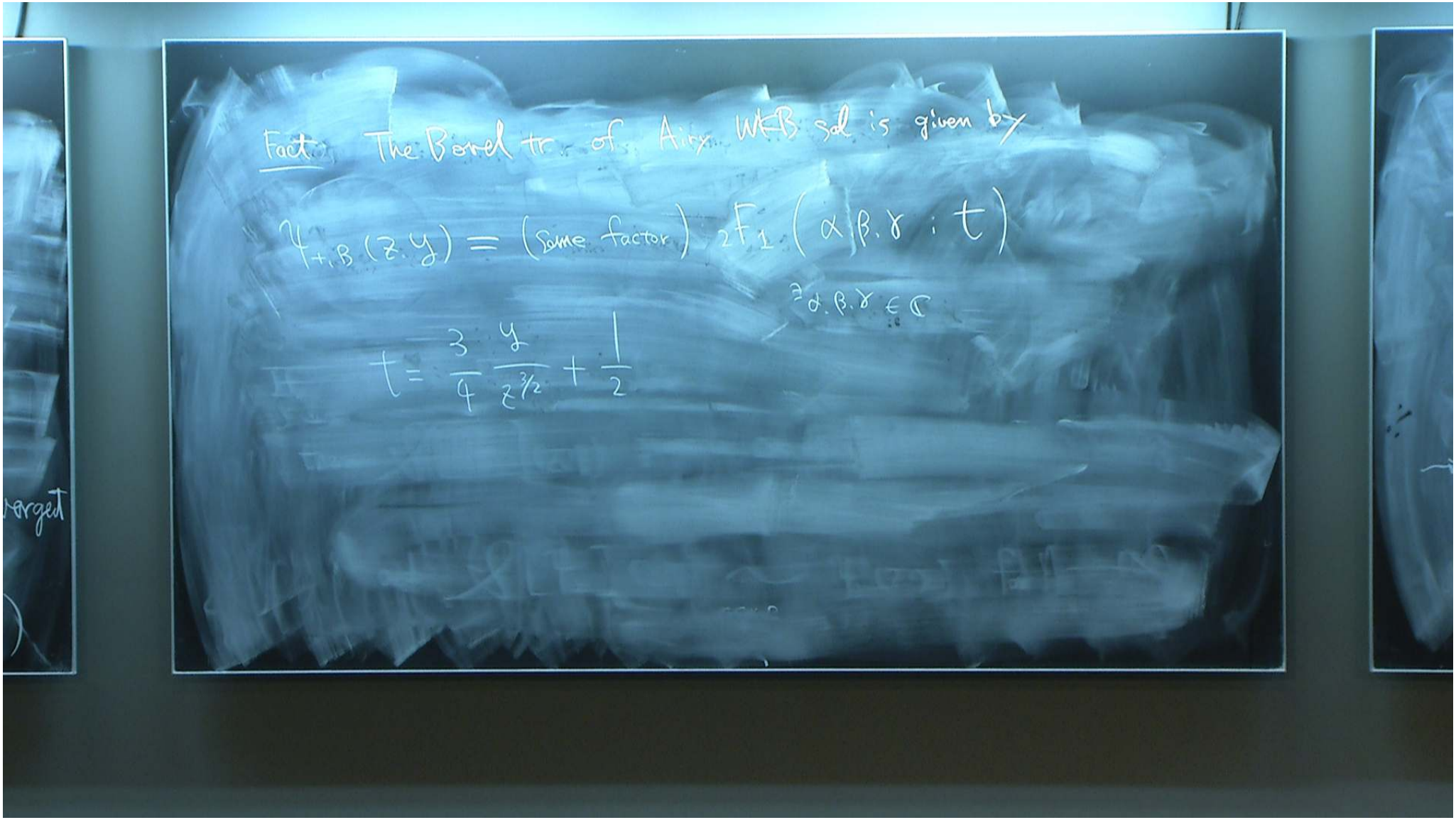
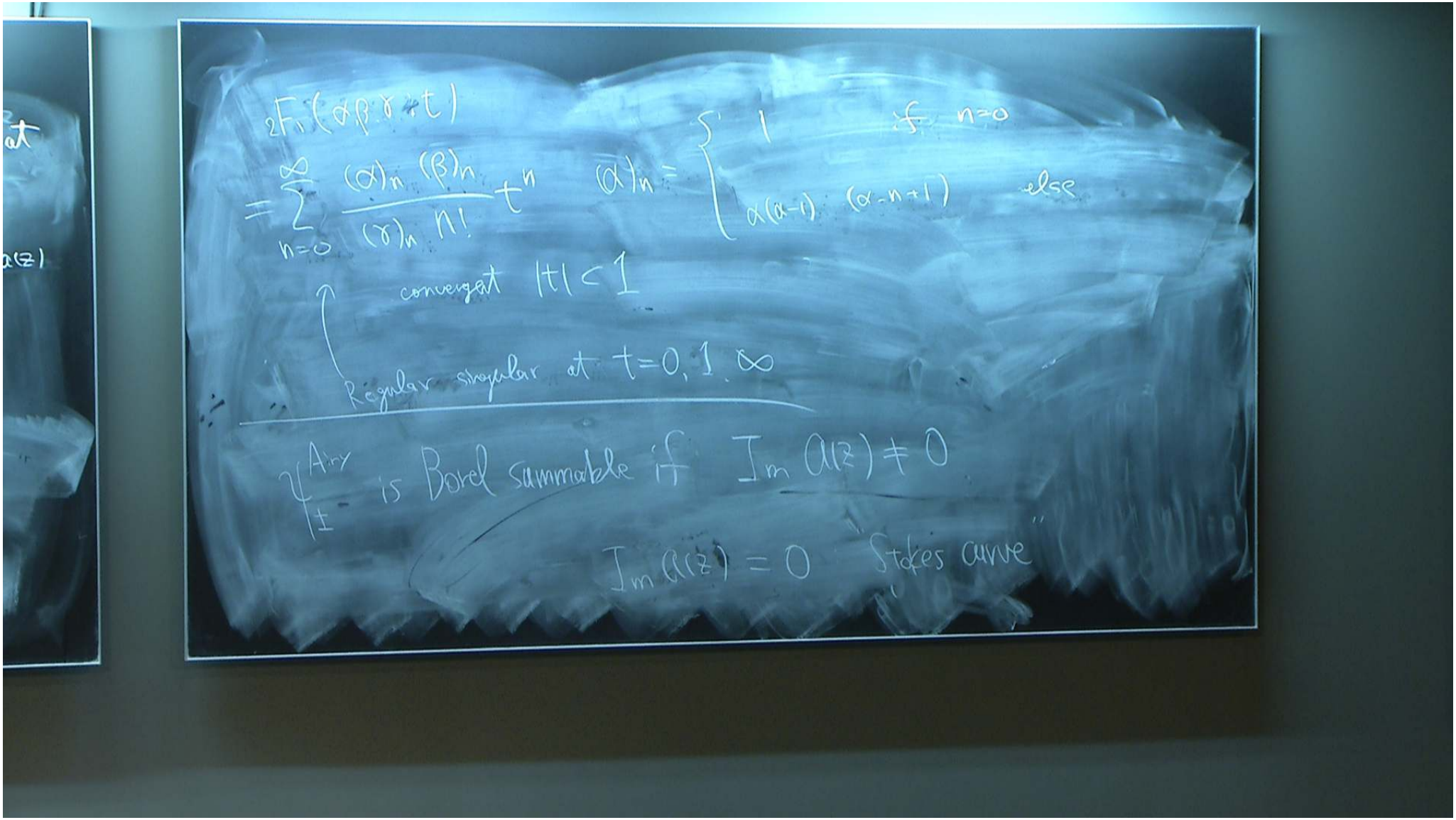
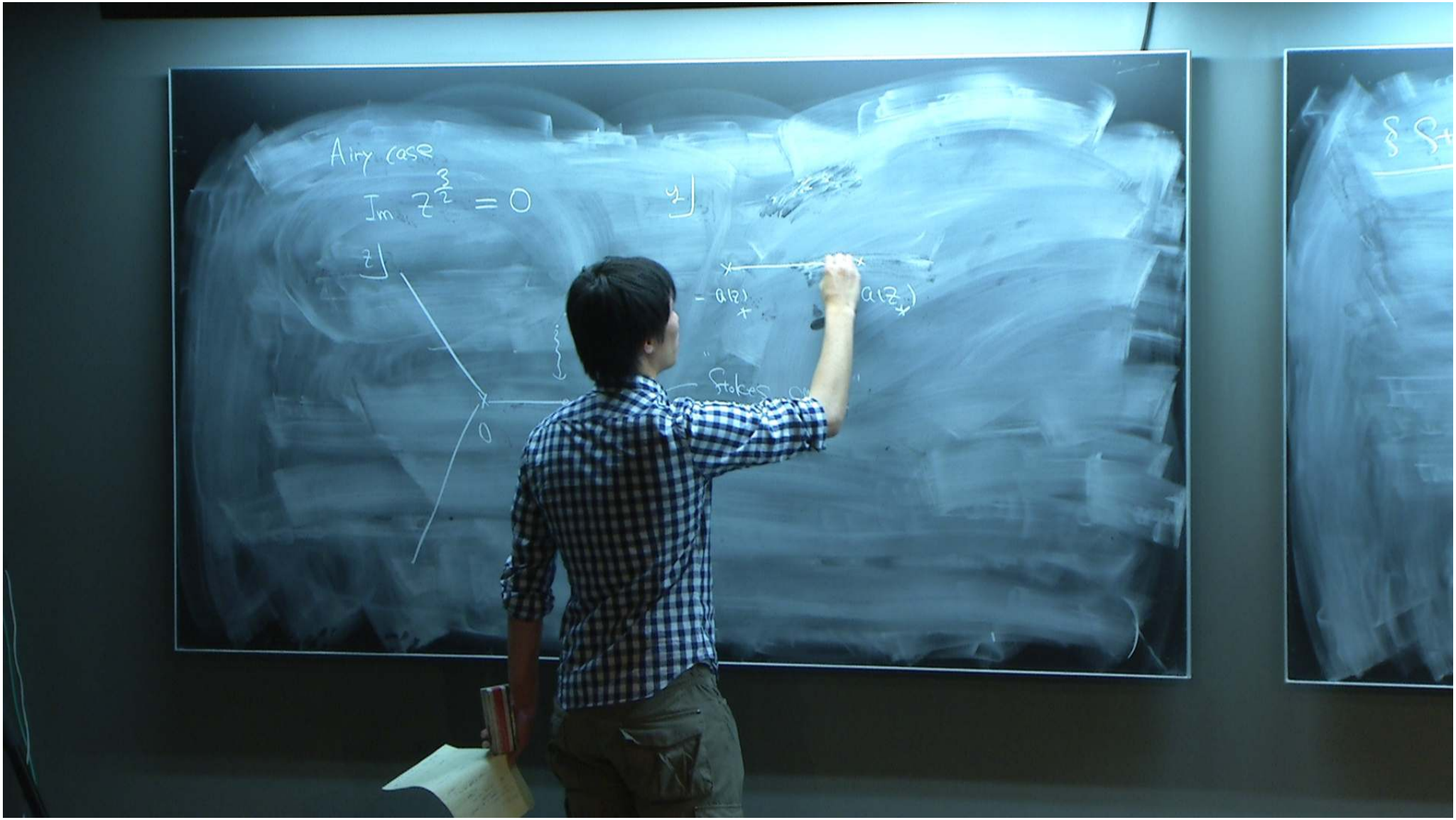
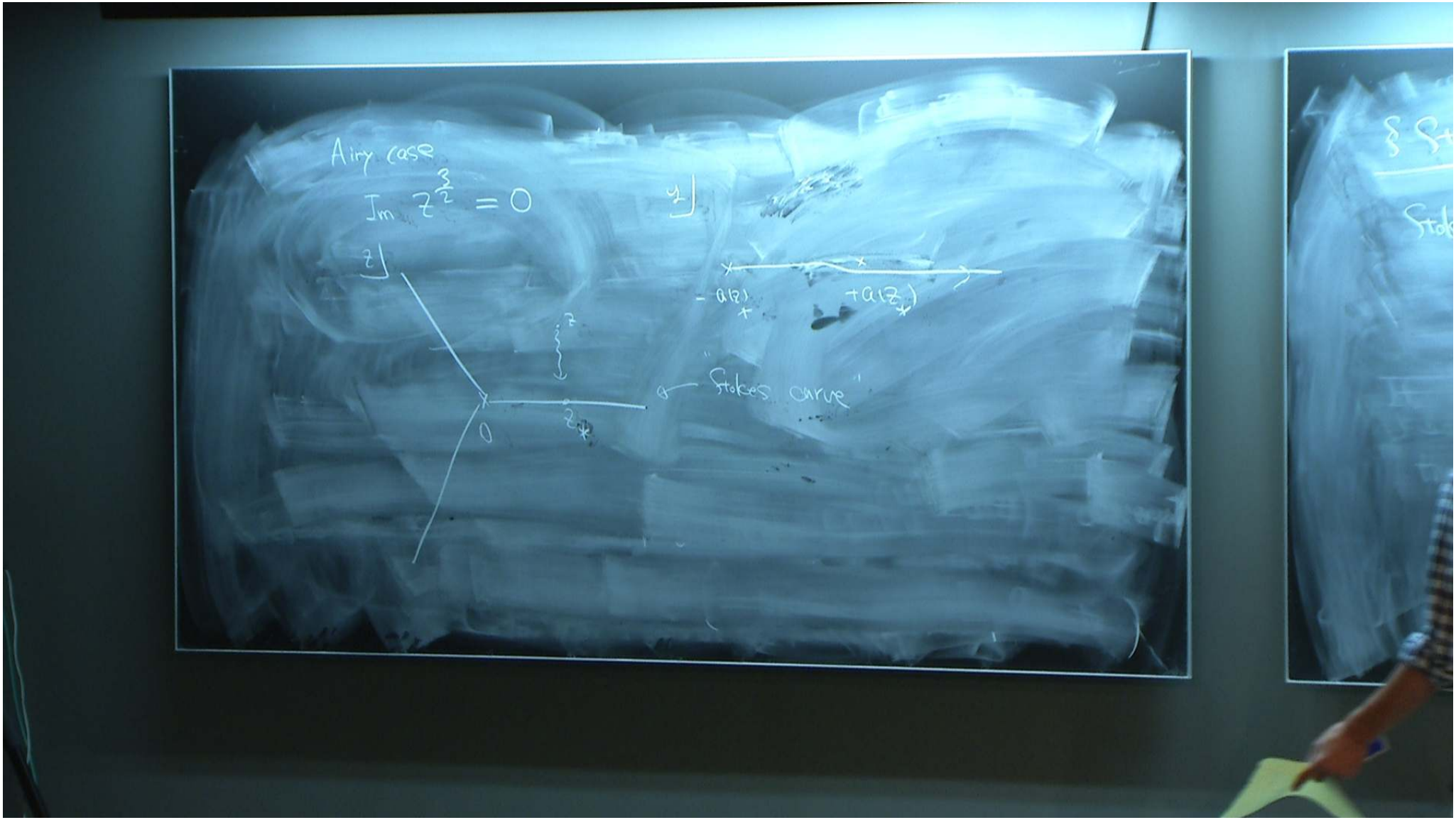
asymp. exp

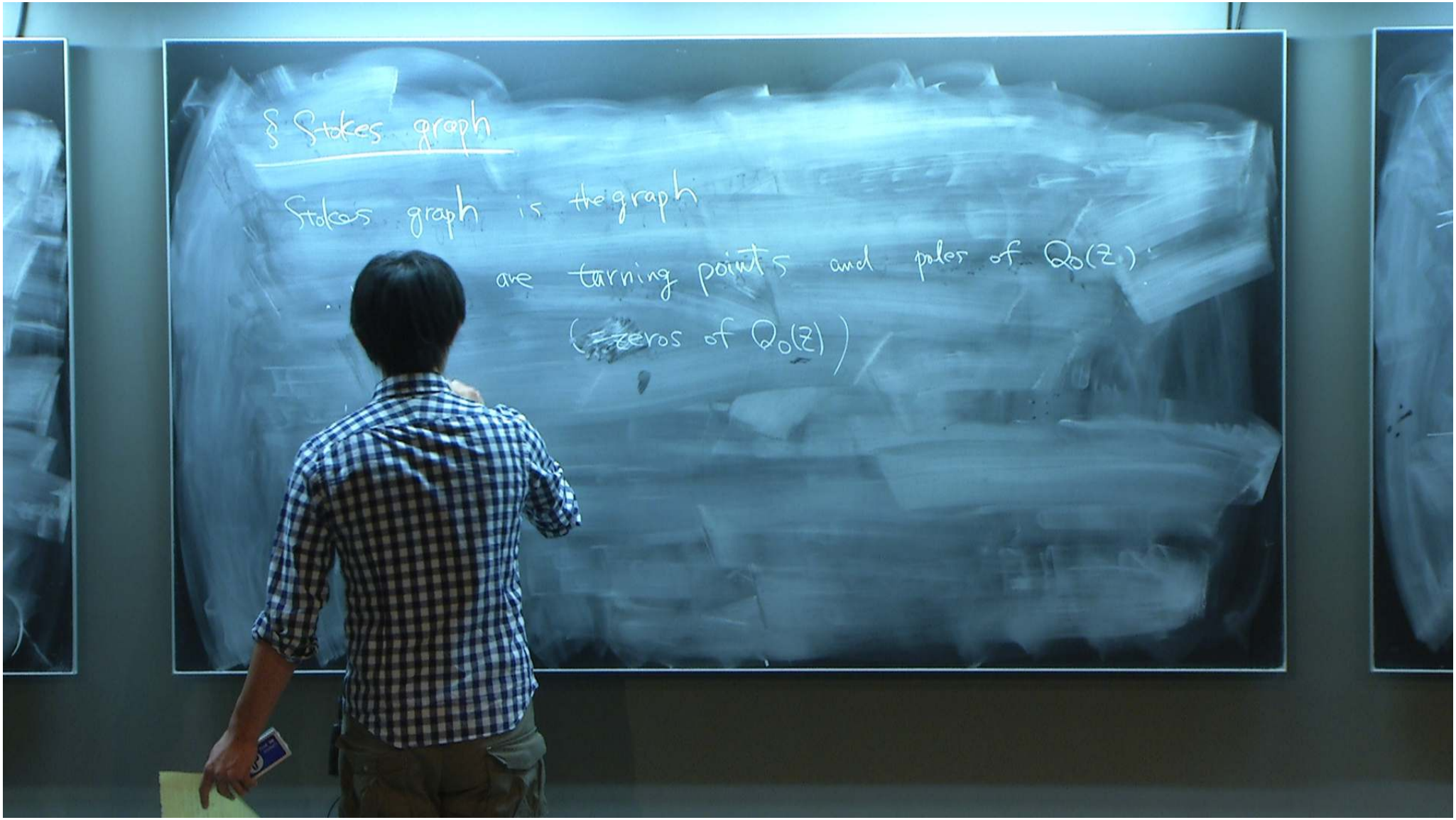

§2 Borel resummation
Remark
is well-defined.
Borel tr. of $\psi_\pm$ (= termwise inverse Laplace tr.)

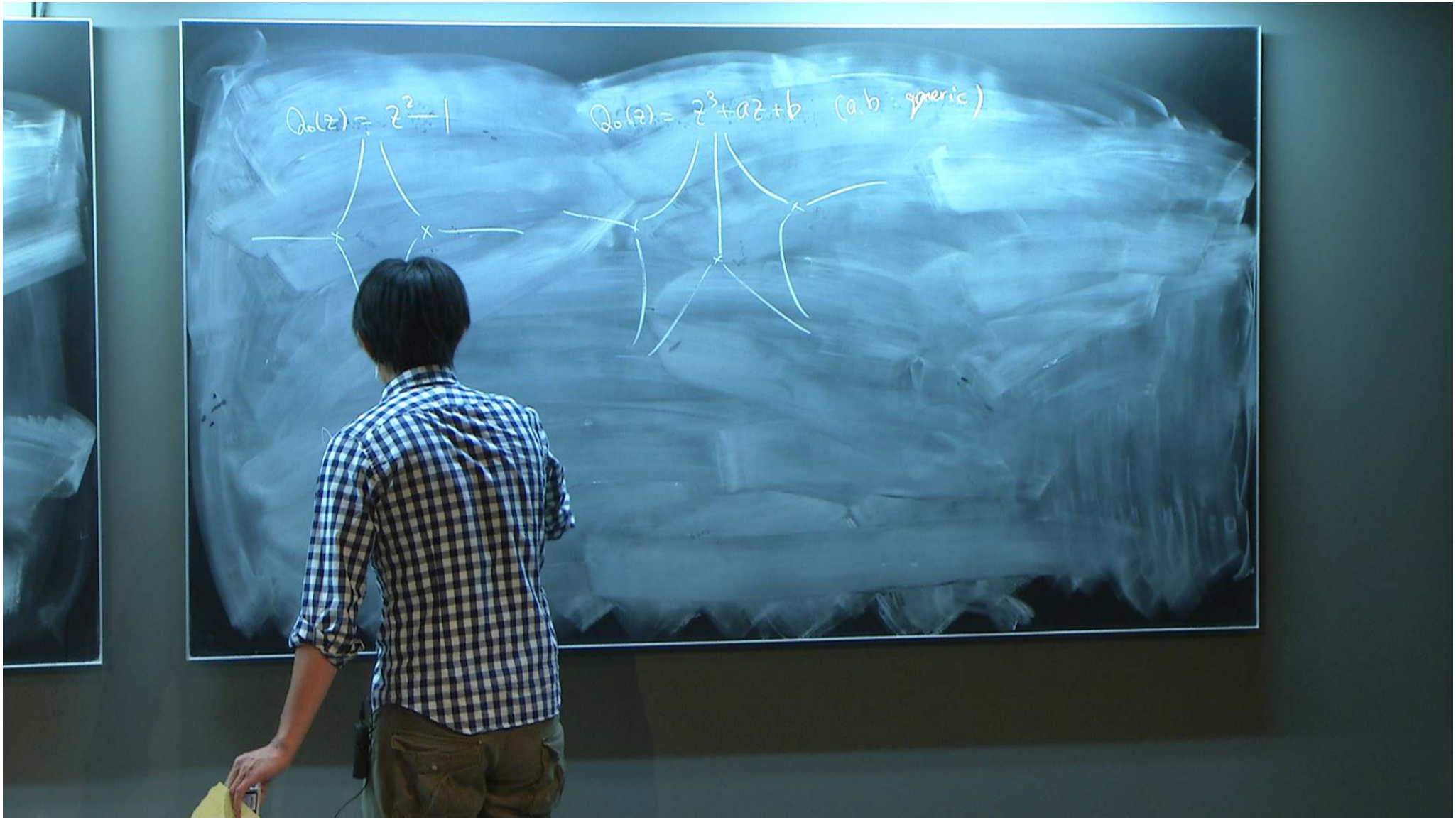

Fact: The Borel tr. of Airy WKB sol is given by
$\psi_{+,\beta}(z,y) = (\text{some factor})\ {}_2F_1(\alpha,\beta,\gamma;t)$
$\exists \alpha,\beta,\gamma \in \mathbb{C}$
$t = \frac{3}{4}\frac{y}{z^{3/2}} + \frac{1}{2}$

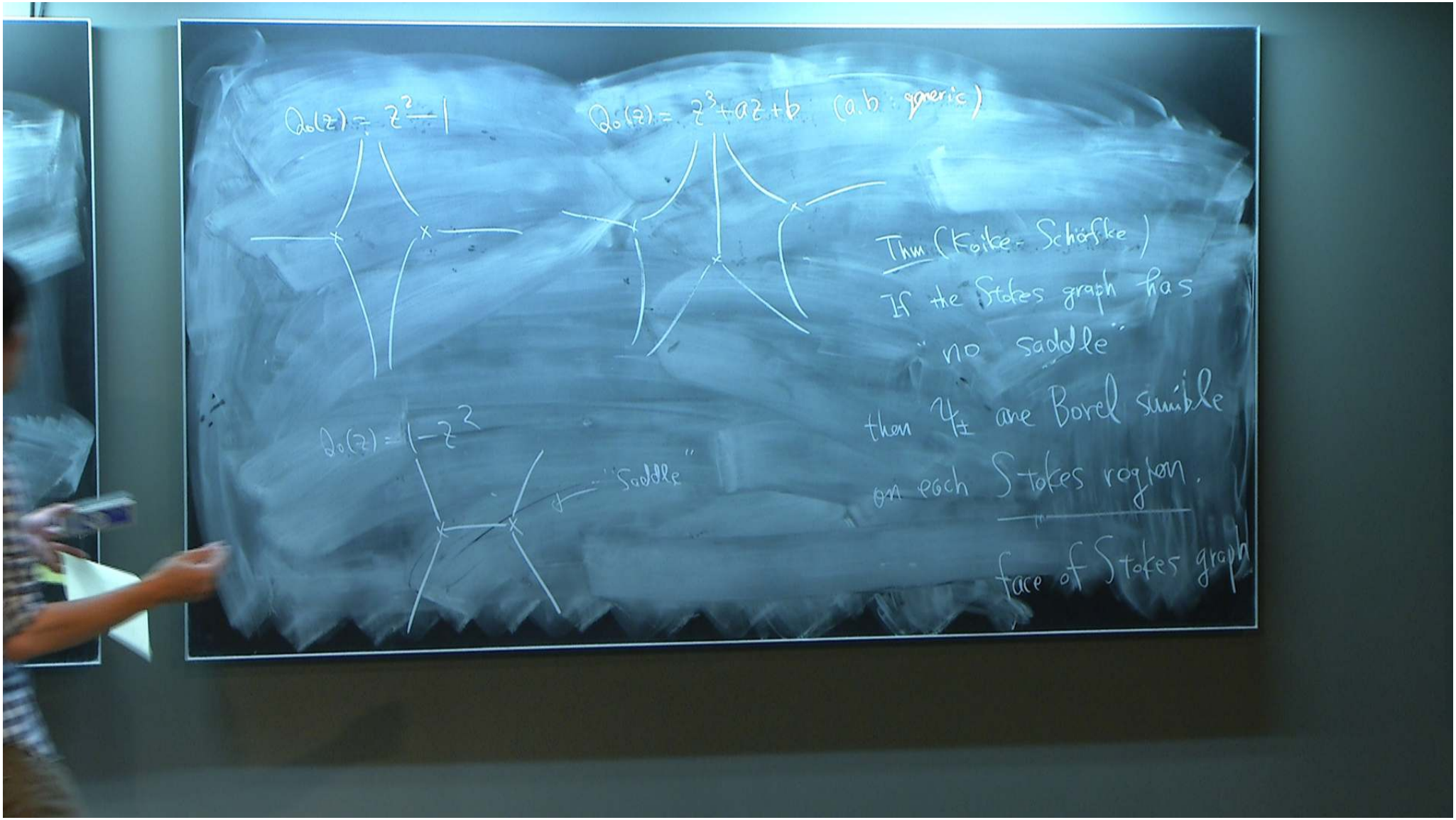

${}_2F_1(\alpha,\beta,\gamma;t)$
$= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} t^n$
$(\alpha)_n = \begin{cases} 1 & \text{if } n=0 \\ \alpha(\alpha-1)\cdots(\alpha-n+1) & \text{else} \end{cases}$
convergent $|t|<1$
Regular singular at $t=0, 1, \infty$
y_I^{Airy} is Borel summable if $\mathrm{Im}\, a(z) \neq 0$
$\mathrm{Im}\, a(z) = 0$ "Stokes curve"


Airy case
$\mathrm{Im}\, z^{\frac{3}{2}} = 0$
z
y
$-a(z)$
$a(z)$
Stokes
0


Airy case
$\mathrm{Im}\, z^{\frac{3}{2}} = 0$
z
y
$-Q(z)$
$+Q(z)$
Stokes curve
0


§ Stokes graph
Stokes graph is the graph
are turning points and poles of $Q_0(z)$
(= zeros of $Q_0(z)$)


$Q_0(z) = z^2 - 1$
$Q_0(z) = z^3 + az + b$ (a, b generic)


$Q_0(z) = z^2 - 1$
$Q_0(z) = z^3 + az + b$ (a, b generic)
$Q_0(z) = -z^3$
"saddle"
Thm (Koike - Schäfke)
If the Stokes graph has
"no saddle"
then $Y_\pm$ are Borel summable
on each Stokes region.
face of Stokes graph