Title: Weyl fermions and QFT anomalies in the hydrodynamic regime

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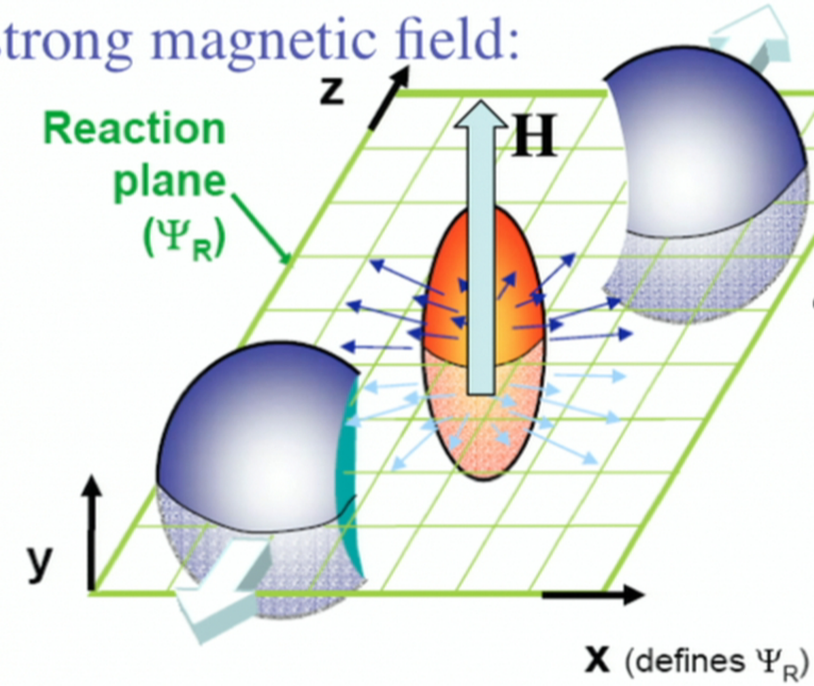
Abstract: <p>I will show how hydrodynamics is modified if the underlying fluid constituents are massless Weyl fermions, which are anomalous at the quantum level. Because of the nondissipative nature of the modification I will construct a partition function which compactly describes the transport properties of the system and I will explain how the anomalous properties can be understood in terms of kinetic theory and heat kernels. Finally I will comment on possible applications to heavy-ion collisions, condensed matter systems such as Weyl-semi metals and potential future questions such as anomalous corrections to entanglement entropy.</p>

MOTIVATION

- Hydrodynamics is a very universal effective field theory used to describe heavy-ion collisions and condensed matter systems
- Kinetic theory with anomalies and applications to condensed matter
- General theory of anomalies at finite temperature (partition functions, generating functionals)
- Inclusion of anomalies of discrete symmetries
- Understanding of Cardy-like properties in higher dimensions
- Applications to Quark-Gluon Plasma and Quantum Hall Effect
- Classical anomalies and applications to soft condensed matter systems

Nucleus-Nucleus Collision

Relativistic ions create a strong magnetic field:



Chiral vortical effect

$$J^\mu = \xi \omega^\mu$$

$$\omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho$$

$$\xi = C_{anom}\left(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\epsilon + p}\right)$$

Chiral magnetic effect

$$J^\mu = \xi_B B^\mu$$

$$B^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} u_\nu F_{\lambda\rho}$$

$$\xi_B = C_{anom}\left(\mu - \frac{1}{2}\frac{n\mu^2}{\epsilon + p}\right)$$

Vorticity

$$\partial_\mu \left(s u^\mu - \frac{\mu}{T} \nu^\mu \right) = -\partial_\mu \left(\frac{\mu}{T} \right) \nu^\mu - \frac{1}{T} \partial_\mu u_\nu \tau^{\mu\nu}$$

The left hand side is then interpreted as the divergence of the entropy current $\partial_\mu J_s^\mu$. When the current is chiral, or when the fundamental theory does not preserve parity, it is possible to construct one additional Lorentz structure that may appear in the current J_μ

$$\omega^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta$$

The new term is consistent with Lorentz symmetry, but its divergence is now:

$$\partial_\mu J_s^\mu = \ldots - \xi(T,\mu) \partial_\mu \left(\frac{\mu}{T} \right) \omega^\mu$$

One has to revisit the entropy current argument

$$J_s^\mu = \ldots + D(T,\mu) \omega^\mu$$

Gravitational anomalies

In the previous calculation we had two integration constants which we cannot constrain by hydrodynamic reasoning. However, we can calculate them using linear response theory in weakly coupled field theory. It turns out these constants emerge as a consequence of gravitational anomalies

$$\xi_M = \lim_{k_n \to 0} \sum_{ij} \epsilon_{ijk} \frac{-i}{2k_n} \left\langle J_M^i T^{0j} \right\rangle \big|_{\omega=0}$$

$$\xi_{MN}^B = \lim_{k_n \to 0} \sum_{ij} \epsilon_{ijk} \frac{-i}{2k_n} \left\langle J_M^i J_N^j \right\rangle \big|_{\omega=0}$$

In the case of vortical conductivity we get T^2 correction

$$\xi_M = \frac{1}{8\pi^2} \sum_{f=1}^{N} T_{M\ f}^{f} \left((\mu^f)^2 + \frac{\pi^2}{3} T^2 \right)$$

Kinetic Theory

Kinetic theory treats the evolution of the one-particle distribution function, which can be associated with the number of on-shell particles per unit phase space

$$f(\vec{p},\vec{x};t) = \frac{dN}{d^3p d^3x}$$

If collisions between particles can be neglected and there is no Berry phase effects, the evolution of $f(\vec{p},\vec{x};t)$ follows from Liouville's theorem

Given this interpretation the particle number density should be proportional to

$$\int d^3p f(\vec{p},\vec{x};t)$$

Summing instead with a weight of particle energy, one expects a result proportional to the product of number density and energy, or energy density, which is a part of the energy-momentum tensor.

HYDRO ⇔ KINETIC THEORY

We can derive hydrodynamic quantities from kinetic theory e.g.

$$T^{\mu\nu} \equiv \int \frac{d^4p}{(2\pi)^3} p^\mu p^\nu \delta(p^\mu p_\mu - m^2) 2\theta(p^0) f(p,x)$$

If we take the distribution function in equilibrium we recover energy-momentum tensor of a perfect fluid. One can derive the correspondence between kinetic theory out of equilibrium and viscous hydrodynamics by considering small departures from equilibrium where

$$f(p^\mu, x^\mu) = f_{\rm eq}\left(\frac{p^\mu u_\mu}{T}\right)\left[1 + \delta f(p^\mu, x^\mu)\right]$$

This procedure allows one to study dissipative effects (first order in the derivatives of fields). Performing the integral one gets perfect fluid contribution plus shear tensor

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + \int \frac{d^4p}{(2\pi)^3} p^\mu p^\nu f_{\rm eq} \delta f = T^{\mu\nu}_{(0)} + \pi^{\mu\nu}$$

ANOMALOUS PART

Solving the Weyl equation we obtain

$$\psi = \int_0^\infty \frac{dE_p}{2\pi}\frac{1}{\sqrt{2E_p}}\left[a_p e^{ip.x} + b_p^\dagger e^{-ip.x}\right]_{p^\mu = E_p\left[u^\mu + \chi_{d=2}\epsilon^{\mu\nu}u_\nu\right]}$$

Populating these states leads to anomalous correction to hydrodynamics

$$T^{\mu\nu} = \sum_{species}\int_0^\infty \frac{dE_p}{2\pi}(f_q + f_{-q})E_p\left[u^\mu + \chi_{d=2}\epsilon^{\mu\alpha}u_\alpha\right]\left[u^\nu + \chi_{d=2}\epsilon^{\nu\lambda}u_\lambda\right]$$
$$= \varepsilon u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu) + q^\mu_{anom}u^\nu + q^\nu_{anom}u^\mu$$

$$J^\mu = \sum_{species}\int_0^\infty \frac{dE_p}{2\pi}(qf_q - qf_{-q})\left[u^\mu + \chi_{d=2}\epsilon^{\mu\alpha}u_\alpha\right]$$
$$= nu^\mu + J^\mu_{anom}$$

← helicity current

$$J^\mu_S = -\sum_{species}\int_0^\infty \frac{dE_p}{2\pi}(\mathcal{H}_q + \mathcal{H}_{-q})\left[u^\mu + \chi_{d=2}\epsilon^{\mu\alpha}u_\alpha\right]$$
$$= su^\mu + J^\mu_{S,anom}$$

GIBBS CURRENT

The above anomalous quantities can be generated from

$$\bar{\mathcal{G}}_{anom} = \sum_F \int_0^\infty \frac{dE_p}{2\pi} g_q \ \chi_{d=2} u$$
$$\bar{J}_{anom} = -\frac{\partial \bar{\mathcal{G}}_{anom}}{\partial \mu} \quad , \quad \bar{J}_{S,anom} = -\frac{\partial \bar{\mathcal{G}}_{anom}}{\partial T}$$
$$\bar{q}_{anom} = \bar{\mathcal{G}}_{anom} + T\bar{J}_{S,anom} + \mu \bar{J}_{anom}$$

where $g_q \equiv -\frac{1}{\beta} \ln \left[1 + e^{-\beta(E_p - q\mu)}\right]$ and we used Hodge duals for simplicity.

We have to evaluate one thermal integral to get

$$\bar{\mathcal{G}}_{anom} = -2\pi \left[\frac{\mu^2}{2!(2\pi)^2} \left(\sum_{species} \chi_{d=2} q^2 \right) + \frac{T^2}{4!} \left(\sum_{species} \chi_{d=2} \right) \right] u$$

Crucial observation : the anomalous contribution is completely proportional to the U(1) anomaly coefficient $\sum_{species} \chi_{d=2} q^2$ and the Lorentz anomaly coefficient $\sum_{species} \chi_{d=2}$

EXAMPLES

The structure of Gibbs functionals in higher dimensions

$$(\bar{\mathcal{G}}_{anom})_{d=2} = -2\pi \sum_{species} \chi_{d=2} \left[\frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2 + \frac{T^2}{4!}\right] u$$

← Cardy entropy formula + first law of thermodynamics

$$(\bar{\mathcal{G}}_{anom})_{d=4} = -2\pi \sum_{species} \chi_{d=4} \left[\frac{1}{3!}\left(\frac{q\mu}{2\pi}\right)^3 + \left(\frac{q\mu}{2\pi}\right)\frac{T^2}{4!}\right] (2\omega) \wedge u$$
$$- 2\pi \sum_{species} \chi_{d=4} \left[\frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2 + \frac{T^2}{4!}\right] (qB) \wedge u$$

$$(\bar{\mathcal{G}}_{anom})_{d=6} = -2\pi \sum_{species} \chi_{d=6} \left[\frac{1}{4!}\left(\frac{q\mu}{2\pi}\right)^4 + \frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2\frac{T^2}{4!} + \frac{7}{8}\frac{T^4}{6!}\right] (2\omega)^2 \wedge u$$
$$- 2\pi \sum_{species} \chi_{d=6} \left[\frac{1}{3!}\left(\frac{q\mu}{2\pi}\right)^3 + \left(\frac{q\mu}{2\pi}\right)\frac{T^2}{4!}\right] (2\omega) \wedge (qB) \wedge u$$
$$- 2\pi \sum_{species} \chi_{d=6} \left[\frac{1}{2!}\left(\frac{q\mu}{2\pi}\right)^2 + \frac{T^2}{4!}\right] \frac{(qB)^2}{2!} \wedge u$$

Berry Phase

Consider a physical system described by a Hamiltonian that depends on time through a set of parameters $R = (R_1, R_2, \dots)$

$$|\Psi_n(t)> = e^{i\gamma_n(t)} e^{-\frac{i}{\hbar}\int_0^t dt' \varepsilon_n(R(t'))} |n(R(t))>$$

Inserting the above expression to the Schroedinger equation and multiplying by bra one finds that the phase factor can be expressed as a path integral in the parameter space

$$\gamma_n = \int_C dR(i < n(R)|\frac{\partial}{\partial R}|n(R)>) \equiv \int_C dR \cdot \mathcal{A}_n(R)$$

where we have defined the Berry connection. We see that in addition to a dynamical phase quantum state will acquire an additional phase during the adiabatic evolution along closed contour.

QUASIPARTICLES

Consider a semiclasical theory of Bloch electrons with a Berry curvature in the presence of weak electromagnetic field. The lagrangian of such a system is given by:

$$\mathcal{L} = \hbar \mathbf{k} \cdot \dot{\mathbf{r}} - \varepsilon_M(\mathbf{k}) + e\phi(\mathbf{r}) - e\dot{\mathbf{r}} \cdot A(\mathbf{r}, t) + \hbar \dot{\mathbf{k}} \cdot \mathcal{A}_n(\mathbf{k})$$

We can derive the EOMs

$$\dot{\mathbf{r}} = \frac{\partial \varepsilon_M(\mathbf{k})}{\hbar \partial \mathbf{k}} - \dot{\mathbf{k}} \times \mathbf{\Omega}(\mathbf{k})$$

$$\hbar \dot{\mathbf{k}} = -e\mathbf{E} - e\dot{\mathbf{r}} \times \mathbf{B}$$

where we have defined the Berry curvature

$$\mathbf{\Omega}^n_{\mu\nu}(R) = \frac{\partial}{\partial R^\mu} \mathcal{A}^n_\nu(R) - \frac{\partial}{\partial R_\nu} \mathcal{A}^n_\mu(R)$$

We see the so-called anomalous Karplus-Luttinger contribution to velocity

DENSITY OF STATES

Liouville's theorem guarantees constant density of states in the classical systems. This is not the case in the presence of the Berry phase and magnetic field. Let us calculate the dynamics of the volume element

$$\frac{1}{\Delta V}\frac{\partial \Delta V}{\partial t} = \nabla_{\mathbf{r}} \cdot \dot{\mathbf{r}} + \nabla_{\mathbf{k}} \cdot \dot{\mathbf{k}}$$

We can solve the above equation

$$\Delta V = \frac{V_0}{(1+(e/\hbar)\mathbf{B}\cdot\mathbf{\Omega})}$$

The fact that the Berry curvature is generally momentum dependent and the magnetic field is position dependent implies that the phase-space volume changes during time evolution. We can introduce a modified density of states

$$D(\mathbf{r},\mathbf{k}) = \frac{1}{(2\pi)^d}(1 + \frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega})$$

such that the number of states in a given volume remains constant in time

PARTITION FUNCTION

We want to have a method to understand hydrodyanmics with various kinds of anomalies. We can calculate the partition functions using heat kernel approach. We need to introduce chiral chemical potential and rotation

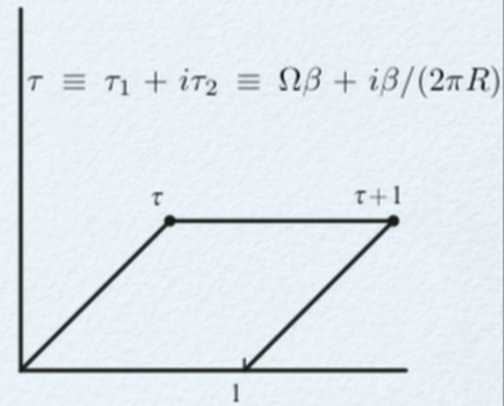
$$Z = \mathrm{Tr}\left[e^{-\beta(H-\Omega_\phi J)}\right]$$

For simplicity we will do that on a manifold which has a compactified spatial direction

$$ds^2 = dr^2 + R^2 d\phi^2 = dt^2 + (2\pi R)^2 dx^2 .$$

The thermal and spatial periodicities are given by

$$(t,x) \sim (t, x+1), \qquad (t,x) \sim (t+\beta, x+\beta\Omega) .$$



HEAT KERNEL EXPANSION (DOES NOT WORK)

Heat kernel technology is valid at one-loop in perturbation theory

$$Z[J] = e^{-\mathcal{L}_{\rm cl}} \det{}^{-\frac{1}{2}}(D) \exp\left(\frac{1}{4} J D^{-1} J\right).$$

Heat kernel is defined as follows

$$K(t; x, y; D) = \langle x | \exp(-tD) | y \rangle.$$

We ignore the classical part of the partition function and focus on the so-called one-loop effective action

$$\ln Z = W = \frac{1}{2} \ln \det(D) = -\frac{1}{2} \int_0^\infty \frac{dt}{t} K(t, D).$$

We used the fact that for each eigenvalue of D we can write down an identity

$$\ln \lambda = -\int_0^\infty \frac{dt}{t} e^{-t\lambda} .$$

2D DIRAC FERMION

We impose antiperiodic boundary conditions both on thermal and spatial circle. We want to find eigenfunctions of

$$i\not{D}\psi_{\vec{n}} = \lambda_{\vec{n}}\psi_{\vec{n}}$$

The solution on a 2D torus is given by

$$\psi^{\pm}_{\vec{n}}(t,\phi) = \frac{1}{\sqrt{\beta}} \exp\left[+2\pi i\left(n_1+\frac{1}{2}\right)\frac{t}{\beta} + 2\pi i\left(n_2+\frac{1}{2}\right)(x - t\Omega)\right] u_{\pm}$$

where

$$u_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the eigenvalues are given by

$$\lambda^+_{\vec{n}} = \frac{(-i)}{R\tau_2}\left[\left(n_2+\frac{1}{2}\right)\bar{\tau} - \left(n_1+\frac{1}{2}\right)\right], \quad \lambda^-_{\vec{n}} = \frac{i}{R\tau_2}\left[\left(n_2+\frac{1}{2}\right)\tau - \left(n_1+\frac{1}{2}\right)\right]$$

Partiction Function of Dirac Fermions

We define the determinant of the Dirac operator to be the squareroot of a laplace-type operator with antiperiodic boundary condition.

$$\det[D] = \sqrt{\prod_{\vec{n}} (\lambda^{+}_{\vec{n}} \lambda^{-}_{\vec{n}})}\,.$$

This quantity has to be regularized and there is various methods developed, such as zeta function regularization.

$$\left[\prod_{n=0}^{\infty} q^{-\frac{1}{2}\left(n+\frac{1}{2}\pm a\right)}\right] = q^{-\frac{1}{2}\zeta(-1,1/2\pm a)} = q^{-1/48+a^2/4}.$$

The result reads

$$W[\tau,\bar{\tau}] \;=\; \frac{1}{2}\log\left| q^{-1/24} \prod_{n=1}^{\infty}\left(1+q^{n-\frac{1}{2}}\right)^2 \right|^2 \quad , \quad q \equiv e^{2\pi i \tau}$$

2D WEYL FERMION

The derimant of the Weyl fermion operator is not well defined, since it mixes left and right chirality modes. Therefore, we need to find a suitable operator that acts on the Dirac space and whose determinant gives the one-loop anomalous effective action

$$i\hat{D} \equiv i\Gamma^{\mu}\left(\partial_{\mu} + A_{\mu}P_{+}\right) = i\left[\not{\partial}_{-} + \not{D}_{+}\right]$$

Since this operator anticommutes with Γ_5 the eigenvalues comes in pairs. Therefore

$$\det[\hat{D}] = \det[(i\not{\partial}_{-})(i\not{D}_{+})]\,.$$

On a torus the solutions can be found exactly. Moreover one can show that chiral coupling can be implemented through the boundary conditions only, thus simplyfying the calculations.

Chiral Coupling to Rotation

We require the chiral coupling to rotations

$$\begin{aligned}
\psi^{\pm}_{\vec{n}}(t,x) &= -\psi^{\pm}_{\vec{n}}(t,x+1) \\
\psi^{-}_{\vec{n}}(t,x) &= -\psi^{-}_{\vec{n}}(t+\beta,x+\beta\Omega) \\
\psi^{+}_{\vec{n}}(t,x) &= -\psi^{+}_{\vec{n}}(t+\beta,x)\,,
\end{aligned}$$

At zero temperature this reproduces Lorentz anomalies. Finite temperature counterpart can be calculated using methods introduced before

$$\lim_{\beta\to 0} W^{\mathrm{chiral}}[\beta,\Omega] = 2\pi i\left(\frac{(1/2)}{24\tau}\right)\,,$$

which, again, is consistent with the Cardy formula.

SUMMARY AND GOALS

- Partition functions are very useful in the analysis of anomalies. Many questions are left unanswered
- Generalization to higher dimensions (elliptic gamma function?)
- Extension to odd dimensions (parity anomalies)
- Anomalies and entanglement entropy
- Manifestations in Weyl-semi metals
- Anomalous swirling vortex solutions