Title: Symbolic dynamics, modular curves, and Bianchi IX cosmologies

Date: Apr 23, 2015 11:00 AM

URL: http://pirsa.org/15040187

Abstract:

Symbolic Dynamics, Modular Curves, and Bianchi IX Cosmologies

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Bianchi IX Cosmologies

Based on:

- Yuri Manin, Matilde Marcolli, *Symbolic Dynamics, Modular Curves, and Bianchi IX Cosmologies*, arXiv:1504.04005 [gr-qc]

Kasner metrics

- real circle in $\mathbb{R}^3$ defined by equations

$$p_a + p_b + p_c = 1, \qquad p_a^2 + p_b^2 + p_c^2 = 1$$

- each point on this circle defines a metric with Minkowskian (or Euclidean) signature

$$\pm dt^2 + a(t)^2 dx^2 + b(t)^2 dy^2 + c(t)^2 dz^2$$

with *scaling factors* a, b, c:

$$a(t) = t^{p_a}, \qquad b(t) = t^{p_b}, \qquad c(t) = t^{p_c}, t > 0.$$

Kasner metric with exponents (p_a, p_b, p_c).

u-parameterization

- Points (p_a, p_b, p_c) on the circle parameterized by a coordinate $u \in [1, \infty]$

$$p_1^{(u)} := -\frac{u}{1+u+u^2} \quad \in [-1/3, 0]$$

$$p_2^{(u)} := \frac{1+u}{1+u+u^2} \quad \in [0, 2/3]$$

$$p_3^{(u)} := \frac{u(1+u)}{1+u+u^2} \quad \in [2/3, 1]$$

- Rearrange the exponents $p_1^{(u)} \leq p_2^{(u)} \leq p_3^{(u)}$ by a bijection $(1, 2, 3) \to (a, b, c)$ (permutation of the 3 space axes)

Mixmaster Universe (1970s)

V. Belinskii, I.M. Khalatnikov, E.M. Lifshitz, *Oscillatory approach to singular point in Relativistic cosmology*. Adv. Phys. 19 (1970), 525–551.

- Anisotropic cosmologies
- Locally described by a Kasner metric
- Sequence of Kasner metrics (Kasner epochs and cycles)
- Within each epoch one direction dominates expansion, the other two oscillate in a series of Kasner cycles
- At the end of each epoch a *bounce* occurs and a possibly different direction becomes responsible for expansion
- Approach: model the dynamics by a discrete dynamical system that determines epochs and cycles

$\Rightarrow$ continued fraction expansion

- Dynamical system: (partial map)
invertible two sided shift

$$\widetilde{T} : [0,1]^2 \to [0,1]^2 \qquad \widetilde{T} : (x,y) \mapsto \left(\frac{1}{x} - \left[\frac{1}{x}\right], \frac{1}{y + [1/x]} \right)$$

- On $[0,1]^2 \cap (\mathbb{R}^2 \smallsetminus \mathbb{Q}^2)$ uniquely defined $k_s \in \mathbb{N}$

$$x = [0, k_0, k_1, k_2, \dots], \qquad y = [0, k_{-1}, k_{-2}, \dots]$$

$$\frac{1}{x} - \left[\frac{1}{x}\right] = [0, k_1, k_2, \dots], \quad \frac{1}{y + [1/x]} = \frac{1}{k_0 + y} = [0, k_0, k_{-1}, k_{-2}, \dots]$$

- On this subset $\widetilde{T}$ bijective with invariant density

$$d\mu(x,y) = \frac{dx\, dy}{\log 2 \cdot (1 + xy)^2}$$

- encode $(x,y) \in [0,1]^2 \cap (\mathbb{R}^2 \smallsetminus \mathbb{Q}^2)$ with doubly infinite sequence $(k) := [\dots k_{-2}, k_{-1}, k_0, k_1, k_2, \dots]$, $k_i \in \mathbb{N}$ where $T(k)_s = k_{s+1}$ invertible shift

Continued fractions and Mixmaster Universe

- *typical* solutions of Einstein equations Bianchi IX type with $SO(3)$–symmetry oscillates (near the initial singularity) close to a sequence of Kasner type solutions
- local logarithmic time $d\Omega := -\frac{dt}{abc}$
- for $\Omega \cong -\log t \to +\infty$:

- increasing sequence of times $\Omega_0 < \Omega_1 < \cdots < \Omega_n < \ldots$
- sequence of irrational real numbers $u_n \in (1, +\infty)$, $n = 0, 1, 2, \ldots$

- semi–interval $[\Omega_n, \Omega_{n+1})$ is n–th Kasner epoch
- start at time Ω_n with a value $u = u_n > 1$:

$$p_1 = -\frac{u}{1+u+u^2}, \quad p_2 = \frac{1+u}{1+u+u^2}, \quad p_3 = \frac{u(1+u)}{1+u+u^2}$$

- Kasner cycles (within same Kasner epoch) $u = u_n - 1,\ u_n - 2, \ldots$, with corresponding Kasner metrics

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- after $k_n := [u_n]$ cycles inside the same Kasner epoch, a jump to the next epoch with new parameter

$$u_{n+1} = \frac{1}{u_n - [u_n]}$$

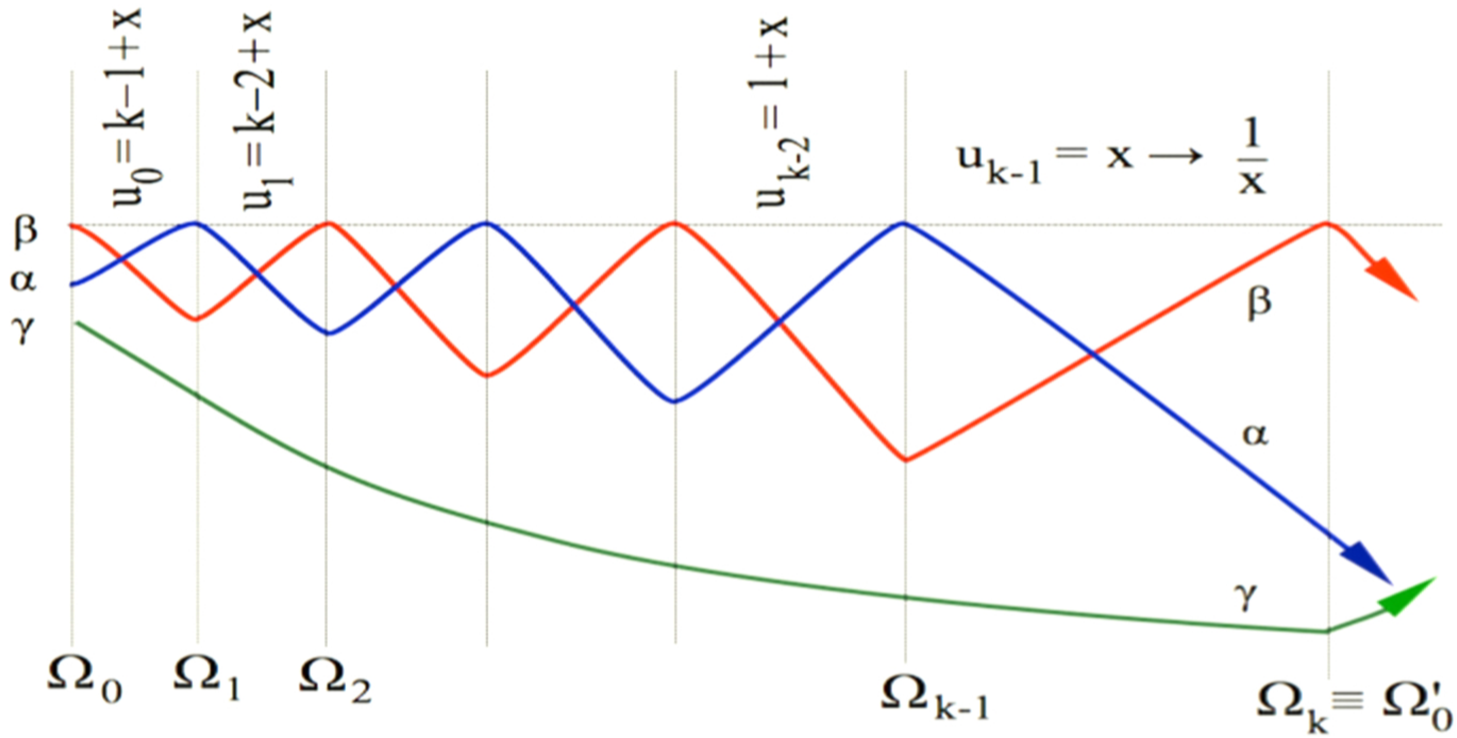
- at the end of each epoch a reshuffling of space axes (also determined by the discrete dynamical system)
- sequence of logarithmic times Ω_n specified by a sequence δ_n

$$\Omega_{n+1} = [1 + \delta_n k_n (u_n + 1/\{u_n\})]\Omega_n$$

- setting $\eta_n = (1 - \delta_n)/\delta_n$ recursion

$$\eta_{n+1} x_n = \frac{1}{k_n + \eta_n x_{n_1}}$$

with $x_n = u_n - k_n$



$u_0 = k-1+x$
$u_1 = k-2+x$
$u_{k-2} = 1+x$
$u_{k-1} = x \longrightarrow \frac{1}{x}$
β
α
γ
β
α
γ
Ω_0 Ω_1 Ω_2
Ω_{k-1}
$\Omega_k \equiv \Omega'_0$

- I. M. Khalatnikov, E. M. Lifshitz, K. M. Khanin, L. N. Shchur, and Ya. G. Sinai. *On the stochasticity in relativistic cosmology.* Journ. Stat. Phys., Vol. 38, Nos. 1/2 (1985), 97–114

- D.H. Mayer, *Relaxation properties of the Mixmaster Universe*, Phys. Lett. A 121 (1987), no. 8,9, 390–394

Conclusion:

- trajectories of mixmaster universe dynamics are parameterized by pairs $(x, y) \in [0,1]^2 \cap (\mathbb{R}^2 \smallsetminus \mathbb{Q}^2)$

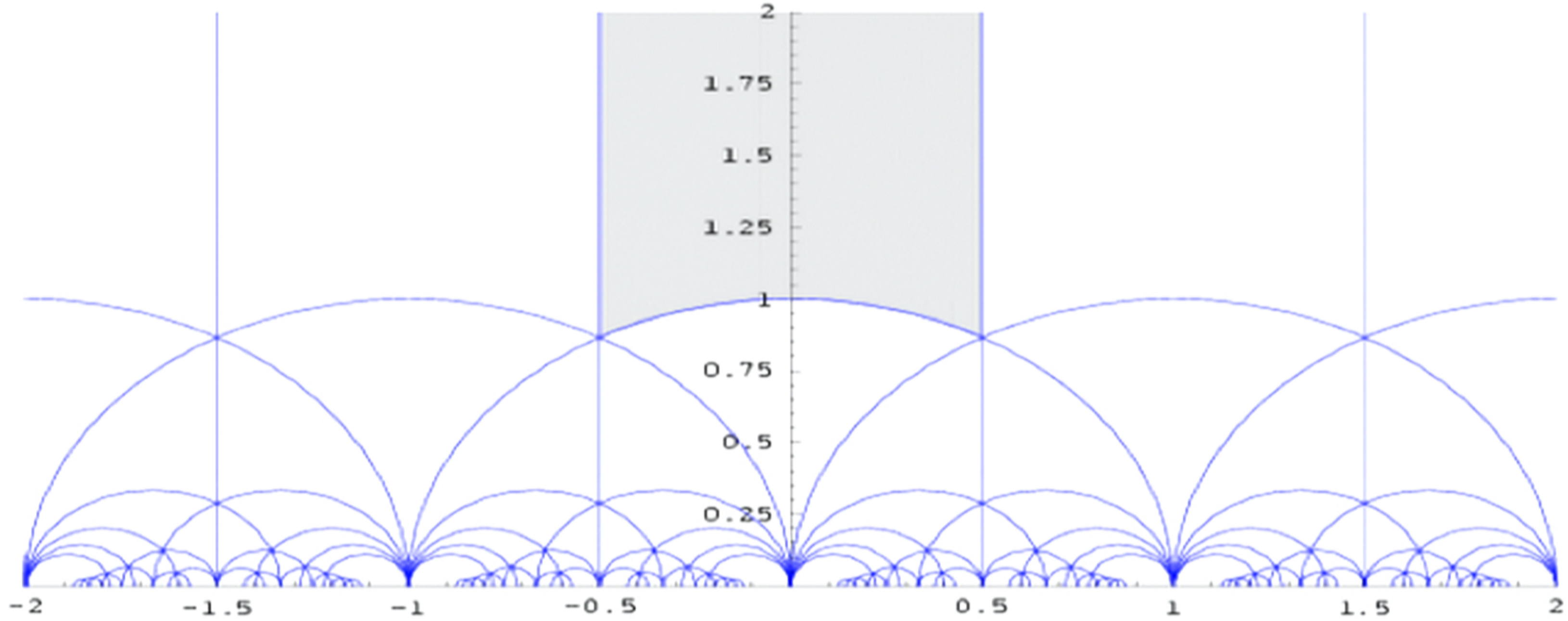
$$x = [0, k_0, k_1, k_2, \ldots], \qquad y = [0, k_{-1}, k_{-2}, \ldots]$$

x specifies number of Kasner cycles in each Kasner epoch, y specifies the Kasner logarithmic times

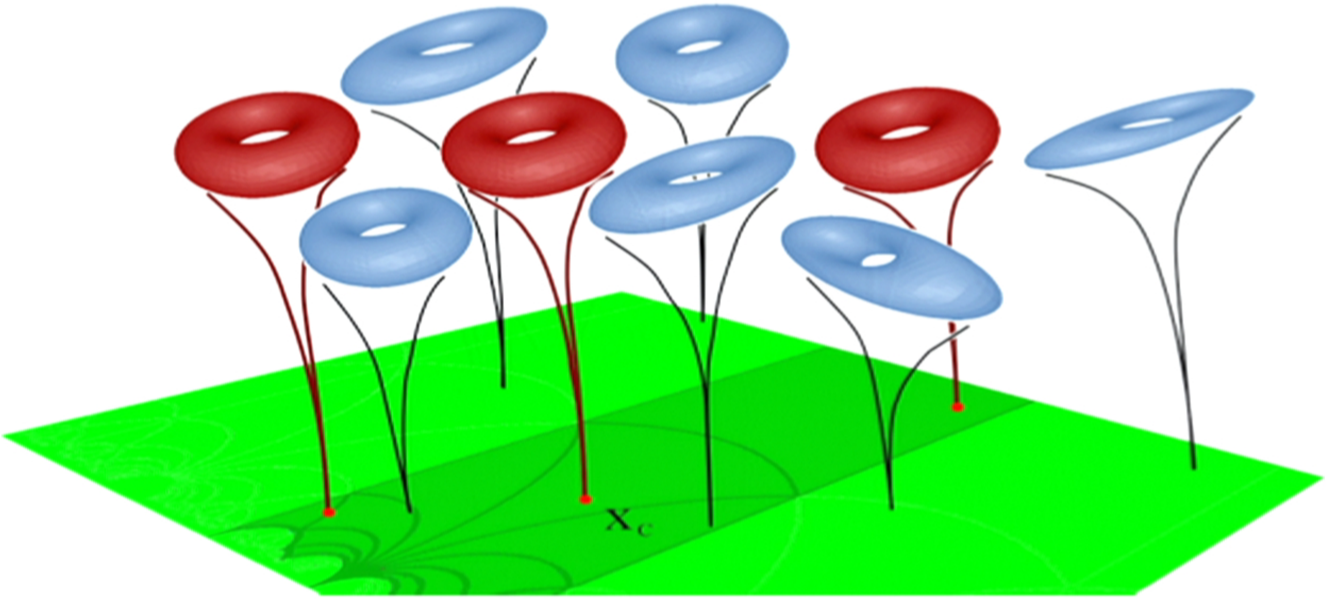
- transition from one Kasner epoch to the next is given by the action of the double sided shift of the continued fraction

$$\widetilde{T} : (x, y) \mapsto \left(\frac{1}{x} - \left[\frac{1}{x} \right], \frac{1}{y + [1/x]} \right)$$

Fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$-action



Elliptic curves and modular curve



(detail from an image by Christian Wuthrich)

Farey tessellation

- adding cusps to the upper half plane: $\overline{\mathbb{H}} := \mathbb{H} \cup \{\mathbb{Q} \cup \{\infty\}\}$
- vertical lines $\Re(z) = n, n \in \mathbb{Z}$, and semicircles in $\overline{\mathbb{H}}$ connecting pairs of finite cusps $(p/q, p'/q')$ with $pq' - p'q = \pm 1$
- these cut $\overline{\mathbb{H}}$ into a union of geodesic ideal triangles: Farey tessellation

- C.Series, *The modular surface and continued fractions*, J. London MS, Vol. 2, no. 31 (1985), 69–80

- coding of geodesics on $\mathcal{M} = \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ using Farey tessellation and continued fraction

- $\mathcal{B}$ set of oriented geodesics β in $\mathbb{H}$ with ideal irrational endpoints $\beta_{-\infty}, \beta_\infty$ in $\mathbb{R}$, such that

$$\beta_{-\infty} \in (-1, 0), \qquad \beta_\infty \in (1, \infty)$$

- continued fraction expansion of endpoints

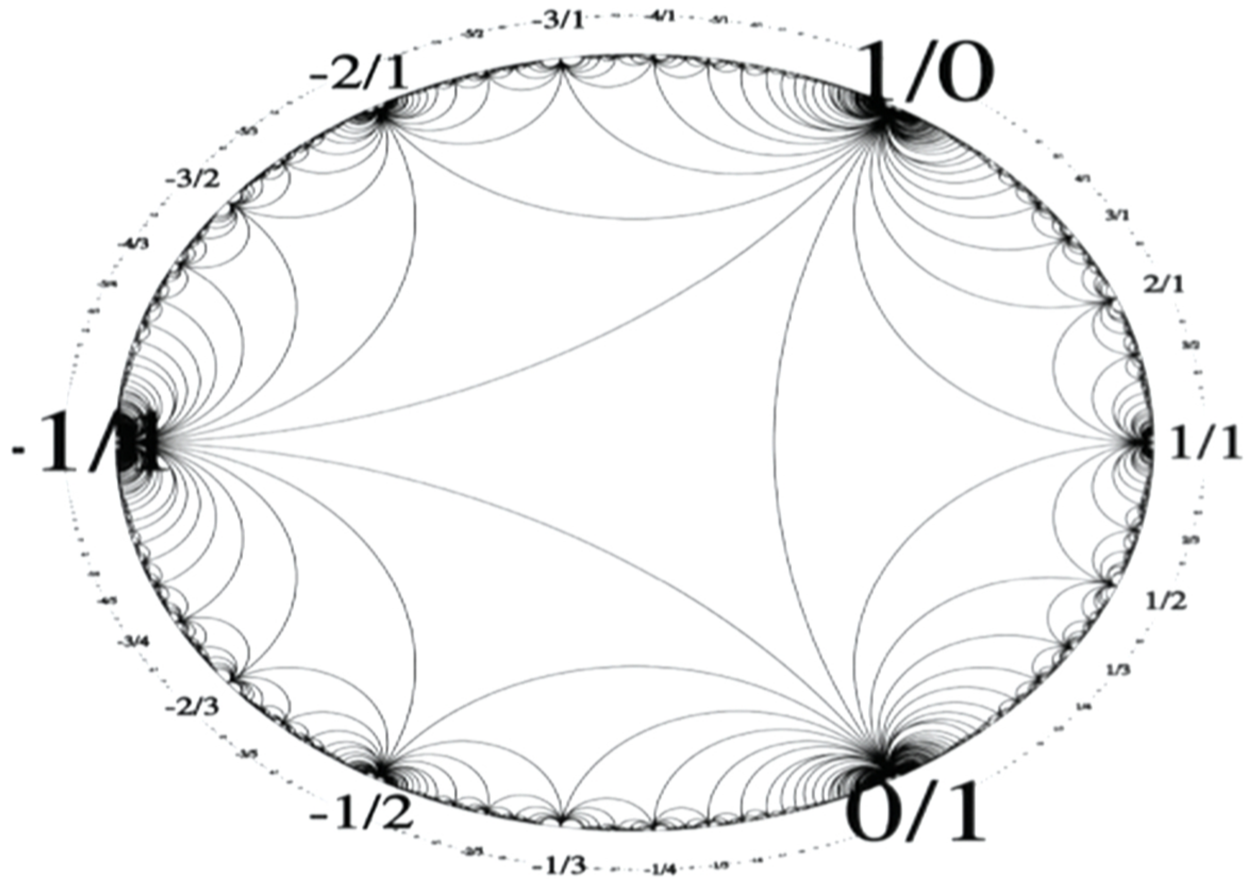
$$\beta_{-\infty} = -[0, k_0, k_{-1}, k_{-2}, \dots], \qquad \beta_\infty = [k_1, k_2, k_3, \dots], \quad k_i \in \mathbb{N},$$

- β determined by endpoints, by doubly infinite sequence of continued fraction digits

$$[\dots k_{-2}, k_{-1}, k_0, k_1, k_2, \dots]$$

- intersection point $x = x(\beta)$ of β with imaginary semiaxis in $\mathbb{H}$
- moving along β: intersect infinite sequence of Farey triangles
- enter each triangle through one side and leave through a different one: the ideal intersection point of these two sides is either to the left or to the right
- infinite sequences in alphabet $\{L, R\}$ (moving in both directions)

$$\dots L^{k_{-3}} R^{k_{-2}} L^{k_{-1}} R^{k_0} \qquad L^{k_1} R^{k_2} L^{k_3} R^{k_4} \dots$$



Farey tessellation in the Poincaré disk model of $\overline{\mathbb{H}}$

• $\mathcal{B}$ set of oriented geodesics β in $\mathbb{H}$ with ideal irrational endpoints $\beta_{-\infty}, \beta_\infty$ in $\mathbb{R}$, such that

$$\beta_{-\infty} \in (-1, 0), \quad \beta_\infty \in (1, \infty)$$

• continued fraction expansion of endpoints

$$\beta_{-\infty} = -[0, k_0, k_{-1}, k_{-2}, \ldots], \quad \beta_\infty = [k_1, k_2, k_3, \ldots], \quad k_i \in \mathbb{N},$$

• β determined by endpoints, by doubly infinite sequence of continued fraction digits

$$[\ldots k_{-2}, k_{-1}, k_0, k_1, k_2, \ldots]$$

• intersection point $x = x(\beta)$ of β with imaginary semiaxis in $\mathbb{H}$

• moving along β: intersect infinite sequence of Farey triangles

• enter each triangle through one side and leave through a different one: the ideal intersection point of these two sides is either to the left or to the right

• infinite sequences in alphabet $\{L, R\}$ (moving in both directions)

$$\ldots L^{k_{-3}} R^{k_{-2}} L^{k_{-1}} R^{k_0} \quad L^{k_1} R^{k_2} L^{k_3} R^{k_4} \ldots$$

• $\mathcal{B}$ set of oriented geodesics β in $\mathbb{H}$ with ideal irrational endpoints $\beta_{-\infty}, \beta_\infty$ in $\mathbb{R}$, such that

$$\beta_{-\infty} \in (-1, 0), \qquad \beta_\infty \in (1, \infty)$$

• continued fraction expansion of endpoints

$$\beta_{-\infty} = -[0, k_0, k_{-1}, k_{-2}, \dots], \qquad \beta_\infty = [k_1, k_2, k_3, \dots], \quad k_i \in \mathbb{N},$$

• β determined by endpoints, by doubly infinite sequence of continued fraction digits

$$[\dots k_{-2}, k_{-1}, k_0, k_1, k_2, \dots]$$

• intersection point $x = x(\beta)$ of β with imaginary semiaxis in $\mathbb{H}$

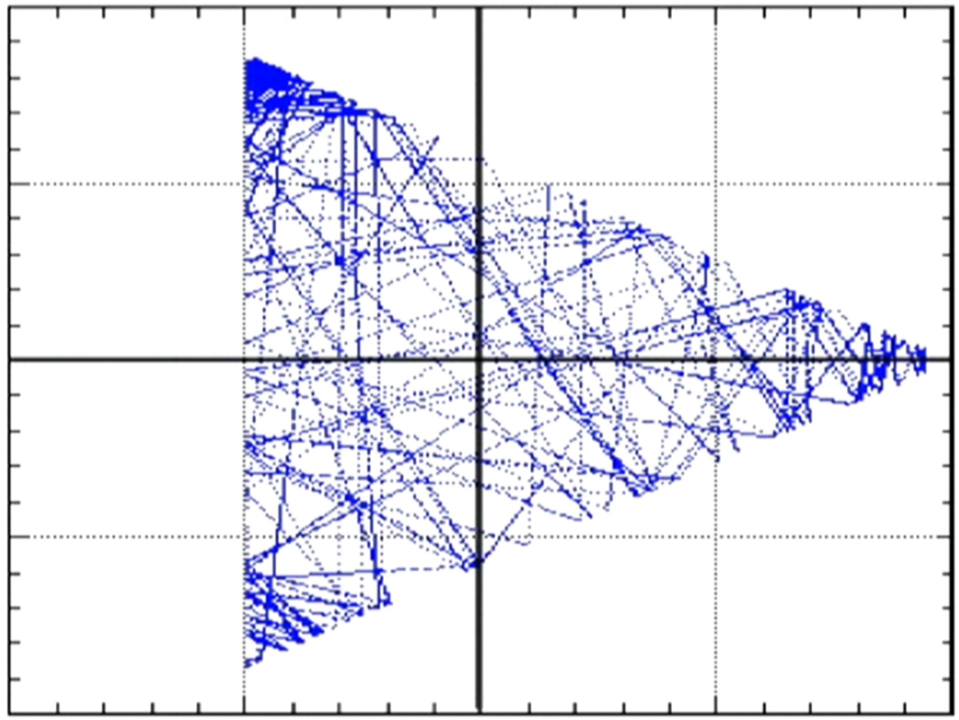
• moving along β: intersect infinite sequence of Farey triangles

• enter each triangle through one side and leave through a different one: the ideal intersection point of these two sides is either to the left or to the right

• infinite sequences in alphabet $\{L, R\}$ (moving in both directions)

$$\dots L^{k_{-3}} R^{k_{-2}} L^{k_{-1}} R^{k_0} \qquad L^{k_1} R^{k_2} L^{k_3} R^{k_4} \dots$$

Other billiard models for Mixmaster dynamics



(from Beverly Berger, "Numerical Approaches to Spacetime Singularities")

Enriched encoding of geodesics and Mixmaster trajectories

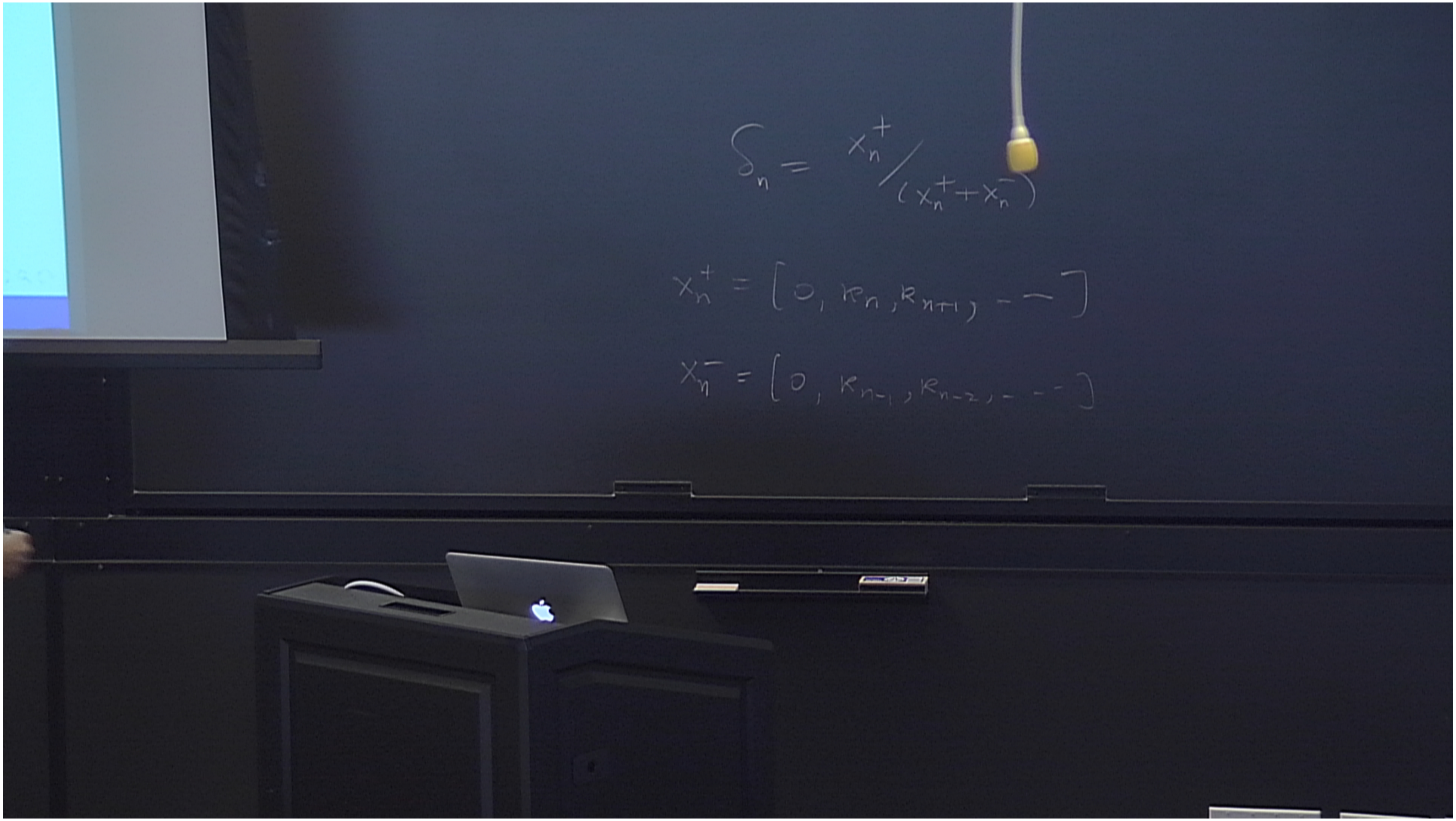
- hyperbolic billiard as above: insert between consecutive powers of L, R the intersection points of β with sides of Farey triangles:

$$\dots L^{k_{-1}} x_{-1} R^{k_0} x_0 L^{k_1} x_1 R^{k_2} x_2 L^{k_3} x_3 R^{k_4} \dots$$

- Result: when $s \to \infty$, $s \in \mathbb{N}$

$$\log \frac{\Omega_{2s}}{\Omega_0} \simeq 2 \sum_{r=0}^{s-1} \mathrm{dist}\,(x_{2r}, x_{2r+1}),$$

dist = hyperbolic distance between consecutive intersection points of the geodesic with sides of the Farey tesselation


$S_n = x_n^+ / (x_n^+ + x_n^-)$
$x_n^+ = [0, k_n, k_{n+1}, \ldots]$
$x_n^- = [0, k_{n-1}, k_{n-2}, \ldots]$

- Sketch of argument: known from mixmaster dynamics that

$$\log \frac{\Omega_{2s}}{\Omega_0} \simeq -\sum_{p=1}^{2s} \log(x_p^+ x_p^-)$$

$$= \sum_{p=1}^{2s} \log([k_{p-1}, k_{p-2}, k_{p-3}, \dots] \cdot [k_p, k_{p+1}, k_{p+2}, \dots])$$

From coding of geodesics also know that

$$\mathrm{dist}(x_0, x_1) = \frac{1}{2} \log([k_0, k_{-1}, k_{-2}, \dots] \cdot [k_1, k_2, \dots] \cdot [k_1, k_0, k_{-1}, \dots] \cdot [k_2, k_3, \dots])$$

and more generally $\mathrm{dist}\,(x_{2r}, x_{2r+1})$ is given by

$$\frac{1}{2} \log([k_{2r}, k_{2r-1}, k_{2r-2}, \dots] \cdot [k_{2r+1}, k_{2r+2}, \dots] \cdot$$

$$\cdot [k_{2r+1}, k_{2r}, k_{2r-1}, \dots] \cdot [k_{2r+2}, k_{2r+3}, \dots])$$

Consequence: identification of distance along geodesic with logarithmic cosmological time

Painlevé VI equations

• *Painlevé transcendents*: solutions of nonlinear second-order ODEs in the plane with *Painlevé property* (the only movable singularities are poles) not solvable in terms of elementary functions; classification in types

• *Painlevé VI*: 4-parameter family $(\alpha, \beta, \gamma, \delta)$

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2$$

$$-\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$

$$+\frac{X(X-1)(X-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right).$$

Painlevé VI and elliptic curves

• Painlevé VI rewritten as (Fuchs)

$$t(1-t)\left[t(1-t)\frac{d^2}{dt^2}+(1-2t)\frac{d}{dt}-\frac{1}{4}\right]\int_{\infty}^{(X,Y)}\frac{dx}{\sqrt{x(x-1)(x-t)}}=$$

$$=\alpha Y+\beta\frac{tY}{X^2}+\gamma\frac{(t-1)Y}{(X-1)^2}+(\delta-\frac{1}{2})\frac{t(t-1)Y}{(X-t)^2}$$

where $(X,Y):=(X(t),Y(t))$ is a section (local and/or multivalued) $P:=(X(t),Y(t))$ of the generic elliptic curve $E=E(t):\ Y^2=X(X-1)(X-t)$

• left-hand-side $\mu(P)$ satisfies $\mu(P+Q)=\mu(P)+\mu(Q)$ for $P+Q$ addition on the elliptic curve E (in particular $\mu(Q)=0$ for points of finite order)

- also have, for $e_i(\tau) = \wp(\frac{T_i}{2}, \tau)$

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau) - e_1(\tau))(\wp(z,\tau) - e_2(\tau))(\wp(z,\tau) - e_3(\tau))$$

so $e_1 + e_2 + e_3 = 0$

- a multivalued solution $z = z(\tau)$ defines a multi–section of the family, which is a covering of $\mathbb{H}$
- is know ramification and monodromy can study behavior over geodesics in $\mathbb{H}$

- Yu.I. Manin, *Sixth Painlevé equation, universal elliptic curve, and mirror of* $\mathbb{P}^2$, in "Geometry of Differential Equations", Amer. Math. Soc. Transl. (2) Vol. 186 (1998) 131–151

• analytic description of the elliptic curve $E_\tau = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, with $\tau \in \mathbb{H}$

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau) - e_1(\tau))(\wp(z,\tau) - e_2(\tau))(\wp(z,\tau) - e_3(\tau))$$

so $e_1 + e_2 + e_3 = 0$

with $(\alpha_0, \ldots, \alpha_3) := (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)$ and $(T_0, T_1, T_2, T_3) := (0, 1, \tau, 1+\tau)$, and

$$\wp(z,\tau) := \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau+n)^2} \right)$$

• also have $e_i(\tau) = \wp(\frac{T_i}{2}, \tau)$

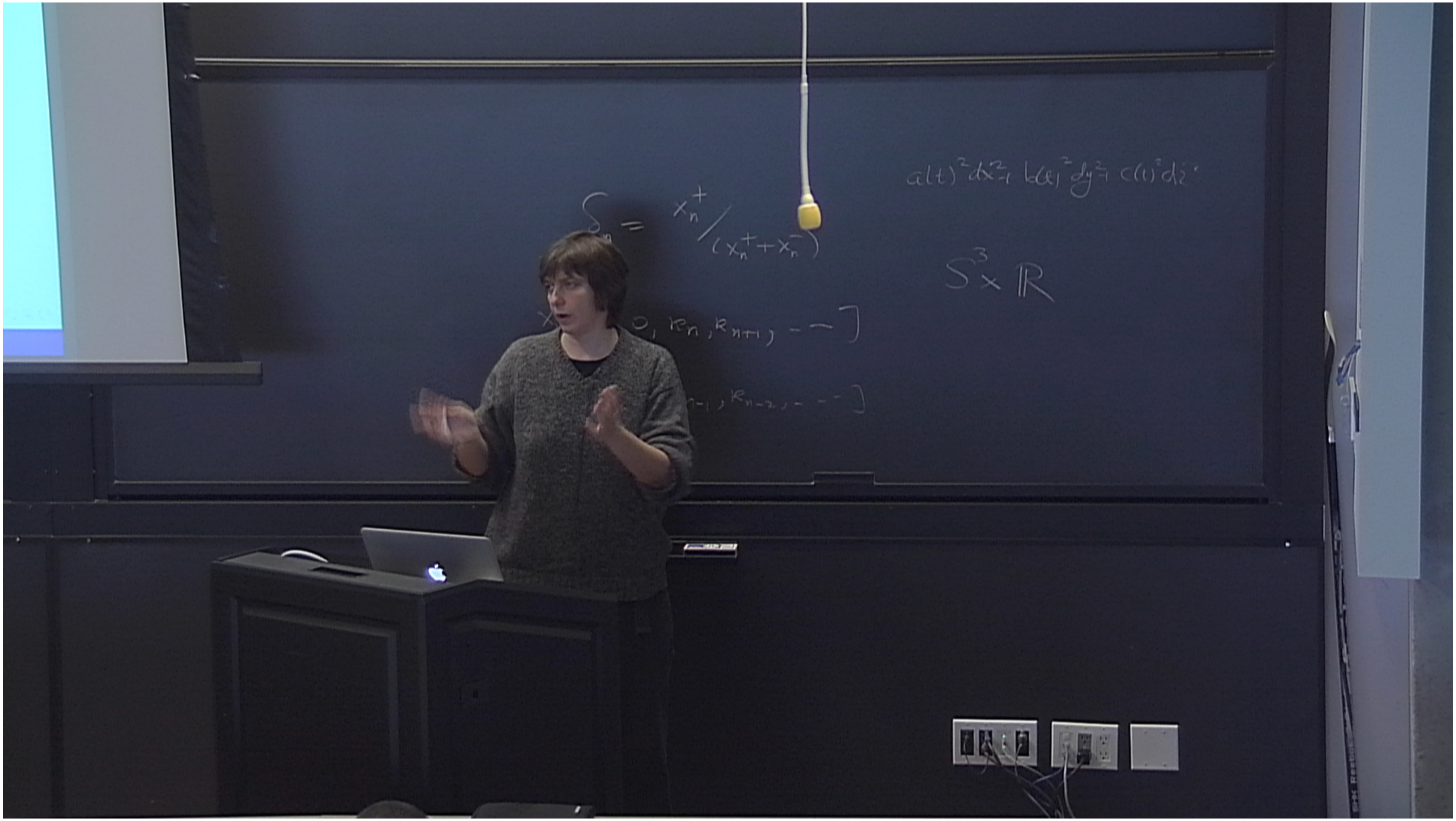
• then Painlevé VI rewritten as (Manin)

$$\frac{d^2 z}{d\tau^2} = \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp_z(z + \frac{T_j}{2}, \tau)$$

• a multivalued solution $z = z(\tau)$ defines a multi-section of the family, which is a covering of $\mathbb{H}$

• its known ramification and monodromy can study behavior over geodesics in $\mathbb{H}$

- Yu.I. Manin, *Sixth Painlevé equation, universal elliptic curve, and mirror of* $\mathbb{P}^2$, in "Geometry of Differential Equations", Amer. Math. Soc. Transl. (2) Vol. 186 (1998) 131–151


$a(t)^2dx^2 + b(t)^2dy^2 + c(t)^2dz^2$
$S^3 \times \mathbb{R}$

- more explicitly

$$\sigma_1 = x_1\,dx_2 - x_2\,dx_1 + x_3\,dx_0 - x_0\,dx_3 = \frac{1}{2}(d\psi + \cos\theta\,d\phi),$$

$$\sigma_2 = x_2\,dx_3 - x_3 dx_2 + x_1\,dx_0 - x_0\,dx_1 = \frac{1}{2}(\sin\psi\,d\theta - \sin\theta\cos\psi\,d\phi),$$

$$\sigma_3 = x_3\,dx_1 - x_1\,dx_3 + x_2\,dx_0 - x_0\,dx_2 = \frac{1}{2}(-\cos\psi\,d\theta - \sin\theta\,\sin\psi\,d\phi),$$

Euler angles $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$ ($SU(2)$ case)

- identifying S^3 with unit quaternions $SU(2)$
- The metrics on S^3

$$\frac{W_2 W_3}{W_1}\,\sigma_1^2 + \frac{W_1 W_3}{W_2}\,\sigma_2^2 + \frac{W_1 W_2}{W_3}\,\sigma_3^2$$

are left-invariants under the action of $SU(2)$ but *not* right-invariant (unlike the round metric on S^3)

• N.J. Hitchin. *Twistor spaces, Einstein metrics and isomonodromic deformations*, J.Diff.Geom., Vol. 42, No. 1 (1995), 30–112.

• K.P. Tod. *Self–dual Einstein metrics from the Painlevé VI equation*, Phys. Lett. A 190 (1994), 221–224.

• S. Okumura. *The self–dual Einstein–Weyl metric and classical solutions of Painlevé VI*, Lett. in Math. Phys., 46 (1998), 219–232.

• M.V. Babich, D.A. Korotkin, *Self–dual* $SU(2)$*–Invariant Einstein Metrics and Modular Dependence of Theta–Functions*. Lett. Math. Phys. 46 (1998), 323–337

Blanchi IX gravitational instantons and Painlevé VI

- Euclidean Bianchi IX metrics with $SU(2)$-symmetry that are
- self-dual (Weyl curvature tensor W self-dual)
- Einstein metrics (Ricci tensor proportional to the metric)

- Self-dual equations for a Riemannian 4-manifold are PDEs; with $SU(2)$-symmetry reduce to ODEs

- This ODE is a Painlevé VI equation with

$$(\alpha, \beta, \gamma, \delta) = (\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$$

Gravitational instantons and theta characteristics

- use notation $\vartheta[p,q] := \vartheta[p,q](0, i\mu)$, and

$$\vartheta_2 := \vartheta[1/2, 0], \qquad \vartheta_3 := \vartheta[0,0], \qquad \vartheta_4 := \vartheta[0, 1/2]$$

- self-dual metrics

$$g = F\,(d\mu^2 + \frac{\sigma_1^2}{W_1^2} + \frac{\sigma_2^2}{W_2^2} + \frac{\sigma_3^2}{W_3^3})$$

with

$$W_1 = -\frac{i}{2}\vartheta_3\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p, q+\frac{1}{2}]}{e^{\pi i p}\vartheta[p,q]}, \quad W_2 = \frac{i}{2}\vartheta_2\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p+\frac{1}{2}, q+\frac{1}{2}]}{e^{\pi i p}\vartheta[p,q]},$$

$$W_3 = -\frac{1}{2}\vartheta_2\vartheta_3 \frac{\frac{\partial}{\partial q}\vartheta[p+\frac{1}{2}, q]}{\vartheta[p,q]},$$

- with non-zero cosmological constant Λ:

$$F = \frac{2}{\pi\Lambda}\frac{W_1 W_2 W_3}{(\frac{\partial}{\partial q}\log\vartheta[p,q])^2}$$

Comments

- Singularities (poles) on the real axis: like Taub-NUT infinity
- Sign changes allowed to get all asymptotics with $W_2 \sim W_3 \neq W_1$ (see Babich, Korotkin)
- instanton analogs of Kasner's solutions with $i\mu \in \Delta \subset \overline{\mathbb{H}}$ in the vicinity of $i\infty$ but not necessarily on the imaginary axis
- behavior $\mu \to \infty$ of these Bianchi IX cosmologies as possible model of (Wick rotated) time at the singularity in algebro-geometric gluing of spacetimes proposed in:

- Yu.I. Manin, M. Marcolli, *Big Bang, blowup, and modular curves: algebraic geometry in cosmology*, SIGMA Symmetry Integrability Geom. Methods Appl. 10 (2014), Paper 073, 20 pp.

Spacetime noncommutativity in the early universe

- noncommutativity hypothesis: near the singularity spacetime coordinates acquire noncommutativity as part of quantum effects
- noncommutative deformation should preserve the metric properties
- Connes–Landi isospectral deformations
- for the 3-sphere S^3 with the round metric: isospectral deformation by making all the tori of the Hopf fibration into noncommutative tori
- do the left-$SU(2)$-invariant Bianchi IX metrics admit similar noncommutative isospectral deformations?
- not always, but yes in the cases that arise as asymptotic behavior at $\mu \to \infty$ of the gravitational instantons

Hopf fibration on S^3

- Hopf coordinates (ξ_1, ξ_2, η)

$$z_1 := x_1 + ix_2 = e^{i(\psi+\phi)} \cos\frac{\theta}{2} = e^{i\xi_1} \cos\eta,$$

$$z_2 := x_3 + ix_0 = e^{i(\psi-\phi)} \sin\frac{\theta}{2} = e^{i\xi_2} \sin\eta.$$

- identifying S^3 with unit quaternions $SU(2)$

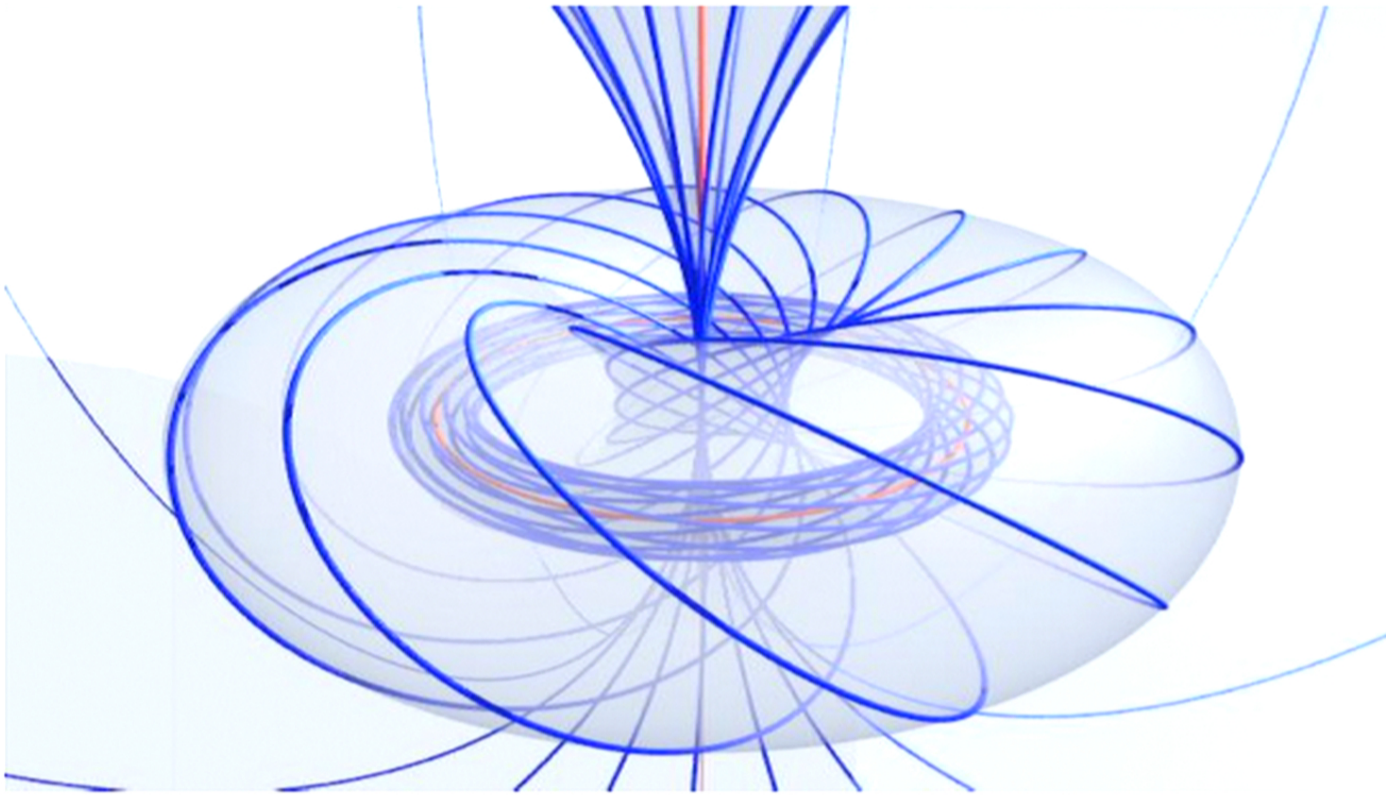
$$q := \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} = \begin{pmatrix} e^{i\xi_1}\cos\eta & e^{i\xi_2}\sin\eta \\ -e^{-i\xi_2}\sin\eta & e^{-i\xi_1}\cos\eta \end{pmatrix}$$

with $|z_1|^2 + |z_2|^2 = 1$ and (ξ_1, ξ_2, η) Hopf coordinates

- Hopf fibration

$$S^1 \hookrightarrow S^3 \to S^2$$

The Hopf fibration of S^3



(image by Benoit Kloeckner)

• Deformed C^*-algebra of functions S^3_θ:
generators $\alpha = U\cos\eta$ and $\beta = V\sin\eta$
relations: $\alpha\beta = e^{2\pi i\theta}\beta\alpha$, $\alpha^*\beta = e^{-2\pi i\theta}\beta\alpha^*$, $\alpha^*\alpha = \alpha\alpha^*$, $\beta^*\beta = \beta\beta^*$ and $\alpha\alpha^* + \beta\beta^* = 1$

• Riemannian geometry in noncommutative setting described by spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, with $\mathcal{A}$ involutive algebra (smooth functions on NC space), $\mathcal{H}$ Hilbert space with representation of $\mathcal{A}$ (spinors), and Dirac operator $\mathcal{D}$

• *Isospectral* deformation X_θ of a manifold X: $\mathcal{A} = C^\infty(X_\theta)$ noncommutative, with $(\mathcal{H}, \mathcal{D}) = (L^2(X, S), \not{D}_X)$ same as for X

• Connes–Landi: if T^2 acts by isometries on X then $\exists\ X_\theta$

• check when have action of T^2 by isometry on the Bianchi IX, compatible with the Hopf fibration of S^3

• in Hopf coordinates T^2 action on S^3

$$(t_1, t_2) : (\xi_1, \xi_2) \mapsto (\xi_1 + t_1, \xi_2 + t_2)$$

Euler angles $(u, v) : (\phi, \psi) \mapsto (\phi + u, \psi + v)$, with $t_1 = (u + v)/2$ and $t_2 = (v - u)/2$

• $U(1)$-action $u : \phi \mapsto \phi + u$ leaves 1-forms σ_i invariant (rotates circles $S^1 \hookrightarrow S^3$ of Hopf fibration)

• the form σ_1 also invariant under other $U(1)$-action $v : \psi \mapsto \psi + v$

$$v^*\sigma_2 = \frac{1}{2}(\sin(\psi + \beta)\, d\theta - \cos(\psi + \beta)\, \sin\theta\, d\phi)$$

$$v^*\sigma_3 = \frac{1}{2}(-\cos(\psi + \beta)\, d\theta - \sin(\psi + \beta)\, \sin\theta\, d\phi),$$

• then $v^*g = g$ for a Bianchi IX metric

$$g = d\mu^2 + a^2\, \sigma_1^2 + b^2\, \sigma_2^2 + c^2\, \sigma_3^2$$

if and only if $b = c$

- This class of Bianchi IX metrics include
  1. Taub-NUT and Eguchi-Hanson gravitational instantons
  2. asymptotic form of the general Bianchui IX gravitational instantons

Dirac operator:

- Berger sphere S^3 with $\lambda^2\sigma_1^2 + \sigma_2^2 + \sigma_3^2$

$$D_B = -i \begin{pmatrix} \frac{1}{\lambda}X_1 & X_2 + iX_3 \\ X_2 - iX_3 & -\frac{1}{\lambda}X_1 \end{pmatrix} + \frac{\lambda^2 + 2}{2\lambda},$$

with $\{X_1, X_2, X_3\}$ basis of the Lie algebra

- on the Bianchi IX (Euclidean) spacetime

$$\mathcal{D} = \frac{1}{W_1^{1/2} W} \left(\gamma^0 \left(\frac{\partial}{\partial\mu} + \frac{1}{2}\left(\frac{\dot{W}}{W} + \frac{1}{2}\frac{\dot{W}_1}{W_1}\right)\right) + W_1 \; D_B|_{\lambda = \frac{W}{W_1}} \right)$$

with $W = W_2 = W_3$

Conclusion: Bianchi IX gravitational instantons are compatible with spacetime noncommutativity (only at $\mu \to \infty$)