Title: Parafermionic phases with symmetry-breaking and topological order

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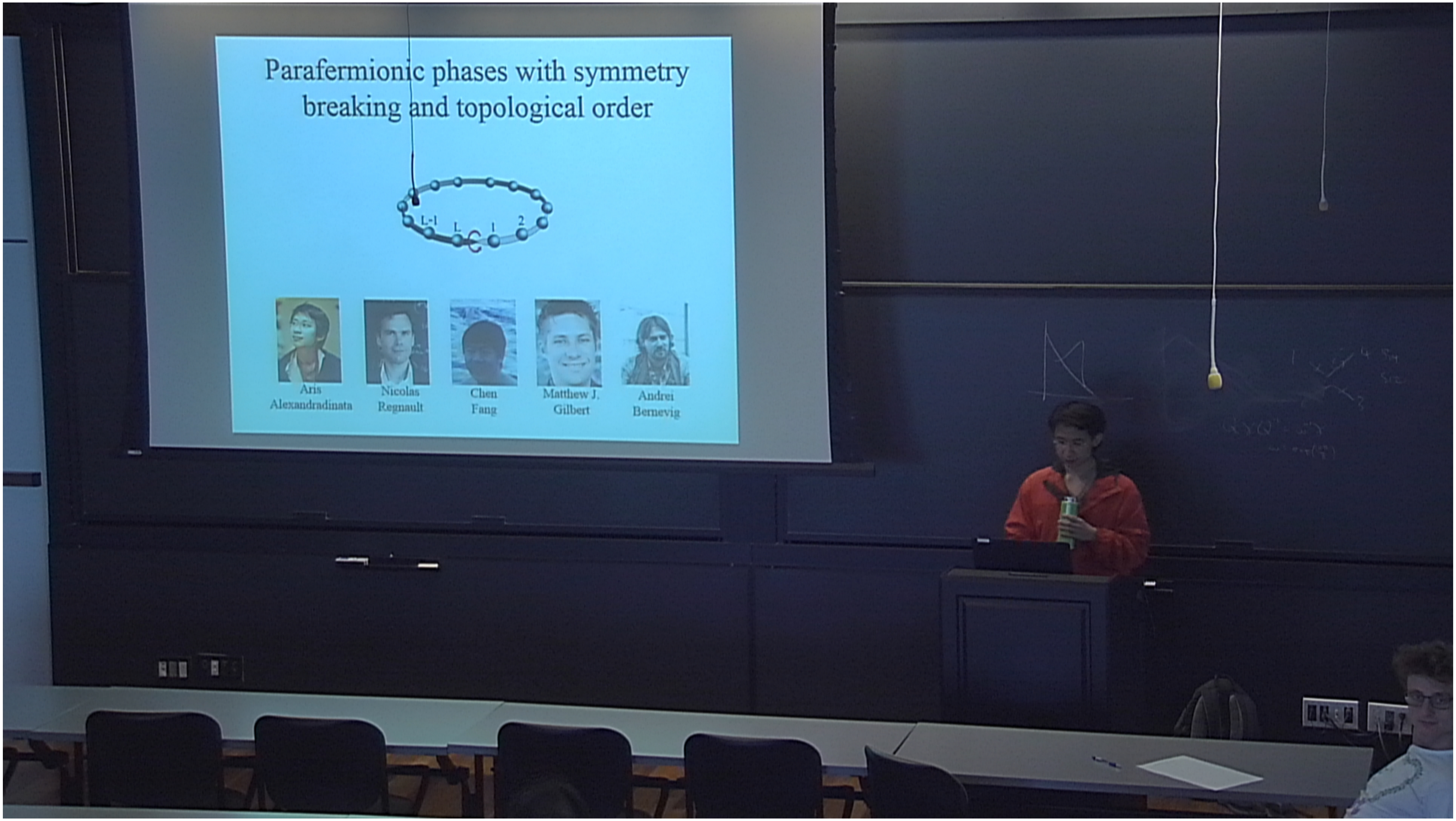
Abstract: <p>Parafermions are the simplest generalizations of Majorana fermions that realize topological order. We propose a less restrictive notion of topological order in 1D open chains, which generalizes the seminal work by Fendley [J. Stat. Mech., P11020 (2012)]. The first essential property is that the groundstates are mutually indistinguishable by local, symmetric probes, and the second is a generalized notion of zero edge modes which cyclically permute the groundstates. These two properties are shown to be topologically robust, and applicable to a wider family of topologically-ordered Hamiltonians than has been previously considered. As an application of these edge modes, we formulate a new notion of twisted boundary conditions on a closed chain, which guarantees that the closed-chain groundstate is topological, i.e., it originates from the topological manifold of the open chain. Finally, we generalize these ideas to describe symmetry-breaking phases with a parafermionic order parameter.</p>

<p>These exotic phases are condensates of parafermion multiplets, which generalizes Cooper pairing in superconductors. The stability of these condensates are investigated on both open and closed chains.</p>


Parafermionic phases with symmetry breaking and topological order
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Aris Alexandradinata
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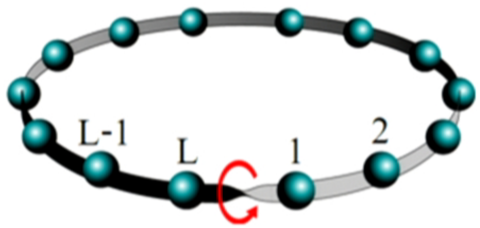


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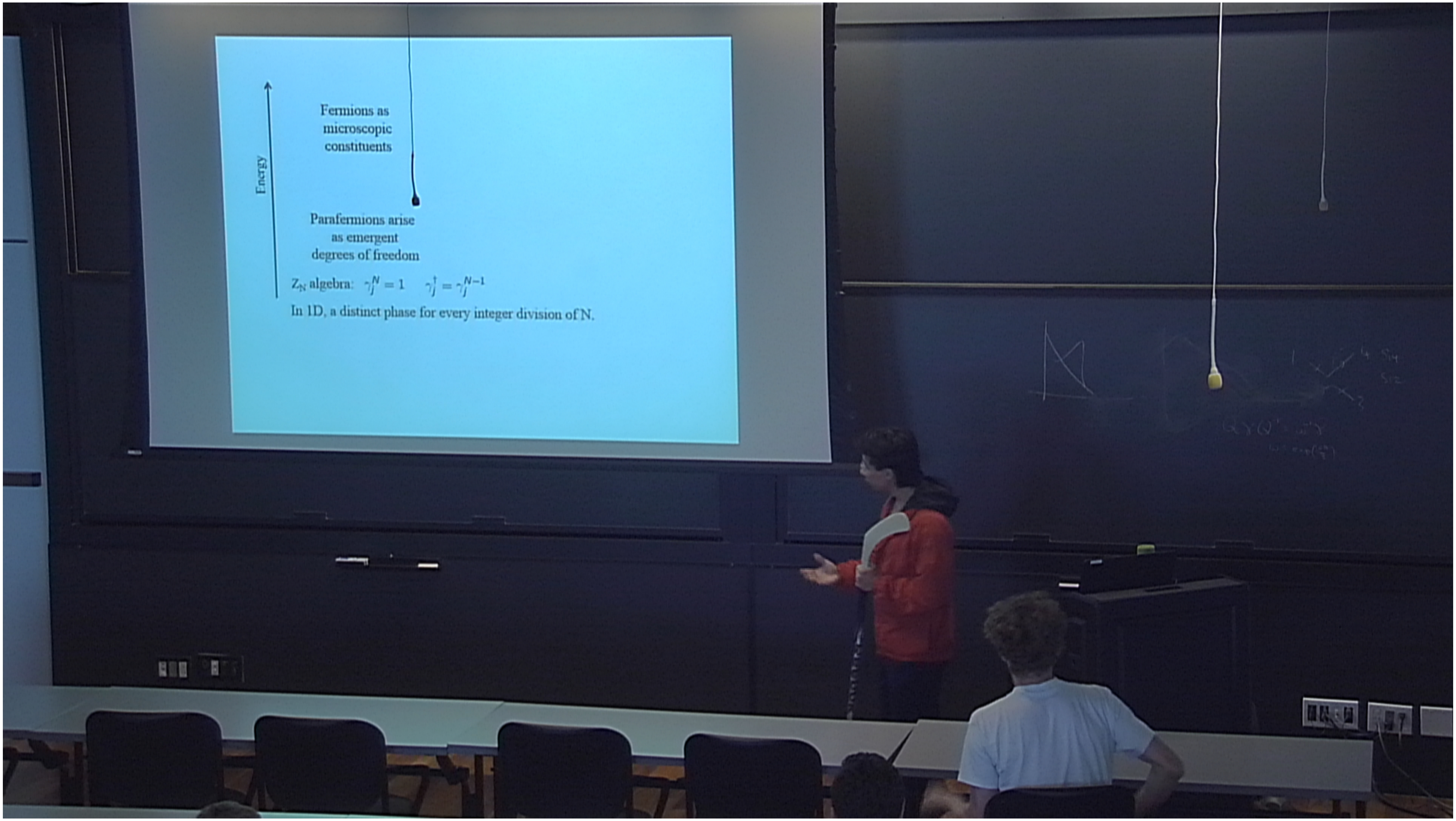
Fermions as microscopic constituents
Energy
Parafermions arise as emergent degrees of freedom
Z_N algebra: γ_j^N = 1   γ_j^† = γ_j^(N−1)
In 1D, a distinct phase for every integer division of N.

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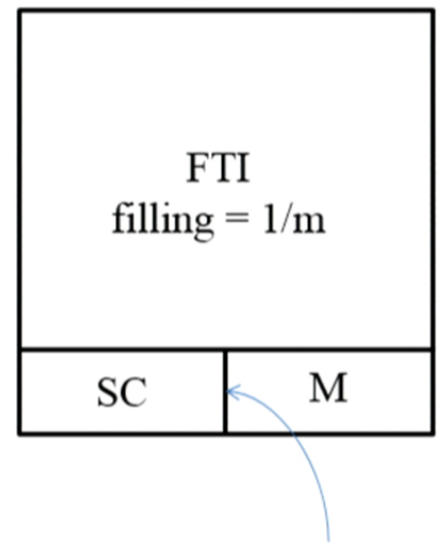
<u>Anomalous origin</u>

(Mengcheng)



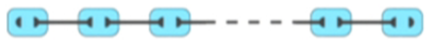
Z_{2m} parafermion

Truly 1D fermions have a Z_2 classification.

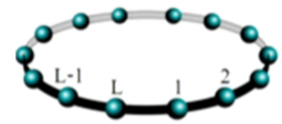
(Fidkowski,Kitaev)

What am I calling topological order?

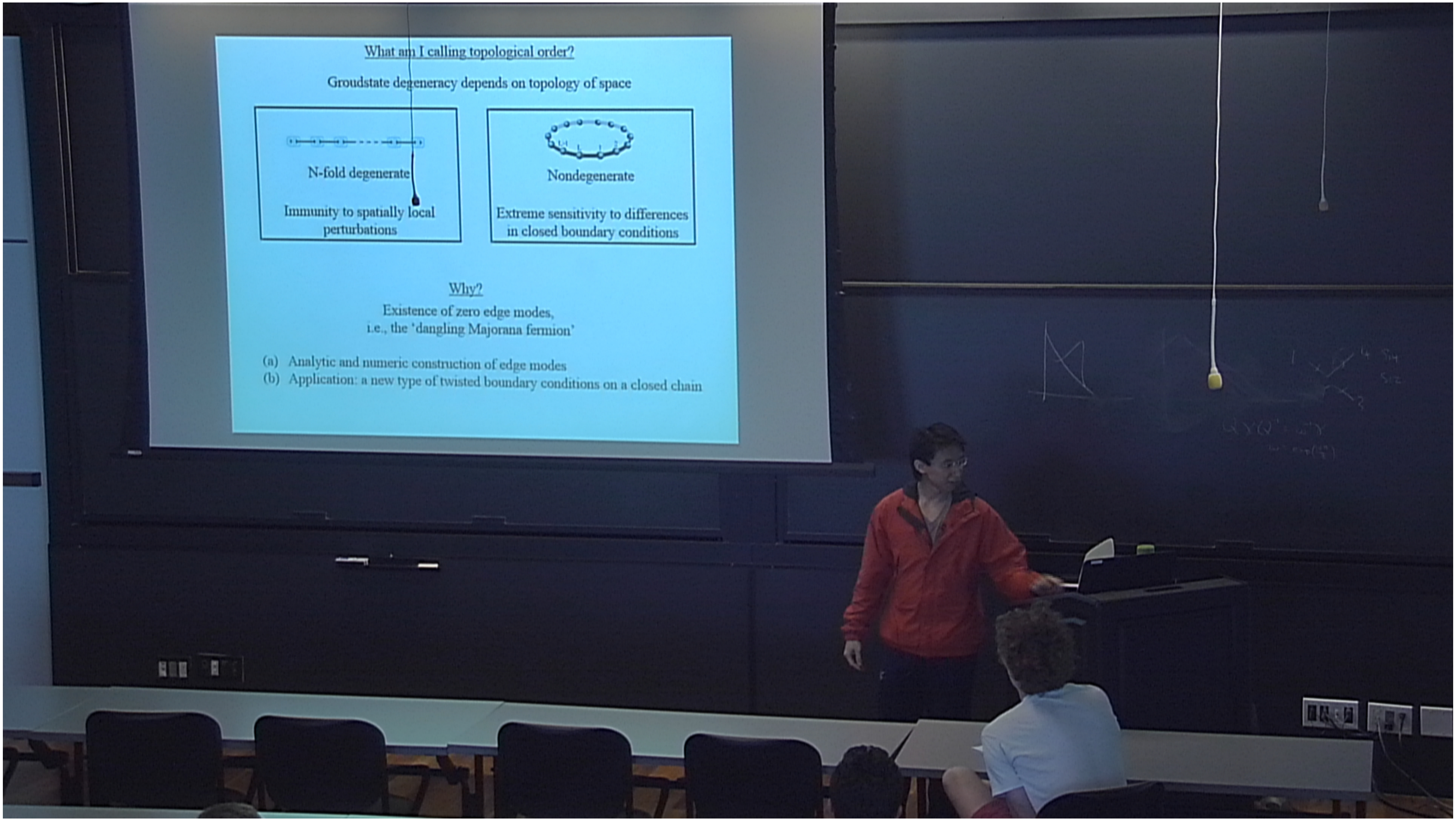
Groudstate degeneracy depends on topology of space

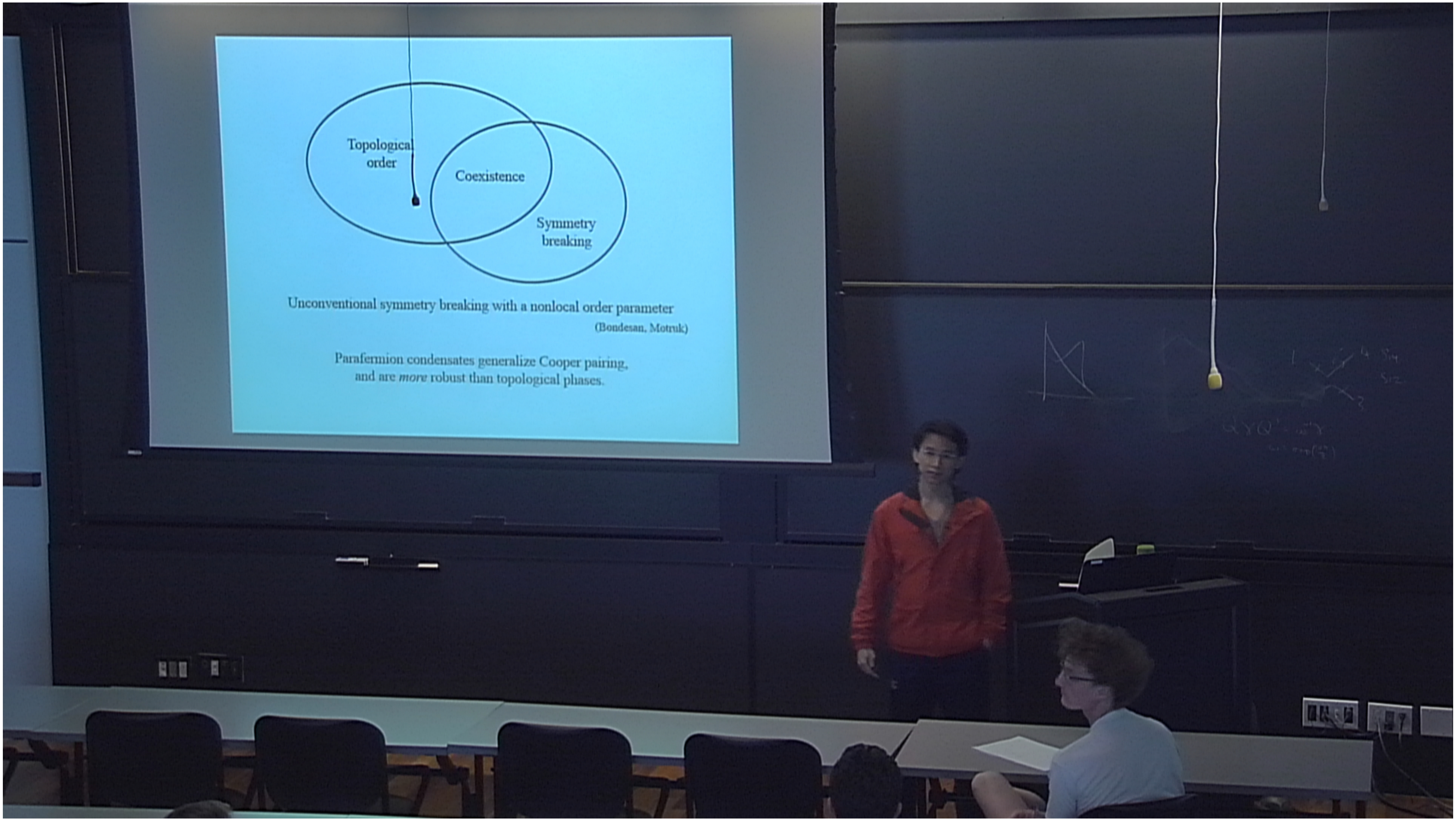


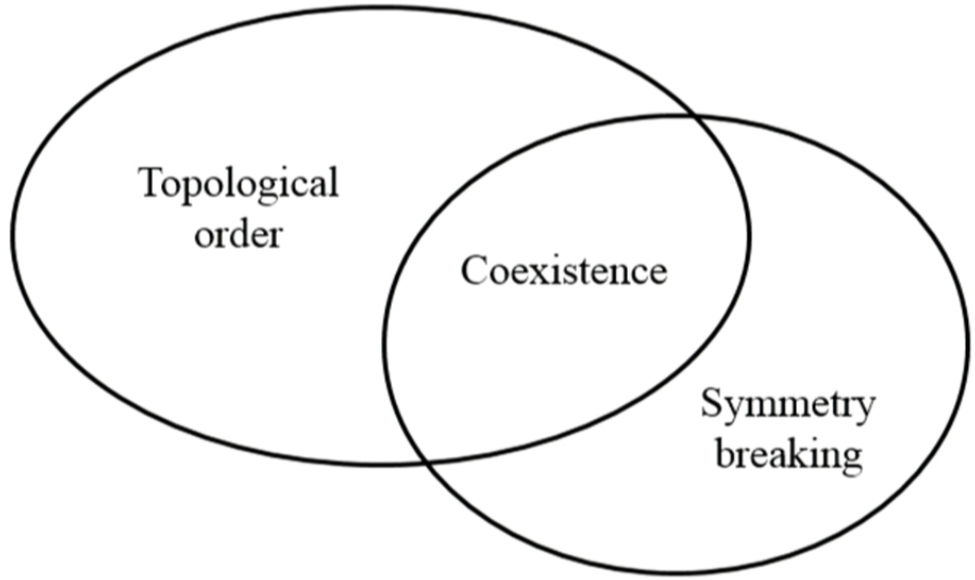
N-fold degenerate



Nondegenerate


What am I calling topological order?
Groudstate degeneracy depends on topology of space
N-fold degenerate
Immunity to spatially local perturbations
Nondegenerate
Extreme sensitivity to differences in closed boundary conditions
Why?
Existence of zero edge modes,
i.e., the 'dangling Majorana fermion'
(a) Analytic and numeric construction of edge modes
(b) Application: a new type of twisted boundary conditions on a closed chain


Topological order
Coexistence
Symmetry breaking
Unconventional symmetry breaking with a nonlocal order parameter
(Bondesan, Motruk)
Parafermion condensates generalize Cooper pairing,
and are *more* robust than topological phases.



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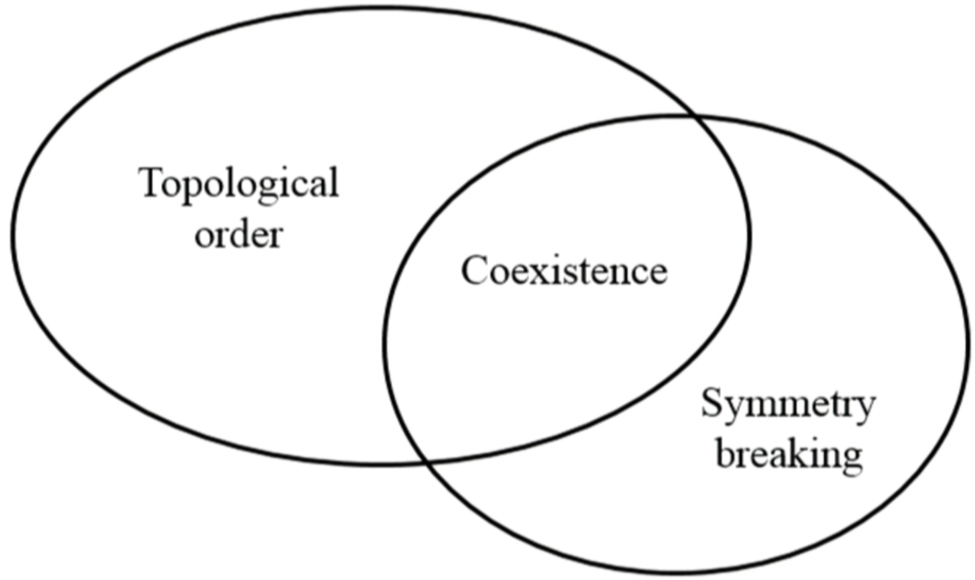

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Introduction to parafermions

$\mathbb{Z}_N$ Parafermions generalize the Majorana algebra

$$\gamma_j^N = 1 \qquad \gamma_j^\dagger = \gamma_j^{N-1} \qquad \gamma_j \gamma_l = \omega^{\mathrm{sgn}[l-j]} \gamma_l \gamma_j \qquad \omega = e^{i2\pi/N}$$

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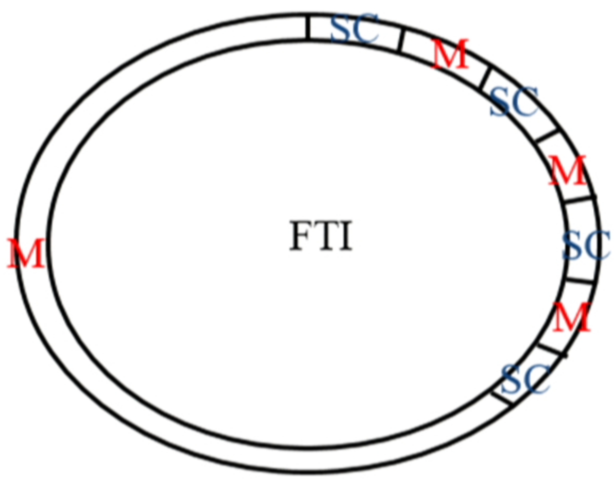
Simplest parafermion model that is topologically-ordered:

$H_{dimer} =$

$$= -\omega^{(N-1)/2} \; \gamma_{2j+1}^\dagger \; \gamma_{2j} + h.c.$$



Z_N symmetry generator Q

$$Q = \prod_{j=1}^{L} \gamma_{2j-1}^{\dagger} \gamma_{2j} \qquad Q^N = I$$

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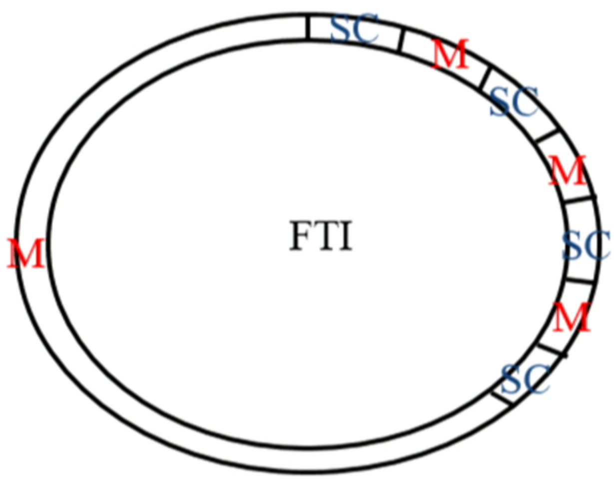

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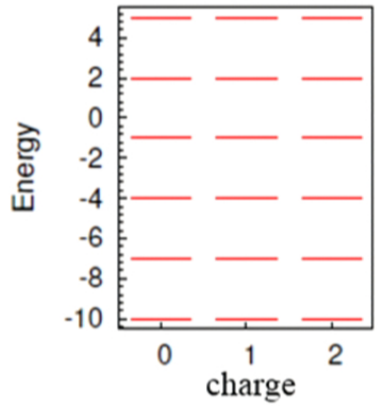
Z_N charge of an operator

$$Q \gamma_j Q^{-1} = \omega^{-1} \gamma_j$$

$$Q \gamma_j^{\dagger} Q^{-1} = \omega \gamma_j^{\dagger}$$

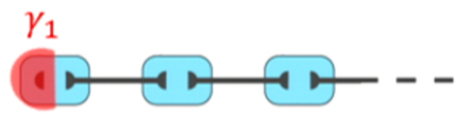
$$Q \gamma_{2j-1}^{\dagger} \gamma_{2j} Q^{-1} = \gamma_{2j-1}^{\dagger} \gamma_{2j}$$

Zero edge modes



$$[\gamma_1, H_{dimer}] = 0$$

$$Q\gamma_1 = \omega\gamma_1 Q$$



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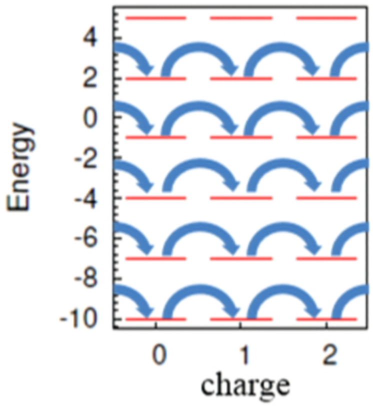

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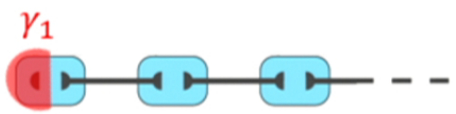

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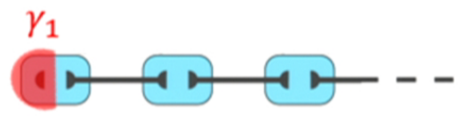
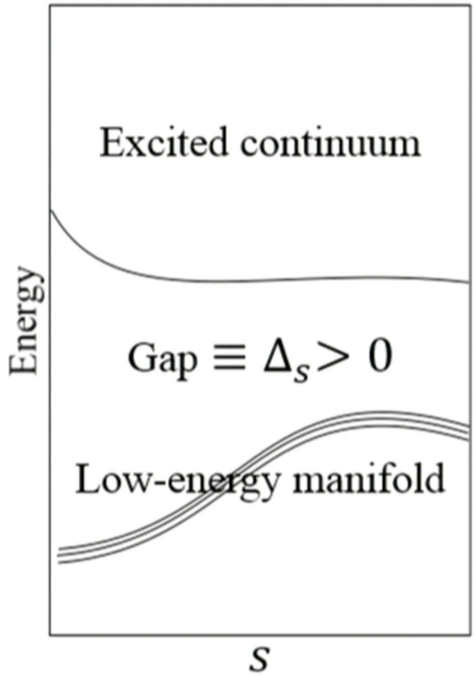


Goal: construct edge modes for $H_s = H_{dimer} + sV$



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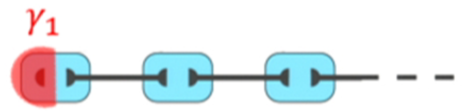

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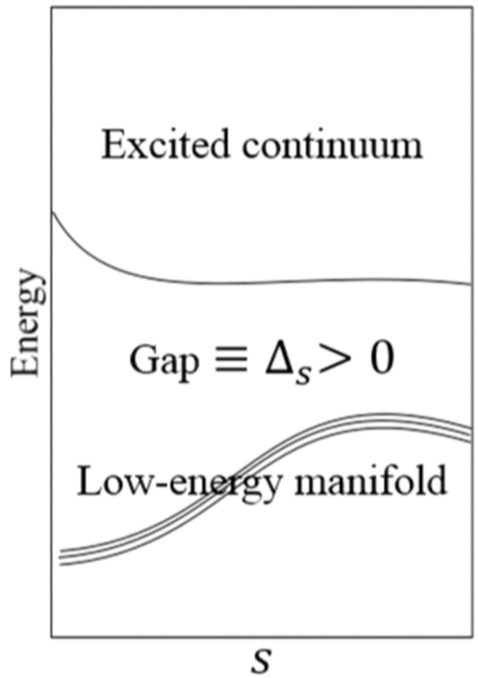
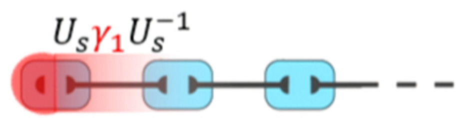


Low-energy manifold $\equiv P_s$

$$U_s P_0 U_s^{-1} = P_s$$

Local, unitary transformation

$$[U_s \gamma_1 U_s^{-1}, P_s H_s P_s] = 0$$



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$$U_s = \exp[-i \int_0^s dt\, D_t]$$

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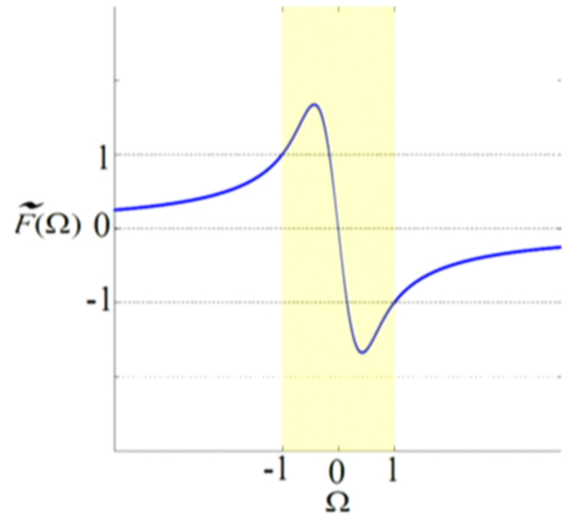
Properties of the filter function

(1) F(t) decays rapidly for large t

(2) Success of filtration depends on Δ_s

(3) $\tilde{F}(\Omega) = -\frac{1}{\Omega}$ for $|\Omega| \geq 1$

$$\Rightarrow\ U_s P_0 U_s^{-1} = P_s$$



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<u>A sketch</u>

$H_s|j,s\rangle = E_{j,s}|j,s\rangle$
labels instantaneous eigenstates.

Matrix element between a ground and excited state:

$$i\langle k,s|D_s|j,s\rangle$$
$$= \int_{-\infty}^{\infty} dt\, F(\Delta_s t)\, e^{i(E_{k,s}-E_{j,s})t}\, \langle k,s|V|j,s\rangle$$
$$= -\frac{\langle k,s|V|j,s\rangle}{E_{k,s}-E_{j,s}}.$$

Example of ***quasi-adiabatic continuation of edge modes***

$$H_s = i\,\gamma_{2j}\,\gamma_{2j+1} + s\,i\,\gamma_{2j-1}\,\gamma_{2j}$$

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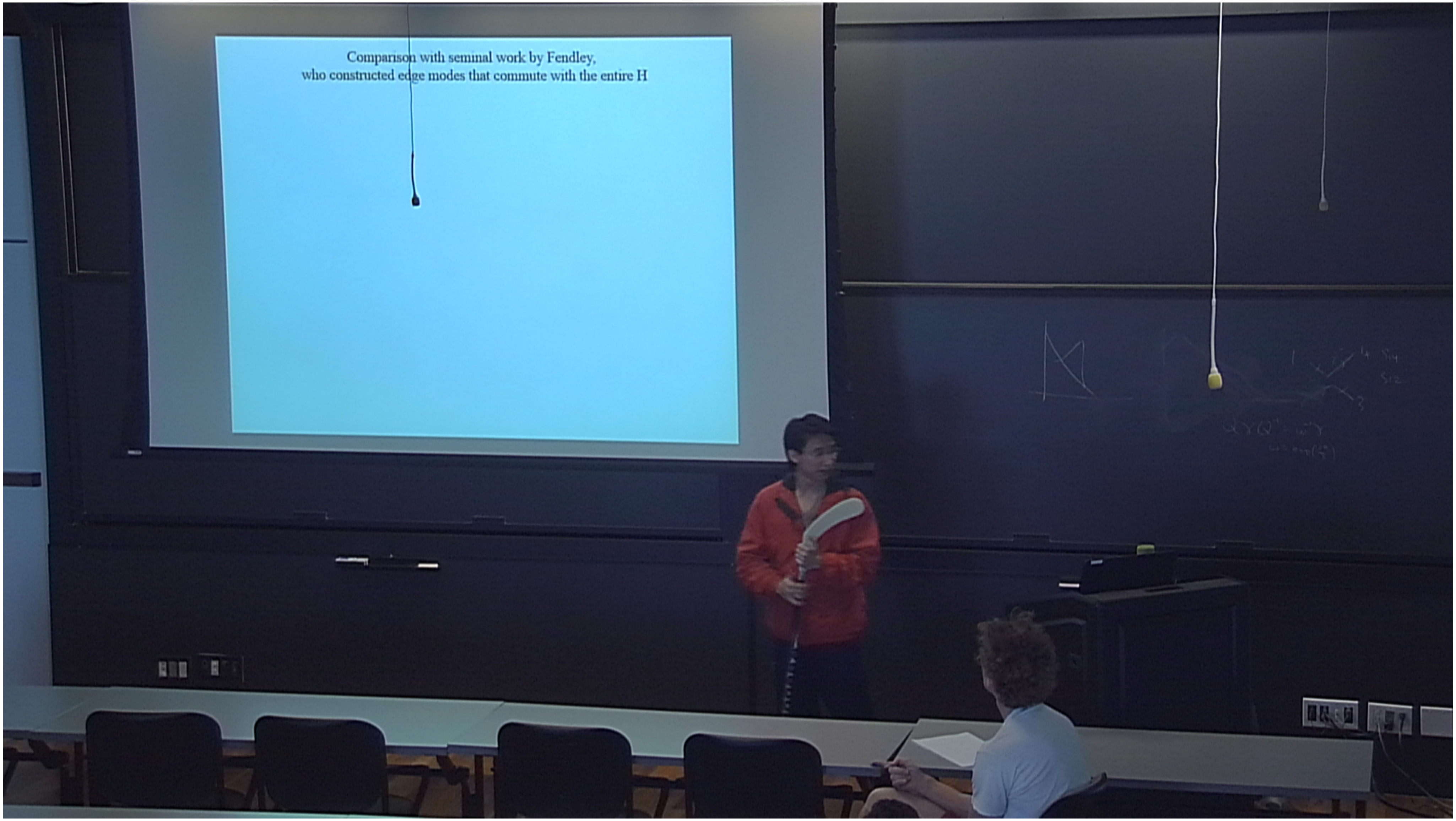
Goal: construct $U_s \gamma_1 U_s^{-1} = \gamma_1 + is\left[D_0, \gamma_1\right] + O(s^2)$

$$i\left[D_0, \gamma_1\right] = \int_{-\infty}^{\infty} dt\; F(2t)\; e^{iH_0 t} \overbrace{\left[V, \gamma_1\right]}^{\gamma_2} e^{-iH_0 t}$$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Fendley's edge mode: $\tilde{\gamma}_F = \gamma_1 + s\,\delta$

$$[H_s, \tilde{\gamma}_F] = s[V, \gamma_1] + s[H_{dimer}, \delta] + O(s^2)$$

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$$\Rightarrow\ \tilde{\gamma}_F = \gamma_1 + s\,\gamma_3 + s^2\gamma_5 + \cdots$$

$[\tilde{\gamma}_F, H_s] = 0$ is stronger than $[\, U_s\gamma_1 U_s^{-1},\ P_s H_s P_s\ ] = 0$

<u>To make our construction seem even less relevant…</u>

For any quadratic Majorana model, we find $[\, U_s\gamma_1 U_s^{-1}, H_s\,] = O(s^2)$.

For an interacting Majorana model, Kells claims a Fendley edge mode exists.

Did we gain anything with our edge-mode construction?

$$H_s = -\omega e^{i\phi}\, \gamma^{\dagger}_{2j+1}\, \gamma_{2j} - s\, \omega^{*}\, \gamma^{\dagger}_{2j-1}\, \gamma_{2j} + h.c.$$

- ϕ is an extra degree of freedom not present in Majoranas.
- $\phi \neq 0$ breaks time-reversal and spatial-inversion symmetries.

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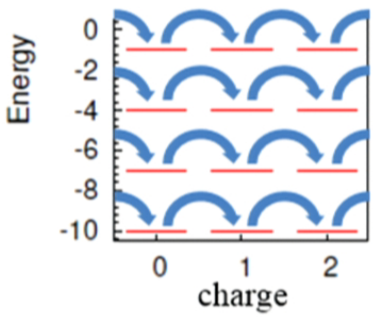


$$\tilde{\gamma}_F = \gamma_1 - \frac{is}{2\sin(3\phi)}\Big[\gamma_2 + e^{i2\phi}\gamma_3 + e^{-i2\phi}\omega\gamma_2^\dagger\gamma_3^\dagger + \omega\gamma_1^\dagger(\gamma_2^\dagger + e^{-i2\phi}\gamma_3^\dagger + e^{i2\phi}\omega\gamma_2\gamma_3)\Big] + \cdots$$

Fendley's edge mode delocalizes as $\phi \to 0$.

symmetries under time reversal symmetry charge conjugation play central roles [16, 17]. Breaking or including these discrete symmetries can indeed change the type of topological order, or eliminate it altogether. The same principle seems to be applicable to this strongly interacting system.

(Fendley, 2012)

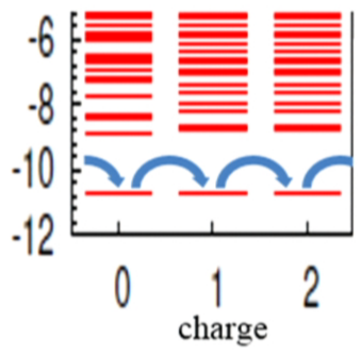
(some confusion cleared by Jermyn et. al., 2014)

$$H_s = -\omega e^{i\phi} \gamma_{2j+1}^{\dagger} \gamma_{2j} - s\,\omega^* \gamma_{2j-1}^{\dagger} \gamma_{2j} + h.c.$$

Our edge mode for $\phi = 0$:

$$U_s \gamma_1 U_s^{-1} = \gamma_1 + \frac{s}{3}\left(\gamma_3 - \omega \gamma_2^{\dagger} \gamma_3^{\dagger} - \omega \gamma_1^{\dagger} \gamma_3^{\dagger} + \omega^* \gamma_1^{\dagger} \gamma_2 \gamma_3\right) + \cdots$$

A genuine many-body edge excitation
(despite a quadratic Hamiltonian!)



Fendley's edge mode	Our edge mode
Worse spatial localization	Maximally localized
$[\tilde{\gamma}_F, H_s] = 0$	$[U_s \gamma_1 U_s^{-1},\ P_s H_s P_s] = 0$

Trade-off between spatial localization and commutivity!

Both edge modes have
the same functional form,
but with different filter functions.

General family of edge modes

$$U_s(\tilde{F})\gamma_1 U_s(\tilde{F})^{-1}$$

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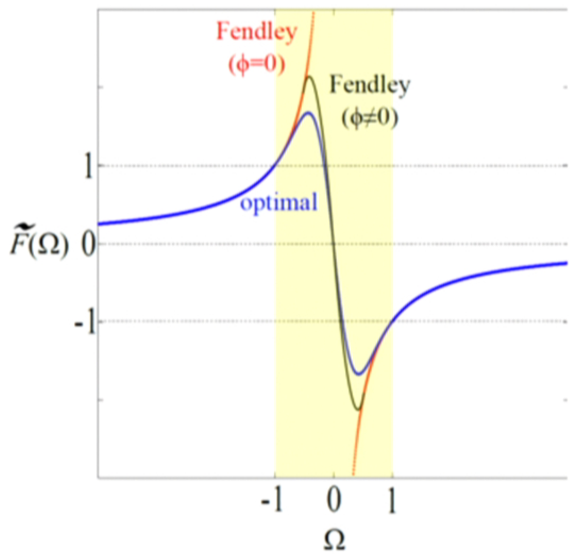
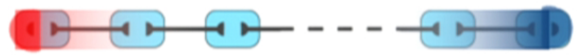


$\Gamma_l = U_s \gamma_1 U_s^{-1}$



$\Gamma_r = U_s \gamma_{2L} U_s^{-1}$

$$\tilde{Q} = \Gamma_l^{\dagger} \, \Gamma_r$$

forms a groundstate representation of the Z_N generator (Q).

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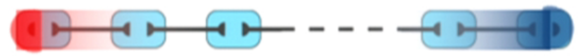

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- ‘Fractionalized’ representation of a symmetry operator (e.g. 1D SPT’s)

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- 'Fractionalized' representation of a symmetry operator (e.g. 1D SPT's)
- Non-commutative algebra in the groundstate space for an open chain

$$\Gamma_r \, \tilde{Q} = \omega \, \tilde{Q} \, \Gamma_r \,, \quad \Gamma_l \, \tilde{Q} = \omega \, \tilde{Q} \, \Gamma_l \,, \quad \Gamma_l \, \Gamma_r = \omega \, \Gamma_r \, \Gamma_l \,.$$

- Useful interpretation as an inter-edge coupling

Polyakov

"A fermion is best thought of as an object with a tail."

Generalized Jordan-Wigner (Fradkin, Kadanoff)

Read the time

$$\sigma = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}$$

Turn the clock

$$\tau = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$$

$$\tau \sigma \tau^{-1} = \omega^* \tau$$

$$\gamma_{2j-1} = \boldsymbol{\sigma_j}\, \tau_{j-1} \tau_{j-2} \ldots =$$

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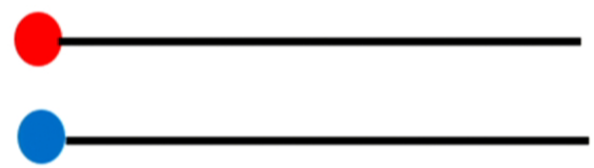

Topological order in parafermions ⟺ Conventional order in clocks

$H_{dimer} = -\sigma_{j+1}^{\dagger}\sigma_j$ Ferromagnetic clock model

Label clock states as: $\sigma_j|\alpha\rangle = \omega^{\alpha}|\alpha\rangle,\qquad Q|\alpha\rangle = |\alpha+1\rangle$

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$$\tilde{\boldsymbol{Q}}_{\boldsymbol{0}} = \gamma_1^{\dagger}\gamma_{2L} = \sigma_1^{\dagger}\sigma_L Q$$

has the same matrix elements in $|\alpha\rangle$ as Q.

Topological order in parafermions $\Longleftrightarrow$ Conventional order in clocks

$$H_{dimer} = -\,\sigma_{j+1}^{\dagger}\sigma_j \quad \text{Ferromagnetic clock model}$$

Label clock states as: $\sigma_j|\alpha\rangle = \omega^{\alpha}|\alpha\rangle, \qquad Q|\alpha\rangle = |\alpha+1\rangle$

$$\widetilde{\boldsymbol{Q}}_{\boldsymbol{0}} = \gamma_1^{\dagger}\gamma_{2L} = \sigma_1^{\dagger}\sigma_L Q$$

has the same matrix elements in $|\alpha\rangle$ as Q.

$$H_s = -\,\sigma_{j+1}^{\dagger}\sigma_j + s\,V \qquad \text{Frustrated ferromagnet}$$

The quasi-adiabatic continuation preserves matrix elements in the groundstate space.

$$\widetilde{\boldsymbol{Q}}_{\boldsymbol{s}} = U_s\widetilde{\boldsymbol{Q}}_{\boldsymbol{0}}U_s^{-1} = \boldsymbol{\Gamma}_{\boldsymbol{l}}^{\dagger}\boldsymbol{\Gamma}_{\boldsymbol{r}}$$

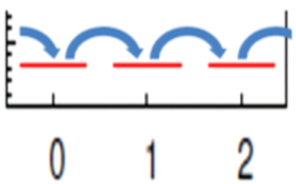
has the same matrix elements in $U_s|\alpha\rangle$ as Q.

The relation between groundstate degeneracy and spatial topology.

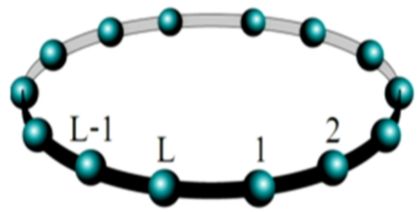
There exist edge modes:

Γ_l and Γ_r,

which generate a non-commutative algebra in the groundstate space.

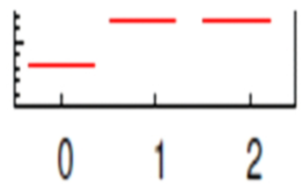


Eigenvalue of $\widetilde{Q}$



A natural way to close the chain

$$\delta H = -\widetilde{Q} + h.c.$$



Eigenvalue of $\widetilde{Q}$

Topological order parameter, in analogy with conventional symmetry-breaking systems.

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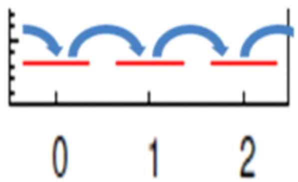

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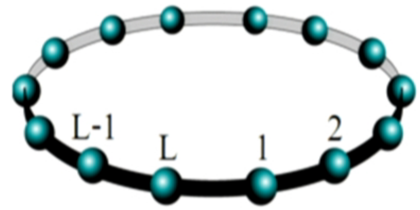

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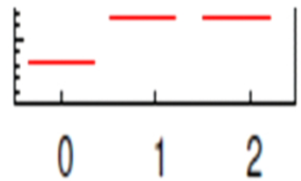

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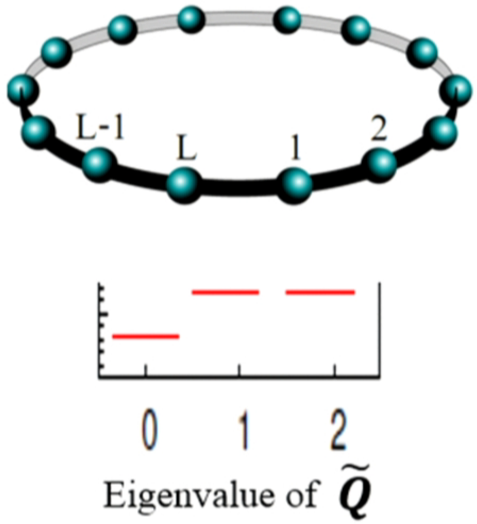

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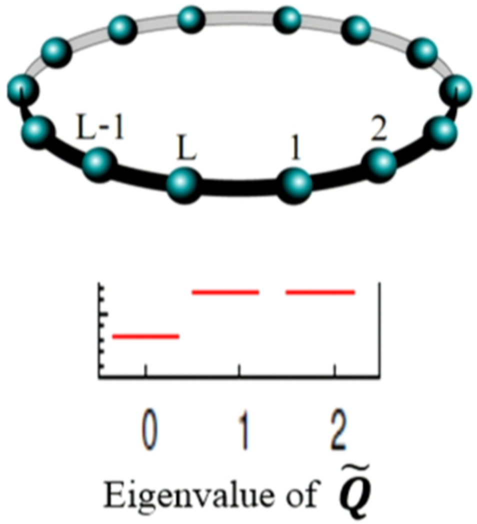
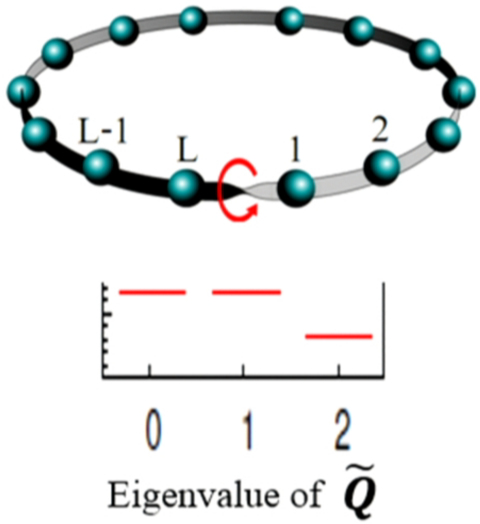
$\alpha = 0$ | $\alpha = 1$





Signature of topological order on a closed chain

Permutation of groundstate charge
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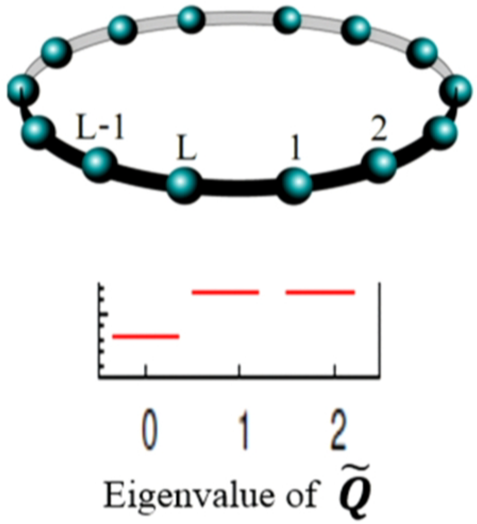

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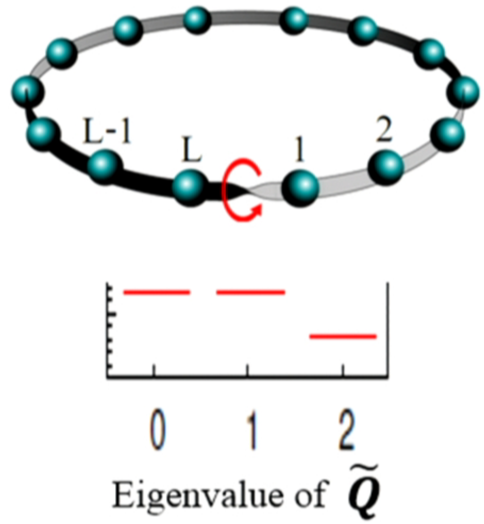

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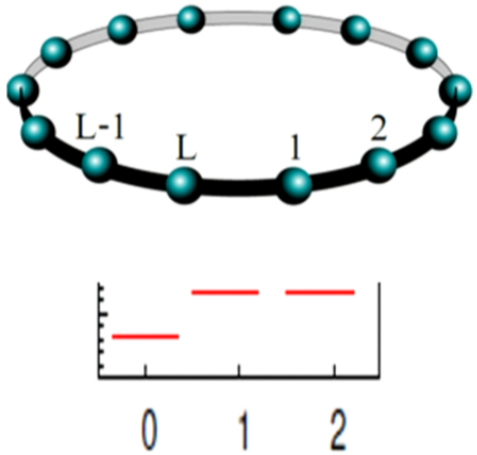

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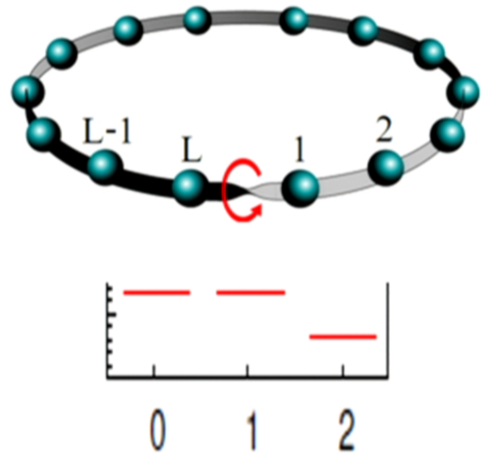

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$$\delta H_{TI}\,(\boldsymbol{\alpha}) = -\omega^{\boldsymbol{\alpha}}(bulk\ term) + h.c.$$

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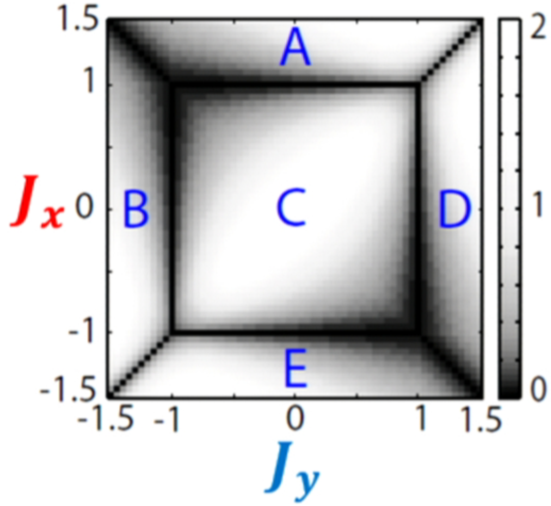
$$H_{XYZ} = \sum_{j=1}^{L} J_x \gamma_{2j-1}\gamma_{2j}\gamma_{2j+1}\gamma_{2j+2} - i J_y \gamma_{2j-1}\gamma_{2j+2} + i\gamma_{2j}\gamma_{2j+1}$$

<u>Open-chain spectral gap</u>



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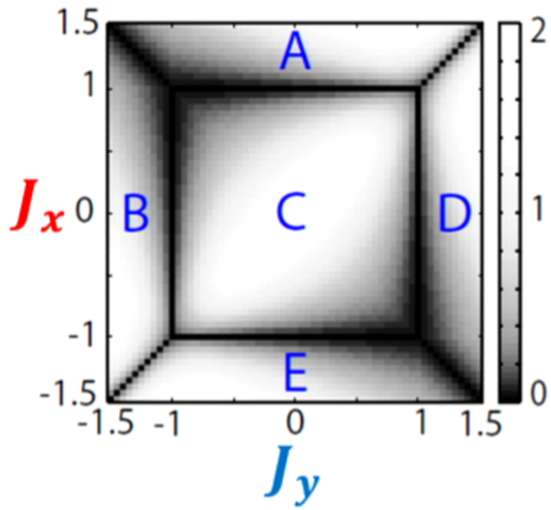

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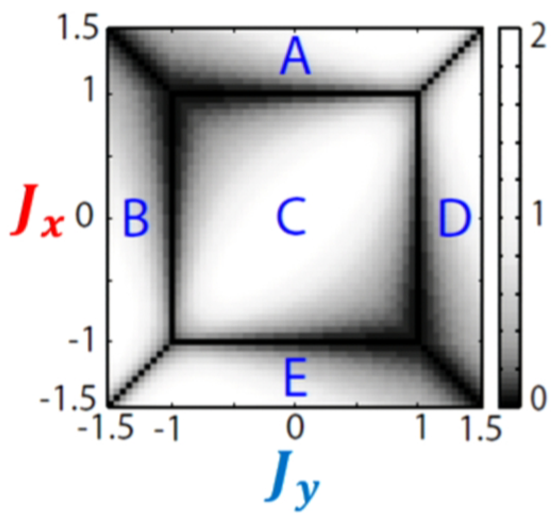

Closed-chain translational-invariant BC

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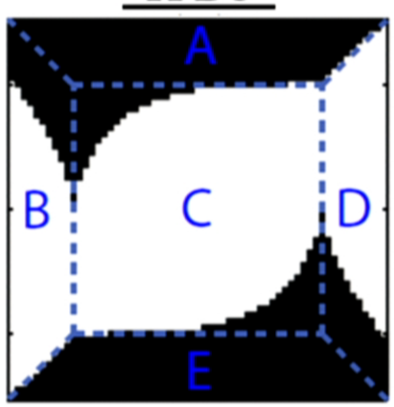
Closed-chain 22-sites
"TI BC"

Criterion for comparison: Sensitivity to boundary conditions



White: groundstate charge changes with twist
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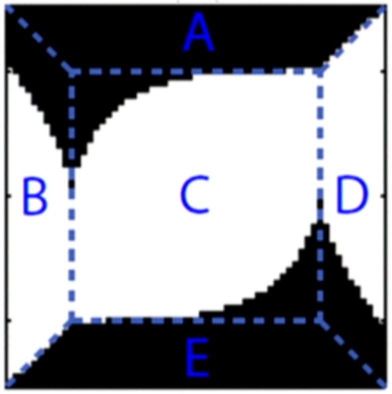

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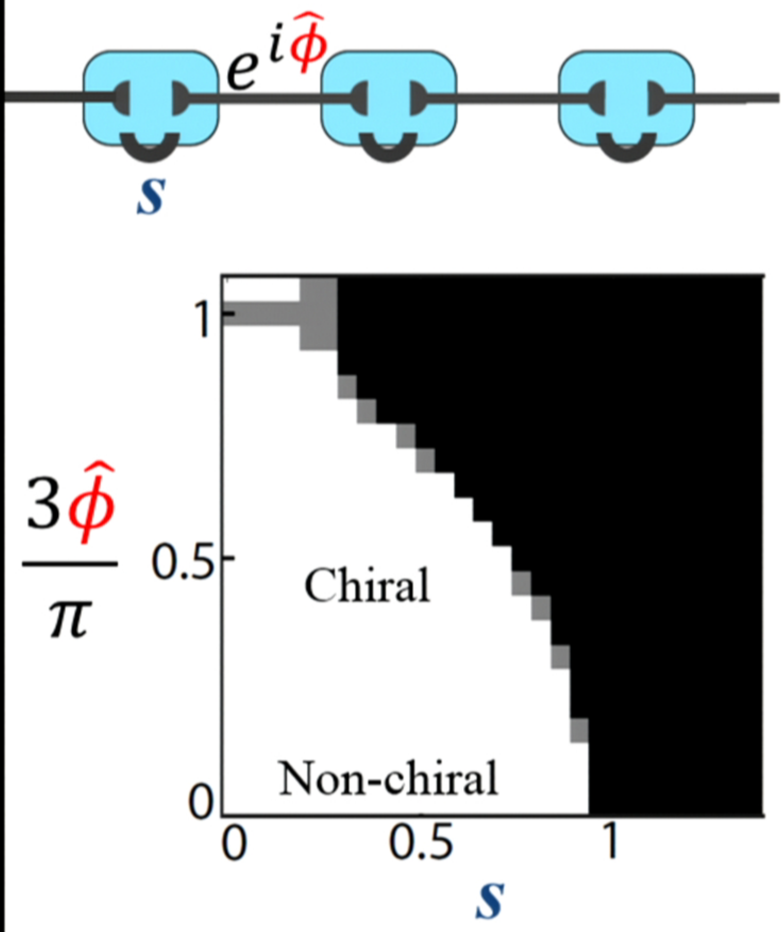
Z_3 phase diagram



White
Groundstate charge changes thrice.
Grey
Groundstate charge changes twice.
Black
Groundstate charge is invariant.

(Nice agreement with entanglement analysis by Ye Zhuang et. al.)

First, we show in the dimerized case that $P_0 O P_0 = c(O) P_0$.

Then apply our technology: $P_s O P_s = U_s P_0 \underbrace{U_s^{-1} O U_s}_{\text{local, neutral}} P_0 U_s^{-1} = c(O') P_0$.

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Local operators have difficulty transforming one polarized state to another.

Expectation of neutral operators cannot distinguish any polarized state.

$$\langle \alpha | \underbrace{O}_{\sigma_j^{n_j} \sigma_{j+1}^{n_{j+1}}} | \alpha \rangle = \langle \alpha + 1 | \, O \, | \alpha + 1 \rangle$$

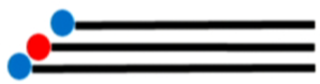
$$\sigma_j^{n_j} \sigma_{j+1}^{n_{j+1}} \to \omega^{\alpha(n_j + n_{j+1})} = 1 \quad \text{since} \quad n_j + n_{j+1} \propto N$$

Refining our notion of locality

Open chain

Any neutral operator that is parafermion-local is also clock-local.

e.g., Z_3

$\gamma_j \gamma_{j+1} \gamma_{j+2} =$  $=$ 

Topological phase

POP=cP applies.

Enlarged groundstate degeneracy.

Duality with conventional order.

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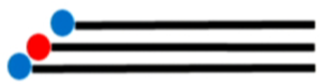
$\gamma_j \gamma_{j+1} \gamma_{j+2} =$  $=$ 

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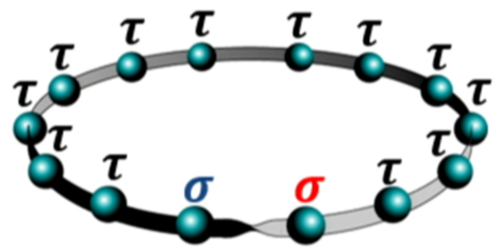

Topological phase

POP=cP does not apply.

Nondegenerate groundstate.

No duality with conventional order.

$Z_4 \to Z_2$ symmetry-broken phase

Parafermion condensate with $\langle GS | \gamma_j^2 | GS \rangle \neq 0$

(Bondesan, Motruk)

Frustration-free model $H_{open} = -\sigma_{j+1}^2 \sigma_j^2 - \tau_j^2$

$|02\rangle = \bigotimes_j \ (|0\rangle_j + |2\rangle_j),$ $\quad |13\rangle = \bigotimes_j \ (|1\rangle_j + |3\rangle_j)$

$\tau_j^2 |02\rangle = |02\rangle$ $\quad \tau_j^2 |13\rangle = |13\rangle$

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Let us show: $\langle 13|O|24\rangle = 0$ even if O is clock-nonlocal.

$$\text{ex. 1: } O = \gamma_1^\dagger \gamma_{2L} = \sigma_1^\dagger \sigma_L Q$$

$$\sigma_1^\dagger \sigma_L Q|24\rangle = \sigma_1^\dagger \sigma_L |13\rangle = \sigma_1^\dagger \otimes_{j=1}^{L-1} (\,|1\rangle_j + |3\rangle_j) \otimes (\,|1\rangle_L - |3\rangle_L).$$

$$\text{ex. 2: } O = \gamma_1^2 \gamma_{2L}^2 \propto \sigma_1^2 \sigma_L^2 Q^2$$

$$\sigma_1^2 \sigma_L^2 Q^2 |24\rangle = \sigma_1^2 \sigma_L^2 |24\rangle = |24\rangle.$$

The general perturbation: $O = O_l \left(\ldots \sigma_{L-1}^{n_{L-1}} \sigma_L^{n_L} \right) Q^{n_L + n_{L-1} + \cdots}$

Summary

<u>Topologically-ordered parafermions</u>

- our edge mode characterizes the most general topological phase on an open chain (e.g., both chiral and nonchiral models)

- Analytic understanding of the trade-off between commutivity and spatial localization (choose your sweet spot)

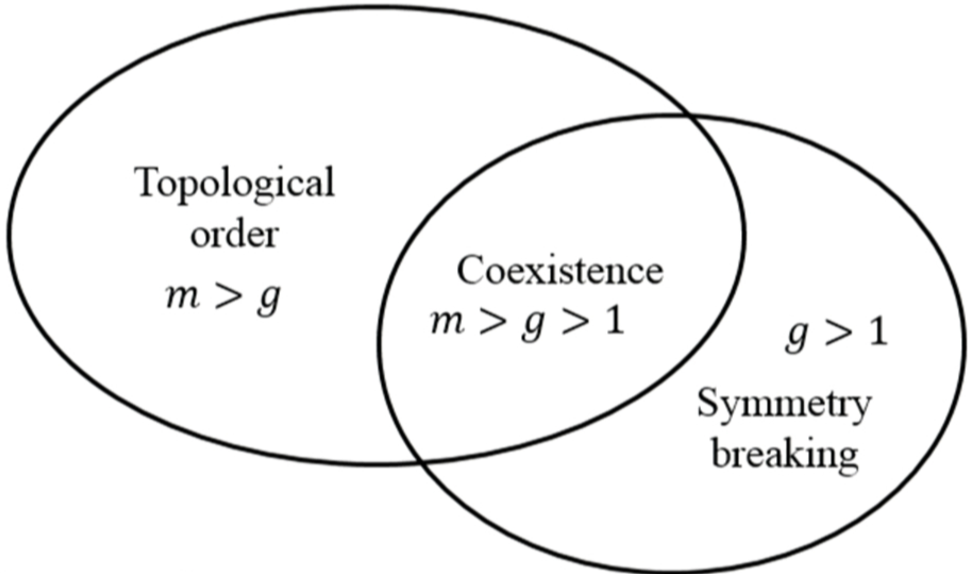
- application: ideal closed-chain Hamiltonians
  - with improved sensitivity to boundary conditions,
  - and are exactly minimized by the open-chain groundstate.

<u>Symmetry-broken parafermions</u>

- proven to be stable under parafermion-local perturbations
  - produces correct closed-chain degeneracy

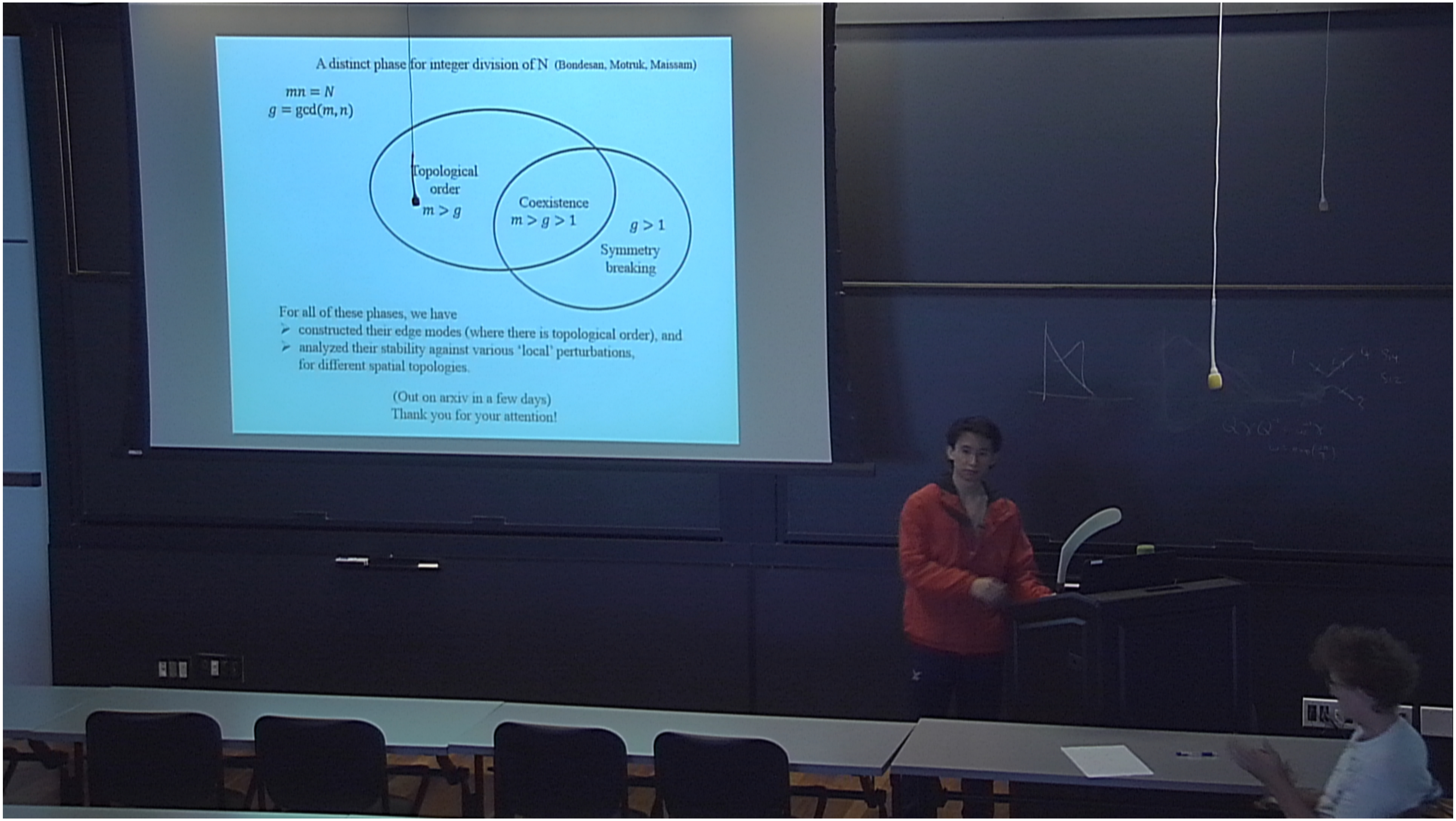
A distinct phase for integer division of N (Bondesan, Motruk, Maissam)

$$mn = N$$
$$g = \mathrm{gcd}(m, n)$$



For all of these phases, we have

- constructed their edge modes (where there is topological order), and
- analyzed their stability against various ‘local’ perturbations,
  for different spatial topologies.


A distinct phase for integer division of N (Bondesan, Motruk, Maissam)
mn = N
g = gcd(m, n)
Topological order
m > g
Coexistence
m > g > 1
g > 1
Symmetry breaking
For all of these phases, we have
constructed their edge modes (where there is topological order), and
analyzed their stability against various 'local' perturbations,
for different spatial topologies.
(Out on arxiv in a few days)
Thank you for your attention!

Introduction to parafermions

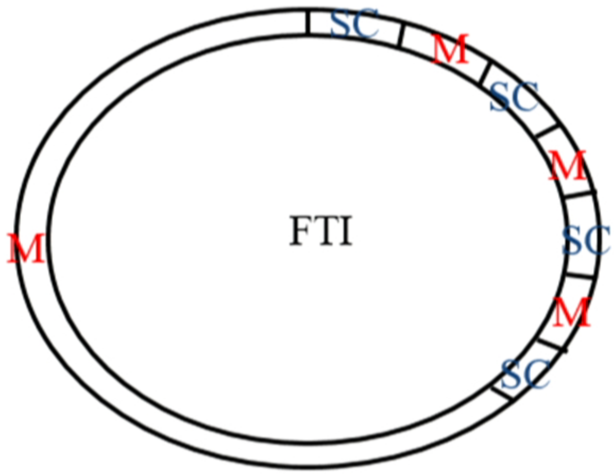
$\mathbb{Z}_N$ Parafermions generalize the Majorana algebra

$$\gamma_j^N = 1 \qquad \gamma_j^\dagger = \gamma_j^{N-1} \qquad \gamma_j \gamma_l = \omega^{\mathrm{sgn}[l-j]} \gamma_l \gamma_j \qquad \omega = e^{i2\pi/N}$$

Simplest parafermion model that is topologically-ordered:

$H_{dimer} =$

$$= -\omega^{(N-1)/2} \; \gamma_{2j+1}^\dagger \; \gamma_{2j} + h.c.$$



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