Title: Defects and degeneracies in supersymmetry protected phases

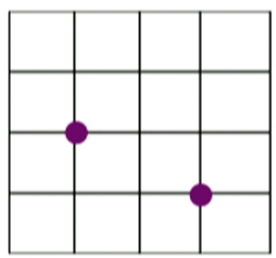
Date: May 19, 2015 03:30 PM

URL: http://pirsa.org/15050037

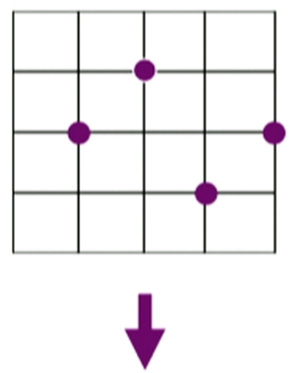
Abstract: <h3>We analyse a class of 1D lattice models, known as M_k models, which are characterised by an order-k clustering of spin-less fermions and by N=2 lattice supersymmetry. We show a fundamental connection to Z_k parafermions and identify a class of (bulk or edge) defects, which are in one-to-one correspondence with so-called spin fields in the parafermion CFT. In the gapped regime, injecting such defects leads to ground state degeneracies that are protected by the supersymmetry. The defects, which are closely analogous to quasi-holes over the fermonic Read-Rezayi quantum Hall states, display characteristic fusion rules, which are of Ising type for k=2 and of Fibonacci type for k=3.</h3>

Fermi phases

challenge: understand quantum phases of strongly repelling lattice fermions at intermediate densities



Fermi liquid



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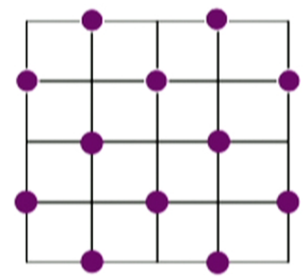
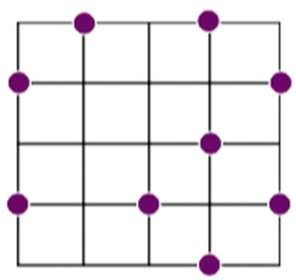


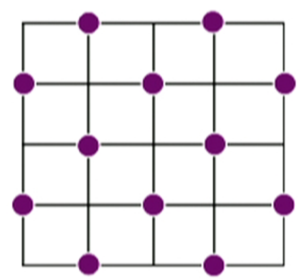
Mott insulator

Fermi phases

prototypical model system: lattice fermions with nearest neighbor exclusion and kinetic terms



Stripe formation?



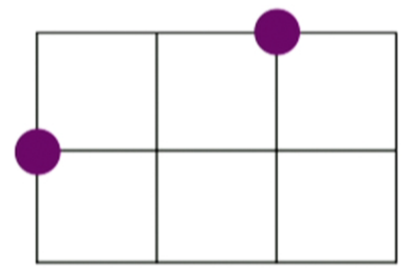
Mott insulator

Zhang–Henley, 2003

Supersymmetric phases

name of the game:

- lattice models for fermions with exclusion and kinetic terms, tuned to be supersymmetric



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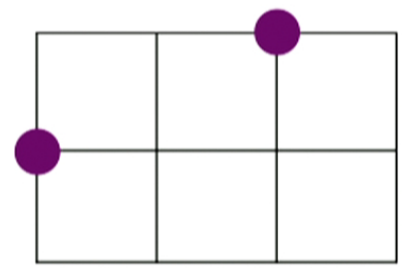

key features:

- supersymmetry implies delicate balance between kinetic and potential terms, leading to interesting ground state structure & degeneracies
- analytic control due to such tools as the Witten index and cohomology techniques

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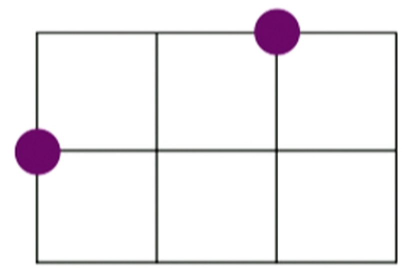

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- analytic control due to such tools as the Witten index and cohomology techniques

Susy phases: degeneracies

# of susy groundstates (M_1 model):

Susy phases: degeneracies

nature of susy groundstates (M_1 model):

- 1D chain, critical, filling 1/3

Susy phases: degeneracies

# of susy groundstates ($\mathbf{M}_k$ model):

- 1D closed chain, $L=(k+2)l$: #=k+1
- 1D open chain: #=0 or #=1

but:

- injecting **boundary or bulk defects** in off-critical M_k model on open chain leads to large number of states with energies $E \sim e^{-\alpha L}$
- structure of the resulting **$\mathbf{M}_k$ quantum registers** set by fusion rules deriving from Z_k parafermion CFT

Thessa Fokkema & KjS, arXiv:1504.02421

Supersymmetric QM: definitions

Supercharges Q^+, $Q^- = (Q^+)^+$ and fermion number N_f

$$(Q^+)^2 = 0, \quad (Q^-)^2 = 0, \quad [N_f, Q^\pm] = \pm Q^\pm$$

Hamiltonian defined as

$$H = \left\{ Q^+, Q^- \right\}$$

satisfies

$$[H, Q^+] = [H, Q^-] = 0, \quad [H, N_f] = 0$$

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Supersymmetric QM: spectrum

- $E \geq 0$ for all states
- $E > 0$ states are paired into **doublets**

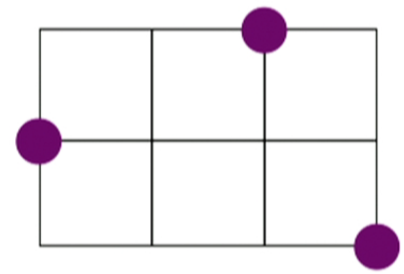
$$\left\{ |\psi>, Q^{+}|\psi> \right\}, \quad Q^{-}|\psi>=0$$

- $E = 0$ iff a state is a **singlet** under supersymmetry

$$Q^{+}|\psi>=Q^{-}|\psi>=0$$

Susy lattice model M_1 , any graph

configurations:
lattice fermions with nearest neighbor exclusion



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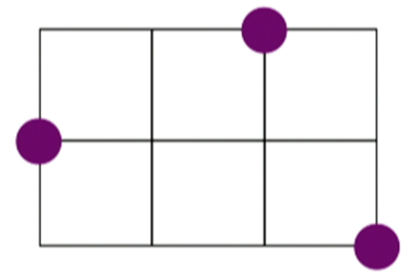


supercharges and hamiltonian

$$Q^+ = \sum_i c_i^\dagger \prod_\delta (1 - n_{i+\delta}), \quad Q^- = (Q^+)^\dagger$$

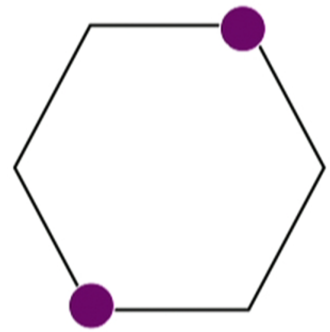
$$n_i = c_i^\dagger c_i$$

$$H = \{Q^+, Q^-\} = H_{kin} + H_{pot}$$

Fendley-KjS-de Boer, 2003

Susy lattice model M_1 , 6 site chain

$$W = \mathrm{Tr}(-1)^{N_f}$$



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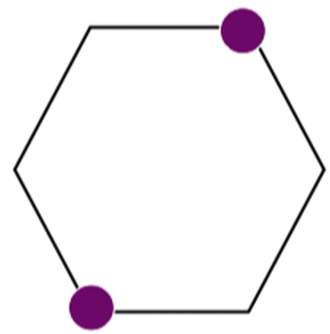


Susy lattice model M_1 , 6 site chain

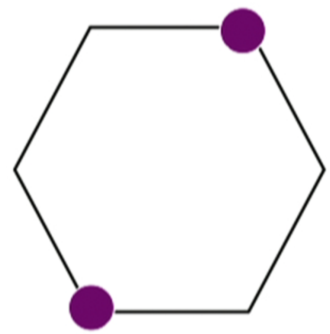
$$W = \mathrm{Tr}(-1)^{N_f}$$

$N_f = 0$: 1 state

$N_f = 1$: 6 states

$N_f = 2$: 9 states

$N_f = 3$: 2 states



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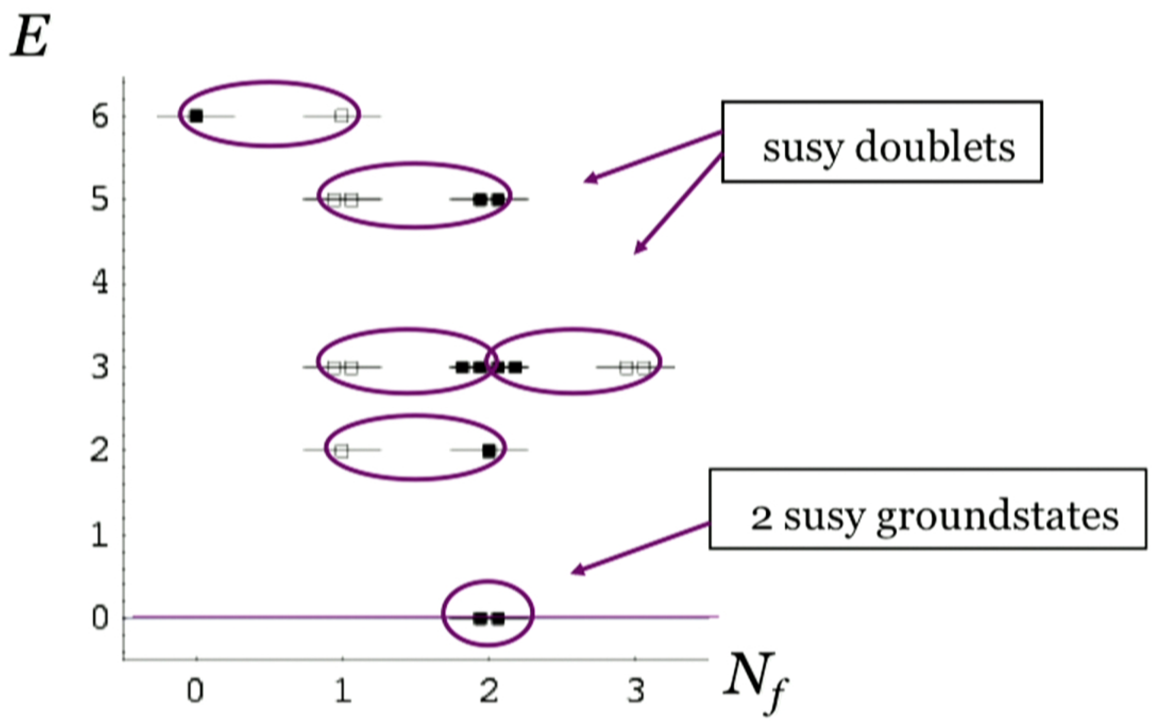

Susy lattice model M_1 , $D=1$

Closed chain

$W=2$ for $L=3l$ sites → find 2 groundstates at filling $f/L=1/3$

Open chain

$W=\pm 1$ for $L=3l, 3l-1$; $W=0$ for $L=3l+1$

Analysis: Bethe Ansatz, mapping to XXZ @ $\Delta=-1/2$

Numerics: ED, finite size spectra

Find that M_1 model is **critical** → CFT

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Susy lattice model M_1 , *D=1*

CFT for open chain spectra

what CFT? → $k=1$ minimal model of
$N=2$ superconformal field theory

$c=1$, write using single scalar field — Waterson, 1986

what modules? → Ramond-sector affine U(1) modules built on charge m vertex operator V_m

$$m = 3f - L - ½$$

Huijse, 2010

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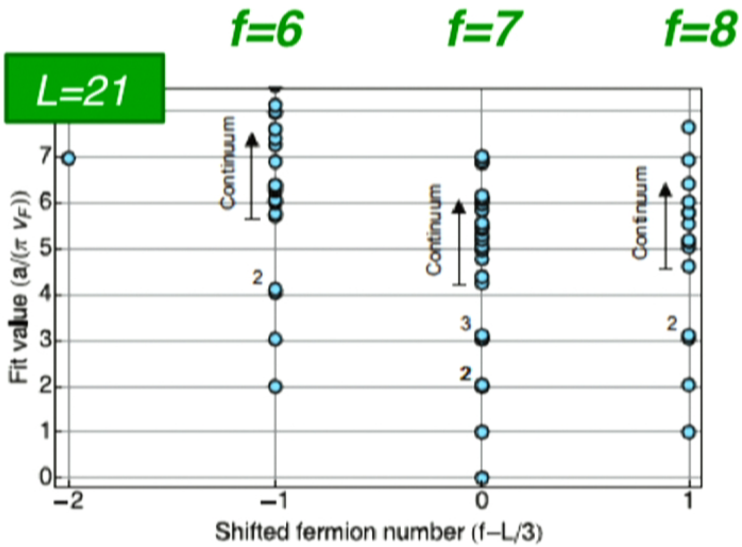
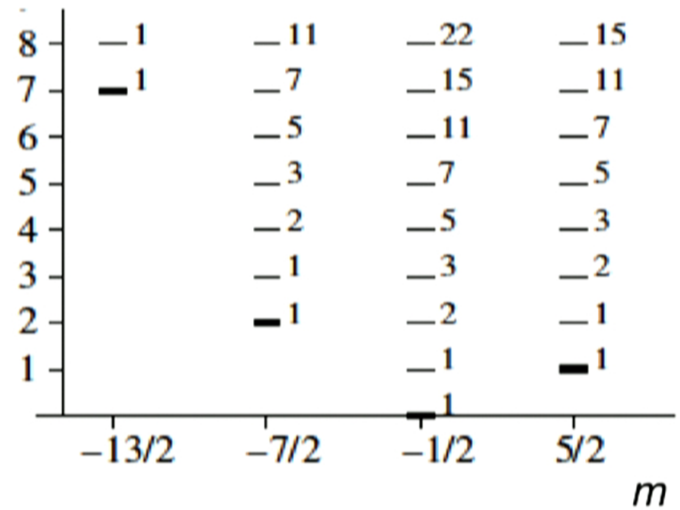


CFT spectrum

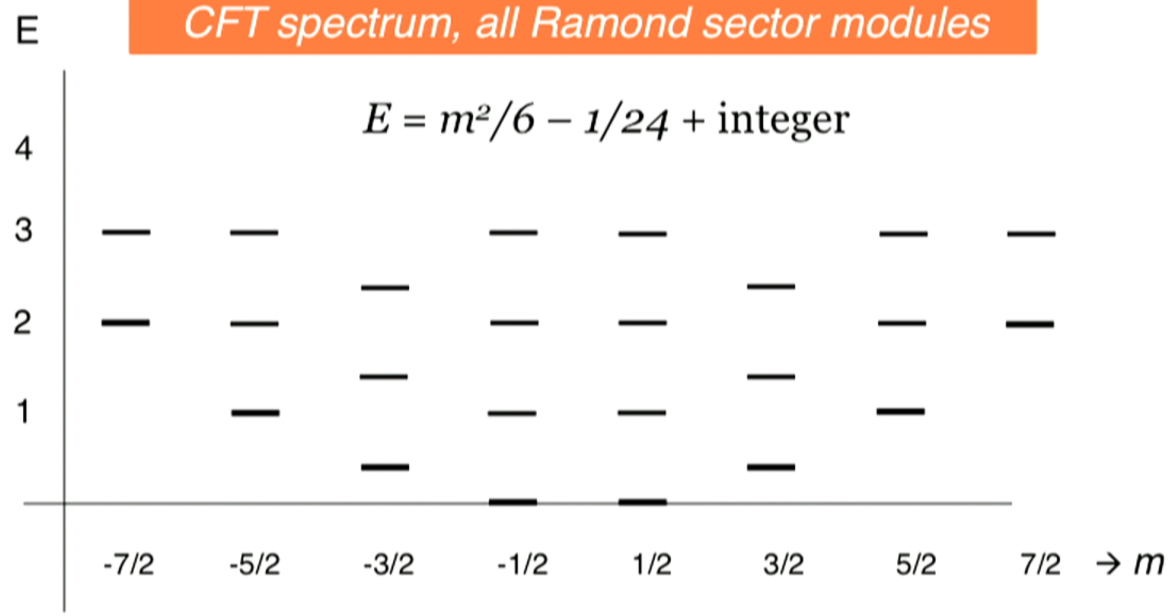


$m = 3f - L - ½$

Susy lattice model M_1 , open chain

CFT spectrum, all Ramond sector modules

$$E = m^2/6 - 1/24 + \text{integer}$$



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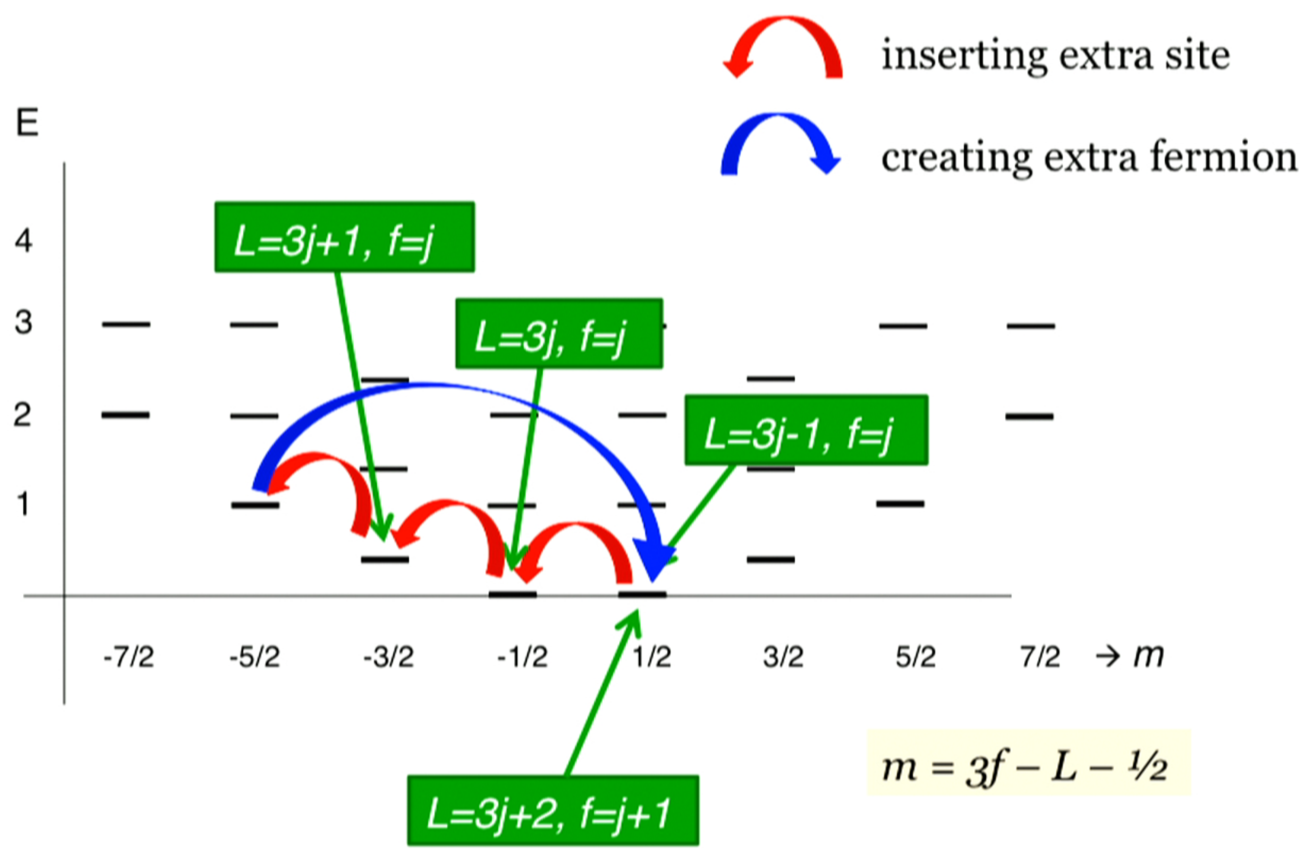

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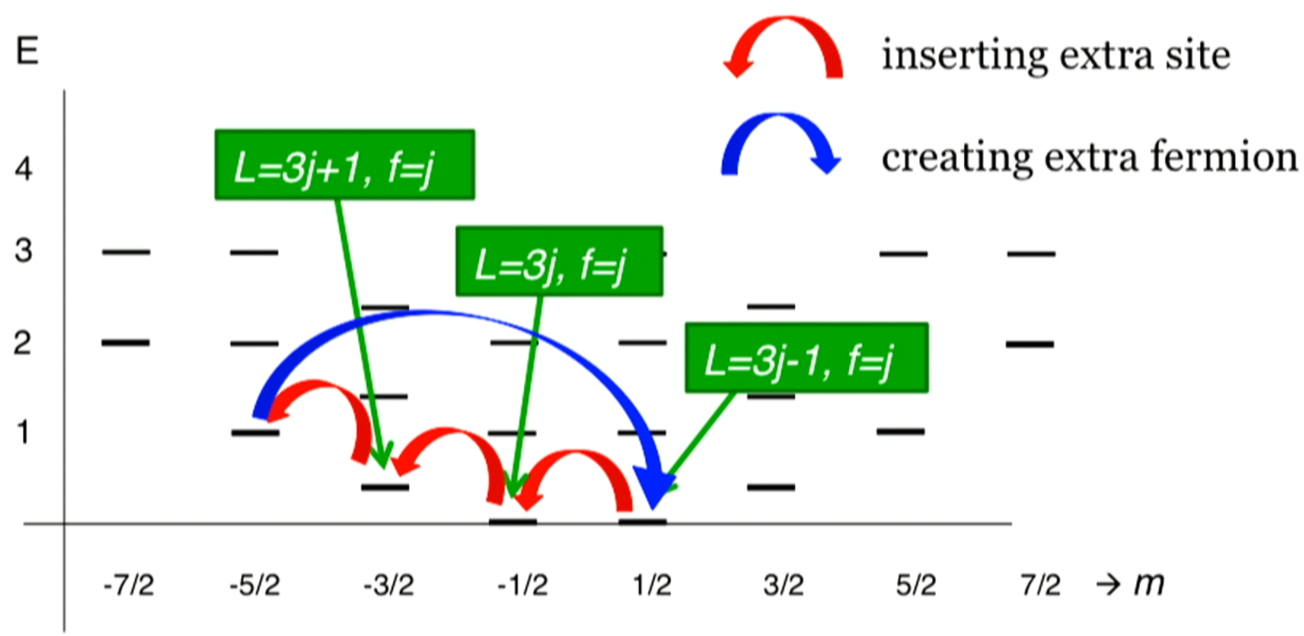
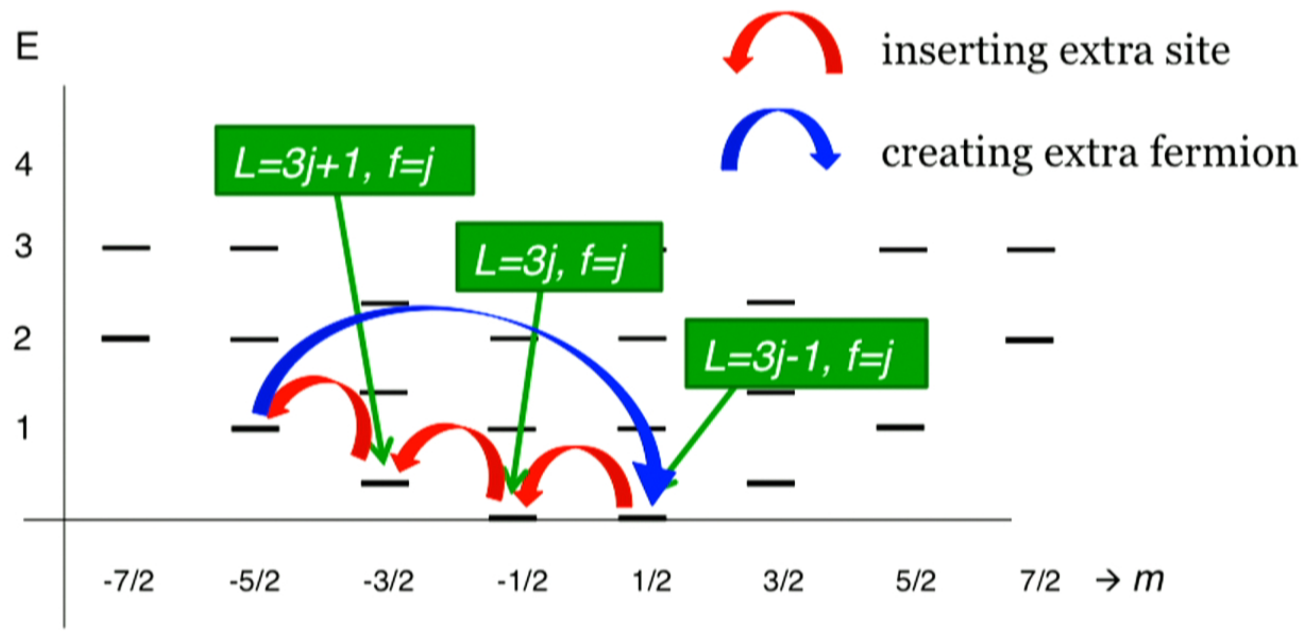


analogy with 1/3 Laughlin state:

extra site → insertion of single flux → $\Delta f = -1/3$ quasi-hole

Susy lattice model M_1, open chain



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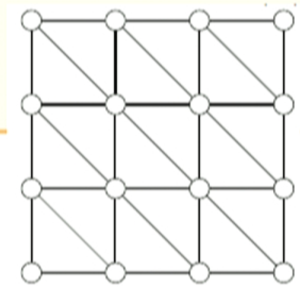
M_1 model, 2D triangular lattice

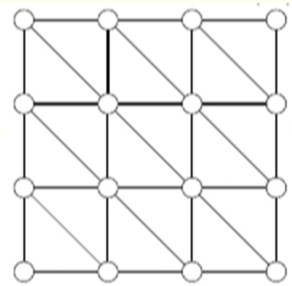


Witten index for $N{\times}M$ sites with periodic BC

	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	-3	-5	1	11	9	-13	-31	-5	57
3	1	-5	-2	7	1	-14	1	31	-2	-65
4	1	1	7	-23	11	25	-69	193	-29	-279
5	1	11	1	11	36	-49	211	-349	811	-1064
6	1	9	-14	25	-49	-102	-13	-415	1462	-4911
7	1	-13	1	-69	211	-13	-797	3403	-7055	5237
8	1	-31	31	193	-349	-415	3403	881	-28517	50849
9	1	-5	-2	-29	881	1462	-7055	-28517	31399	313315
10	1	57	-65	-279	-1064	-4911	5237	50849	313315	950592
11	1	67	1	859	1651	12607	32418	159083	499060	2011307
12	1	-47	130	-1295	-589	-26006	-152697	-535895	-2573258	-3973827
13	1	-181	1	-77	-1949	67523	330331	-595373	-10989458	-49705161
14	1	-87	-257	3641	12611	-139935	-235717	5651377	4765189	-232675057
15	1	275	-2	-8053	-32664	272486	-1184714	-1867189	134858383	-702709340

van Eerten, 2005

M_1 model, 2D triangular lattice



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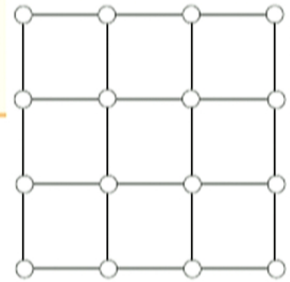
`superfrustration'

van Eerten, 2005

M_1 model, 2D square lattice



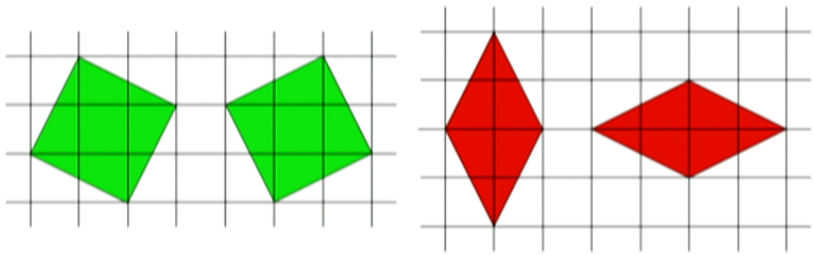
Witten index for *N*×*M* sites with periodic BC

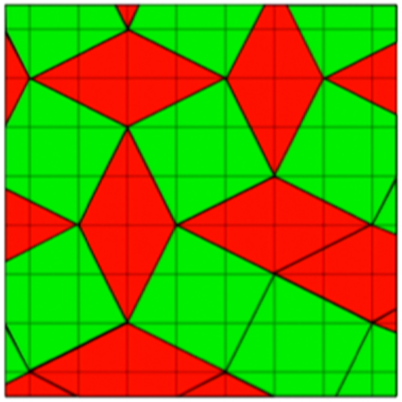
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	−1	1	3	1	−1	1	3	1	−1	1	3	1	−1	1	3	1	−1	1	3
3	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1
4	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1	7
5	1	1	1	1	−9	1	1	1	1	11	1	1	1	1	−9	1	1	1	1	11
6	1	−1	4	3	1	14	1	3	4	−1	1	18	1	−1	4	3	1	14	1	3
7	1	1	1	1	1	1	1	1	1	1	1	1	1	−27	1	1	1	1	1	1
8	1	3	1	7	1	3	1	7	1	43	1	7	1	3	1	7	1	3	1	47
9	1	1	4	1	1	4	1	1	40	1	1	4	1	1	4	1	1	76	1	1
10	1	−1	1	3	11	−1	1	43	1	9	1	3	1	69	11	43	1	−1	1	13
11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	1	3	4	7	1	18	1	7	4	3	1	166	1	3	4	7	1	126	1	7
13	1	1	1	1	1	1	1	1	1	1	1	1	−51	1	1	1	1	1	1	1
14	1	−1	1	3	1	−1	−27	3	1	69	1	3	1	55	1	451	1	−1	1	73
15	1	1	4	1	−9	4	1	1	4	11	1	4	1	1	174	1	1	4	1	11

Fendley-KjS -van Eerten, 2005

M_1 model, 2D square lattice

Witten index related to rhombus tilings of the lattice





Theorem [Jonsson 2005]

$$W_{\vec{u},\vec{v}} = t_{even} - t_{odd} - (-1)^{d_-}\,\theta_{d_-}\,\theta_{d_+}$$

with $d_\pm = \gcd(u_1 \pm u_2, v_1 \pm v_2)$, $\theta_{3p} = 2$, $\theta_{3p\pm1} = -1$

$M_k[\lambda]$, definitions

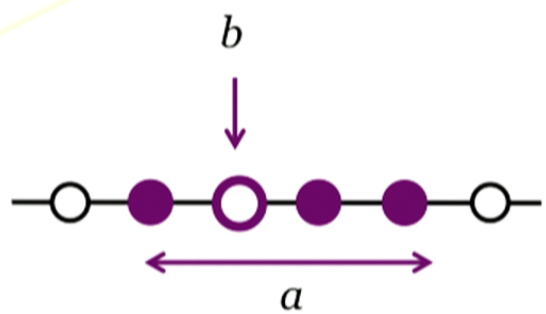
Fendley-Nienhuis-KjS, 2003

configurations:
lattice fermions **up to *k***
nearest neighbors occupied

supercharges

$$Q^+ = \sum_{j=1}^{L} \sum_{a,b} \lambda_{[a,b],j} d^+_{[a,b],j}$$

creates string of length a,
new particle at position b



$M_k[\lambda]$, definitions

M_2 : staggering modulo 4

$\lambda_{[1,1],j}$:	...	$\sqrt{2}$	$\sqrt{2}\lambda$	$\sqrt{2}$	$\sqrt{2}\lambda$	...
$\lambda_{[2,1],j}$:	...	1	λ	1	λ	...
$\lambda_{[2,2],j}$:	...	1	λ	1	λ	...

M_3, staggering mod 5, $\lambda \ll 1$

$\lambda_{[1,1],j}$:	...	1	$\sqrt{2}$	$\sqrt{2}\lambda$	$\sqrt{2}$	1	...
$\lambda_{[2,1],j}$:	...	1	$\sqrt{2}$	λ	$\sqrt{2}$	λ	...
$\lambda_{[2,2],j}$:	...	λ	$\sqrt{2}$	λ	1	1	...
$\lambda_{[3,1],j}$:	...	1	λ	λ	1	$\frac{\lambda}{\sqrt{2}}$	...
$\lambda_{[3,2],j}$:	...	$\frac{\lambda}{\sqrt{2}}$	1	$\frac{\lambda^2}{\sqrt{2}}$	1	$\frac{\lambda}{\sqrt{2}}$	...
$\lambda_{[3,3],j}$:	...	$\frac{\lambda}{\sqrt{2}}$	1	λ	λ	1	...
				$\uparrow$			

M_k, staggering mod (k+2)

- integrable for all λ
- critical for $\lambda = 1$

Hagendorf-Fokkema-Huijse, 2014
Hagendorf-Huijse, 2015

$M_k[\lambda]$, properties

Closed chain: $W=k+1$ for $L=l(k+2)$,

→ $k+1$ groundstates at filling $f/L = k/(k+2)$

Fendley-Nienhuis-KjS 2003, Hagendorf-Huijse 2015

$\boldsymbol{\lambda=1}$: CFT_k, k^{th} minimal model of $N=2$ SCFT,

$U(1) \times Z_k$ parafermions

$\boldsymbol{\lambda<1}$: relevant perturbation, RG flow to massive integrable

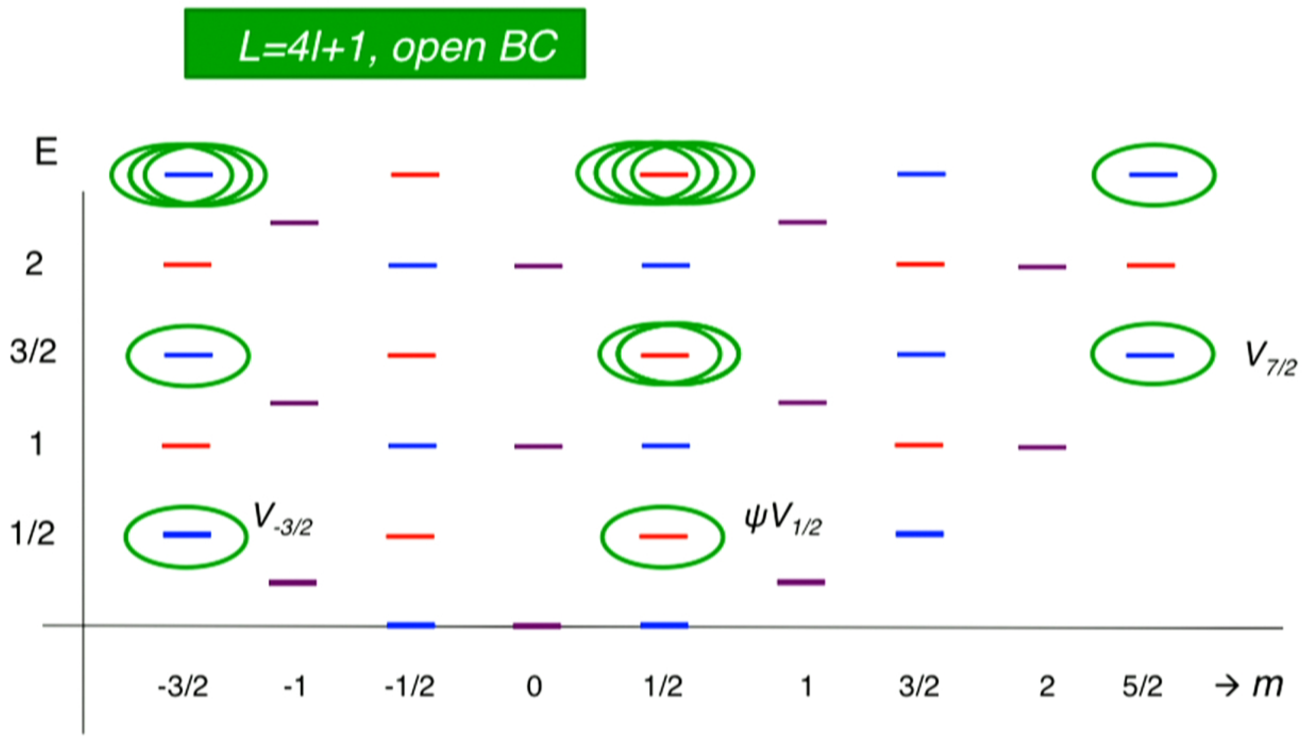
$N=2$ QFT with LG potentials of Chebyshev form

Fendley-Nienhuis-KjS 2003, Hagendorf-Fokkema-Huijse 2014

$M_2[\lambda]$ on open chains

Thessa Fokkema & KjS, arXiv:1504.02421

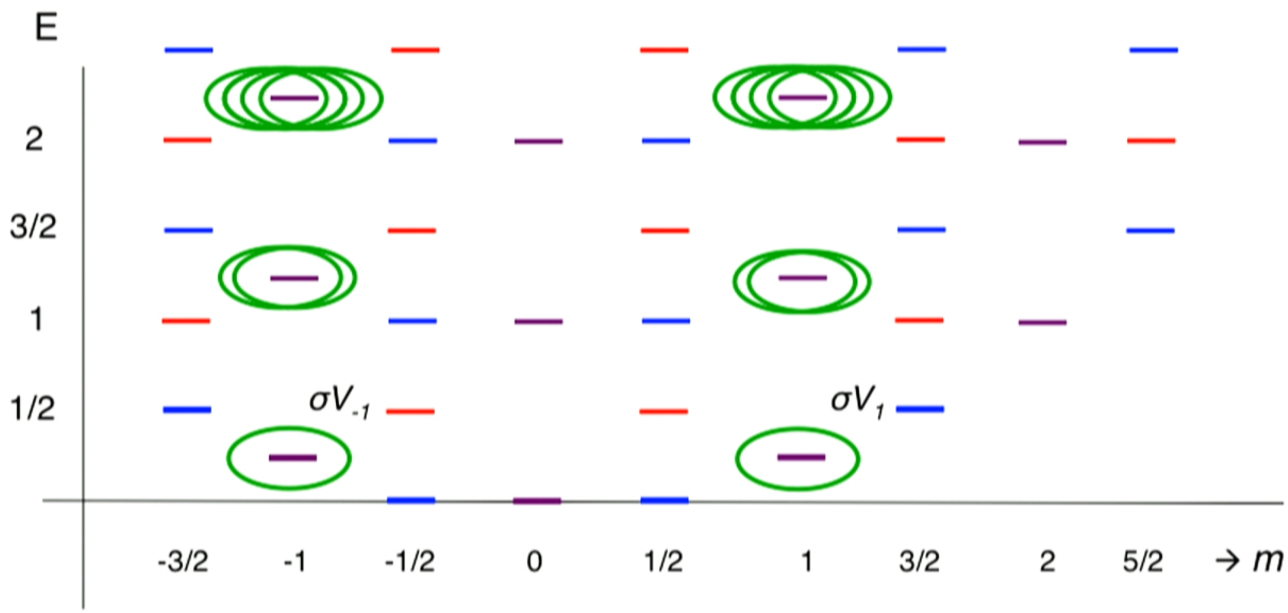
$M_2[\lambda=1]$, open chain, CFT spectra



CFT modules: V_m for f even, ψV_m for f odd, $m = 2f - L - 1/2$

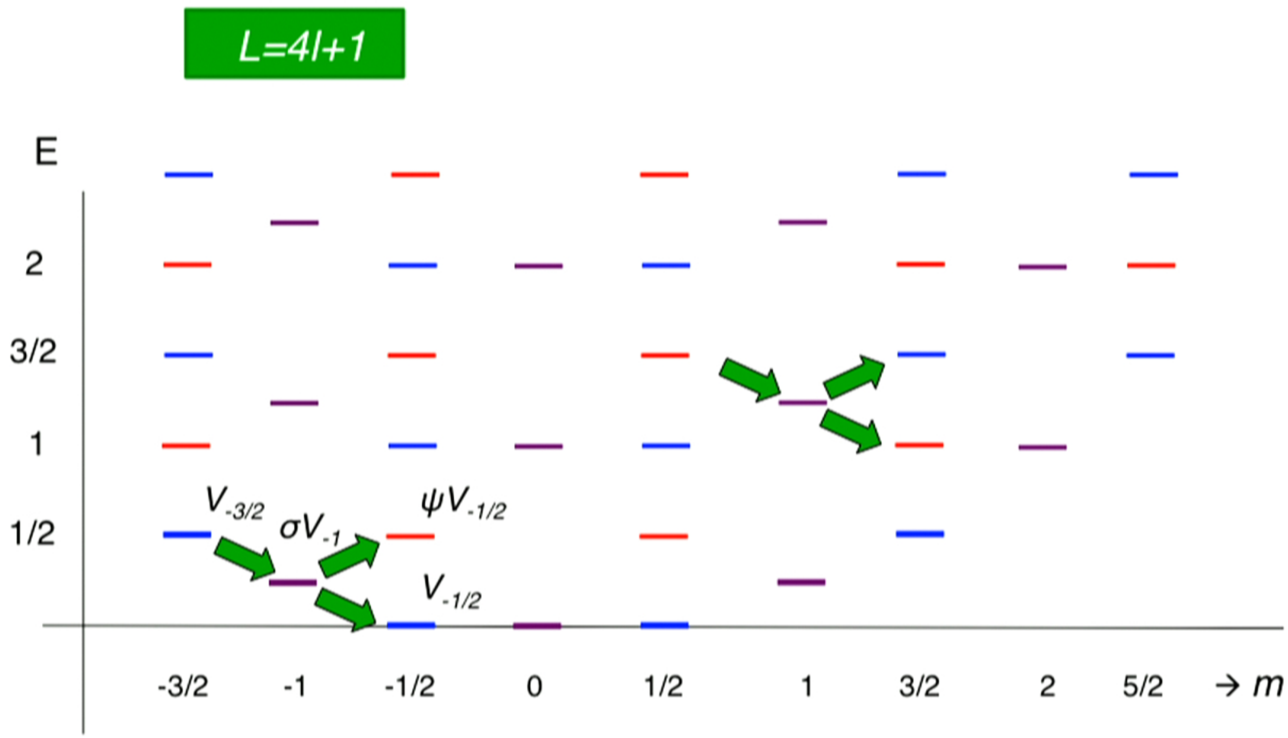
$M_2[\lambda=1]$, open chain, CFT spectra

L=4l+1, σ / open BC



CFT modules: σV_m, $m = 2f - L$

$M_2[\lambda=1]$, open chain, CFT spectra

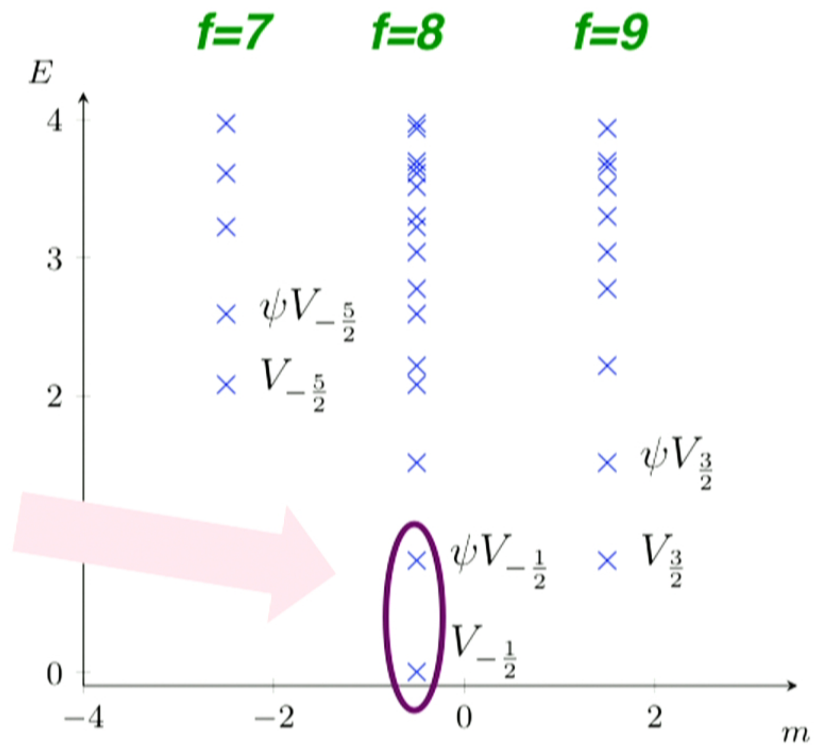


σ-type BC have fusion rule $\sigma\times\sigma = 1+\psi$

$M_2[\lambda=1]$, σ/σ boundary conditions

now follow the $L=17, f=8$ spectrum with σ-type boundary defects on both ends as a function of $\lambda<1$

expect that lowest two states form `susy qubit' due to localization of Majorana modes carried by the σ-defects

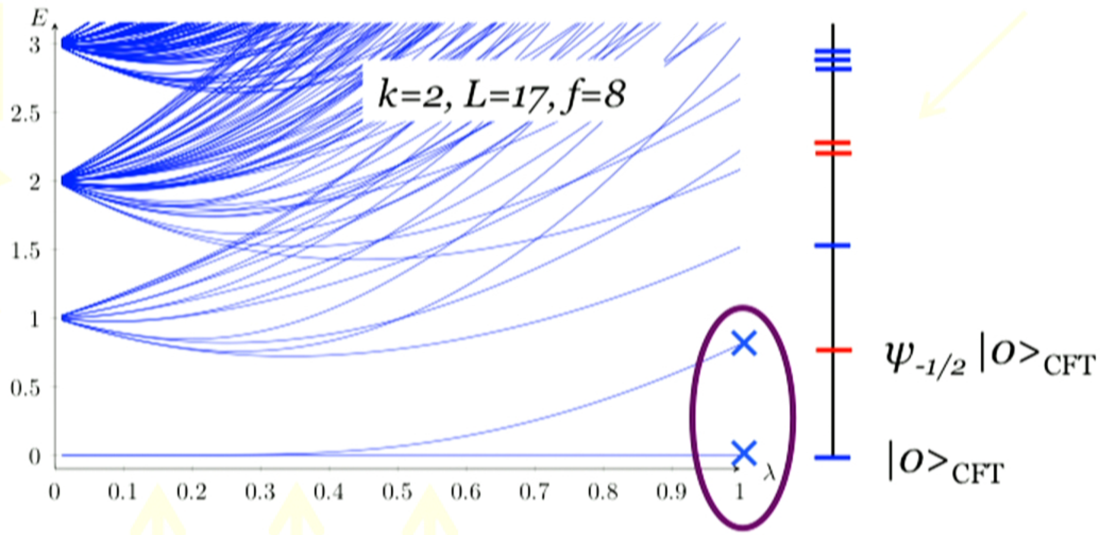


Staggering the M_2 model

M_2 model, σ/σ boundary defects

CFT spectrum:
$Z=1+q^{1/2}+q+2q^{3/2}+3q^2+\ldots$

(multi-)kink states, $E=1,2,..$



two degenerate groundstates, $E \sim \lambda^{(L-1)/2} \sim e^{-\alpha L}$

Extreme staggering for M_2

In extreme staggering (λ=0), the three supersymmetric E=0 groundstate of the infinite system are

$$|-\rangle = \ldots 0(\bullet 1\bullet)0(\bullet 1\bullet)0\ldots$$
$$|+\rangle = \ldots 1\bullet)0(\bullet 1\bullet)0(\bullet 1\ldots$$
$$|0\rangle = \ldots 10\,1\,0\,1\,0\,1\,01\ldots$$

with

$$(\bullet 1\bullet) = (110) + (011)$$

Excited states are kink/anti-kinks.

Staggered M_2 model, σ/σ BC

In extreme staggering, lowest $L=4l+1, f=2l$ states become, with $(\bullet 1 \bullet) = (110) + (001)$

$$|-\rangle_{\sigma,\sigma} = {}_{\sigma}[0(\bullet 1 \bullet)0(\bullet 1 \bullet)0 \ldots (\bullet 1 \bullet)0]_{\sigma}$$
$$|+\rangle_{\sigma,\sigma} = {}_{\sigma}[100(\bullet 1 \bullet)0 \ldots 0(\bullet 1 \bullet)001]_{\sigma}$$

Energies for $\lambda << 1$

$$E_{-}(\lambda) = 0, \quad E_{+}(\lambda) \approx \lambda^{(L-1)/2}$$

The supercharge Q^{+} pairs $|+\rangle_{\sigma,\sigma}$ with

$$|0\rangle_{\sigma,\sigma} = {}_{\sigma}[101010 \ldots 010101]_{\sigma}$$

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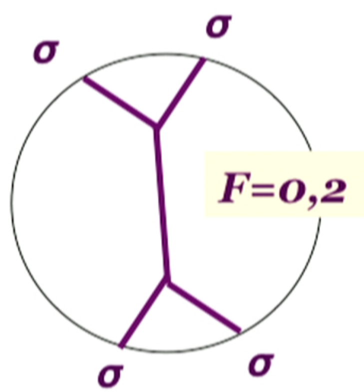
M_2 qubit vs MR qubit

M_2 \`qubit' at extreme staggering

$$|-\rangle_{\sigma,\sigma} = {}_{\sigma}[0(\bullet 1\bullet)0(\bullet 1\bullet)0\ldots(\bullet 1\bullet)0]_{\sigma}$$
$$|+\rangle_{\sigma,\sigma} = {}_{\sigma}[100(\bullet 1\bullet)0\ldots0(\bullet 1\bullet)001]_{\sigma}$$

Compare with MR \`qubit' (thin torus)
[four $q=e/4$ quasi-holes on sphere,
two at north-pole, two at south-pole,
$F=0,\ 2$ label the fusion channels]

$$F=0: \quad [01100\ldots00110]$$
$$F=2: \quad [10011\ldots11001]$$

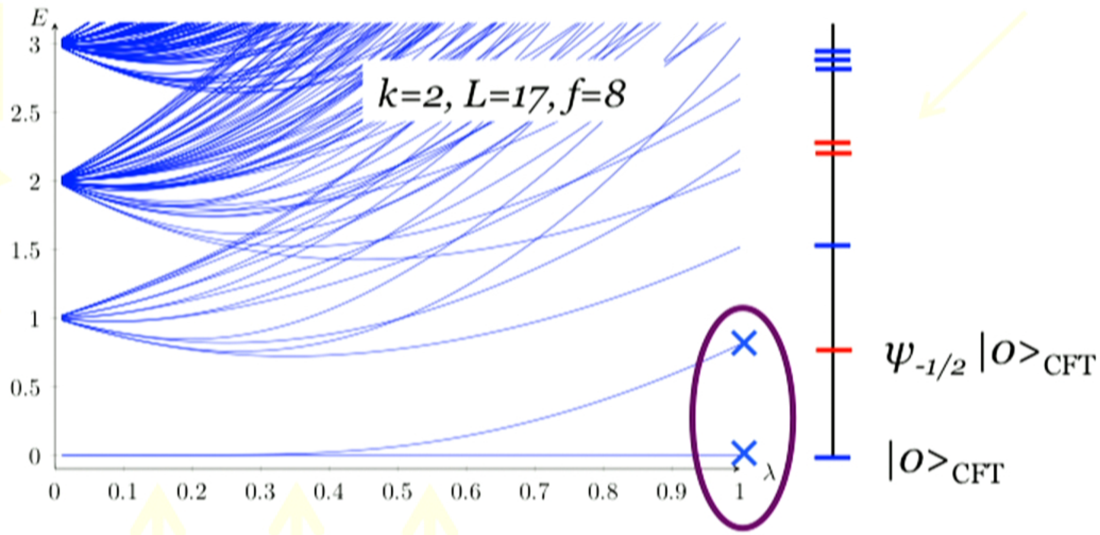


Staggering the M_2 model

M_2 model, σ / σ boundary defects

CFT spectrum:
$Z = 1 + q^{1/2} + q + 2q^{3/2} + 3q^2 + ...$

(multi-)kink states, $E = 1, 2, ..$



two degenerate groundstates, $E \sim \lambda^{(L-1)/2} \sim e^{-\alpha L}$

M_2 quantum register

- extend definition of σ-type defects to bulk
- simplest defect connects vacuum |0> coming in from left with |+> or |-> or |0> extending to right, or v.v.
- n (bulk or edge) defects open up register of dimension (at minimal fermion number f)

$$D_n = 2^{n/2} \quad \Rightarrow \quad \text{quantum dimension } \; d_\sigma = \sqrt{2}$$

- exponential degeneracies protected as long as $N=2$ supersymmetry and pairing condition respected

$M_3[\lambda]$ model : U(1) × Z_3 parafermions

Operators in the CFT$_3$ $\{V_m\}$ $\{1,\psi_1,\psi_2\}$ $\{\sigma_1,\sigma_2,\varepsilon\}$

$\times$	σ_1	σ_2	ε	ψ_1	ψ_2
σ_1	$\psi_1+\sigma_2$				
σ_2	$\mathbf{1}+\varepsilon$	$\psi_2+\sigma_1$			
ε	$\psi_2+\sigma_1$	$\psi_1+\sigma_2$	$\mathbf{1}+\varepsilon$		
ψ_1	ε	σ_1	σ_2	ψ_2	
ψ_2	σ_2	ε	σ_1	$\mathbf{1}$	ψ_1

$k=3$ Read-Rezayi state, $\nu=3/5$, built on CFT$_3$ operators

$$\Psi_{\text{electron}} = \psi_1 V_{5/2} \qquad \Phi_{\text{quasi-hole}} = \sigma_1 V_{\pm 1/2}$$

M_3 model defects

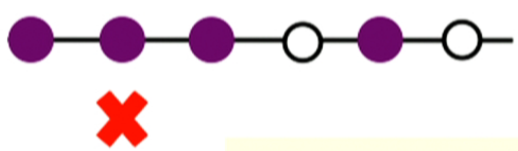
lattice operators vs CFT3 vs Read-Rezayi qH state

supercharge Q^+ : $\psi_1 V_{5/2}$, creates particle, Δf=+1

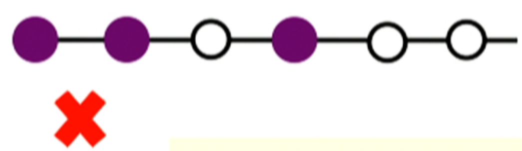
extra site : $V_{-3/2}$, quasi-hole of charge Δf=-3/5

σ_1-type defect : $\sigma_1 V_{1/2}$, quasi-hole of charge Δf=-1/5

σ_2-type defect : $\sigma_2 V_1$, quasi-hole of charge Δf=-2/5



σ_1-type defect



σ_2-type defect

$M_3[\lambda]$ model : U(1) × Z_3 parafermions

Operators in the CFT_3 $\{V_m\}$ $\{1,\psi_1,\psi_2\}$ $\{\sigma_1,\sigma_2,\varepsilon\}$

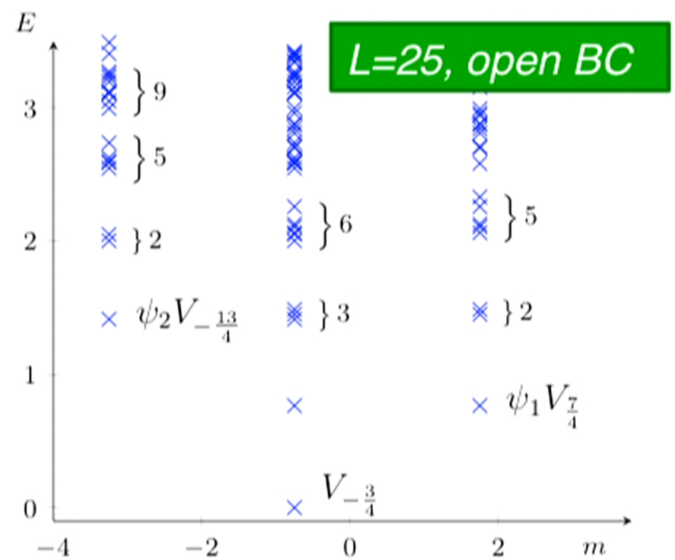
$\times$	σ_1	σ_2	ε	ψ_1	ψ_2
σ_1	$\psi_1+\sigma_2$				
σ_2	$\mathbf{1}+\varepsilon$	$\psi_2+\sigma_1$			
ε	$\psi_2+\sigma_1$	$\psi_1+\sigma_2$	$\mathbf{1}+\varepsilon$		
ψ_1	ε	σ_1	σ_2	ψ_2	
ψ_2	σ_2	ε	σ_1	$\mathbf{1}$	ψ_1

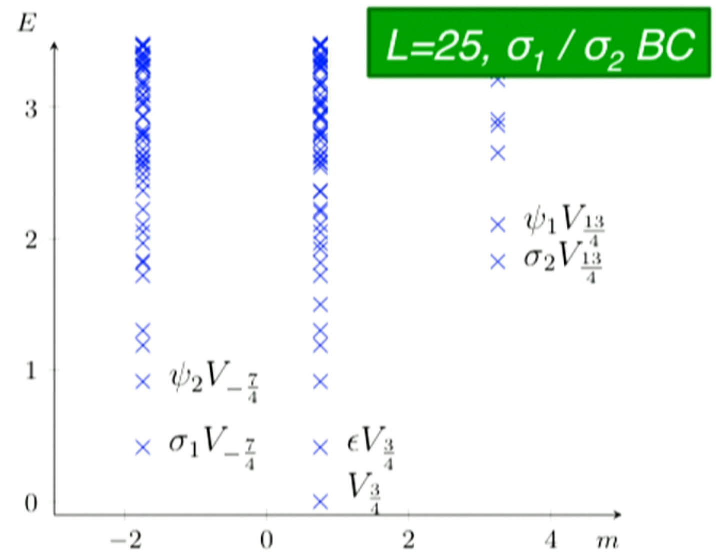
k=3 Read-Rezayi state, *ν=3/5*, built on CFT_3 operators

$$\Psi_{\text{electron}} = \psi_1 V_{5/2} \qquad \Phi_{\text{quasi-hole}} = \sigma_1 V_{\pm 1/2}$$

M_3 model : open chain spectra

Example: *L=25, f=14, 15, 16*. Injecting boundary defects σ_1, σ_2 sends $V_{-3/4} \rightarrow V_{3/4} + \varepsilon V_{3/4}$





Extreme staggering for M_3

In extreme staggering (λ=0), the four supersymmetric E=0 groundstate of the infinite system are

$$|1\rangle = \dots 1\underset{\uparrow}{1}100\dots \qquad \left|\tfrac{1}{2}\right\rangle = \dots(\bullet\underset{\uparrow}{1}\bullet\bullet\bullet)\dots$$

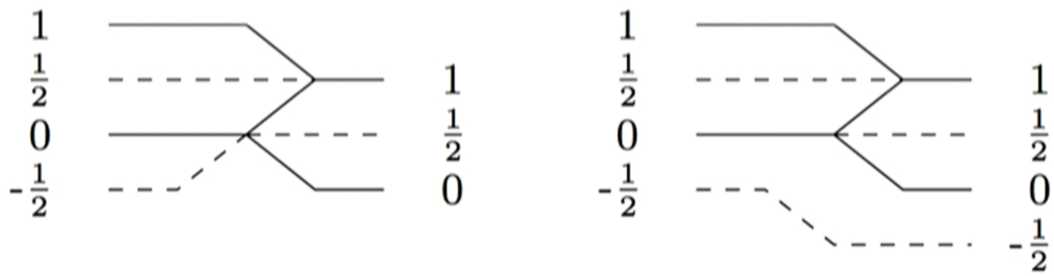
$$|0\rangle = \dots 0\underset{\uparrow}{1}011\dots \qquad \left|-\tfrac{1}{2}\right\rangle = \dots \underset{\uparrow}{0}(\bullet 11\bullet)\dots$$

with

$$(\bullet 1\bullet\bullet\bullet) = 01101 - 01110 + 11001 - 11010$$

$$(\bullet 11\bullet) = 1110 - 0111$$

M_3 : bulk defect fusion rules



(a) σ_2 defect (b) σ_1 defect

- n (bulk or edge) σ_2 defects open up register of dimension (at maximal fermion number f)

$$D_n = \text{Fibo}_{n+1} \quad \Rightarrow \quad \text{quantum dimension } \; d_{\sigma_2} = (1+\sqrt{5})/2$$

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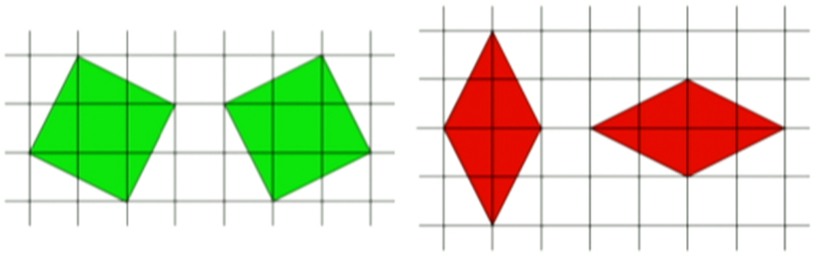
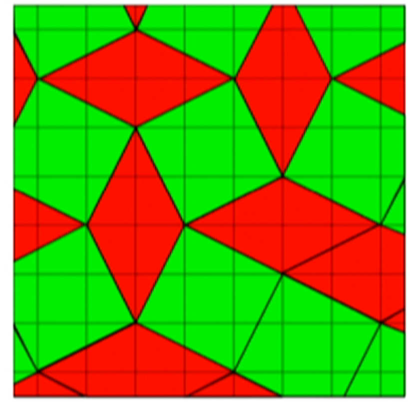
M_1 model, 2D square lattice

Number of gs related to rhombus tilings of the lattice, with $N_f = N_t$





Theorem [Fendley, Huijse-KjS 2009]

$$\# \text{ GS } = t_{even} + t_{odd} - (-1)^{(\theta_m+1)p}\theta_{d_-}\theta_{d_+}$$

with $d_\pm = \gcd(u_1 \pm u_2, v_1 \pm v_2)$, $\theta_{3p} = 2$, $\theta_{3p\pm1} = -1$