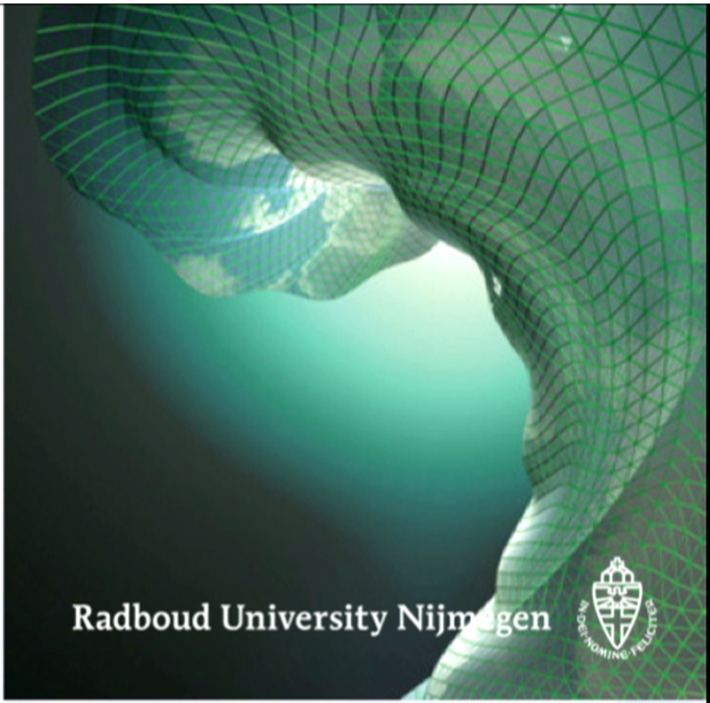
Title: Wilson Loops in CDT Quantum Gravity

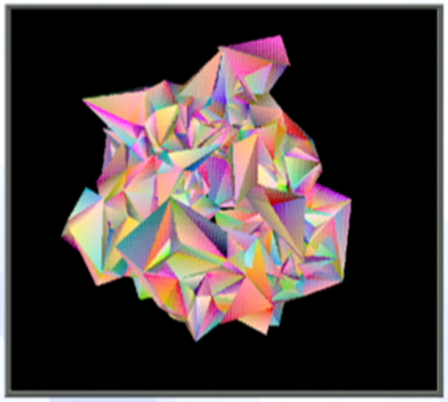
Date: May 21, 2015 02:30 PM

URL: http://pirsa.org/15050018

Abstract: <p>By explicit construction, I will show that one can in a simple way introduce and measure gravitational holonomies and Wilson loops in lattice formulations of nonperturbative quantum gravity based on (Causal) Dynamical Triangulations.</p>

Wilson Loops in Causal Dynamical Triangulations



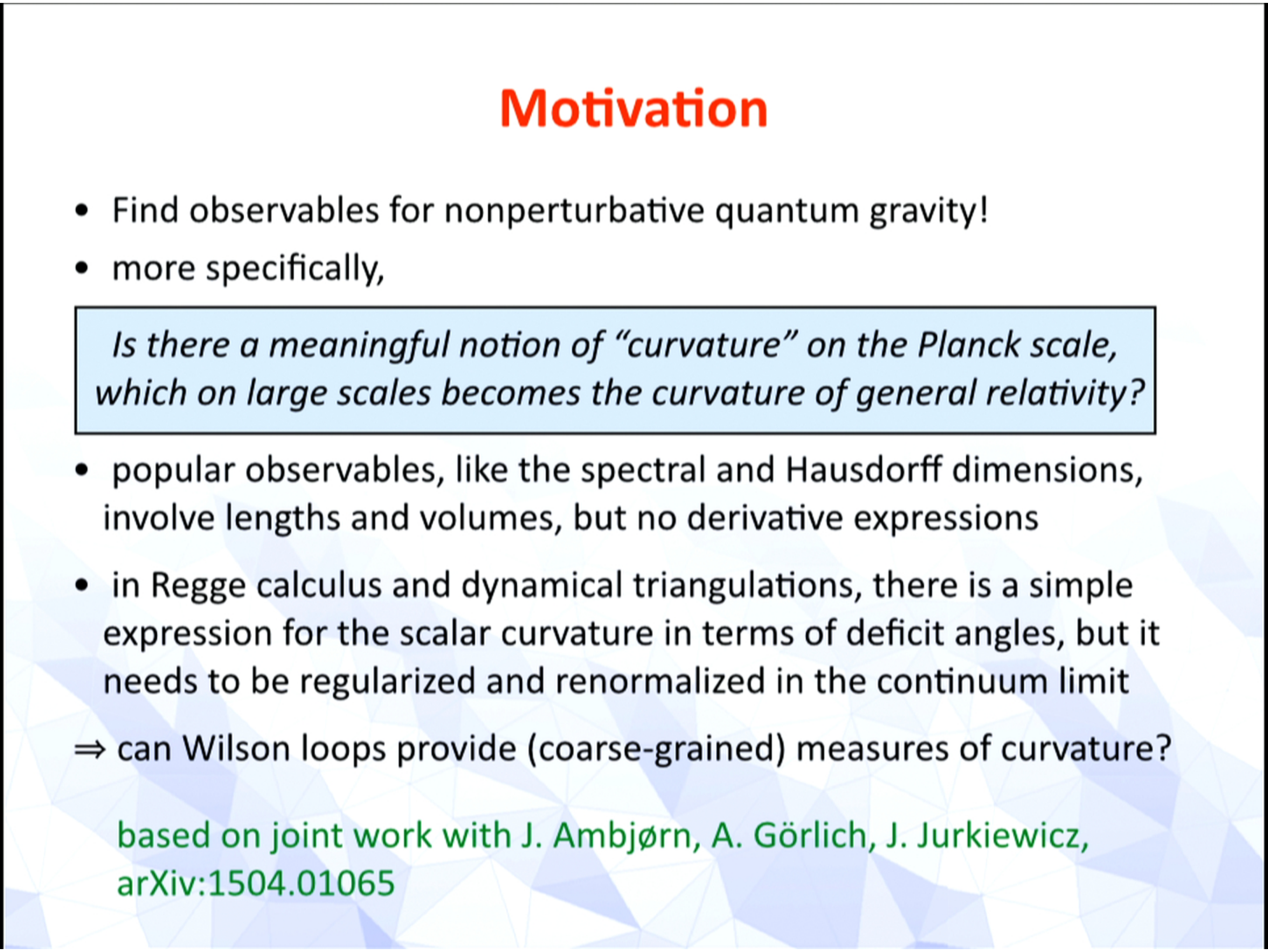


triangulated model of quantum space

Renate Loll

Institute for Mathematics, Astrophysics and
Particle Physics (IMAPP),
Radboud University Nijmegen (NL)

Perimeter Institute for Theoretical Physics 21 May 2015



Motivation

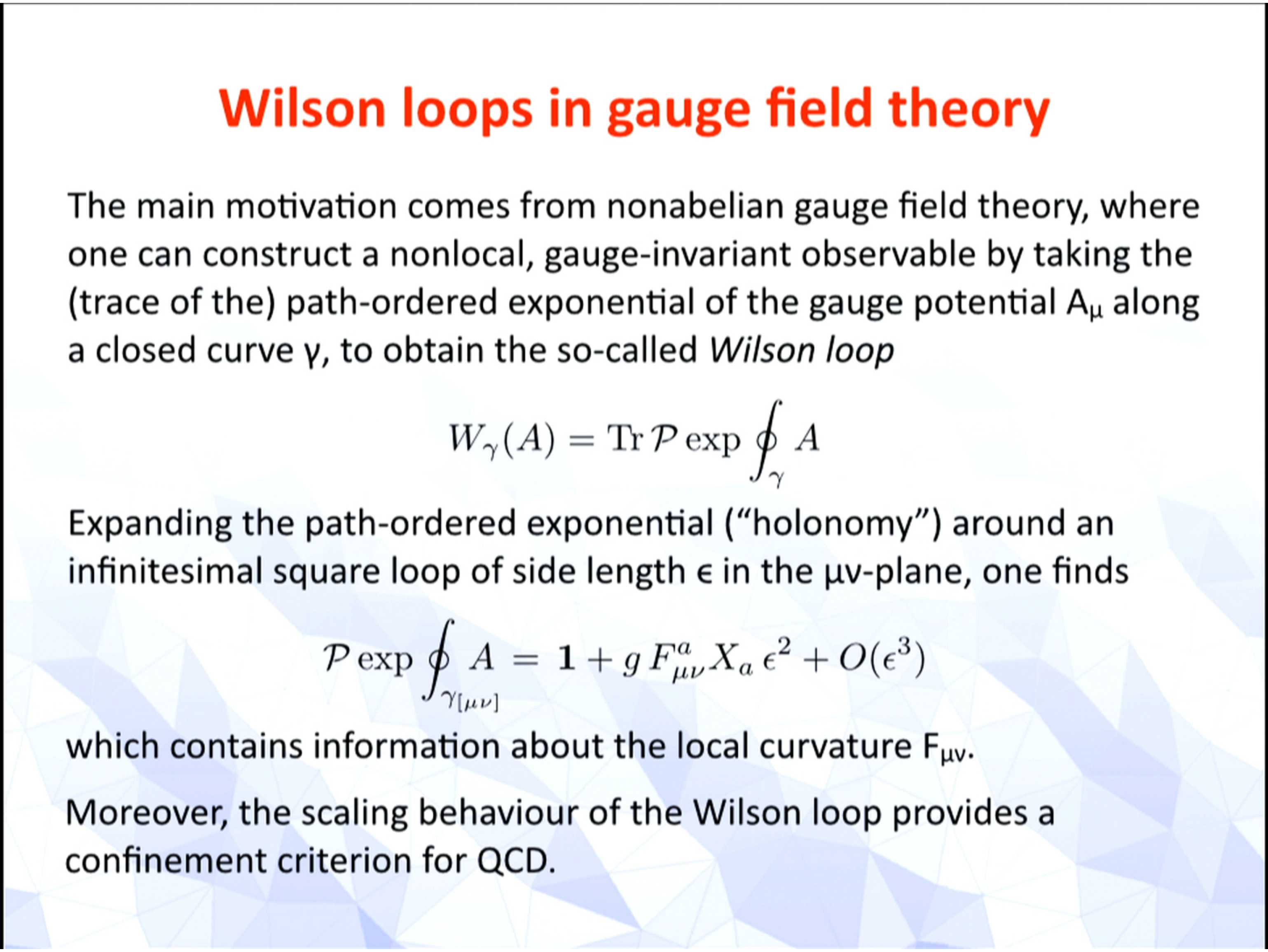
- Find observables for nonperturbative quantum gravity!
- more specifically,

> *Is there a meaningful notion of "curvature" on the Planck scale, which on large scales becomes the curvature of general relativity?*

- popular observables, like the spectral and Hausdorff dimensions, involve lengths and volumes, but no derivative expressions
- in Regge calculus and dynamical triangulations, there is a simple expression for the scalar curvature in terms of deficit angles, but it needs to be regularized and renormalized in the continuum limit

⇒ can Wilson loops provide (coarse-grained) measures of curvature?

based on joint work with J. Ambjørn, A. Görlich, J. Jurkiewicz, arXiv:1504.01065



Wilson loops in gauge field theory

The main motivation comes from nonabelian gauge field theory, where one can construct a nonlocal, gauge-invariant observable by taking the (trace of the) path-ordered exponential of the gauge potential A_μ along a closed curve γ, to obtain the so-called *Wilson loop*

$$W_\gamma(A) = \mathrm{Tr}\,\mathcal{P}\exp\oint_\gamma A$$

Expanding the path-ordered exponential (“holonomy”) around an infinitesimal square loop of side length ϵ in the μν-plane, one finds

$$\mathcal{P}\exp\oint_{\gamma_{[\mu\nu]}} A = \mathbf{1} + g\,F^a_{\mu\nu}X_a\,\epsilon^2 + O(\epsilon^3)$$

which contains information about the local curvature $F_{\mu\nu}$.

Moreover, the scaling behaviour of the Wilson loop provides a confinement criterion for QCD.

Wilson loops for (quantum) gravity?

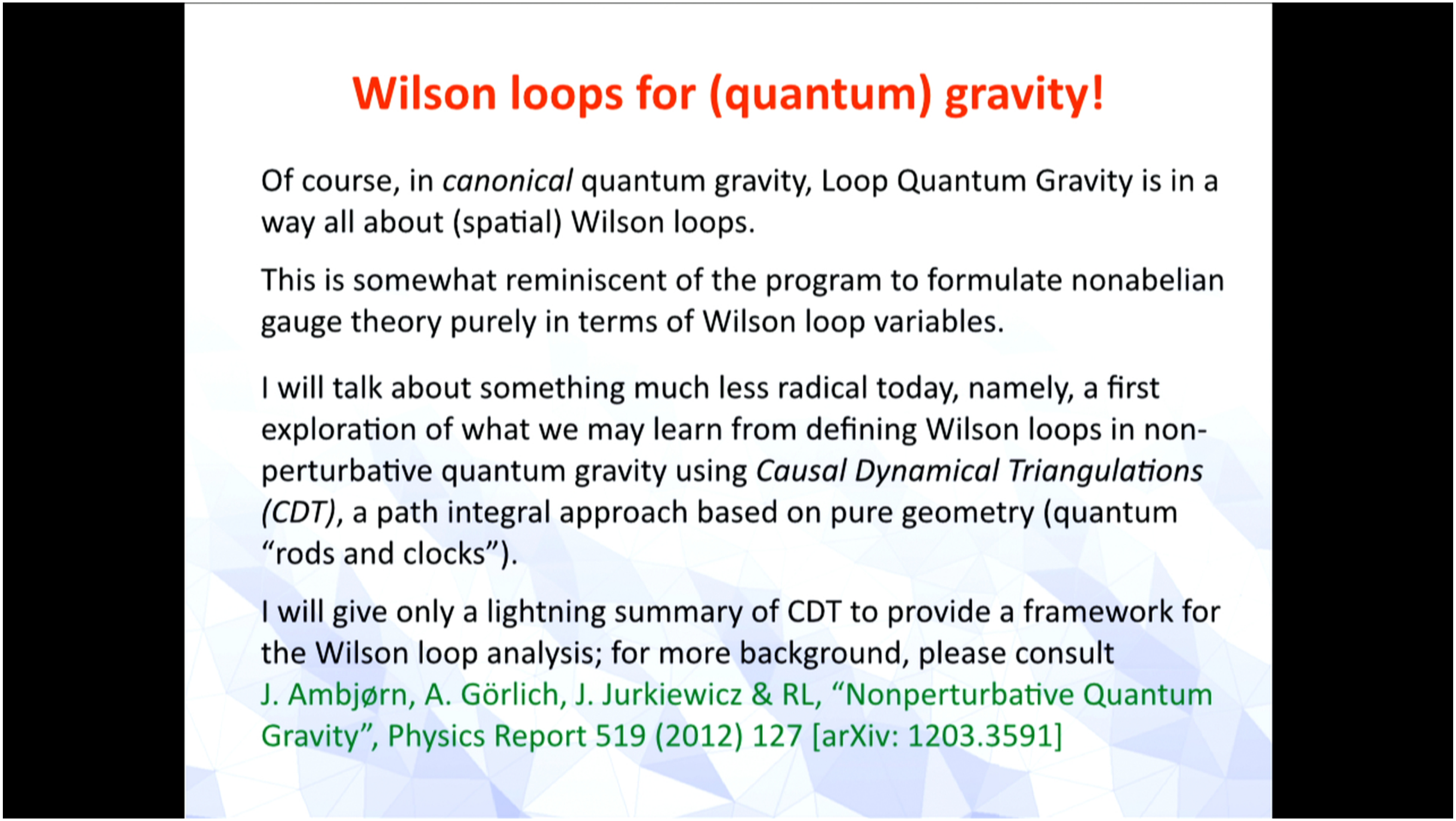
In gravity, one can use the Levi-Civita connection $\Gamma^{\mu}{}_{\nu\kappa}$ to construct holonomies and analogues of Wilson loops.

The path-ordered exponential of Γ along a path defines a notion of parallel transport of tangent vectors.

All physical information contained in the Riemann curvature tensor $R^{\mu}{}_{\nu\kappa\lambda}$ can be retrieved from infinitesimal holonomies.

However, Wilson loops are not diffeomorphism-invariant. On spacetime, they have been little studied or used, with the exception of the work by G. Modanese in perturbative quantum gravity (see, e.g. G. Modanese, PRD 49 (1994) 6534).

There are attempts to study Wilson loops in quantum Regge calculus (H. Hamber, R. Williams, PRD 76 (2007) 084008, 81 (2010) 084048).



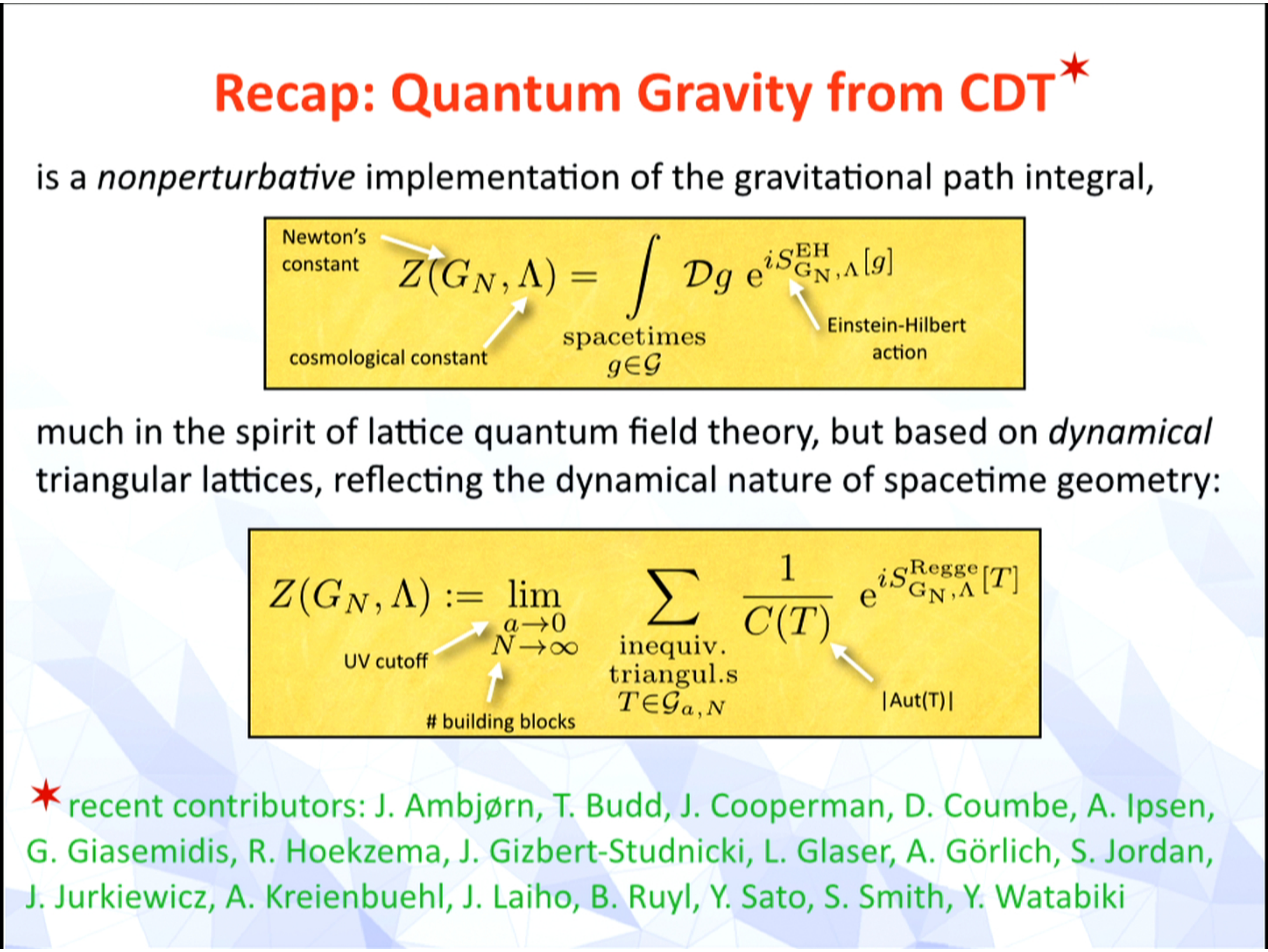
Wilson loops for (quantum) gravity!

Of course, in *canonical* quantum gravity, Loop Quantum Gravity is in a way all about (spatial) Wilson loops.

This is somewhat reminiscent of the program to formulate nonabelian gauge theory purely in terms of Wilson loop variables.

I will talk about something much less radical today, namely, a first exploration of what we may learn from defining Wilson loops in non-perturbative quantum gravity using *Causal Dynamical Triangulations (CDT)*, a path integral approach based on pure geometry (quantum "rods and clocks").

I will give only a lightning summary of CDT to provide a framework for the Wilson loop analysis; for more background, please consult J. Ambjørn, A. Görlich, J. Jurkiewicz & RL, "Nonperturbative Quantum Gravity", Physics Report 519 (2012) 127 [arXiv: 1203.3591]



Recap: Quantum Gravity from CDT✶

is a *nonperturbative* implementation of the gravitational path integral,

Newton's constant

$$Z(G_N,\Lambda) = \int_{\substack{\text{spacetimes}\\ g\in\mathcal{G}}} \mathcal{D}g\; \mathrm{e}^{iS^{\mathrm{EH}}_{\mathrm{G_N},\Lambda}[g]}$$

cosmological constant

Einstein-Hilbert action

much in the spirit of lattice quantum field theory, but based on *dynamical* triangular lattices, reflecting the dynamical nature of spacetime geometry:

$$Z(G_N,\Lambda) := \lim_{\substack{a\to 0\\ N\to\infty}} \sum_{\substack{\text{inequiv.}\\ \text{triangul.s}\\ T\in\mathcal{G}_{a,N}}} \frac{1}{C(T)}\; \mathrm{e}^{iS^{\mathrm{Regge}}_{\mathrm{G_N},\Lambda}[T]}$$

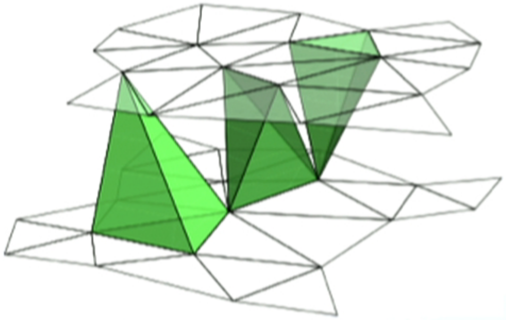
UV cutoff

building blocks

|Aut(T)|

✶ recent contributors: J. Ambjørn, T. Budd, J. Cooperman, D. Coumbe, A. Ipsen, G. Giasemidis, R. Hoekzema, J. Gizbert-Studnicki, L. Glaser, A. Görlich, S. Jordan, J. Jurkiewicz, A. Kreienbuehl, J. Laiho, B. Ruyl, Y. Sato, S. Smith, Y. Watabiki

Key properties and ingredients of the CDT approach to quantum gravity



piece of causal triangulation

- CDT uses *few* ingredients/priors:
  - path integral/quantum superposition principle
  - locality and causal structure (this is *not* Euclidean quantum gravity)
  - notion of (proper) time; can be relaxed
  - Wick rotation
  - standard tools of quantum field theory (and standard QT!)
- “conservative” configuration space: curved spacetimes of GR are represented by piecewise flat geometries (the triangulations)
- phase space spanned by *few* free parameters (Λ, G_N, Δ)
- universal properties (contributes to uniqueness!)
- Crucial: nonperturbative computational tools to extract quantitative results

Why does CDT quantum gravity matter?

- “as simple as it can be, but not simpler”
- it has come a long way, most recent results: 2nd-order phase transitions; background-independent implementation of RG
- explicit framework; concrete, quantitative results: falsifiable!
- it may actually tell us what quantum gravity *is,* exhibiting its universal properties
- CDT QG is a perfect test bed for implementing and measuring *observables*, including observables involving Wilson loops

possible objections:

- Wilson loops are all about connections and coordinate frames, whereas CDT quantum gravity is coordinate-free!?
- CDT works only with two types of simplicial building blocks, and there will be very strong discretization effects

Holonomies in gravity

Given a Riemannian manifold *M* with metric $g_{\mu\nu}(x)$, the Levi-Civita connection $\Gamma^{\mu}{}_{\nu\kappa}$ defines the parallel transport of a vector V^{μ} along a curve $\gamma^{\mu}(\lambda)$.

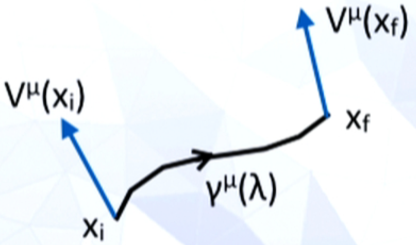
path ordering

$$V^{\mu}(x_{\rm f}) = \left(\mathcal{P}\,{\rm e}^{-\int_{\lambda_{\rm i}}^{\lambda_{\rm f}} \Gamma_{\kappa}\dot{\gamma}^{\kappa}(\lambda)d\lambda}\right)^{\mu}{}_{\nu}\,V^{\nu}(x_{\rm i}), \qquad \left(\Gamma_{\kappa}\right)^{\mu}{}_{\nu} = \Gamma^{\mu}_{\kappa\nu}$$

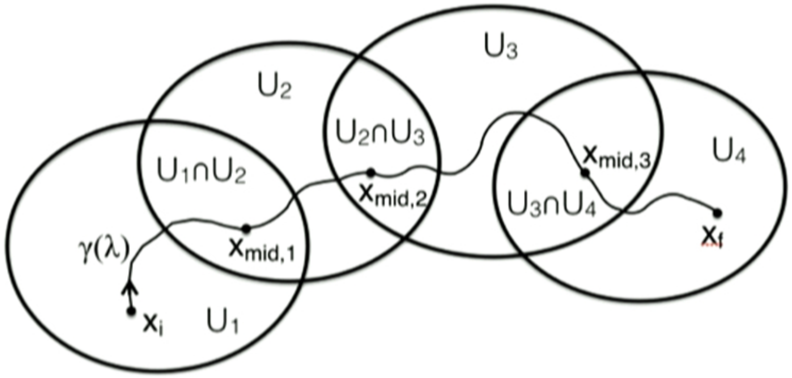
Under a coordinate transformation $x \to \tilde{x}(x)$,

with $M^{\mu}{}_{\nu}(x) = \dfrac{\partial \tilde{x}^{\mu}(x)}{\partial x^{\nu}}$

the path-ordered integral transforms as



$$\left(\mathcal{P}\,{\rm e}^{-\int_{\lambda_{\rm i}}^{\lambda_{\rm f}} \tilde{\Gamma}_{\kappa}\dot{\tilde{\gamma}}^{\kappa}(\lambda)d\lambda}\right)^{\mu}{}_{\nu} = M^{\mu}{}_{\alpha}(x_{\rm f})\left(\mathcal{P}\,{\rm e}^{-\int_{\lambda_{\rm i}}^{\lambda_{\rm f}} \Gamma_{\kappa}\dot{\gamma}^{\kappa}(\lambda)d\lambda}\right)^{\alpha}{}_{\beta}(M^{-1}(x_{\rm i}))^{\beta}{}_{\nu}$$



Generally, the path γ(λ) passes through several coordinate patches U_k, with coordinates x_k^μ. We then compute the path-ordered integral in every patch, with a transition matrix $M(x_{\text{mid},k})$ inserted at each arbitrary midpoint $x_{\text{mid},k}$ in the *k*th overlap region $U_k \cap U_{k+1}$. For a *closed* loop γ(λ) based at x_i, its <u>*holonomy*</u> is

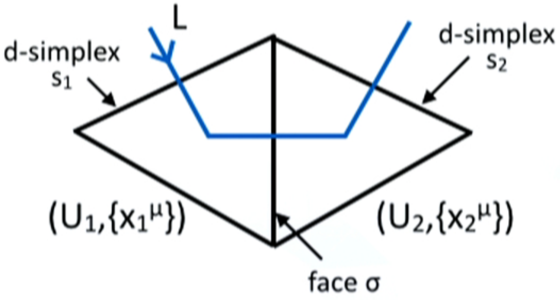
$$\left(\mathcal{P}\mathrm{e}^{-\oint_\gamma \Gamma}\right)_{x_\mathrm{i}} = \left(\mathcal{P}\,\mathrm{e}^{-\int_{x_{\mathrm{mid},n+1}}^{x_\mathrm{f}} \Gamma_0}\right) M(x_{\mathrm{mid},n+1})\cdot \prod_{k=1}^{n} \left(\left(\mathcal{P}\,\mathrm{e}^{-\int_{x_{\mathrm{mid},k}}^{x_{\mathrm{mid},k+1}} \Gamma_k}\right) M(x_{\mathrm{mid},k})\right) \left(\mathcal{P}\,\mathrm{e}^{-\int_{x_\mathrm{i}}^{x_{\mathrm{mid},1}} \Gamma_0}\right),$$

which still transforms under coordinate changes at the base point,

$$\left(\mathcal{P}\mathrm{e}^{-\oint_\gamma \tilde{\Gamma}}\right)_{\tilde{x}_\mathrm{i}} = M(x_\mathrm{i}) \left(\mathcal{P}\mathrm{e}^{-\oint_\gamma \Gamma}\right)_{x_\mathrm{i}} M^{-1}(x_\mathrm{i}), \quad \text{with} \quad M^\mu{}_\nu(x) = \frac{\partial \tilde{x}^\mu(x)}{\partial x^\nu}.$$

Holonomies in Euclidean(ized) DT

In *d* dimensions, introduce the same coordinates $\{x_k{}^{\mu}\}$ on each flat, equilateral *d*-simplex s_k. Since $R \equiv 0$ *inside* s_k, for any closed path totally inside s_k we have $\mathcal{P}(\exp \oint \Gamma) = \mathbb{1}$.



- w.l.o.g. use piecewise straight paths L between the centres of neighbouring simplices when constructing holonomies
- can use Cartesian coordinate frames where the metric is constant (to be specified; in the end, nothing will depend on it)
- have transition matrix $M^{\mu}{}_{\nu} = \frac{\partial x_2^{\mu}}{\partial x_1^{\nu}}|_{\sigma} =: R(s_2, s_1)^{\mu}{}_{\nu} \in SO(4)$
- the holonomy of a closed, oriented path L, starting and ending at x_1 is

$$R_L = R(s_1, s_n) R(s_n, s_{n-1}) \cdots R(s_2, s_1) \in SO(4)$$

- the holonomy R_L still transforms under a change of the Cartesian coordinates $x \to \tilde{x}(x)$ in the simplex *s₁*, with some $\Lambda \in SO(4)$:

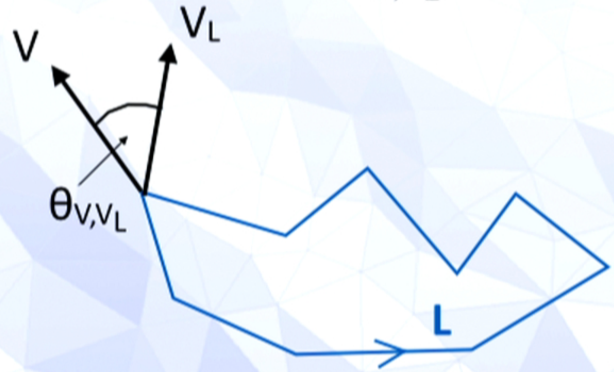
$$R_L \to \Lambda R_L \Lambda^T, \quad \Lambda^\mu{}_\nu = \frac{\partial \tilde{x}^\mu(x)}{\partial x^\nu}$$

- parallel-transporting a vector V^μ around loop L gives a rotated vector

$$V_L = R_L V$$

- the angle between V and V_L is independent of Λ,

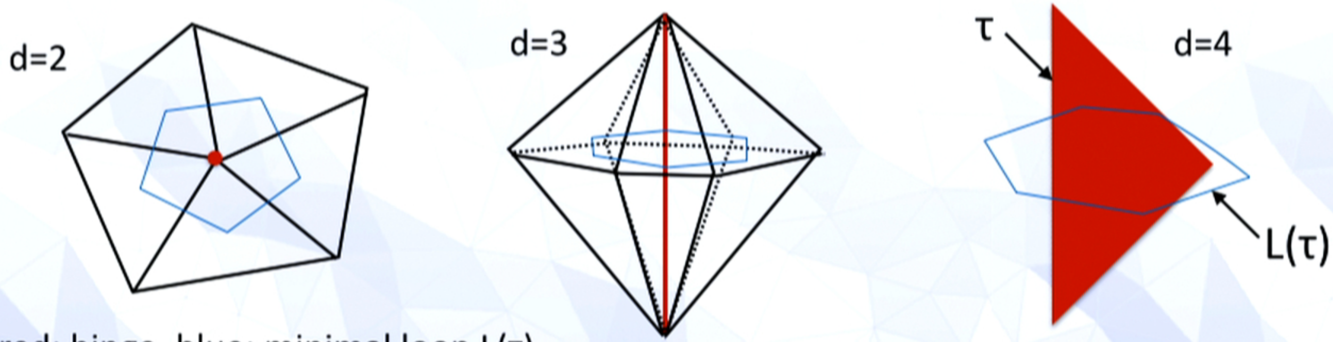
$$\theta_{V,V_L} := \arccos\left(\frac{V \cdot V_L}{\sqrt{V \cdot V}\sqrt{V_L \cdot V_L}}\right)$$



Relation with local curvature in Regge calculus

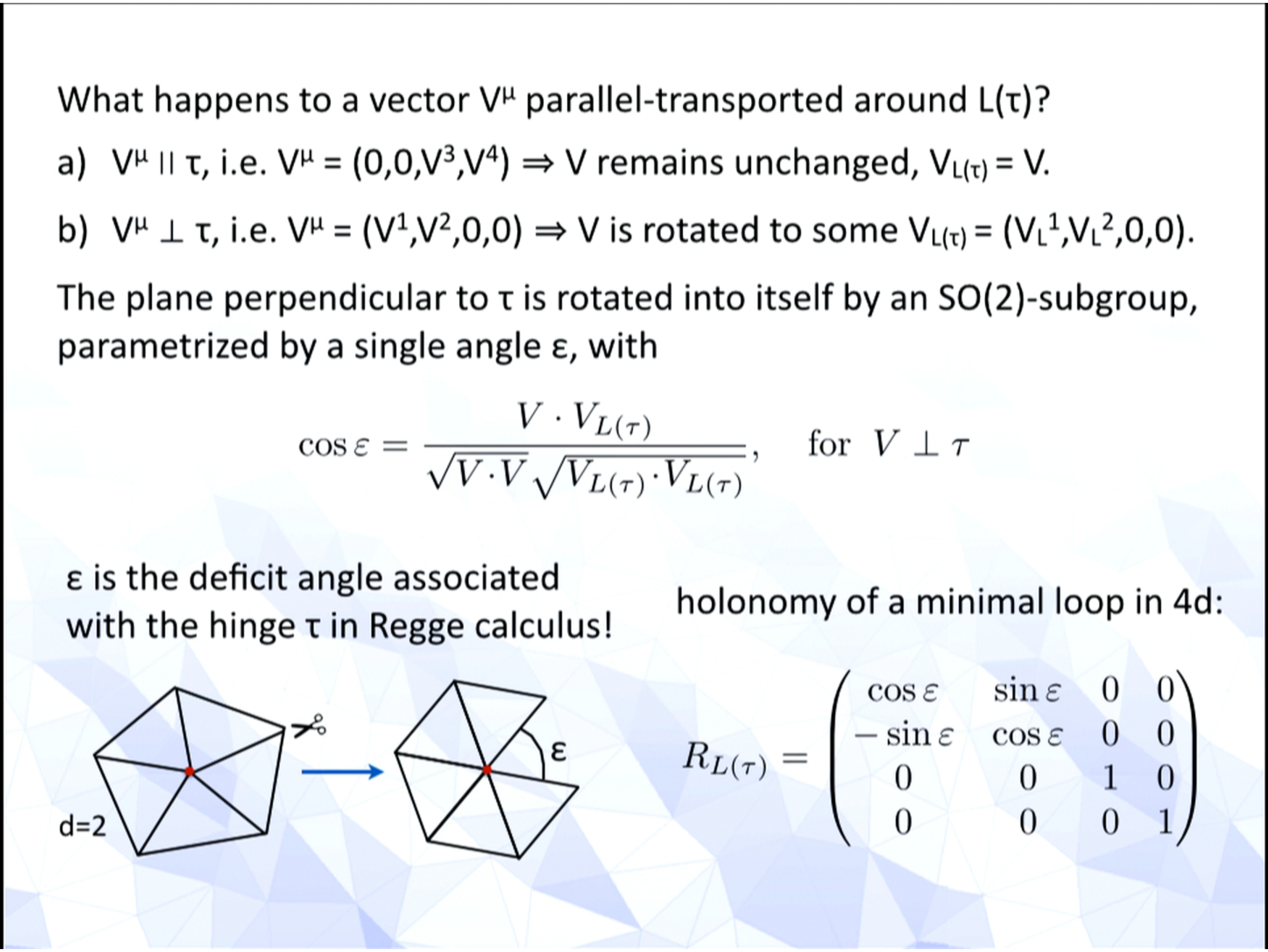
Curvature is located at “hinges” = subsimplices of dimension d−2. In four dimensions it is located at triangles τ.

The Gaussian curvature of a small surface perpendicular to the hinge τ is obtained by parallel-transporting a vector along a minimal loop around τ.



red: hinge, blue: minimal loop L(τ)

In d=4, each hinge τ is shared by a ring of four-simplices (not shown). In each of them we can choose coordinates s.t. τ lies in the 3-4 plane.



What happens to a vector V^μ parallel-transported around $L(\tau)$?

a) $V^\mu \parallel \tau$, i.e. $V^\mu = (0,0,V^3,V^4) \Rightarrow$ V remains unchanged, $V_{L(\tau)} = V$.

b) $V^\mu \perp \tau$, i.e. $V^\mu = (V^1,V^2,0,0) \Rightarrow$ V is rotated to some $V_{L(\tau)} = (V_L^{\,1},V_L^{\,2},0,0)$.

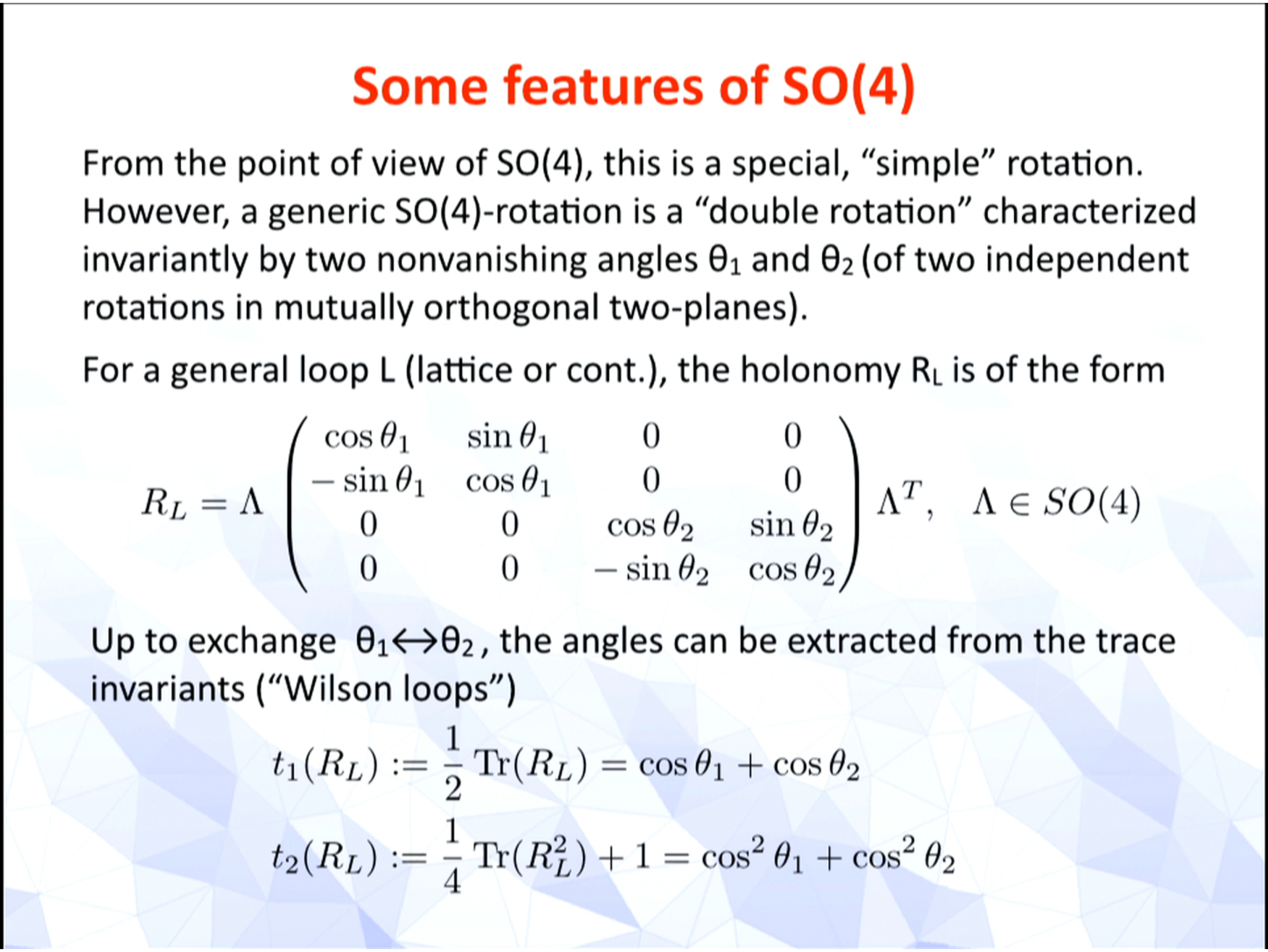
The plane perpendicular to τ is rotated into itself by an SO(2)-subgroup, parametrized by a single angle ε, with

$$\cos\varepsilon = \frac{V\cdot V_{L(\tau)}}{\sqrt{V\cdot V}\sqrt{V_{L(\tau)}\cdot V_{L(\tau)}}}, \quad \text{for } \; V\perp\tau$$

ε is the deficit angle associated with the hinge τ in Regge calculus!

holonomy of a minimal loop in 4d:

$$R_{L(\tau)} = \begin{pmatrix} \cos\varepsilon & \sin\varepsilon & 0 & 0 \\ -\sin\varepsilon & \cos\varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Some features of SO(4)

From the point of view of SO(4), this is a special, “simple” rotation. However, a generic SO(4)-rotation is a “double rotation” characterized invariantly by two nonvanishing angles θ_1 and θ_2 (of two independent rotations in mutually orthogonal two-planes).

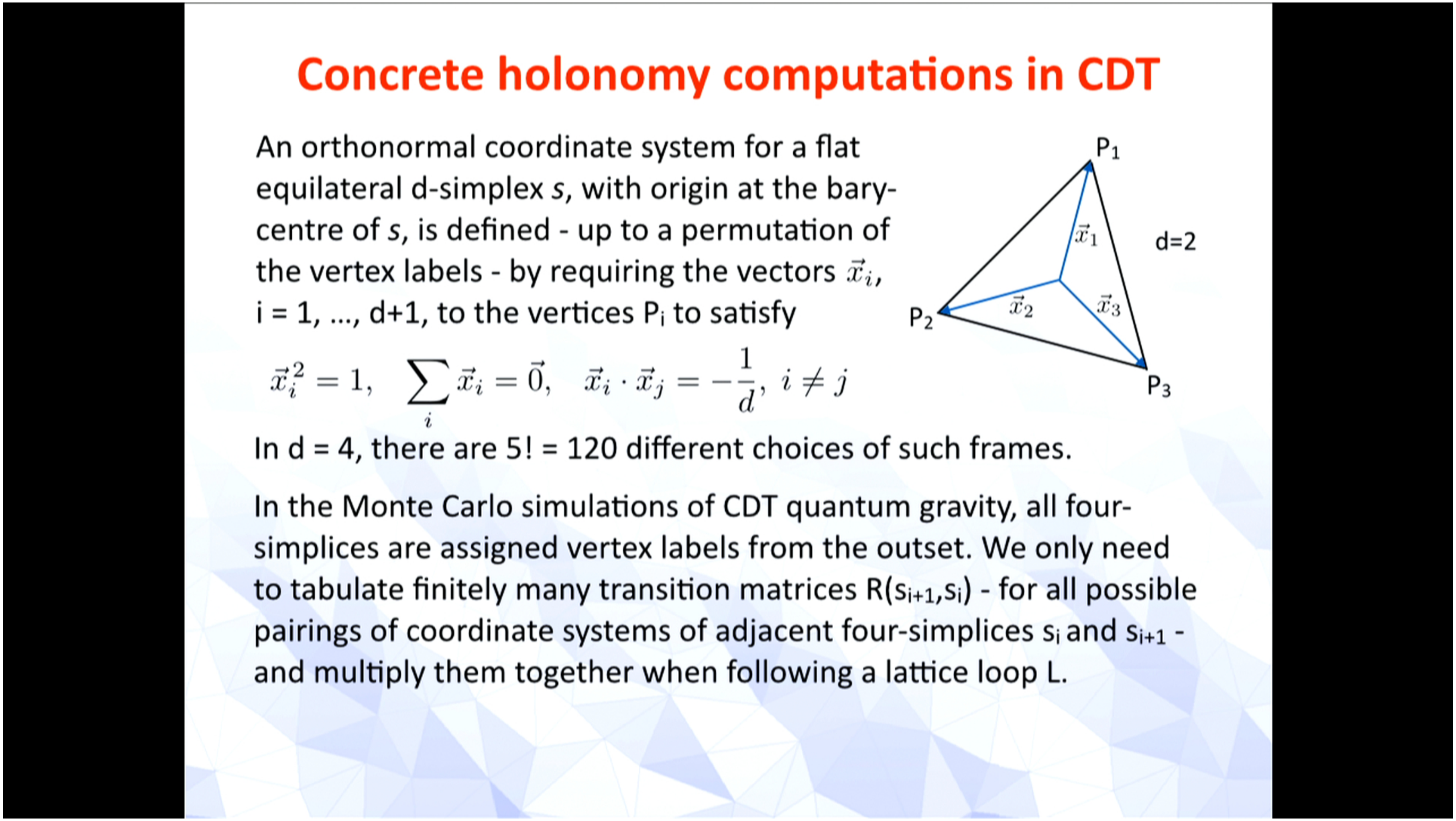
For a general loop L (lattice or cont.), the holonomy R_L is of the form

$$R_L = \Lambda \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & 0 & -\sin\theta_2 & \cos\theta_2 \end{pmatrix} \Lambda^T, \quad \Lambda \in SO(4)$$

Up to exchange $\theta_1 \leftrightarrow \theta_2$, the angles can be extracted from the trace invariants (“Wilson loops”)

$$t_1(R_L) := \frac{1}{2}\,\mathrm{Tr}(R_L) = \cos\theta_1 + \cos\theta_2$$

$$t_2(R_L) := \frac{1}{4}\,\mathrm{Tr}(R_L^2) + 1 = \cos^2\theta_1 + \cos^2\theta_2$$



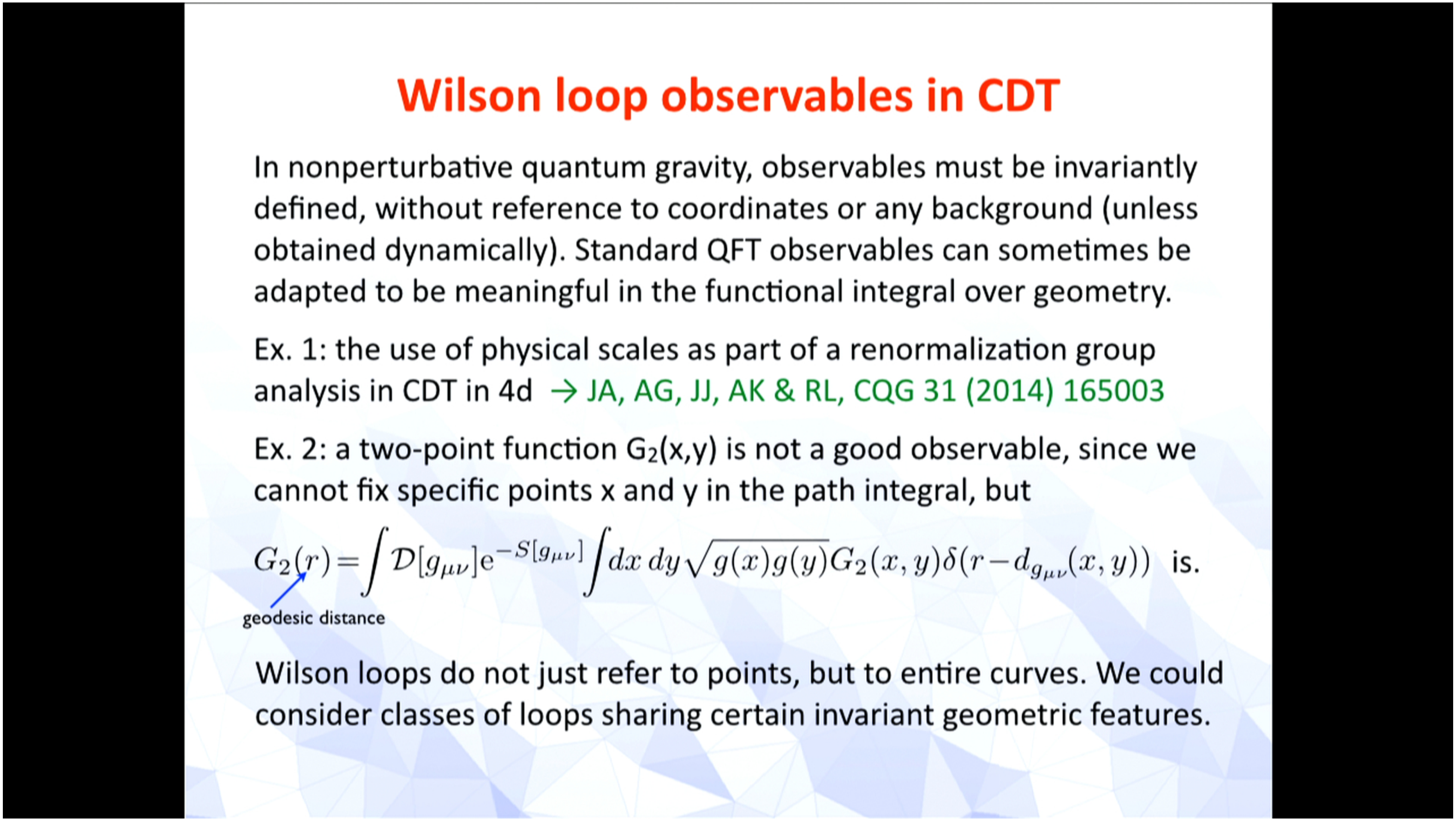
Concrete holonomy computations in CDT

An orthonormal coordinate system for a flat equilateral d-simplex *s*, with origin at the barycentre of *s*, is defined - up to a permutation of the vertex labels - by requiring the vectors $\vec{x}_i$, i = 1, …, d+1, to the vertices P$_i$ to satisfy

$$\vec{x}_i^{\,2} = 1, \quad \sum_i \vec{x}_i = \vec{0}, \quad \vec{x}_i \cdot \vec{x}_j = -\frac{1}{d}, \; i \neq j$$

In d = 4, there are 5! = 120 different choices of such frames.

In the Monte Carlo simulations of CDT quantum gravity, all four-simplices are assigned vertex labels from the outset. We only need to tabulate finitely many transition matrices R(s_{i+1},s_i) - for all possible pairings of coordinate systems of adjacent four-simplices s_i and s_{i+1} - and multiply them together when following a lattice loop L.



Wilson loop observables in CDT

In nonperturbative quantum gravity, observables must be invariantly defined, without reference to coordinates or any background (unless obtained dynamically). Standard QFT observables can sometimes be adapted to be meaningful in the functional integral over geometry.

Ex. 1: the use of physical scales as part of a renormalization group analysis in CDT in 4d → JA, AG, JJ, AK & RL, CQG 31 (2014) 165003

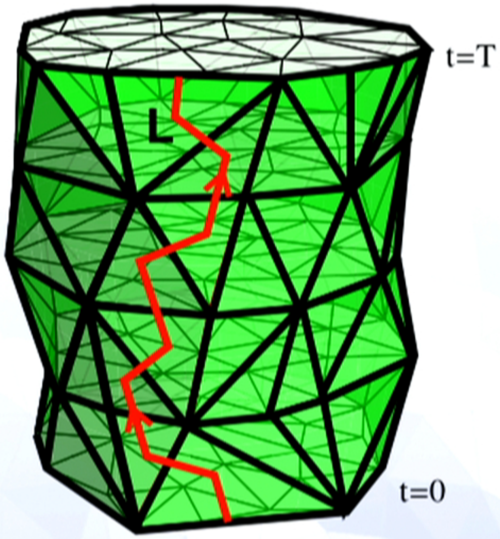
Ex. 2: a two-point function $G_2(x,y)$ is not a good observable, since we cannot fix specific points x and y in the path integral, but

$$G_2(r) = \int \mathcal{D}[g_{\mu\nu}] \mathrm{e}^{-S[g_{\mu\nu}]} \int dx\, dy \sqrt{g(x)g(y)}\, G_2(x,y)\, \delta(r - d_{g_{\mu\nu}}(x,y)) \quad \text{is.}$$

geodesic distance

Wilson loops do not just refer to points, but to entire curves. We could consider classes of loops sharing certain invariant geometric features.

Defining physical Wilson loops in CDT



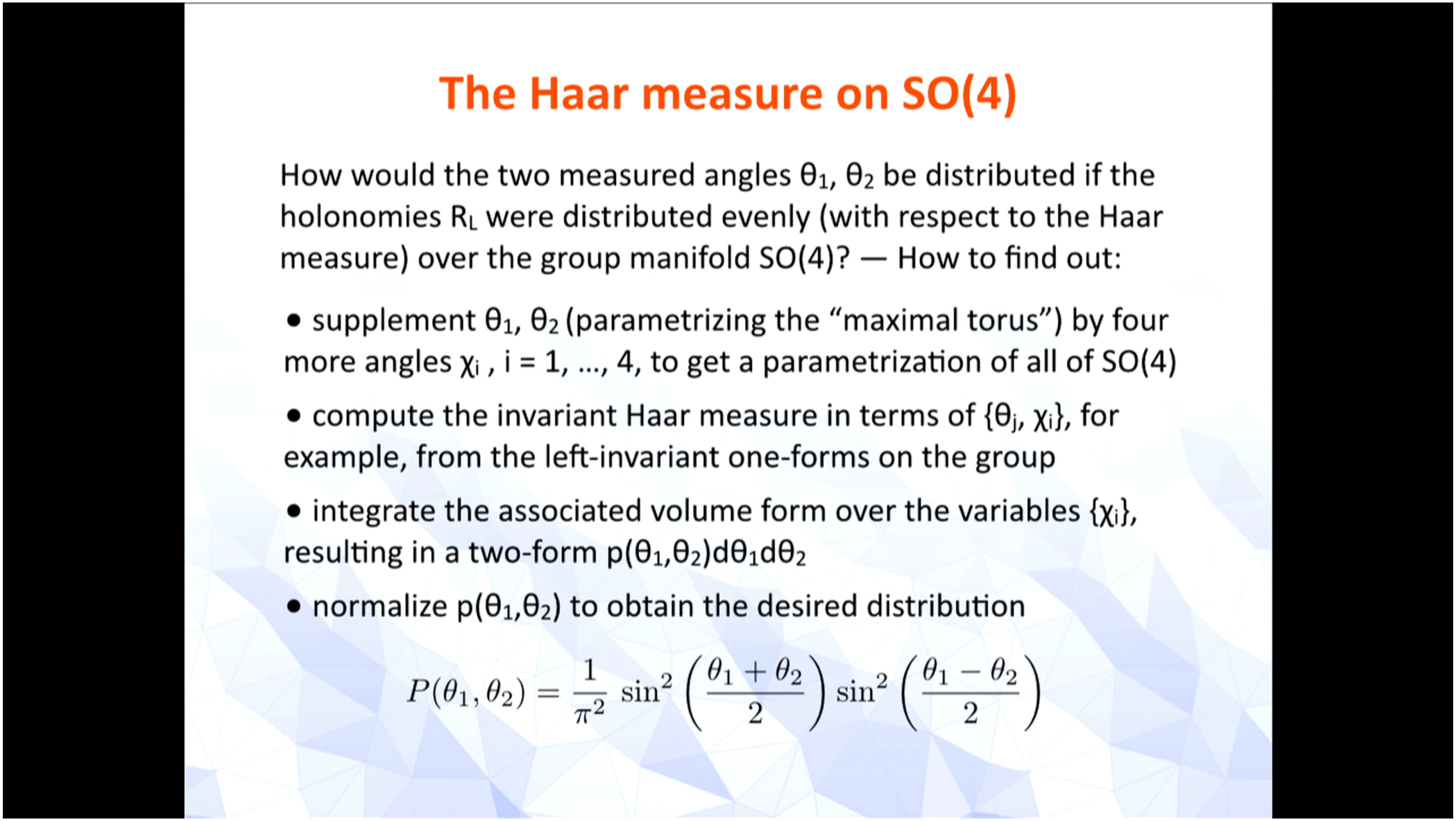
We let the loop L coincide with the world line of a particle moving forward in time. The loops wind once around the compactified time direction of the triangulated spacetimes, which have topology $S^1 \times S^3$.

Correspondingly, we add to the action a term for a free massive point particle

$$S^{\text{p.p.}} = m \int dl \quad \rightarrow \quad S^{\text{p.p.}}_{\text{CDT}} = m_0 N_L$$

where N_L = number of four-simplices along L.

We then perform Monte Carlo simulations for the combined gravity-particle system, updating both, for time extension T = 80, and spacetime volume N_4 = 20.000. We measure the trace invariants $t_1(R_L)$ and $t_2(R_L)$ introduced earlier, and extract the two angles θ_1 and θ_2.



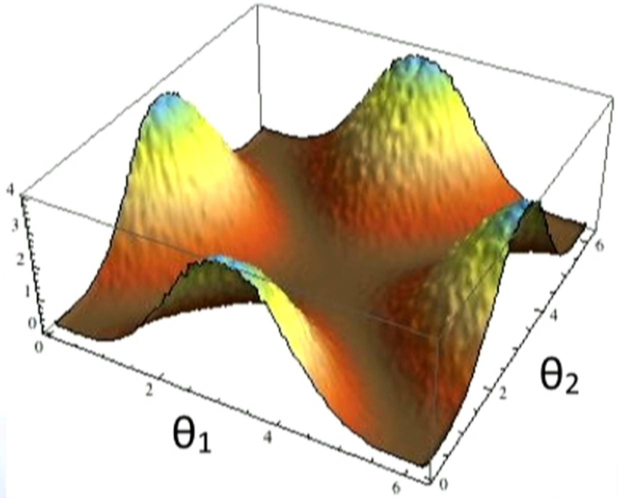
The Haar measure on SO(4)

How would the two measured angles θ_1, θ_2 be distributed if the holonomies R_L were distributed evenly (with respect to the Haar measure) over the group manifold SO(4)? — How to find out:

- supplement θ_1, θ_2 (parametrizing the "maximal torus") by four more angles χ_i , i = 1, …, 4, to get a parametrization of all of SO(4)
- compute the invariant Haar measure in terms of $\{\theta_j, \chi_i\}$, for example, from the left-invariant one-forms on the group
- integrate the associated volume form over the variables $\{\chi_i\}$, resulting in a two-form $p(\theta_1,\theta_2)d\theta_1 d\theta_2$
- normalize $p(\theta_1,\theta_2)$ to obtain the desired distribution

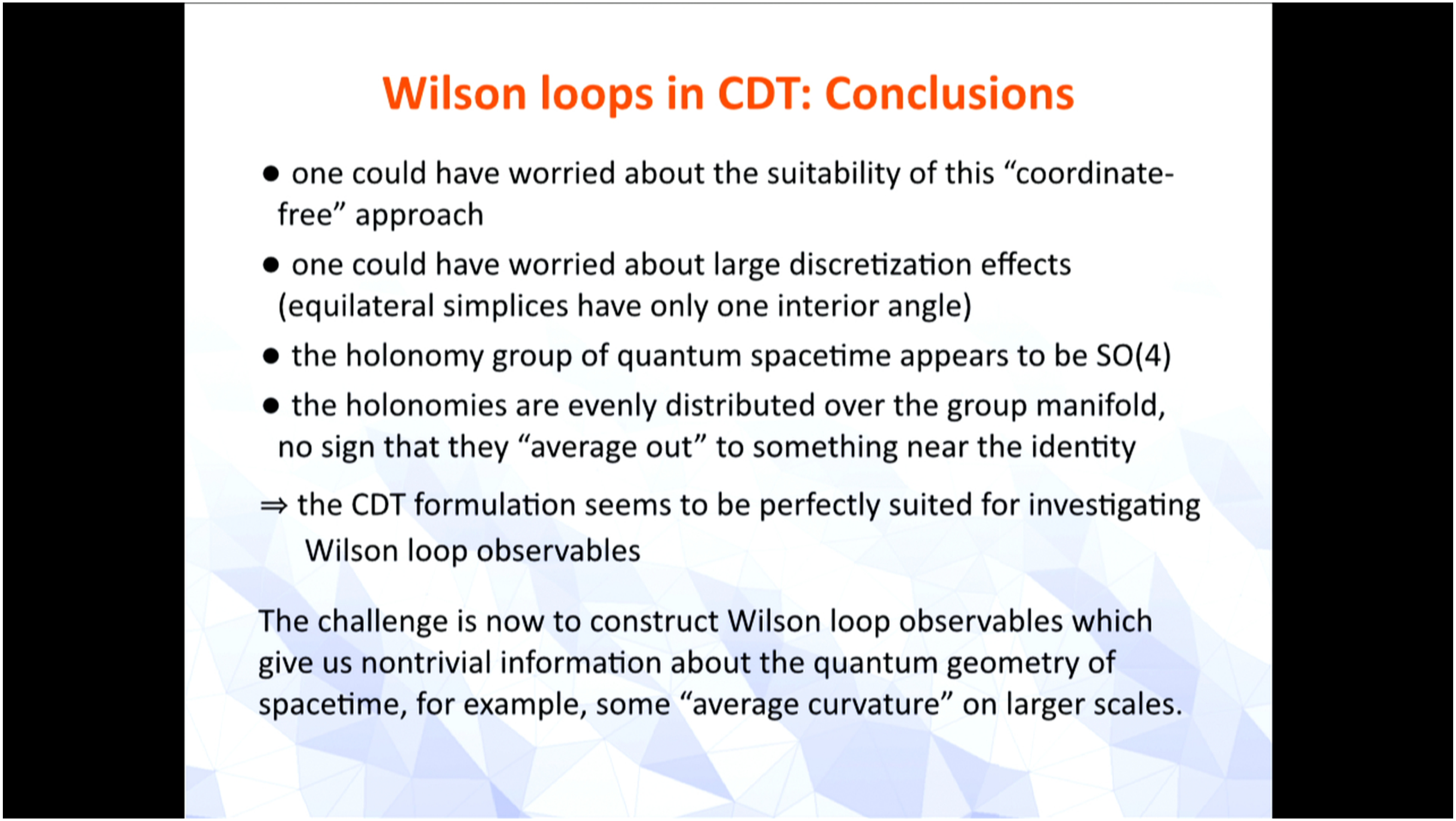
$$P(\theta_1,\theta_2) = \frac{1}{\pi^2}\sin^2\left(\frac{\theta_1+\theta_2}{2}\right)\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)$$

The measured distribution $P(\theta_1,\theta_2)$



The outcome of our Monte Carlo measurements is shown here and is in almost perfect agreement with the theoretical “even” distribution

$$P(\theta_1,\theta_2) = \frac{1}{\pi^2}\,\sin^2\left(\frac{\theta_1+\theta_2}{2}\right)\sin^2\left(\frac{\theta_1-\theta_2}{2}\right).$$

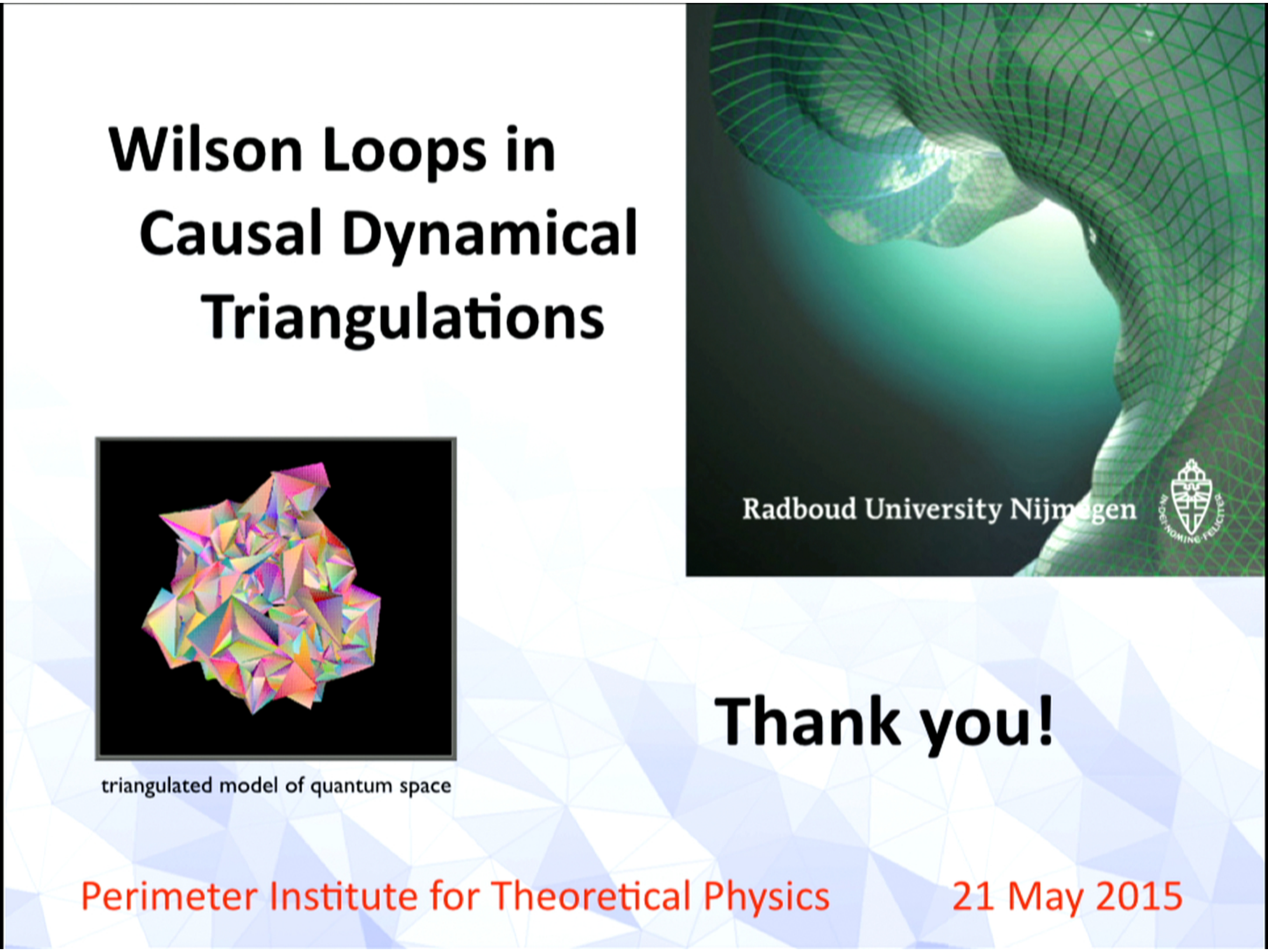


Wilson loops in CDT: Conclusions

- one could have worried about the suitability of this “coordinate-free” approach
- one could have worried about large discretization effects (equilateral simplices have only one interior angle)
- the holonomy group of quantum spacetime appears to be SO(4)
- the holonomies are evenly distributed over the group manifold, no sign that they “average out” to something near the identity

⇒ the CDT formulation seems to be perfectly suited for investigating Wilson loop observables

The challenge is now to construct Wilson loop observables which give us nontrivial information about the quantum geometry of spacetime, for example, some “average curvature” on larger scales.



Wilson Loops in Causal Dynamical Triangulations

triangulated model of quantum space

Radboud University Nijmegen

Thank you!

Perimeter Institute for Theoretical Physics 21 May 2015